Positivity and higher Teichmüller theory

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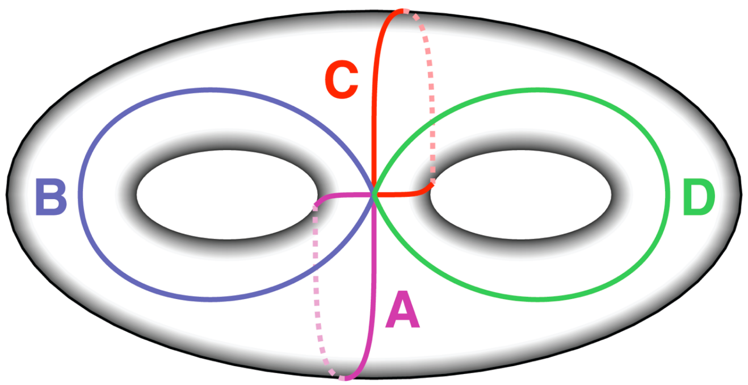
IHES, September 26, 2016

Teichmüller space



S oriented topological surface, genus $g>1$.

$\mathrm{Teich}(S) = \{$ marked conformal structures on S $\}/\sim$

Teichmüller space is an important object in mathematics and physics

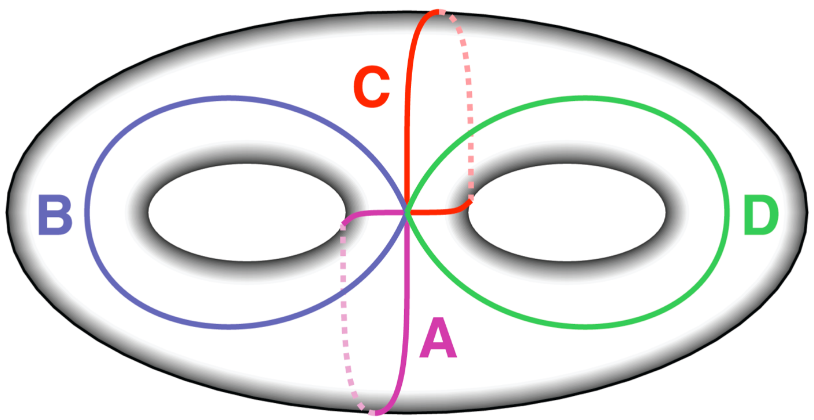
- universal cover of Riemann moduli space
- proper action of mapping class group
- important in study of 3-manifolds

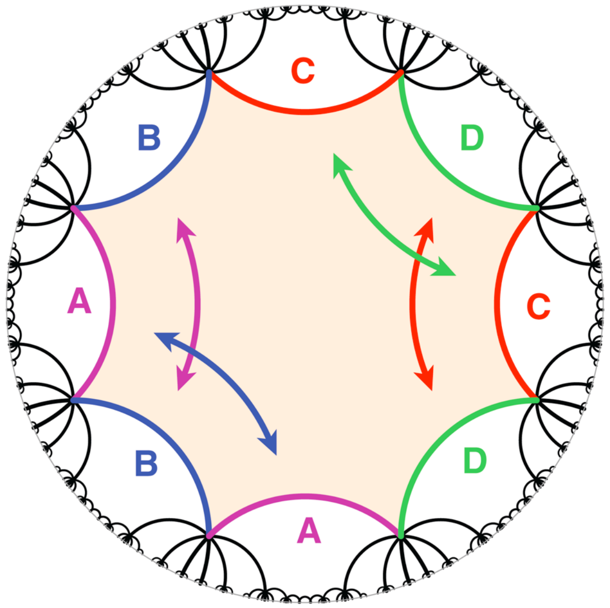
It has many facets, and carries interesting structures itself.
analysis, algebra, dynamics, geometry, number theory, ...

From conformal to hyperbolic

conformal structure $\longrightarrow$ hyperbolic structure

Teich(S) = Hyp(S) = {marked hyperbolic structures on S}/~





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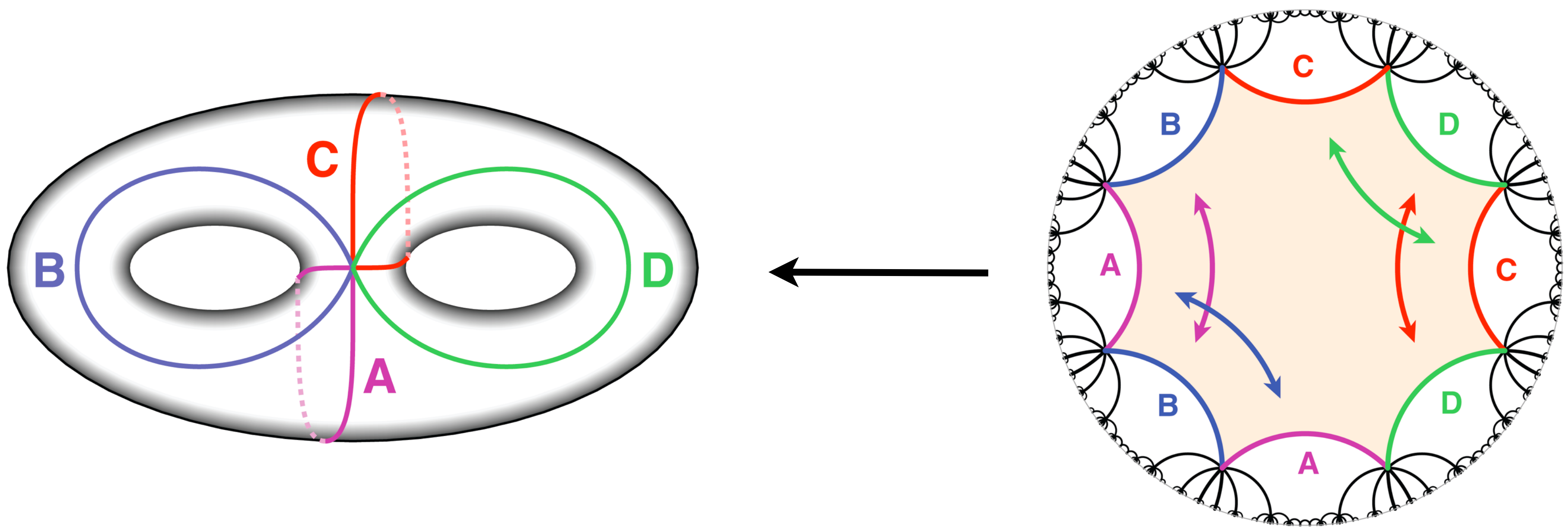

The fundamental group $\pi_1(\mathrm{S})$ acts on the Poincaré disk by isometries

$$hol : \mathrm{Hyp(S)} \longrightarrow \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

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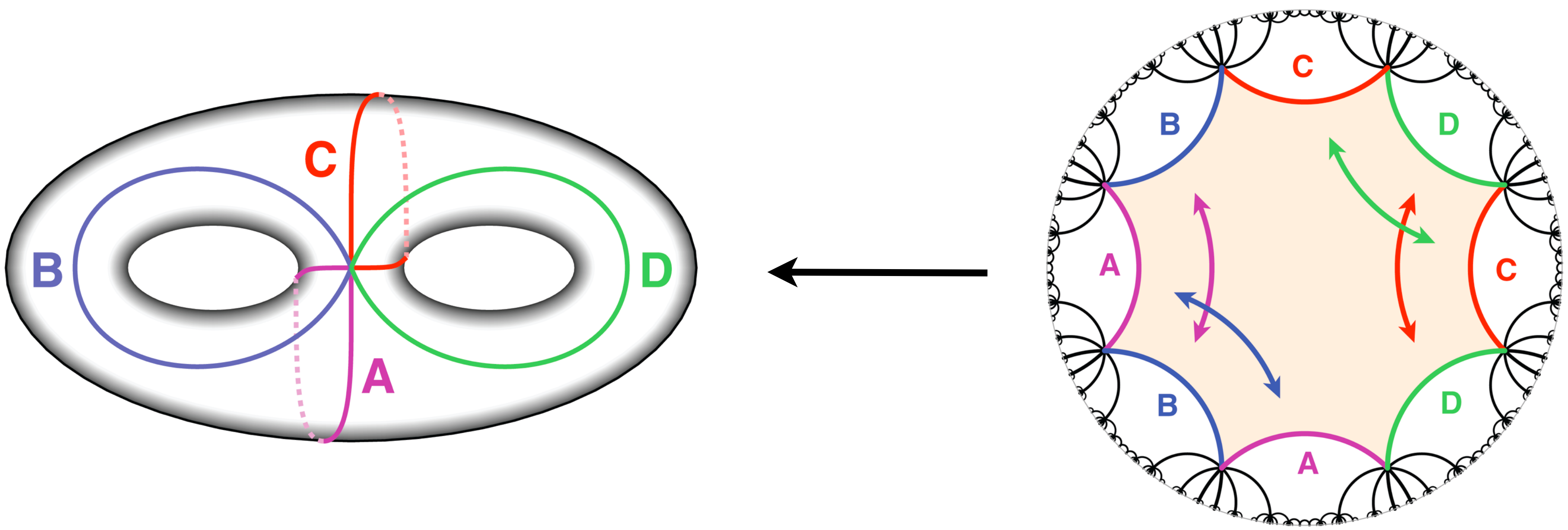

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From conformal to hyperbolic

conformal structure $\longrightarrow$ hyperbolic structure $\longrightarrow$ representations

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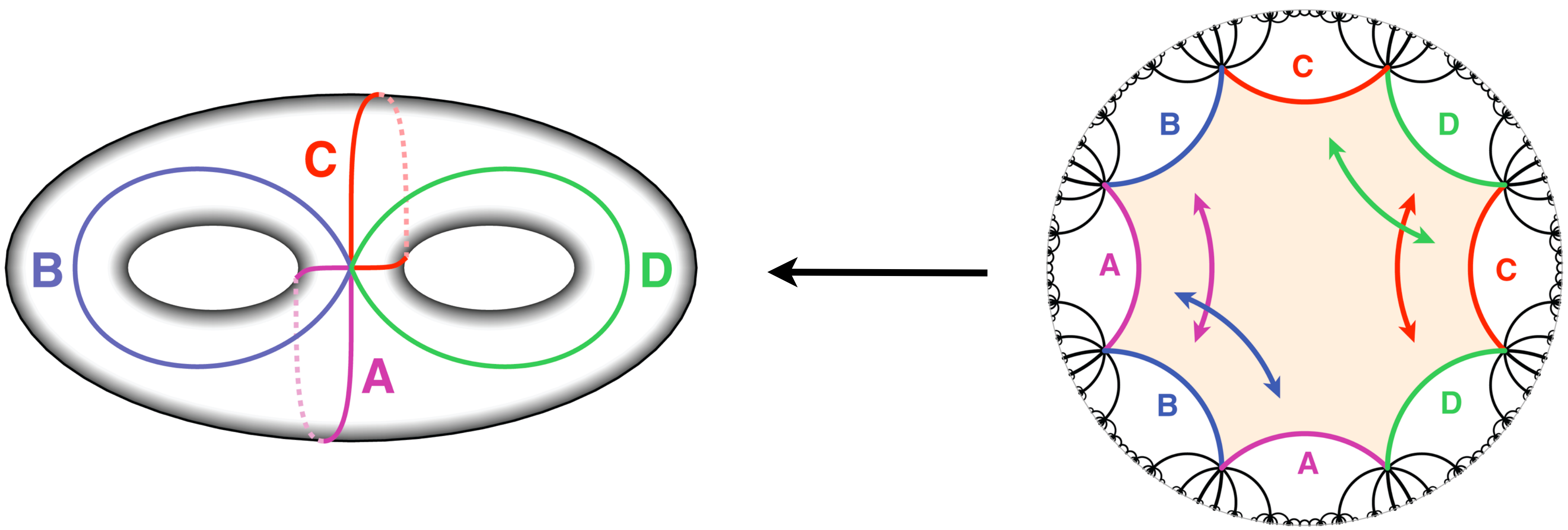

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Higher Teichmüller spaces

Teichmüller space is identified with a subset

$$\mathrm{Teich(S)} \subset \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$$

which is a connected component consisting of discrete and faithful representations.

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$\boxed{\text{Pass from } \mathrm{PSL}(2,\mathbb{R}) \text{ to simple Lie group } \mathrm{G}}$

Higher Teichmüller spaces are subsets

$$\mathcal{T}(\mathrm{S},\mathrm{G}) \subset \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G}$$

which are union of connected components consisting entirely of discrete and faithful representations.

Families of higher Teichmüller spaces

Hitchin components

G is a split real Lie group

$\mathrm{SL}(n,\mathbb{R})$ $\quad$ $\mathrm{Sp}(2n,\mathbb{R})$

$\mathrm{SO}(n,n+1)$ $\quad$ $\mathrm{SO}(n,n)$

Maximal representations

G is a Hermitian Lie group

$\mathrm{Sp}(2n,\mathbb{R})$ $\quad$ $\mathrm{SU}(n,m)$

$\mathrm{SO}(2,n)$ $\quad$ $\mathrm{SO}^*(2n)$

When $G = \mathrm{PSL}(2,\mathbb{R})$ the Hitchin component and the space of maximal representations agree with Teichmüller space.

For other groups they resemble classical Teichmüller space in many ways:

shear coordinates, Fenchel-Nielsen coordinates, cross-ratios, Weil-Petersson metric, collar lemma, geometric structures,…

[Hitchin, Goldman, Labourie, Fock-Goncharov, Burger-Iozzi-W, Bonahon-Dreyer, Labourie-McShane Gaiotto-Moore-Neitzke, Bridgman-Canary-Labourie-Sambarino, Zhang, Lee-Zhang, Guichard-W, Le, Burger-Pozzetti, …]

Hitchin component

Introduced in 1992 by Hitchin via Higgs bundles and Hitchin fibration for split real Lie groups G

$$\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{G}) \subset \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G}$$

It is the connected component containing the principal representation

$$\pi_1(\mathrm{S}) \longrightarrow \mathrm{PSL}(2, \mathbb{R}) \longrightarrow \mathrm{G}$$

The Hitchin component is homeomorphic to a ball. [Hitchin]

It consists entirely of discrete and faithful representations.

[Choi-Goldman, Labourie, Fock-Goncharov]

There is a mapping class group invariant projection (if rank = 2)

$$\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{G}) \longrightarrow \mathrm{Teich}(\mathrm{S})$$ [Labourie]

Maximal representations

Maximal representations are defined for Lie groups G of Hermitian type by a characteristic number

$$\mathrm{T} : \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G} \longrightarrow \mathbb{Z}$$

which satisfies $|\mathrm{T}(\rho)| \leq (2g-2)\mathrm{r_G}$.

$$\mathcal{T}_{max}(\mathrm{S}, \mathrm{G}) = \mathrm{T}^{-1}((2g-2)\mathrm{r_G} \subset \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G}$$

Every maximal representation is discrete and faithful.

[Goldman, Burger-Iozzi-W]

The space of maximal representations consist of several connected components. [Gothen, Bradlow-GarciaPrada-Gothen, Guichard-W]

For $\mathrm{G} = \mathrm{Sp}(4, \mathbb{R})$ some components contain only Zariski-dense representations. [Gothen, Bradlow-GarciaPrada-Gothen, Guichard-W]

For the only group which is split and Hermitian

$$\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{Sp}(2n, \mathbb{R})) \subseteq \mathcal{T}_{max}(\mathrm{S}, \mathrm{Sp}(2n, \mathbb{R}))$$

Why are there higher Teichmüller spaces?

Hitchin components and maximal representations are defined and investigated by very different methods, but we see similar results.

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Why are there higher Teichmüller spaces exactly for split real Lie groups and Lie groups of Hermitian type?

Are there higher Teichmüller spaces for other Lie groups?

What is the common underlying structure?

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What is the common underlying structure?

Common characterization of Hitchin components and maximal representations in terms of positive structures in flag varieties.

The positive reals

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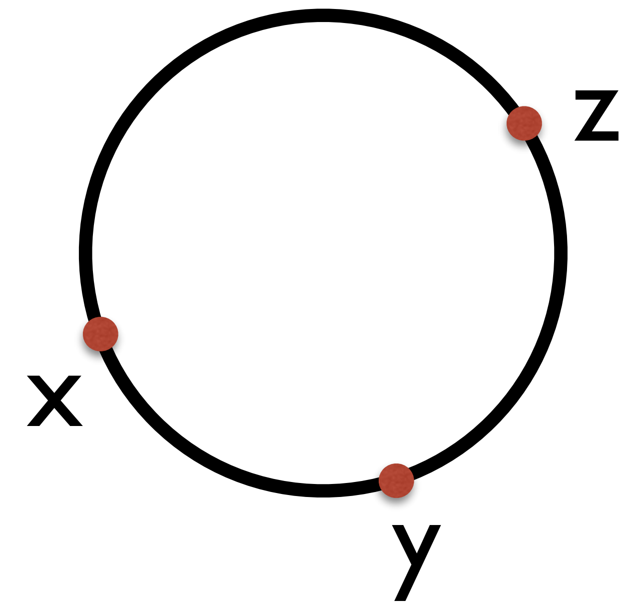

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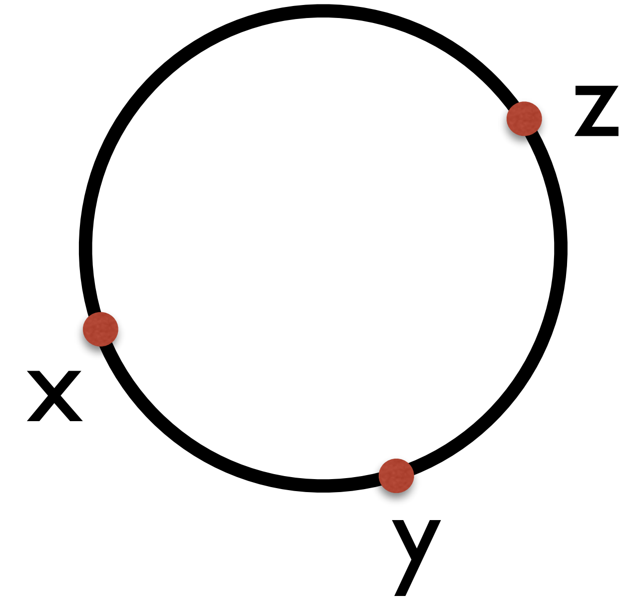

Positive oriented triples on $S^1 = \mathbb{RP}^1$

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot x\,, \;\; t > 0$$

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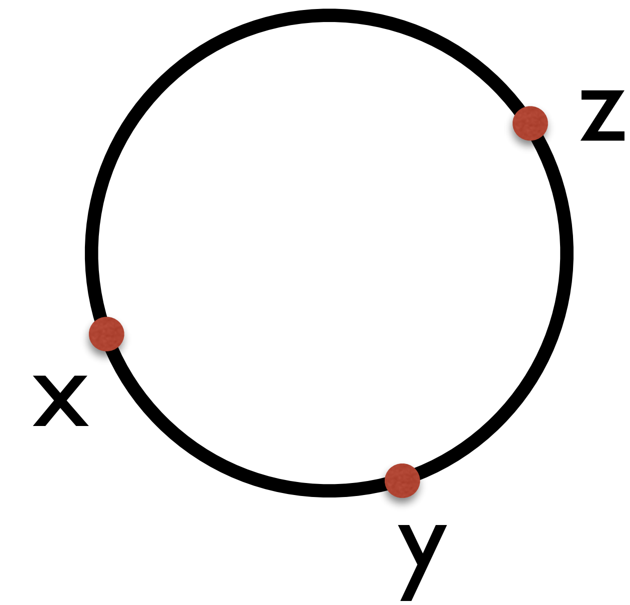

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A representation $\rho : \pi_1(S) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$
is the holonomy of a hyperbolic structure if and only if there is a continuous equivariant map $\phi : \partial\pi_1(S) = S^1 \longrightarrow S^1$
sending positively oriented triples to positively oriented triples.

Total positivity

An invertible matrix is totally positive if every minor is positive.

A totally positive lower triangular matrix is one, where every possibly nonzero minor is positive.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a, b, c, d > 0 \, , \; ad - bc > 0 \qquad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \; t > 0$$

Total positivity has played an important role in stochastic processes, representation theory, combinatorics, cluster algebras, …

Lusztig generalized total positivity to all split real Lie groups.

Recently total positivity has been recast by Fock and Goncharov in a more geometric setting - in the context of higher Teichmüller theory.

Positivity in the flag variety

Flag variety: $\mathcal{F} = \{(F_1, \cdots, F_n) \,|\, F_i \subset F_{i+1}, dim(F_i) = i\}$

F^+, F^- standard flags: $F_i^+ = \langle e_1, \cdots, e_i \rangle$, $F_i^- = \langle e_n, \cdots, e_{n-i+1} \rangle$

Any other flag F transverse to F^- is the image of F^+

by a lower triangular unipotent matrix $F = l_F \cdot F^+$

Triple of flags (F^+, F, F^-) is positive if and only if l_F is totally positive.

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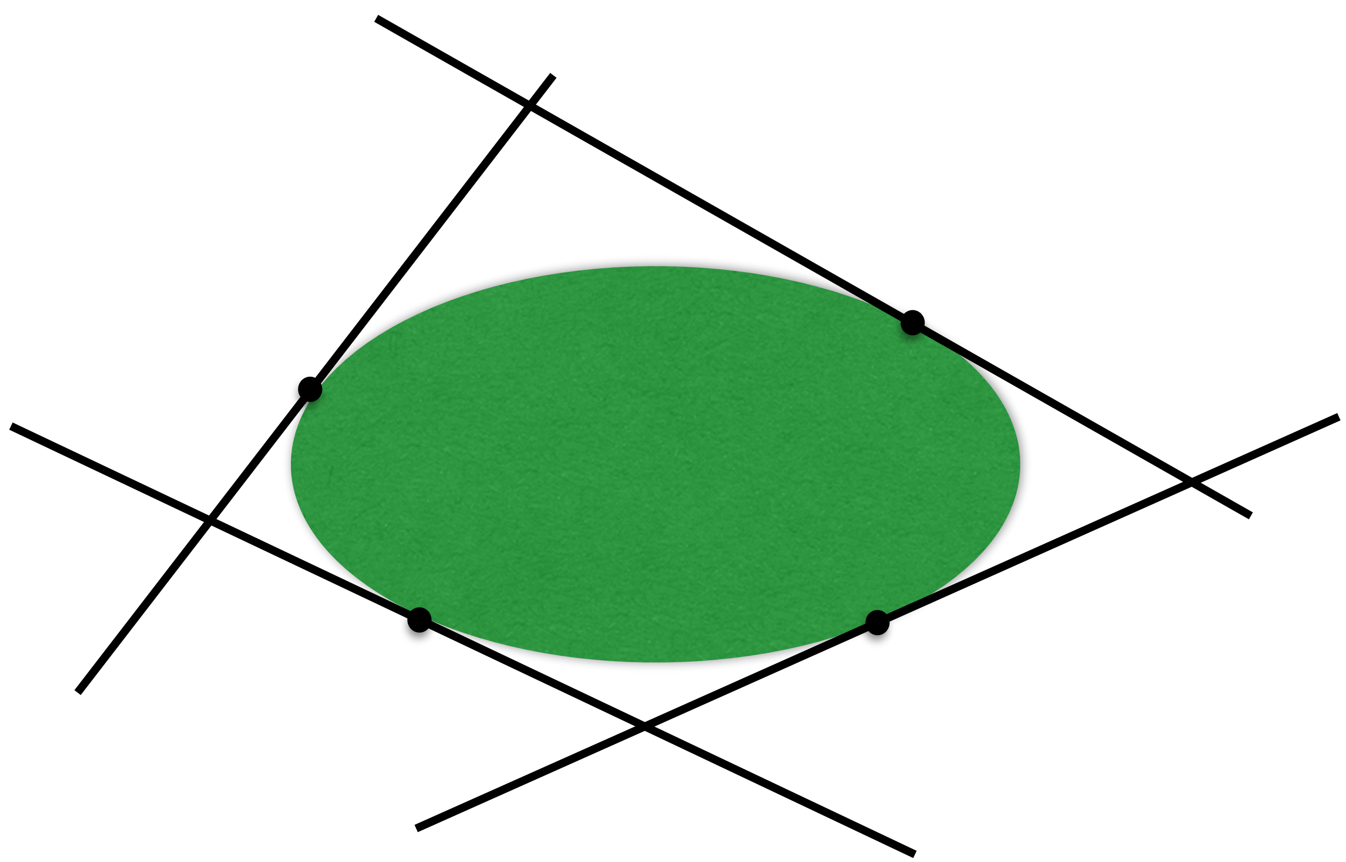

Characterization of Hitchin components

A representation $\rho : \pi_1(S) \longrightarrow G$ lies in the Hitchin component if and only if there exists a continuous equivariant map $\phi : S^1 \longrightarrow \mathcal{F}$ sending positively oriented triples to positive triples of flags.

[Labourie, Guichard, Fock-Goncharov]

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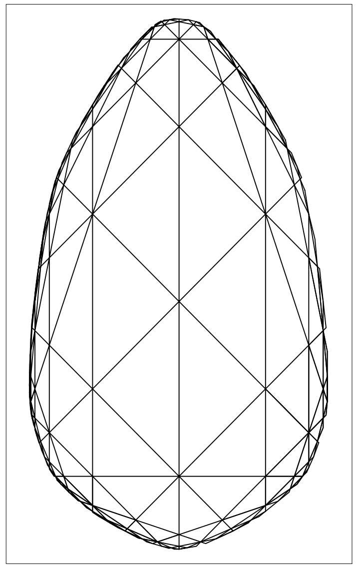
The Hitchin component $\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{PSL}(3, \mathbb{R}))$ parametrizes convex real projective structures on S. [Goldman, Choi-Goldman]

The Hitchin component $\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{PSL}(4, \mathbb{R}))$ parametrizes convex foliated projective structures on $T^1 S$. [Guichard-W]



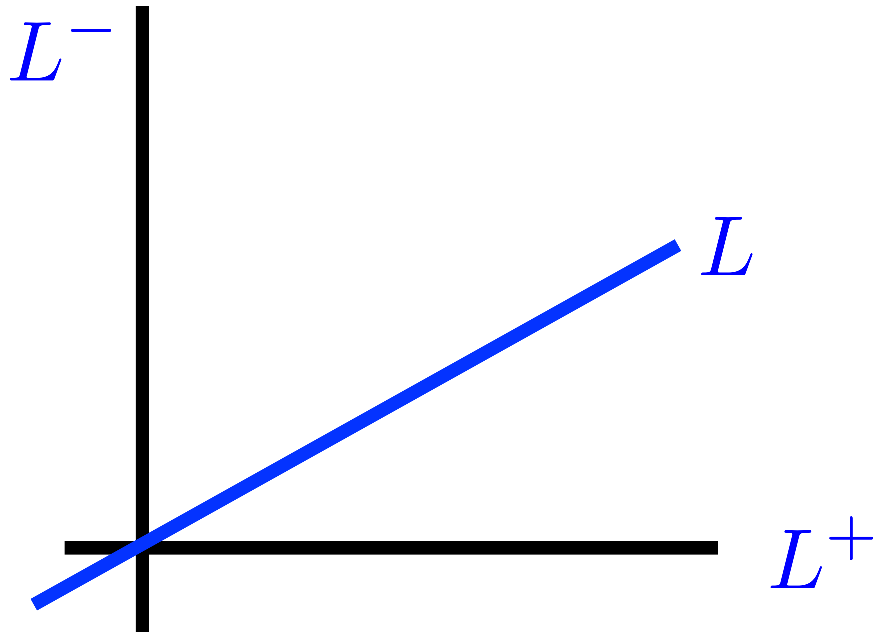
Positivity in the space of Lagrangians

Space of Lagrangians $\mathcal{L} = \{L \subset \mathbb{R}^{2n} \,|\, \dim(L) = n,\, \omega|_{L\times L} = 0\}$

$L^+,\, L^-$ standard Lagrangians: $L^+ = \langle e_1, \cdots, e_n \rangle$, $L^- = \langle f_1, \cdots, f_n \rangle$

They define a quadratic form $\alpha_{L^+,L^-}(v) = \omega(v^+, v^-)$ on $\mathbb{R}^{2n}$.

Triple of Lagrangians (L^+, L, L^-) is positive if and only if this quadratic form is positive definite on L: $\alpha_{L^+,L^-}|_L > 0$



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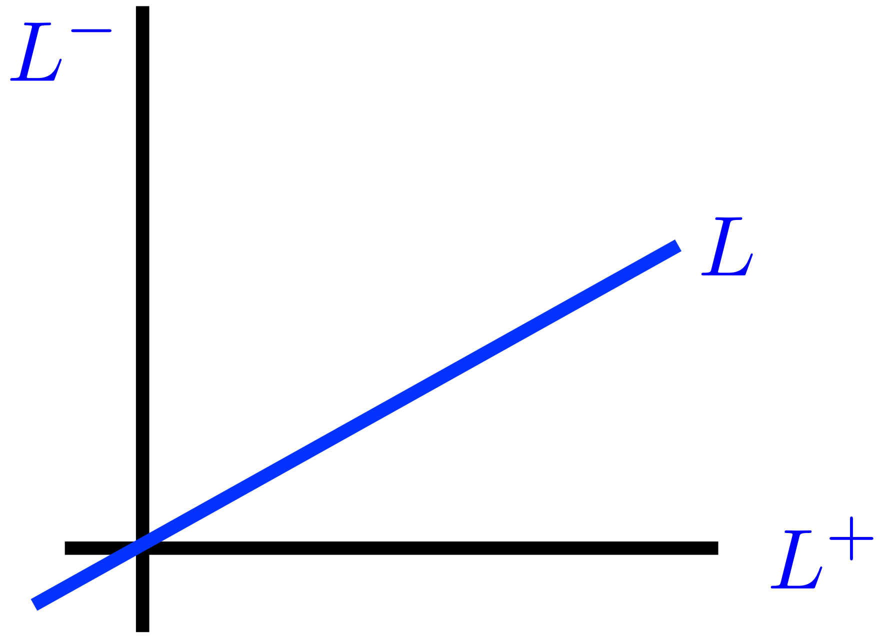

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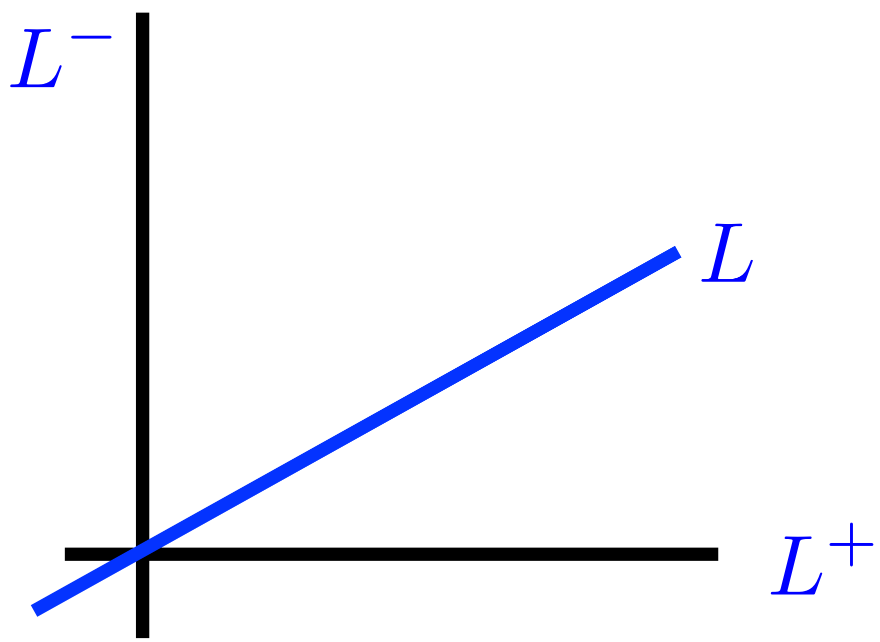

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Equivalently $L = \begin{pmatrix} \mathrm{Id} & 0 \\ M & \mathrm{Id} \end{pmatrix} \cdot L^+$

where M is positive definite.

Characterization of maximality

A representation $\rho : \pi_1(S) \longrightarrow G$ is maximal if and only if there exists a continuous equivariant map $\phi : S^1 \to \mathcal{L}$ sending positively oriented triples to positive triples of Lagrangians.

[Burger-Iozzi-W]

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[Burger-Iozzi-W]

Maximal representations $\mathcal{T}_{max}(\mathrm{S}, \mathrm{Sp}(2n, \mathbb{R}))$ parametrize projective structures on $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundles over S.

This also holds for the Hitchin component $\mathcal{T}_{Hit}(\mathrm{S}, \mathrm{PSL}(2n, \mathbb{R}))$.

[Guichard-W]

Generalizing Lusztig positivity

What is common to Lusztig total positivity and positivity in Hermitian Lie groups?

Observation: In both cases the set of positive triples is a connected component of the intersection of two open Bruhat cells,

$$\Omega_{F^-} \cap \Omega_{F^+}$$

and this connected component has the structure of a semigroup.

We say that a simple Lie group G admits a positive structure in a generalized flag variety $\mathcal{F} = \mathrm{G}/\mathrm{P}$ if for every pair of transverse flags F^+ and F^- there is a connected component of the intersection of two open Bruhat cells $\Omega_{F^-} \cap \Omega_{F^+}$, which has the structure of a semigroup.

[Guichard-W]

This generalizes at the same time Lusztig total positivity and positivity in Hermitian Lie groups.

New positive structures

There are four families of Lie groups which admit a positive structure:

G split real Lie group, and $\mathcal{F} = \mathrm{G}/\mathrm{B}$ full flag variety

G Hermitian Lie group, and $\mathcal{F} = \mathrm{G}/\mathrm{Q}$ Shilov boundary

$\mathrm{G} = \mathrm{SO}(p,q),\ p < q$, and $\mathcal{F} = \mathcal{F}_{1,\cdots,p-1}$ partial flag variety

G with reduced root system F_4, and $\mathcal{F} = \mathcal{F}_{\alpha_1,\alpha_2}$ partial flag variety

[Guichard-W]

The positive structure on $\mathcal{F}$ leads to positive semigroup in G, and arises from a semigroup in the unipotent radical of P, where $\mathcal{F} = \mathrm{G}/\mathrm{P}$.

There is an explicit parametrization of the positive semigroup in the unipotent radical of P, given a (special) reduced expression of the longest element in the Weyl group.

Every element in the positive semigroup in G acts proximally on $\mathcal{F}$.

[Guichard-W]

The case of SO(p,q)

Let Q be a quadratic form of signature (3,q)

$$Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ -K & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -Id_{q-3} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Partial flag variety $\mathcal{F}_{1,2} = \left\{ V_1 \subset V_2 \,|\, \dim(V_i) = i, Q|_{V_2 \times V_2} = 0 \right\}$

The unipotent $U_\Theta = \{U(x,v,a,w) \,|\, a, x \in \mathbb{R}, v, w \in \mathbb{R}^{q-1}\}$

$$U(x,v,a,w) = \begin{pmatrix} 1 & x & w + x\frac{v}{2} & a & au - q_J(w + \frac{v}{2}) \\ 0 & 1 & v & q_J(v) & a - 2b_J(v,w) \\ 0 & 0 & Id_{q-1} & J^t v & -J^t w + xJ^t\frac{v}{2} \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The positive unipotent

Parametrization of totally positive unipotent matrices in SL(3,R)

$$u_1(a)u_2(b)u_1(c) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Parametrization of the positive unipotent semigroup in U_Θ

$$x_{\alpha_1}(x) = \begin{pmatrix} 1 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & Id_{q-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad x_{\alpha_2}(v) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & v & q_J(v) & 0 \\ 0 & 0 & Id_{q-1} & J^t v & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{U_\Theta^{>0} = \{x_{\alpha_1}(x_1)x_{\alpha_2}(v_1)x_{\alpha_1}(x_2)x_{\alpha_2}(v_2)\}}$$

$$x_1, x_2 \in \mathbb{R}^+ \text{ and } v_1, v_2 \in \{v \in \mathbb{R}^{q-1} \,|\, q_J(v) > 0\,,\, \mathrm{sign}(v^{(1)}) > 0\}$$

Positive representations

Assume G admits a positive structure on $\mathcal{F}$.

A representation $\rho : \pi_1(\mathrm{S}) \longrightarrow \mathrm{G}$ is positive if there exists an equivariant continuous map $\phi : S^1 \longrightarrow \mathcal{F}$ sending positively oriented triples to positive triples of generalized flags.

Theorem: [Guichard-Labourie-W]

Every positive representations is discrete and faithful (Anosov).
The set of positive representations is open.

Conjecture: [Guichard-Labourie-W]

The set of positive representations is closed.

Examples: irreducible representation in SO(p,p+1) < SO(p,q)
irreducible representation in SO(p,p-1) < SO(p,q)

Predictions

There are two other families of higher Teichmüller spaces:

$$\mathcal{T}_{pos}(\pi_1(\mathrm{S}), \mathrm{G})$$

for $\mathrm{G} = \mathrm{SO}(p, q),\ p < q$ and G special real form of $\mathrm{F}_4,\ \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8$.

The variety of representations

$$\mathrm{Hom}(\pi_1(S), G)/G$$

contains additional connected components coming from positive representations.

This prediction has been partially confirmed in the case of SO(p,q) using the theory of Higgs bundles. [Collier, Gothen et. al.]

Is there a non-commutative version of cluster algebras associated to these new positive structure?