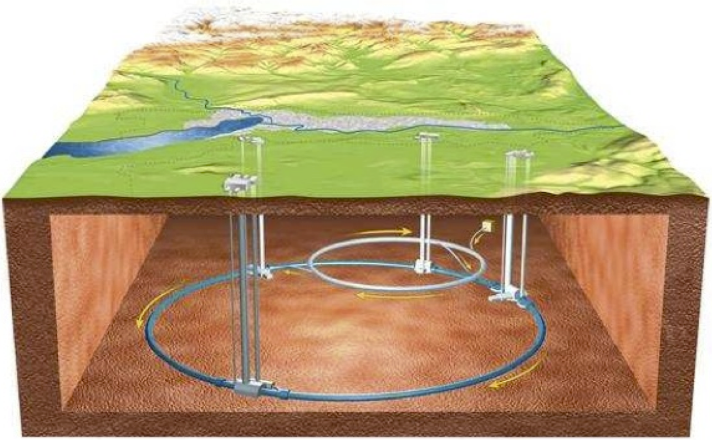
Modular forms in high-energy physics

Francis Brown
All Souls College, Oxford

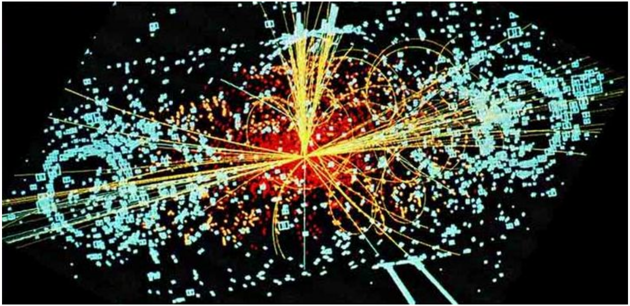
Abel Symposium
27th May 2016

I. Particle physics

LHC



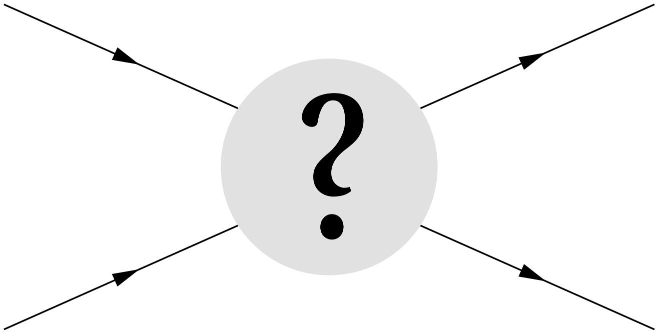
Collision of beam particles



Test the laws of physics by analysing particle tracks.

Perturbative Quantum Field theory

General framework describing fundamental forces and particles.



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Feynman *amplitude* is a complex probability assigned to G.

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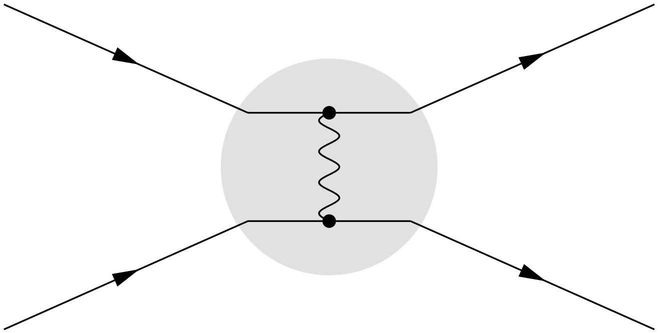

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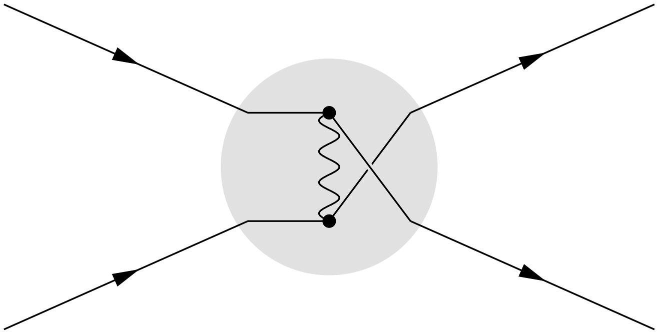

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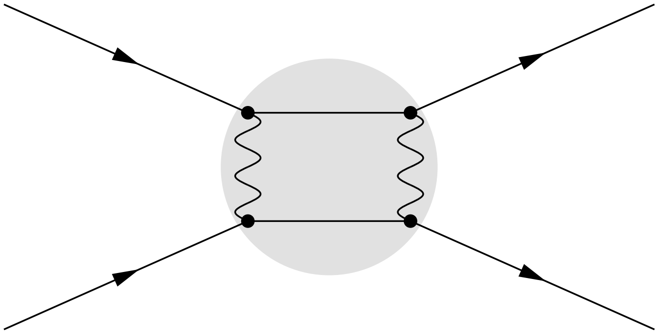

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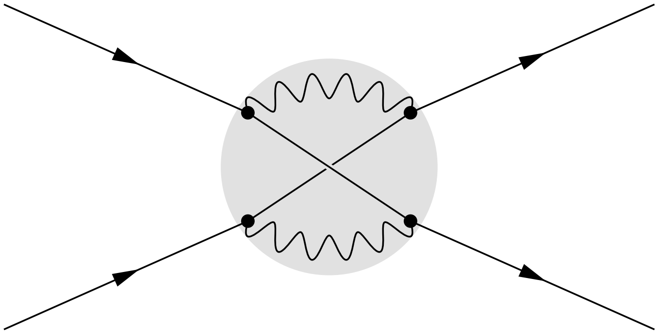

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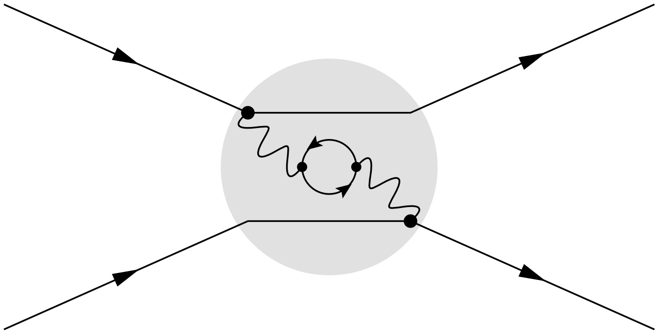

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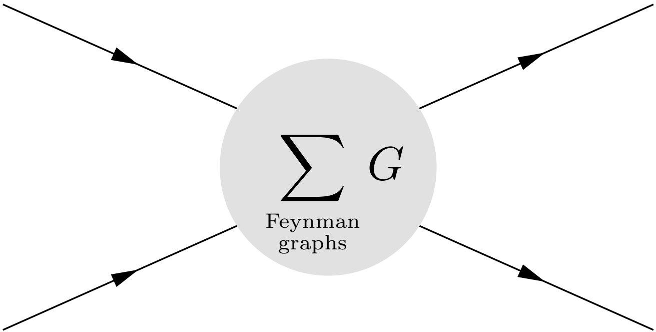

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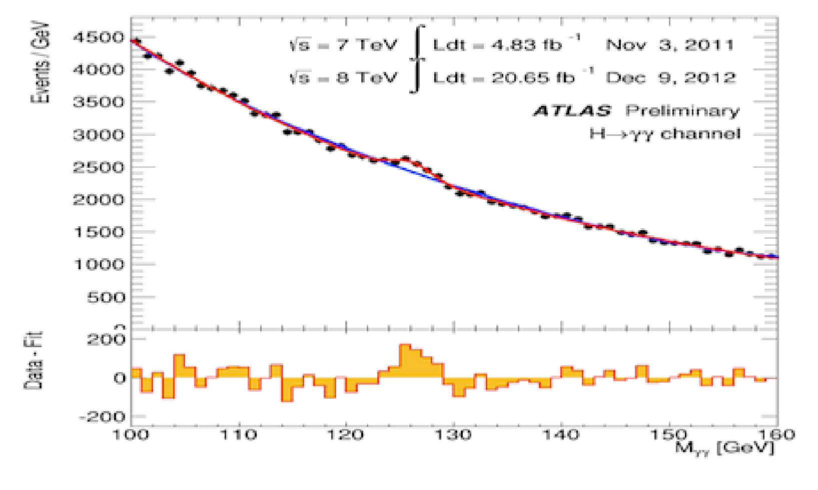
Higgs boson



The blue line (background) requires calculating a huge number of Feynman amplitudes.

II. Graphs and Numbers

Graph polynomials (Kirchhoff 1847)

Let $G = (V_G, E_G)$ be a connected graph. The *graph polynomial*

$$\Psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]$$

is a sum over spanning trees T of G

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$$

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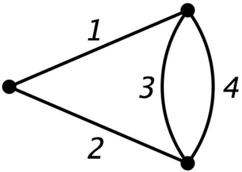
A tree $T \subset G$ is *spanning* if $V_T = V_G$.

Example



$\Psi_G = ?$

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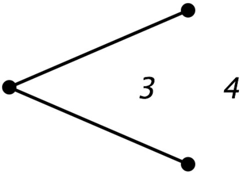

$$\Psi_G = \alpha_3\alpha_4$$

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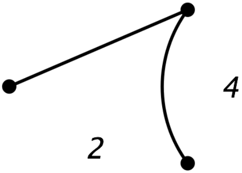

$$\Psi_G = \alpha_3\alpha_4 + \alpha_2\alpha_4$$

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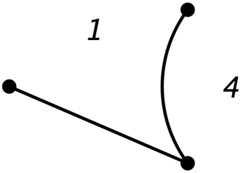

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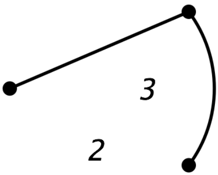

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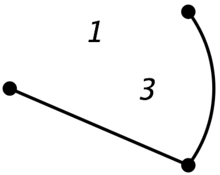

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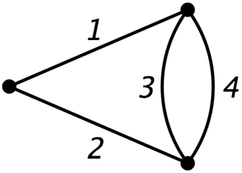

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In general, Ψ_G is homogeneous of degree h_G ('loop number').

$$\deg \Psi_G = h_G \qquad N_G = \#E(G)$$

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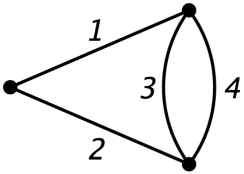

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Physically relevant graphs have vertices of degree ≤ 4. ('G in ϕ^4').

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- $N_G = 2h_G$
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$$\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G - 1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$

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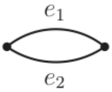

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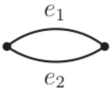

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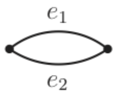

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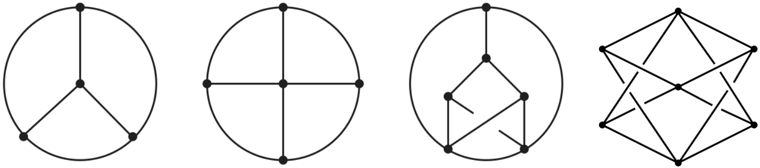
Compute the integral on the chart $\alpha_2 = 1$:

$$I_G = \int_\sigma \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} = \int_{\alpha_1 \geq 0} \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1$$

The Zoo



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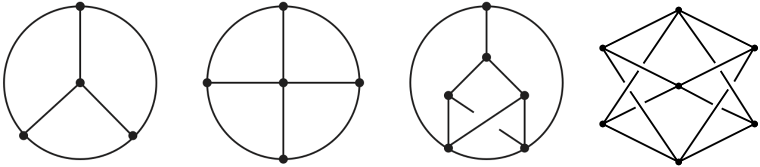

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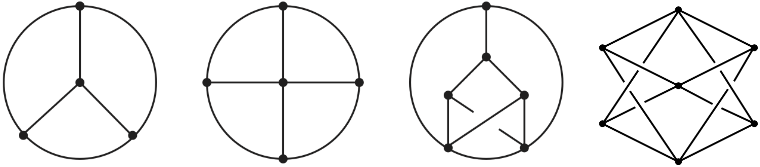

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Multiple Zeta Values, defined for $n_1, \ldots, n_{r-1} \geq 1$, and $n_r \geq 2$:

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \quad \in \mathbb{R}$$

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Folklore conjecture 90's

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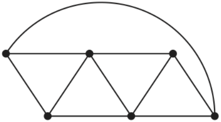
Z_5  $\propto \quad \zeta(2n-3)$

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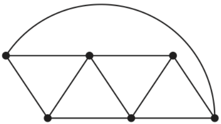
Z_5  $\propto \quad \zeta(2n-3)$

In general, very hard to compute the integrals even numerically because they are highly singular.

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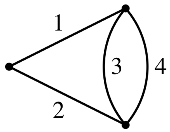

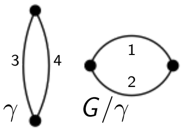

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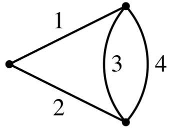

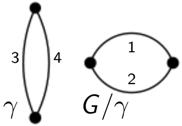

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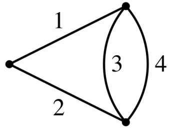

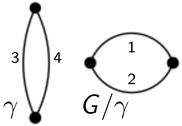

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Determines Ψ_G essentially uniquely.

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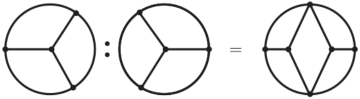

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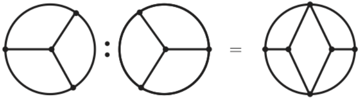

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and planar duals, completion (Fourier transform), ...

III. Point-counting

Points over finite fields

Let $f_1, \ldots, f_n \in \mathbb{Z}[x_1, \ldots, x_N]$. Let X denote the algebraic variety (affine scheme over $\mathbb{Z}$) defined by

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For example,

$$[X]_p = \sharp\{(x_1, \ldots, x_N) : x_i \in \mathbb{F}_p, f_i(x_1, \ldots, x_N) \equiv 0 \mod p \text{ for all } i\}$$

Some general results

Serre: if $[X]_p = [Y]_p$ for a set of primes p of density 1, then

$$[X]_{p^e} = [Y]_{p^e}$$

for all $e \geq 1$, provided $p \geq p_0$ sufficiently large.

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Grothendieck-Lefschetz trace formula:

$$[X]_q = \sum_i (-1)^i Tr(F : H^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$$

Dwork, Deligne.

Graph hypersurfaces

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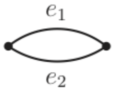
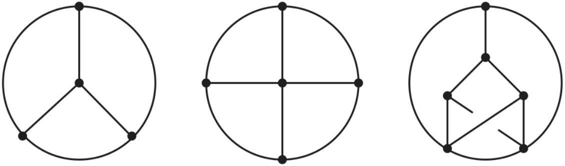


G

$$\Psi_G = \alpha_1 + \alpha_2 \qquad , \qquad [G]_q = q$$

Examples



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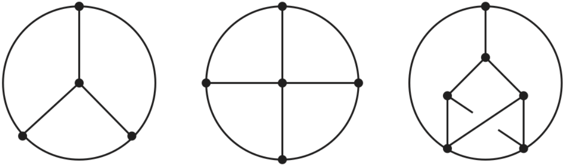
$20\zeta(5)$	$q^7 + 3q^5 - 6q^4 + 4q^3 - q^2$
$36\zeta(3)^2$	$q^9 + 4q^7 - 7q^6 + 3q^5$

Examples



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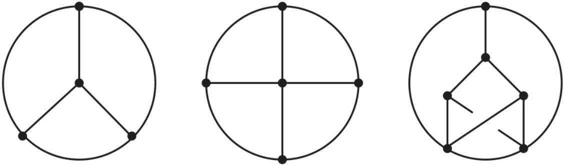

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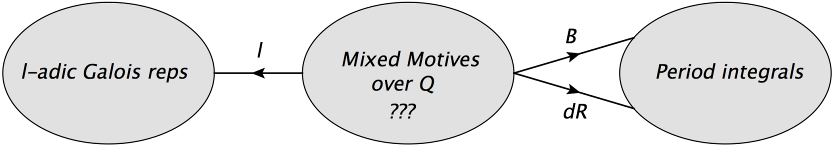
Question: is $[X_G]_q$ always a polynomial in q?

Philosophy


l-adic Galois reps
l
Mixed Motives
over Q
???
B
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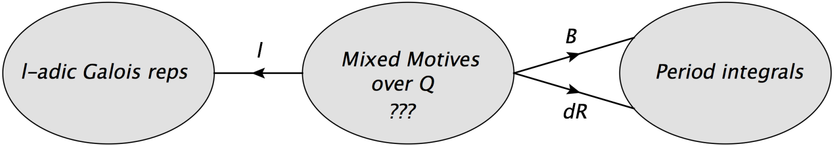
Period integrals

Philosophy



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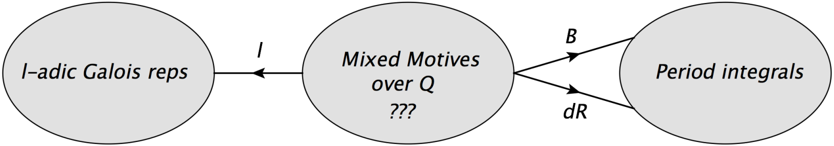

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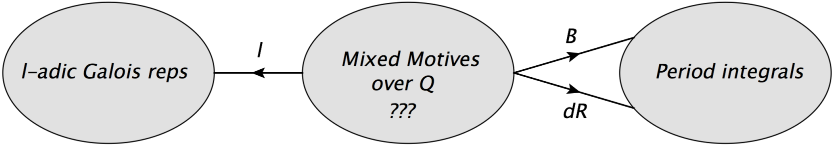

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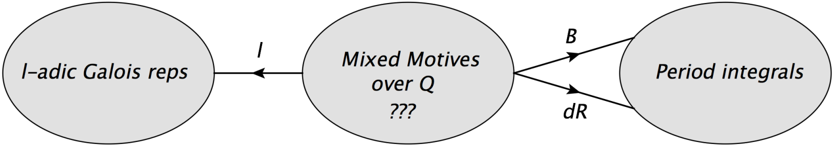

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Their point-counting functions are polynomials in q.

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The graphs G_i have vertices of huge degrees. But physics demands that the vertices be of degree at most 4.

IV. Modularity

Point counts over $\mathbb{F}_p$ modulo p

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2. (Chevalley-Warning theorem). If degree $f < N$ then

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$$c_G(p) := \frac{[G]_p}{p^2} \mod p$$

c-invariants of graphs

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Call such a sequence *constant*.

c-invariant examples

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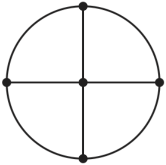

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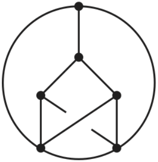

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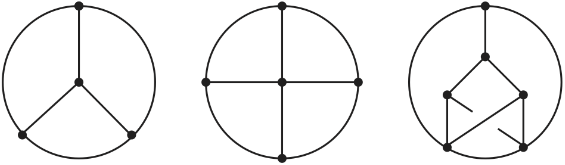

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Graph	$[G]_p$	c_G
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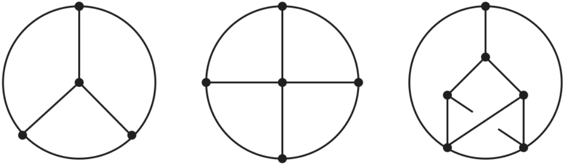

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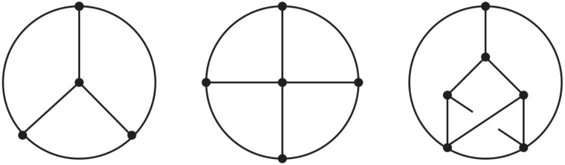

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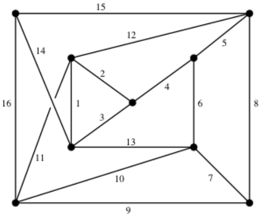
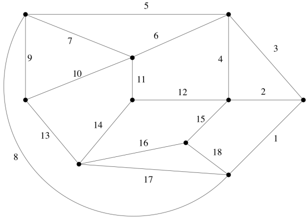
Conjecture: If $I_G = I_{G'}$ then $c_G = c_{G'}$.

For each of the two (convergent, ϕ^4) graphs:





Modular graphs (w/ O. Schnetz, 2012)

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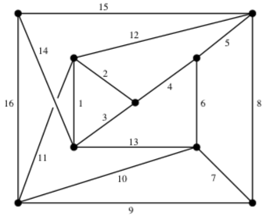

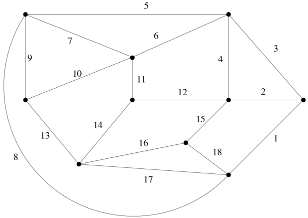

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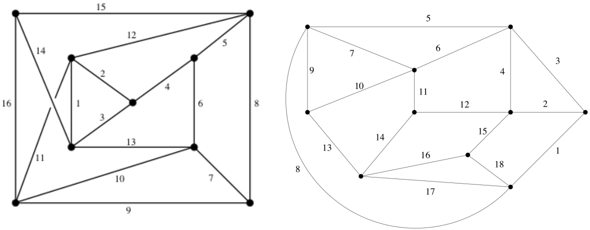

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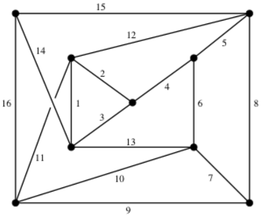

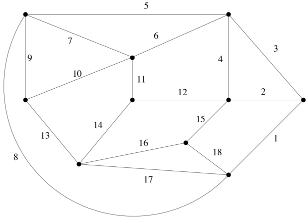

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$$\begin{aligned} \sum a_n z^n &= z \prod_{n \geq 1} \left((1-z^n)(1-z^{7n})\right)^3 \\ &= z - 3z^2 + 5z^4 - 7z^7 - 3z^8 + \dots \end{aligned}$$

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Ψ_G of degree 8 in 16 variables, and 3785 terms.

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$$F = b(a+c)(ac+bd) - ad(b+c)(c+d)$$

The zero locus of F defines a singular K_3 surface.

Modularity in dimensions 2 and 3

- Singular K3 surfaces (maximal Picard rank 20) over $\mathbb{Q}$ are modular. Modular forms of weight 3 with CM by $\mathbb{Q}(\sqrt{-d})$, and rational coefficients. Follows from Livné (1995), modularity of two-dimensional CM Galois representations. Elkies and Schütt: they all arise in this way (2013).

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- Rigid Calabi-Yau three-folds over $\mathbb{Q}$ are modular (..., Gouvêa-Yui (2010)). Uses proof of Serre's modularity conjecture by Khare and Wintenberger.

V. Questions

More modular counter-examples in ϕ^4 (O. Schnetz)

weight	2	3	4	5	6	7	8
level	11	$\boxed{7}_{8}$	$\boxed{5}_{8}$	$\boxed{4}_{9}$	$\boxed{3}_{8}$	$\boxed{3}_{9}$	$\boxed{2}_{10}$
	14	$\boxed{8}_{8}$	$\boxed{6}_{9}$	7	$\boxed{4}_{9}$	7	3
	15	11	$\boxed{7}_{10}$	8	5	8	$\boxed{5}_{10}$
	17	$\boxed{12}_{9}$	8	11	6	11	6
	19	15	9	12	$\boxed{7}_{9}$	15	7
	20	15	10	15	8	15	8
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No obvious relation between M_{pt} and M_{int}!

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The folklore conjecture would be false.

Amplitudes are much more complicated than expected.

What numbers should we expect for these graphs?

Pure motive	Pure periods	Hodge type	Mixed motives	Mixed periods
$\mathbb{Q}(-n)$	$(2i\pi)^n$	(p,p)	Mixed Tate over $\mathbb{Z}$	$\zeta(n),\ n \geq 2$ $\zeta(n_1,\ldots,n_r)$

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How do we construct realisations of motives of mixed modular type? What are their period integrals?

Which numbers and functions for quantum field theory?

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The numbers and functions generated by a single space:

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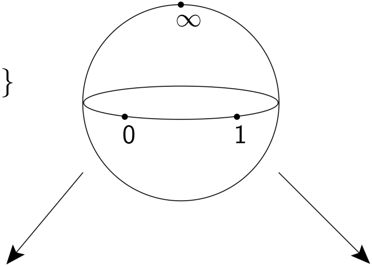

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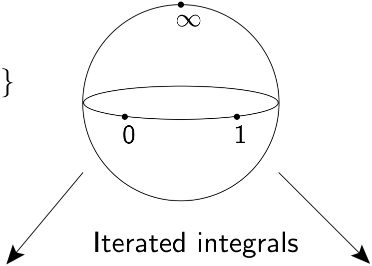

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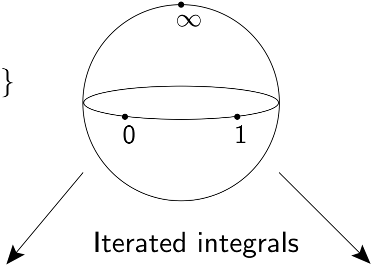

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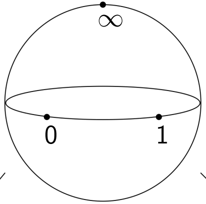

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Iterated integrals

Multiple zeta values

Polylogarithms

generate all amplitudes up to a certain number of loops, and infinite families of amplitudes in $N = 4$ SYM, ϕ^4, QCD, QED, ...

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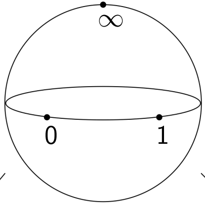

Grothendieck, Deligne, Ihara, Drinfeld, ...

Iterated integrals

Multiple zeta values

Polylogarithms

generate all amplitudes up to a certain number of loops, and infinite families of amplitudes in $N = 4$ SYM, ϕ^4, QCD, QED, ...

Modular examples beyond this regime (e.g. also with masses)

Which numbers and functions for quantum field theory?

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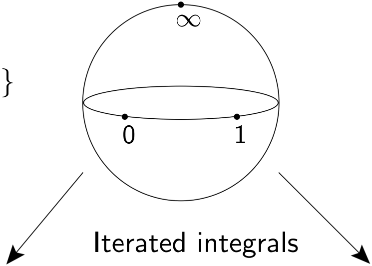

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What are the geometric objects which describe QFT in general?