European Research Council

Amplitudes in ϕ^4

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Overview:

1. Parametric Feynman integrals

2. Graph polynomials

3. Symbol calculus

4. Point-counting over finite fields

5. Modular forms in ϕ^4 theory

6. Galois coaction

7. Renormalization in parametric space

Joint work with D. Doryn, D. Kreimer, O. Schnetz, K. Yeats. Applications in progress: C. Bogner, F. Wissbrock. ERC grant 257638 'PAGAP'.

Why massless ϕ_4^4 theory?

Universal: changing the theory will only affect *numerators* of integrals - hence the particular linear combinations of basic amplitudes which occur, not the amplitudes themselves.

ϕ_4^4 gives interesting scalar quantities independent of renormalization scheme. But many results will also hold for more general kinematics.

The *parametric* representation enables us to apply methods from algebraic geometry (motivic philosophy), and leads to new insights.

Goals

1. A qualitative understanding of the nature of the numbers which occur as Feynman amplitudes and their relation to topology.
2. Practical tools for the exact (symbolic) computation of Feynman amplitudes.

Reminders on Parametric Feynman integrals

Work in $\mathbb{R}^D$. Scalar Feynman integral of graph G with edges E and vertices V is

$$\int_{\mathbb{R}^D} \prod_{e \in E} \frac{d^D k_e}{k_e^2 + m_e^2} \prod_{v \in V} \delta(\sum p_E)$$

Apply the Schwinger trick

$$\frac{1}{x} = \int_0^\infty e^{-\alpha x} dx$$

by introducing auxilliary parameters α_e, for each edge. This converts the product into a sum:

$$\prod_{e \in E^I} \frac{d^D k_e}{k_e^2 + m_e^2} = \int_0^\infty e^{-\sum_e \alpha_e (k_e^2 + m_e^2)} \prod_e d\alpha_e$$

Next use the Gauss integral

$$\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$$

and do all the momentum integrations by completing the square in the k_e.

After integrating out overall logarithmic divergence, and omitting Γ factors: the general unrenormalized parametric Feynman integral is:

$$\int_0^\infty \frac{\Psi_G^{-(\ell+1)D/2}}{(\Psi_G \sum_{e\in E} m_e^2 \alpha_e - \Phi_G)^{-\ell D/2}} \delta(\sum_{e\in E} \alpha_e - 1)$$

where ℓ is the loop number, and the *graph polynomial* or first Symanzik is:

$$\Psi_G = \sum_{T\subset G} \prod_{e\notin E_T} \alpha_e$$

which is a sum over all spanning trees of G, and the second Symanzik is a sum over all cut spanning trees S:

$$\Phi_G = \sum_S \prod_{e\notin S} \alpha_e (q^S)^2$$

where q^S is the moment flow through the cut.

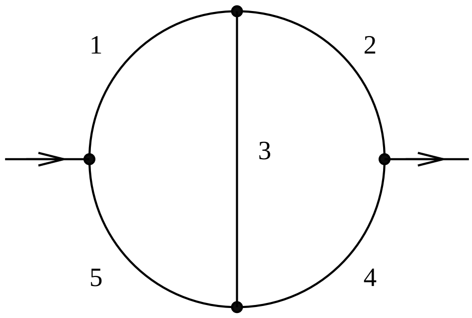
Important point: the integrand is completely algebraic (polynomial). Amplitudes belong to algebraic geometry ("periods of motives").

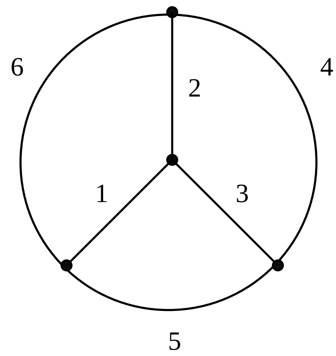
We only consider the case of graphs in massless ϕ^4_4 with trivial momentum dependence. In this case, there is a single momentum scale q:

$$\Phi_G(q) = q^2 \overline{\Phi}_G(\alpha_e)$$

and factors out. Let $\widetilde{G}$ be the 'closed-up graph' obtained by joining the two external legs.

Example: Master 2-loop 2-point function.





Amplitude (in Mom) is

$$I_{\widetilde{G}} \log\left(q^2/\mu^2\right)$$

where the coefficient (residue) is

$$I_{\widetilde{G}} = \int_0^\infty \frac{d\alpha_1 \dots d\alpha_6}{\Psi^2_{\widetilde{G}}} \delta(\sum \alpha_e - 1)$$

and $\Psi_{\widetilde{G}} = \alpha_1\alpha_2\alpha_4 + \ldots + \alpha_4\alpha_5\alpha_6$

From now on consider the closed up graph (now denoted G). To ensure convergence, assume that G is overall logarithmically divergent but free of subdivergences (a primitive graph):

- $\ell = 2|E_G|$
- $\ell_\gamma < 2|E_\gamma|$ for all strict subgraphs $\gamma \subset G$.

In this case the residue can be simply written

$$I_G = \int \frac{\prod_e d\alpha_e}{\Psi_G^2} \delta(\sum \alpha_e - 1)$$

which converges absolutely. By a change of variables, this can be written

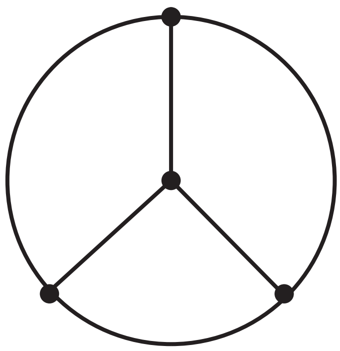
$$I_G = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \ldots d\alpha_{N-1}}{\Psi_G^2\Big|_{\alpha_N = 1}}$$

for some choice of edge N.

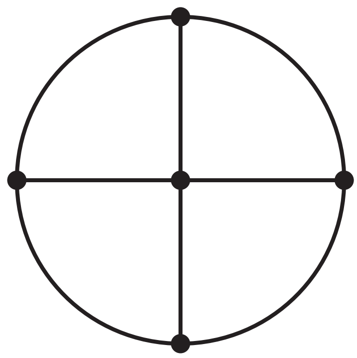
Main point: Changing the theory, or raising propagators to $1 + n_i\varepsilon$, only affects the *numerator* of the integral. The underlying geometry comes from the denominator.

The zoo

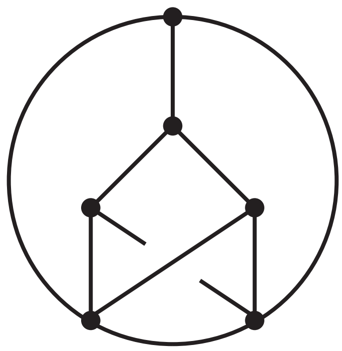
Broadhurst-Kreimer: All (numerically) computed I_G are multiple zeta values. All graphs with $\ell \leq 6$ known, some cases at $\ell = 7$ unknown.



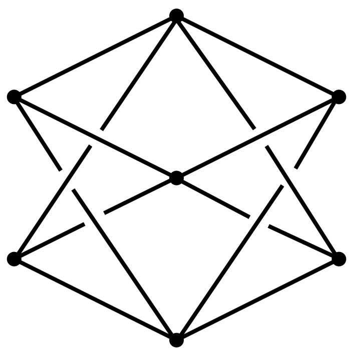
$6\zeta(3)$



$20\zeta(5)$



$36\zeta(3)^2$



$\frac{27}{5}\zeta(5,3)+\frac{45}{4}\zeta(5)\zeta(3)-\frac{261}{20}\zeta(8)$

Expected weight is $2\ell - 3$. But last two examples have weight $2\ell - 4$. This is *weight drop*.

Each graph encodes many master integrals and give *scheme-independent* contributions to β.

Schnetz extended computations (see 'census'). Mixture of Gegenbauer + accelerated convergence + lattice reduction algorithms.

Questions:

Do all primitive log-divergent graphs in ϕ^4 theory give multiple zeta values?

Can we predict weight-drop?

Why do these strange linear combinations of MZV's appear?

Methods for exact symbolic computation?

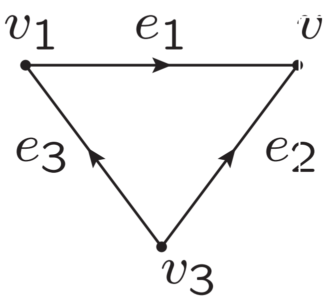
More on Graph polynomials

External edges play no role. Choose orientation of G; let $\mathcal{E}_G$ be the (edge x vertex) incidence matrix of G, after deleting any vertex.

$$M_G = \left(\begin{array}{ccc|c} \alpha_1 & & & \\ & \ddots & & \mathcal{E}_G \\ & & \alpha_{e_G} & \\ \hline & -{}^T\mathcal{E}_G & & 0 \end{array}\right)$$

Then $\Psi_G = \det M_G$.

Example:



	v_1	v_2	v_3
e_1	-1	1	0
e_2	0	1	-1
e_3	-1	0	1

Delete the v_3 column:

$$M_G = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 & 1 \\ 0 & \alpha_2 & 0 & 0 & 1 \\ 0 & 0 & \alpha_3 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

We have $\Psi_G = \det(M_G) = \alpha_1 + \alpha_2 + \alpha_3$.

We need to generalise: define, for any subsets of edges I, J, K of G such that $|I| = |J|$,

$$\Psi_{G,K}^{I,J} = \det M_G(I,J)\Big|_{\alpha_k=0, k\in K}$$

were $M_G(I,J)$ denotes the matrix M_G with rows I and columns J removed. We call $\Psi_{G,K}^{I,J}$ the *Dodgson polynomials* of G.

The key to computing the Feynman integrals is to exploit the many identities between the polynomials $\Psi_{G,K}^{I,J}$. We have:

- The *contraction-deletion formula*:

$$\Psi_{G,K}^{I,J} = \Psi_{G,K}^{Ie,Je}\alpha_e + \Psi_{G,Ke}^{I,J}$$

  We have $\Psi_{G,K}^{Ie,Je} = \pm\Psi_{G\backslash e,K}^{I,J}$ (deletion of e), and $\Psi_{G,Ke}^{I,J} = \pm\Psi_{G/e,K}^{I,J}$ (contraction of e).

- General *determinantal identities* such as:

$$\Psi^{I,J}_{G,Kabx}\Psi^{Iax,Jbx}_{G,K}-\Psi^{Ix,Jx}_{G,Kab}\Psi^{Ia,Jb}_{G,Kx}=\Psi^{Ia,Jx}_{G,Kb}\Psi^{Ix,Jb}_{G,Ka}$$

  or Plücker-type identities such as:

$$\Psi^{ij,kl}_{G,K}-\Psi^{ik,jl}_{G,K}+\Psi^{il,jk}_{G,K}=0$$

- *Graph-specific identities*. If, for example, K contains a loop, then $\Psi^{I,J}_{G,K}=0$, and many more complicated examples.

So there is a kind of dictionary:

Topology of $G \longleftrightarrow$ Relations amongst $\Psi^{I,J}_{G,K}$

The calculus of graph-type polynomials has consequences for amplitudes that is very far from being fully exploited at present.

Method for computing amplitudes

Compute the Feynman integral in parametric form by integrating out one variable at a time.

$$I_G = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \ldots d\alpha_{N-1}}{\Psi_G^2\Big|_{\alpha_N=1}}$$

In more general situations the numerator will be a polynomial in α_i (or even $\log \alpha_i$) and does not affect the method significantly.

By the contraction-deletion formula, we can write $\Psi = \Psi^{1,1}\alpha_1 + \Psi_1$. Therefore

$$I_G = \int_0^\infty \frac{d\alpha_1 \ldots d\alpha_{N-1}}{\Psi^2}$$

can be written

$$\int_0^\infty \frac{d\alpha_1 \ldots d\alpha_{N-1}}{(\Psi^{1,1}\alpha_1 + \Psi_1)^2} = \int_0^\infty \frac{d\alpha_2 \ldots d\alpha_{N-1}}{\Psi^{1,1}\Psi_1}$$

By contraction-deletion, the polynomials $\Psi^{1,1}$ and Ψ_1 are linear in the next variable, α_2:

$$\begin{aligned} \Psi^{1,1} &= \Psi^{12,12}\alpha_2 + \Psi^{1,1}_2, \\ \Psi_1 &= \Psi^{2,2}_1\alpha_2 + \Psi_{12} \end{aligned}$$

We can write the previous integral $\int \frac{1}{\Psi^{1,1}\Psi_1}$as

$$\int_0^\infty \frac{d\alpha_2 \dots d\alpha_{N-1}}{(\Psi^{12,12}\alpha_2 + \Psi^{1,1}_2)(\Psi^{2,2}_1\alpha_2 + \Psi_{12})}$$

Decompose into partial fractions and integrate out α_2. This leaves an integrand of the form

$$\frac{\log \Psi^{1,1}_2 + \log \Psi^{2,2}_1 - \log \Psi^{12,12} - \log \Psi_{12}}{\Psi^{1,1}_2\Psi^{2,2}_1 - \Psi^{12,12}\Psi_{12}}$$

At this point, we should be stuck since the denominator is quadratic in every variable. Miraculously, there is an identity due to Dodgson:

$$\Psi^{1,1}_2\Psi^{2,2}_1 - \Psi^{12,12}\Psi_{12} = (\Psi^{1,2})^2$$

So after two integrations we have $\int \frac{d\alpha_1 d\alpha_2}{\Psi^2}$

$$= \frac{\log \Psi_2^{1,1} + \log \Psi_1^{2,2} - \log \Psi^{12,12} - \log \Psi_{12}}{(\Psi^{1,2})^2}$$

We can then write $\Psi^{1,2} = \Psi^{13,23}\alpha_3 + \Psi_3^{1,2}$ and keep integrating out variables. . .

As long as we can find a Schwinger coordinate α_i in which all the terms in the integrand are *linear*, then we can always perform the next integration. This requires choosing a good order on the edges of G.

In this case, the integral is expressible as multiple polylogarithms:

$$\mathrm{Li}_{n_1,\ldots,n_r}(x_1, \ldots, x_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}}$$

typically evaluated at arguments $\Psi^{I,J}_{G,K}$. When this process terminates, the Feynman integral is expressed as values of multiple polylogarithms evaluated at 1 (or roots of unity).

When and why it works

- Can predict *in advance* when the integration will go through. Just compute the discriminant loci for each integration step:

$$S_0 = \{\Psi_G\} \tag{1}$$
$$S_1 = \{\Psi_G^{1,1}, \Psi_{G,1}\} \tag{2}$$
$$S_2 = \{\Psi_G^{12,12}, \Psi_{G,2}^{1,1}, \Psi_{G,1}^{2,2}, \Psi_{G,12}, \Psi_G^{1,2}\} \tag{3}$$
$$\vdots$$

  The singularities S_i are resultants of pairs of polynomials in the previous stage S_{i-1}. Integration works when elements of S_i are linear in the next integration variable α_{i+1}.

- This also holds if we have external momenta or masses. At the end we obtain polylogarithms in the kinematic variables.

- Do not need to compute with actual functions : the calculation is purely algebraic by replacing functions with symbols.

"Symbols"

(Highly simplified version of K.T. Chen's theory of iterated integrals $\sim$ 70's).

Let M be a smooth manifold, and $\omega_1, \ldots, \omega_n$ closed 1-forms on M. A symbol is a linear combination of multilinear elements

$$\xi = \sum_{I=(i_1,\ldots,i_n)} c_I[\omega_{i_1}|\ldots|\omega_{i_n}]$$

which satisfy the integrability condition

$$\sum_{I=(i_1,\ldots,i_n)} c_I\,\omega_{i_1}\otimes\ldots\otimes(\omega_{i_k}\wedge\omega_{i_{k+1}})\otimes\ldots\otimes\omega_{i_n} = 0$$

for all $k = 1, \ldots, n-1$. Given a base point $x \in M$, we get a (multivalued) function

$$F(z) = \int_x^z \xi$$

by iterated integration. A version of Chen's π_1-de Rham theorem gives isomorphism

$$\{\text{Symbols}\} \longrightarrow \{\text{homotopy invariant iterated integrals on M}\}$$

Example: If $M = \mathbb{C}\backslash\{0,1\}$, we get an isomorphism between symbols in $\frac{dx}{x}$ and $\frac{dx}{1-x}$ and generalized multiple polylogarithms.

However, for general M, it is *not enough* to consider only closed forms ω_i (general definition of symbol is more complicated).

In any case, by Chen's theorem we can work with the symbols instead of the functions. Differential operations on integrals are replaced with algebraic manipulations on symbols:

truncation of a letter $\longleftrightarrow$ differentiation

affixing a letter $\longleftrightarrow$ integration

deconcatentation coproduct $\longleftrightarrow$ monodromy

We get effective *algebraic* algorithms for the manipulation of certain families of multivalued functions (in progress with C. Bogner).

Results

By computing the sets S_i we can prove for all graphs with low loop orders ($\ell \leq 6$) that I_G is a multiple zeta value.

We can write down some non-trivial *infinite families* of graphs for which the sets S_i consist entirely of Dodgson polynomials, and hence these give multiple zeta values too.

Each stage of integration is expressible as a symbol in the differential forms df/f, where $f \in S_i$. So for graphs with kinematic parameters, one can write down an Ansatz for the amplitude as a function of external momenta.

But for a general G, linearity of S_i will *fail*...

Denominator reduction

Now try to identify the *worst* contribution to the singularities S_i to try to find non-linear obstructions to integration method.

If $k \geq 5$, at the k^{th} stage of integration, the answer is a symbol over a *single denominator*:

$$\frac{\sum c_I[\omega_{i_1}|\ldots|\omega_{i_{k-2}}]}{D_k}$$

The denominators D_k can be computed easily:

$$D_5 = \pm \det \begin{pmatrix} \Psi_5^{12,34} & \Psi^{125,345} \\ \Psi_5^{13,24} & \Psi^{135,245} \end{pmatrix}$$

If α_{k+1} is the next integration variable, and

$$D_k = (A\alpha_{k+1} + B)(C\alpha_{k+1} + D)$$

factorizes (very often does), then

$$D_{k+1} = \pm(AD - BC) \ .$$

The *denominator reduction* is the sequence $D_5, D_6, \ldots, D_n$ up to the point n where D_n vanishes, or no longer factorizes.

A graph G is *denominator reducible* if there exists an ordering on its edges, so that the denominators D_n exist for all n. All graphs with < 8 loops are denominator reducible.

The denominator reduction is a microscope to analyse qualitative features of the Feynman integral I_G at higher loop orders.

Example: Weight-drop.

If a denominator D_n vanishes, then the graph has weight-drop. Conversely, all known weight-drops in ϕ^4 have a vanishing D_n.

Using this, with Karen Yeats, we obtained combinatorial criteria for graphs to have weight-drop (2-vertex reducible graphs, or graphs obtained from these by splitting triangles).

Point-counting

Let $q = p^n$ prime power. G any graph, Ψ_G its graph polynomial. Idea (Kontsevich) is to count the number of solutions to the equation

$$\Psi_G(\alpha_1, \ldots, \alpha_N) = 0$$

where $\alpha_i \in \mathbb{F}_q$, the finite field with q elements. We get a function $[G]_q$

$$[G] : \{\text{prime powers } q\} \longrightarrow \mathbb{N}$$

Examples: Consider the 3,4,5 loop graphs for $6\zeta(3), 20\zeta(5), 36\zeta(3)^2$ we had earlier.

$$\begin{aligned} [G_3] &= q^2(q^3 + q - 1) \\ [G_4] &= q^2(q^5 + 3q^3 - 6q^2 + 4q - 1) \\ [G_5] &= q^5(q^4 + 4\,q^2 - 7q + 3) \end{aligned}$$

The functions $[G]$ are *polynomials* in q.

Notice already that G_5, the weight-drop graph, has vanishing coefficient of q^2.

The theory of mixed Tate motives suggests that "I_G is a multiple zeta value" should be (loosely) related to "$[G]$ is polynomial". In fact, this relation is *a posteriori* rather good.

Analogue of conjecture that I_G are MZVs:

Conjecture 1. (Kontsevich '97) For any graph G, the function $[G]$ is a polynomial in q.

Stembridge showed this to be true for all graphs with ≤ 12 edges. But Belkale-Brosnan proved this is false: the functions $[G]$ can be very general. No explicit counter-examples, unphysical (would have huge numbers of edges).

With O. Schnetz, we proved Kontsevich's conjecture is true for some infinite classes of graphs:

Theorem 2. *All graphs with 'vertex-width ≤ 3' have polynomial point counts.*

$\underline{c_2 \text{ invariants}}$ (with O. Schnetz)

Observation: the point-count is divisible by q^2

$$[G]_q \equiv 0 \mod q^2 \ .$$

Thus we can define

$$c_2(G)_q = q^{-2}[G]_q \mod q$$

The invariant $c_2(G)$ maps any prime power q to a number in $\mathbb{Z}/q\mathbb{Z}$. When $[G]$ is a polynomial, $c_2(G)$ is just the coefficient of q^2.

The $c_2(G)$ contains all the essential info:

Conjecture 3. If G_1, G_2 two graphs whose residues are equal: $I_{G_1} = I_{G_2}$, then $c_2(G_1) = c_2(G_2)$.

and is very easy to compute:

Theorem 4. $c_2(G)_q \equiv (-1)^n[D_n]_q \mod q$

Just count points on one of the denominator polynomials D_n! From this we deduce that $c_2(G)$ has very good combinatorial properties.

Weight-drop can also be read off the c_2-invariant: it corresponds to $c_2(G) = 0$.

To falsify the conjecture it suffices to find graphs G for which $c_2(G)$ is *non-constant*.

To simplify further, restrict to counting points over $\mathbb{F}_p$ with p prime. Let $a_p = p^{-2}[G]_p$ and $\overline{a}_p = a_p \mod p$ its reduction mod p. Define

$$\widetilde{c}_2(G) = (\overline{a}_2, \overline{a}_3, \overline{a}_5, \ldots) \in \mathbb{F}_2 \times \mathbb{F}_3 \times \cdots$$

Enough to exhibit G with non-constant $\overline{a}_p$. Even with just 6 primes, every graph G gives an element $(\overline{a}_2, \ldots, \overline{a}_{13})$ in a large set $\mathbb{F}_2 \times \ldots \times \mathbb{F}_{13}$. Probability of a false positive is $\sim 0.0033\%$.

Huge gain in computational difficulty compared to the original Feynman integral. It suffices to take the final polynomial D_n of the denominator reduction, and count points of $D_n = 0$ over finite fields $\mathbb{F}_p$ for small p.

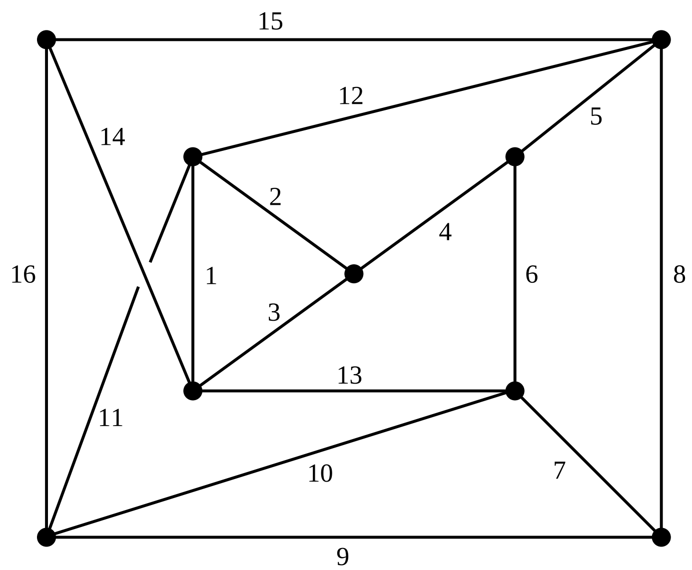
Counter-examples (with O. Schnetz)

Theorem 5. *The following 8-loop graph has* $\widetilde{c}_2(G) = (\overline{a}_2, \overline{a}_3, \overline{a}_5, \ldots)$ *where* a_n *are the Fourier coefficients of the modular form*

$$\sum a_n z^n = \Big(\eta(z)\eta(z^7)\Big)^3$$

of weight 3 and level 7, where

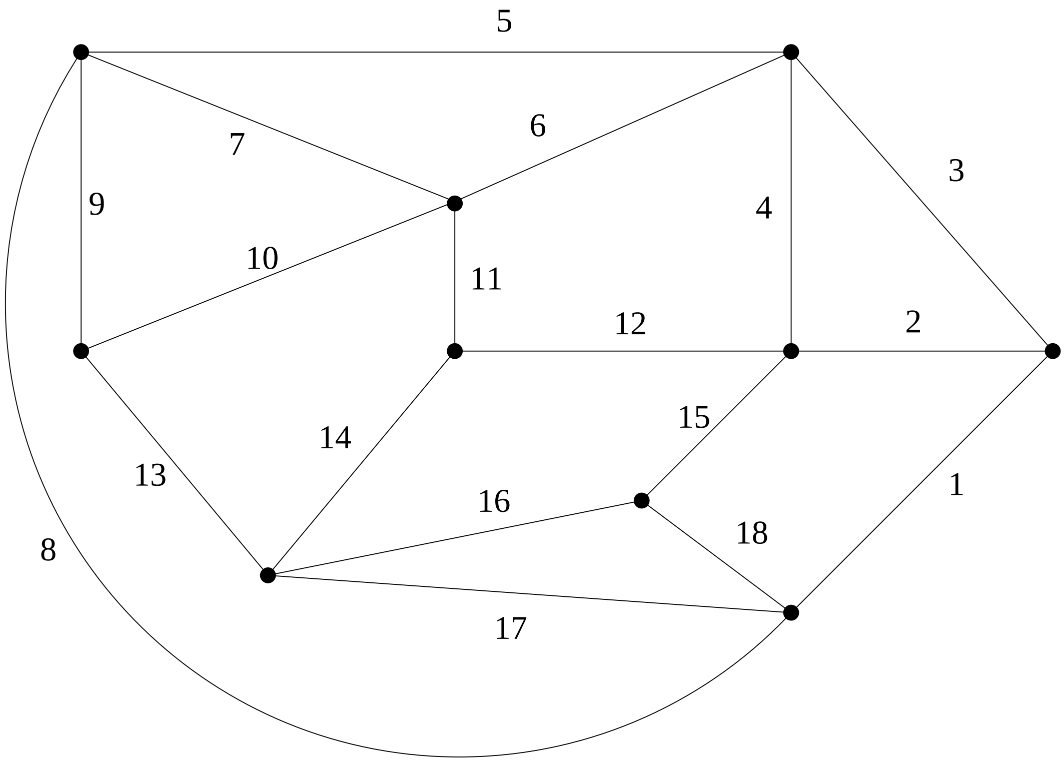
$$\eta(z) = z^{1/24} \prod_{n\geq 1} (1 - z^n).$$



(In this case we know $\overline{a}_p$ for all primes p). The theorem implies $[G]$ is very far from polynomial in q, and hence I_G is probably not an MZV.

...as if that wasn't bad enough...

consider the following *planar* graph at 9 loops:



It is also modular, with the same modular form as in the previous theorem.

Furthermore, its vertex width = 4.

More modular forms in ϕ^4 (with O. Schnetz)

Up to 6 loops: $[G]$ always polynomial

$\ell = 7$. New functions $[X^2 + X + 1]$, $[X^2 - 1]$

$\ell = 8$. We get 4 modular forms

$\ell = 9, 10$. Another 12 modular forms (but no weight 2!), and many unidentified $\widetilde{c}_2(G)$.

wt	2	3	4	5	6	7	8
level	11	$\boxed{7}_8$	$\boxed{5}_8$	$\boxed{4}_9$	$\boxed{3}_8$	$\boxed{3}_9$	$\boxed{2}_{10}$
	14	$\boxed{8}_8$	$\boxed{6}_9$	7	$\boxed{4}_9$	7	3
	15	11	$\boxed{7}_{10}$	8	5	8	$\boxed{5}_{10}$
	17	$\boxed{12}_9$	8	11	6	11	6
	19	15	9	12	$\boxed{7}_9$	15	7
	20	15	10	15	8	15	8
	21	16	12	15	9	16	8
	24	19	$\boxed{13}_9$	19	$\boxed{10}_{10}$	19	9
	26	20	⋮	20	10	20	10
	26	20	$\boxed{17}_{10}$	20	10	20	12

Conclusion: the general integrals I_G are exotic, but still seem to be highly constrained.

Motivic Galois coaction

What can we say about the Feynman integrals I_G which *do* evaluate to multiple zetas?

If we pass to 'motivic' multiple zeta values $\zeta^{\mathfrak{m}}$, there is a hidden structure: the action of motivic derivations for each $2n+1$:

$$\partial_3, \partial_5, \partial_7, \ldots$$

We have $\partial_{2n+1}\xi = 0$ for all $n \geq 1$ if and only if ξ is a linear combination of simple zetas $\zeta^{\mathfrak{m}}(m)$.

The action is highly non-trivial: e.g.

$$\begin{aligned} \partial_3\, \zeta^{\mathfrak{m}}(7,3) &= 17\, \zeta^{\mathfrak{m}}(7) \\ \partial_5\, \zeta^{\mathfrak{m}}(7,3) &= 6\, \zeta^{\mathfrak{m}}(5) \\ \partial_7\, \zeta^{\mathfrak{m}}(7,3) &= \zeta^{\mathfrak{m}}(3) \end{aligned} \tag{4}$$

and respects all motivic (and conjecturally, all possible) $\mathbb{Q}$-algebraic relations between MZVs.

Thus there is the period map

$$per : \zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \rightarrow \zeta(n_1, \ldots, n_r)$$

which respects all the relations.

(with O. Schnetz). Experimentally, the known multiple zeta values which occur in ϕ^4 theory are highly constrained by how the operators ∂_{2n+1} act upon them. This partially explains the strange linear combinations of multiple zetas which occur in ϕ^4.

Define a linear combination of (motivic) multiple zeta values to be *abelian* if it is polynomial in single zeta values:

$$P(\zeta^{\mathfrak{m}}(2), \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \ldots, \zeta^{\mathfrak{m}}(2n+1))$$

with P a polynomial with rational coefficients.

Incredibly, we find that the multiple zeta values which occur in ϕ^4 are only slightly more general than abelian multiple zetas.

Holy grail: would be to compute the action of the ∂_{2n+1} from the topology of a graph.

The motivic philosophy suggests that *all* amplitudes carry an action of a motivic Galois Lie algebra (P. Cartier's 'cosmic Galois group').

Renormalization (joint with D. Kreimer)

We can treat graphs with subdivergences in a similar way (BPHZ and Mom).

Using the Hopf algebra of renormalization, separate Feynman integral into single-scale part and purely angular, convergent part. The renormalization can be done graph by graph:

$$\phi^{ren} = \sum \phi^{ren}_{1-scale}(S)\phi_{fin}(\theta)$$

Leads to explicit parametric representations for the renormalized amplitudes, which can then be treated by the general methods above.

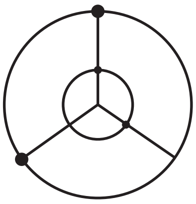
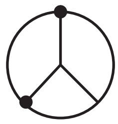
We get simple proofs of

1). Finiteness of renormalized amplitudes.

2). Graph-by-graph Callan-Symanzik equations.

Uses only simple properties of graph polynomials and the Hopf algebra of renormalization.

<u>Example</u>: 3-loop graph inserted into itself:

Γ  γ  Γ/γ 

For the single-scale contribution, the renormalization group equation gives

$$I_\Gamma \log(\tfrac{q^2}{\mu^2}) + \tfrac{1}{2} I_\gamma I_{\Gamma/\gamma} \log^2(\tfrac{q^2}{\mu^2})$$

with $I_\gamma = I_{\Gamma/\gamma} = 6\zeta(3)$. I_Γ can be written as an integral over graph polynomials involving auxilliary graphs obtained from $\Gamma, \gamma, \Gamma/\gamma$.

$$I_\Gamma = \int \Big(\frac{1}{\Psi_\Gamma^2} - \frac{\Psi_{\Gamma^\bullet/\gamma}}{\Psi_{\Gamma/\gamma}^2 \Psi_\gamma \Psi_{c(\gamma,\Gamma/\gamma)}} \Big) < \infty.$$

This should be computable by the symbolic integration method (in progress).

Theorem 6. *(with O. Schnetz and K. Yeats):* $c_2(\Gamma) = 0$ *for any* $\Gamma \in \phi^4$ *with subdivergences.*

1. Standard conjectures in algebraic geometry suggest that the amplitudes of the modular 8-loop graphs are not multiple zeta values (multiple elliptic polylogs come next?).

2. By theorem 6, we expect graphs with subdivergences to have weight drop. The subdivergences should also make the amplitudes *easier* to compute: so the number theory content of 1-scale divergent graphs is simpler.

3. Because of their sparsity, there is nothing for these exotic 'modular' amplitudes to cancel against. Thus we expect them to survive in the sum over all graphs. This suggests there is no way to change the renormalization scheme to get only MZVs.

4. What about $N = 4$ SYM?

There is some hope: all the non-MZV examples have a weight drop in the sense of Hodge theory.

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