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Anatomy of the motivic Lie algebra

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The motivic Lie algebra

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$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\widehat{\pi}_1(\mathbb{P}^1 \backslash \{0,1,\infty\}))$$

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$$\mathfrak{g}^{\mathfrak{m}} \underset{\text{non-can}}{\cong} \mathrm{Lie}_{\mathbb{Q}}\langle \sigma_3, \sigma_5, \sigma_7, \ldots \rangle$$

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Goal: construct a kind of 'Taylor expansion' for the σ_{2n+1}.

The motivic Lie algebra (II)

Various versions of $\pi_1^{un}(\mathbb{P}^1\backslash\{0,1,\infty\})$ give rise to algebras

$$\mathrm{Lie}_{\mathbb{Q}}\langle e_0, e_1\rangle \subset T(e_0\mathbb{Q} \oplus e_1\mathbb{Q}) \subset \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$$

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Example 1

The element $\sigma_3 \in \mathfrak{g}^{\mathfrak{m}}$ maps to $[e_0,[e_0,e_1]]+[e_1,[e_1,e_0]]$

The elements $\sigma_3,\sigma_5,\sigma_7,\sigma_9$ are **canonical**. The element σ_{11} is only well-defined up to $\mathbb{Q}\{\sigma_3,\{\sigma_5,\sigma_3\}\}$. ‘Large’ torsor.

An aside: canonical generators

We can construct rational motivic associators as follows. Let

$$\Phi^{\mathfrak{m}} = \sum_{w \in \{e_0, e_1\}^{\times}} \zeta^{\mathfrak{m}}(w) w \in \mathcal{H}\langle\langle e_0, e_1 \rangle\rangle$$

be the motivic Drinfel'd associator, where $\mathcal{H}$ is the algebra of motivic MZV's. We know that $\mathcal{H}$ has as basis the Hoffman elements

$$B = \{\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \text{ where } n_i \in \{2, 3\}\} .$$

Define a map $\tau : \mathcal{H} \to \mathbb{Q}$ by sending $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \in B$ to 0 if at least one $n_i = 3$, and $\tau(\zeta^{\mathfrak{m}}(2^{\{n\}})) = (2n+1)!2^{2n}$.

Theorem 2

$\tau(\Phi^{\mathfrak{m}}) \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ *is a rational associator.*

A similar construction using Hoffman-Lyndon elements gives canonical σ_{2n+1} for all $n \geq 1$.

Today's goal is **NOT** to define rational associators!

Depth filtration

The *depth filtration* $\mathfrak{D}$ is the decreasing filtration:

$$\mathfrak{D}^r\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle = \{S \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle : S_w = 0 \text{ if } \deg_{e_1} w < r\} \ ,$$

which counts the number of e_1's in a word.

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which counts the number of e_1's in a word. Every series $S \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ admits a decomposition by depth

$$S = \sum_{r\geq 0} S^{(r)} \quad \text{with} \quad S^{(r)} \in \mathrm{gr}^r_{\mathfrak{D}}\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$$

The component $S^{(r)}$ consists of words with exactly r letters e_1.

One can show that $\mathfrak{D}$ is motivic: i.e., the induced filtration $\mathfrak{D}^r\mathfrak{g}^{\mathfrak{m}}$ on the motivic Lie algebra is preserved by $\{\, , \}$.

Words and power series

We shall replace *non-commutative formal power series* in e_0, e_1 with *commutative power series* in many variables x_i (Ecalle, Zagier).

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There is a $\mathbb{Q}$-linear isomorphism

$$\begin{array}{ccc} \mathrm{gr}^r_{\mathfrak{D}} \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle & \xrightarrow{\sim} & \mathbb{Q}[[x_0, x_1, \ldots, x_r]] \\ e_0^{a_0} e_1 e_0^{a_1} \ldots e_1 e_0^{a_r} & \mapsto & x_0^{a_0} x_1^{a_1} \ldots x_r^{a_r} \ . \end{array} \tag{1}$$

All power series that we consider will be **translation-invariant**. Therefore no information will be lost in setting $x_0 = 0$.

A formal power series $\Phi \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ is completely determined by its depth r-components $\Phi^{(r)} \in \mathbb{Q}[[x_0, \ldots, x_r]]$, for $r \geq 0$.

Examples

Example 3

We have $\sigma_3 = [e_0, [e_0, e_1]] + [e_1, [e_1, e_0]]$. Therefore

$$\begin{aligned} \sigma^{(1)} &= [e_0, [e_0, e_1]] &\longrightarrow& \quad (x_1 - x_0)^2 \\ \sigma^{(2)} &= [e_1, [e_1, e_0]] &\longrightarrow& \quad x_2 - 2x_1 + x_0 \end{aligned}$$

Translation invariant, so set $x_0 = 0$. Represent σ_3 as

$$\sigma_3 = (x_1^2, x_2 - 2x_1, 0, \ldots)$$

The depth one component is always

$$\sigma_{2n+1}^{(1)} = \mathbf{x}_1^{2n}$$

In depths $d \geq 3$, $\sigma_{2n+1}^{(d)}$ depends on choice of σ_{2n+1}.

Examples II

Example 4

The element σ_9 is canonical. We can compute its depth 3 component:

$$\sigma_9^{(3)} = -\frac{25}{6}x_1^6 + \frac{1159}{72}x_1^5x_2 + \frac{551}{36}x_1^5x_3 - \frac{559}{24}x_1^4x_2^2 - \frac{\mathbf{3319}}{\mathbf{72}}x_1^4x_2x_3 + \cdots$$

plus 23 more terms. It clearly has **non-trivial denominators**.

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Each coefficient in σ_9 is the coefficient of $\zeta^{\mathfrak{m}}(2n+1)$ in a decomposition of the corresponding multiple zeta value:

$$\zeta^{\mathfrak{m}}(5,2,2) = \frac{\mathbf{-3319}}{\mathbf{72}}\zeta^{\mathfrak{m}}(9) + \frac{2}{3}\zeta^{\mathfrak{m}}(3)^3 + 31\,\zeta^{\mathfrak{m}}(7)\zeta^{\mathfrak{m}}(2) - \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(4) - \frac{25}{6}\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(6)\ .$$

Standard relations

The simplest way to get at the generators σ_{2n+1} is via relations.

$\{$Motivic associators$\} \subset \{$Drinfel'd associators$\} \overset{\text{Furushou}}{\subset} \{$Solutions to Dbsh$\}$

where Dbsh are double shuffle equations. On the level of Lie algebras:

$$\mathfrak{g}^{\mathfrak{m}} \subset \mathfrak{grt} \subset \mathfrak{dmr} .$$

$\mathfrak{dmr}$ = solutions to double shuffle equations mod products (Racinet).

Standard Conjectures

$\mathfrak{g}^{\mathfrak{m}} \cong \mathfrak{dmr}$ (Zagier) ($\Rightarrow \mathfrak{g}^{\mathfrak{m}} \cong \mathfrak{grt}$ (Drinfel'd))

Focus on $\mathfrak{dmr}$ since it is well-adapted to depth filtration.

Double shuffle equations I

The double shuffle equations are relations satisfied by MZV's:

$$\zeta(n_1,\ldots,n_r) = \sum_{1\leq k_1<\ldots<k_r} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}} \quad , \quad n_r \geq 2$$

Recall that the **weight** is the $n_1+\ldots+n_r$. The **depth** is r.
Consider the (regularized!) generating series in depth r

$$Z^{(r)}(x_1,\ldots,x_r) = \sum_{n_1,\ldots,n_r\geq 1} \zeta(n_1,\ldots,n_r)\, x_1^{n_1-1}\ldots x_r^{n_r-1}$$

Relations between MZV's $\longleftrightarrow$ Functional equations for $Z^{(\bullet)}$

Double shuffle equations II

The *stuffle* equation is an equation of the type

$$\zeta(m)\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$$

Taking the generating series (and ignoring issues of divergence):

The depth two stuffle eqation

$$Z^{(1)}(x_1)Z^{(1)}(x_2) = Z^{(2)}(x_1,x_2) + Z^{(2)}(x_2,x_1) + \frac{Z^{(1)}(x_1) - Z^{(1)}(x_2)}{x_1 - x_2}$$

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The *shuffle* equation comes from representation of MZV's as integrals.

The depth two shuffle eqation

$$Z^{(1)}(x_1)Z^{(1)}(x_2) = Z^{(2)}(x_1,x_1+x_2) + Z^{(2)}(x_2,x_1+x_2)$$

The linearized double shuffle equations

We shall consider two simplified variants of these equations:

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Double shuffle equations modulo products

Kill all products on the left-hand side:

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The linearized double shuffle equations (Zagier, Ihara, Kaneko)

Kill all terms of lower depth as well

$$\begin{aligned} 0 &= Z^{(2)}(x_1,x_2) + Z^{(2)}(x_2,x_1) \\ 0 &= Z^{(2)}(x_1,x_1+x_2) + Z^{(2)}(x_2,x_1+x_2) \end{aligned}$$

Double shuffle equations II

Notation

$\mathfrak{dmr}$ is the set of $\mathbb{Q}$-solutions to the Dsh equations mod products:

$$\Phi = (\Phi^{(1)}(x_1), \Phi^{(2)}(x_1, x_2), \dots) \qquad \Phi^{(r)} \in \mathbb{Q}[x_1, \dots, x_r]$$

$\mathfrak{ls}_r$ is the set of $\mathbb{Q}$-solutions to the linearized equations in depth r. Let

$$\mathfrak{ls} = \bigoplus_{r \geq 1} \mathfrak{ls}_r$$

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Definition

Let $\mathfrak{dg}^{\mathfrak{m}}_{\bullet} = \mathrm{gr}^{\bullet}_{\mathfrak{D}} \mathfrak{g}^{\mathfrak{m}}$ be the depth-graded motivic Lie algebra.

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Relation with the motivic Lie algebra

$$\mathfrak{g}^{\mathfrak{m}} \subset \mathfrak{dmr} \qquad \mathfrak{dg}^{\mathfrak{m}} \subset \mathfrak{ls}$$

Zagier-Ihara-Kaneko have conjectured that $\mathfrak{dg}^{\mathfrak{m}} = \mathfrak{ls}$.

Ihara action

The Ihara bracket can be written explicitly

$$\mathbb{Q}[x_0,\ldots,x_r]\otimes\mathbb{Q}[x_0,\ldots,x_s]\longrightarrow\mathbb{Q}[x_0,\ldots,x_{r+s}]$$

$$f,g\mapsto\{f,g\}$$

where

$$\{f,g\}=\sum_{\mu\in D_{r+s+1}}\varepsilon(\mu)\mu\big(f(x_0,\ldots,x_r)g(x_r,\ldots,x_{r+s})\big)$$

Here $D_{r+s+1}=\langle\sigma,\tau\rangle$ is the dihedral group generated by

$$\begin{aligned}\tau(f)&=f(x_0,x_r,\ldots,x_1)\\ \sigma(f)&=(-1)^r f(-x_r,\ldots,-x_1,-x_0)\end{aligned}$$

and $\varepsilon: D_{r+s+1}\to\{\pm1\}$ is sign representation.

Solving the linearized double shuffle equations

Variant of Racinet's theorem

The vector space of solutions to the linearized double shuffle equations is a bigraded Lie algebra for the Ihara bracket:

$$\{\ ,\ \} : \mathfrak{ls} \wedge \mathfrak{ls} \longrightarrow \mathfrak{ls}$$

In depth one, the only elements in $\mathfrak{ls}_1$ are

$$\mathbf{x_1^{2n}} \qquad \text{for} \qquad n \geq 1$$

These are precisely the depth one components of the σ_{2n+1}

$$\begin{array}{rcl} \mathrm{gr}_{\mathfrak{D}} : \mathfrak{g}^{\mathfrak{m}} & \longrightarrow & \mathfrak{d}\mathfrak{g}^{\mathfrak{m}} \subset \mathfrak{ls} \\ \sigma_{2n+1} & \mapsto & x_1^{2n} \end{array}$$

'Depth-graded motivic multiple zeta values', arXiv:1301.3053

Relations

The theorem means that any iterated bracket

$$\{x_1^{2n_1}, \{x_1^{2n_2}, \{x_1^{2n_3}, \ldots, \} \cdots \} \in \mathfrak{d}\mathfrak{g}^{\mathfrak{m}}$$

Example 5

Ihara discovered the relation $\{x_1^2, x_1^8\} - 3\{x_1^4, x_1^6\} = 0$.

There are many quadratic relations. A relation

$$\sum_{i<j} \lambda_{i,j} \{x_1^{2i}, x_1^{2j}\} = 0$$

holds if and only if the polynomial

$$P(X, Y) = \sum_{i,j} \lambda_{ij} X^{2i} Y^{2j}$$

is an even period polynomial (Ihara-Takao, Goncharov, Schneps)

Period polynomials

Even period polynomials

Let $n \geq 1$ and let $\mathrm{W}^o_{2n} \subset \mathbb{Q}[x,y]$ denote the vector space of homogeneous polynomials $P(x,y)$ of degree $2n-2$ satisfying $P(1,0)=0$,

$$\begin{aligned} P(x,y)+P(y,x) &= 0 \\ P(\pm x, \pm y) - P(x,y) &= 0 \\ P(x,y)+P(x-y,x)+P(-y,x-y) &= 0\,. \end{aligned}$$

The Eichler-Shimura-Manin theorem implies that

$$\dim_{\mathbb{Q}} \mathrm{W}^o_{2k-2} = \dim_{\mathbb{Q}} S_{2k}(PSL_2(\mathbb{Z}))$$

where $S_{2k}(PSL_2(\mathbb{Z}))$ is the space of cusp forms of weight $2k$.

Exceptional solutions

Since $\mathrm{Lie}_{\mathbb{Q}}\langle x_1^2, x_1^4, x_1^6, \ldots\rangle$ is not free, the depth-graded motivic Lie algebra must contain some extra generators in higher depth!

Therefore we should expect a map $\mathsf{W}^o \longrightarrow (\mathfrak{ls})^{ab}$. Surprisingly:

Theorem 6

For every even period polynomial $f \in \mathsf{W}^o$, there is an element

$$\overline{\mathbf{e}}_f \in \mathbb{Q}[x_1, x_2, x_3, x_4]$$

which is defined explicitly, such that $\overline{\mathbf{e}}_f$ is a solution to the (four) linearized double shuffle equations in depth 4.

We actually get a map $\mathsf{W}^o \hookrightarrow \mathfrak{ls}_4$ which is defined over $\mathbb{Z}$.

Not known if the $\overline{\mathbf{e}}_f$ are motivic, i.e., lie in $\mathfrak{dg}_4^{\mathfrak{m}}$.

Motivic Broadhurst-Kreimer-Zagier-Ihara-Kaneko conjecture

If we believe the Broadhurst-Kreimer conjecture we are led to:

Conjecture 2

$$\begin{aligned} H_1(\mathfrak{ls};\mathbb{Q}) &\cong \bigoplus x_1^{2n}\mathbb{Q} \oplus \overline{\mathbf{e}}(\mathsf{W}^o) \\ H_2(\mathfrak{ls};\mathbb{Q}) &\cong \mathsf{W}^o \\ H_i(\mathfrak{ls};\mathbb{Q}) &= 0 \quad \text{for all} \quad i \geq 3 \ . \end{aligned}$$

Suggests that the depth-graded motivic Lie algebra $\mathfrak{dg}^{\mathfrak{m}}$ has:

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Theorem 7

Conjecture 2 $\Rightarrow$ Broadhurst-Kreimer conjecture for motivic MZV's & $\mathfrak{dg}^{\mathfrak{m}} = \mathfrak{ls}$ & Zagier, Drinfel'd conjecture $\mathfrak{g}^{\mathfrak{m}} = \mathfrak{grt} = \mathfrak{dmr}$.

We have:

A complete conjectural description of solutions to the linearized double shuffle equations. $\mathfrak{ls} \longleftrightarrow \mathfrak{dg}^{\mathfrak{m}}$.

Now:

We shall try to construct all solutions to the double shuffle equations modulo products explicitly. $\mathfrak{dmr} \longleftrightarrow \mathfrak{g}^{\mathfrak{m}}$.

Double shuffle equations with poles

Idea[1] is to try solve the double shuffle equations **with poles**. Define

$$\mathcal{O}_d = \mathbb{Q}\Big[x_1, \ldots, x_d, \frac{1}{x_1}, \ldots, \frac{1}{x_d}, \Big(\frac{1}{x_i - x_j}\Big)_{1 \leq i < j \leq d}\Big] .$$

Taking the product over all d, we define

$$\mathcal{O} = \prod_{d \geq 1} \mathcal{O}_d$$

Wish to find elements $\Phi = (\Phi^{(1)}, \Phi^{(2)}, \ldots) \in \mathcal{O}$ which solve the double shuffle equations mod products. We have

$$\mathfrak{g}^{\mathfrak{m}} \subset \mathfrak{dmr} \subset \mathfrak{pdmr} \subset \mathcal{O}$$

where $\mathfrak{pdmr}$ means **'polar solutions to double shuffle'.**

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Miracle: we can write down explicit solutions in $\mathfrak{pdmr}$.

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Canonical Polar Associators

For every $n, d \geq 1$, define

$$\psi_{2n+1}^{(d)} = \frac{1}{2}\sum_{i=1}^{d}\Big(\frac{(x_i - x_{i-1})^{2n}}{x_{\{0,..,i-2\},\{i-1\}}\, x_{\{i+1,..,d\},\{i\}}} + \frac{x_d^{2n}}{x_{\{1,..,i-1\},\{0\}}\, x_{\{i,..,d-1\},\{d\}}}\Big)$$
$$+ \frac{1}{2}\sum_{i=1}^{d-1}\Big(\frac{(x_1 - x_d)^{2n}}{x_{\{2,..,i\},\{1\}}\, x_{\{i+1,..,d-1,0\},\{d\}}} - \frac{x_{d-1}^{2n}}{x_{\{d,1,..,i-1\},\{0\}}\, x_{\{i,..,d-2\},\{d-1\}}}\Big)$$

where for $A, B \subset \{0, \ldots, d\}$, write $x_{A,B} = \prod_{a\in A, b\in B}(x_a - x_b)$,

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For every $n, d \geq 1$, define

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$$+ \frac{1}{2}\sum_{i=1}^{d-1}\Big(\frac{(x_1 - x_d)^{2n}}{x_{\{2,..,i\},\{1\}}\, x_{\{i+1,..,d-1,0\},\{d\}}} - \frac{x_{d-1}^{2n}}{x_{\{d,1,..,i-1\},\{0\}}\, x_{\{i,..,d-2\},\{d-1\}}}\Big)$$

where for $A, B \subset \{0, \dots, d\}$, write $x_{A,B} = \prod_{a\in A, b\in B}(x_a - x_b)$,

Theorem 8

For all $n \geq 1$, the elements $\psi_{2n+1} = (\psi^{(\bullet)}_{2n+1})$ satisfy the double shuffle equations modulo products.

Canonical Polar Associators II

We think of ψ_{2n+1} as the 'polar version' of σ_{2n+1}, since

$$\psi_{2n+1}^{(1)} \quad = \quad x_1^{2n} \quad = \quad \sigma_{2n+1}^{(1)}$$

It corresponds to $\zeta(2n+1)$. So we have *integral* polar generators

$$\psi_3, \psi_5, \psi_7, \ldots$$

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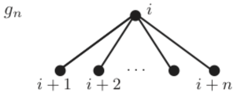
However, we cannot construct elements of $\mathfrak{g}^{\mathfrak{m}}$ out of these polar solutions. Idea from Quantum Field Theory: we need to construct a 'pure pole' solution in $\mathfrak{pdmr}$ and subtract counter-terms to cancel the poles.

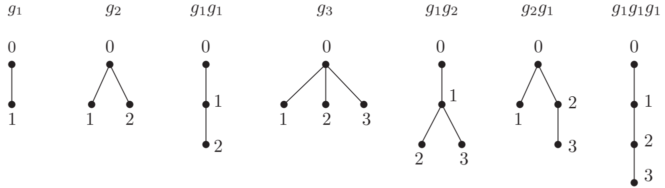
Vineyards

A *bunch of n grapes* g_n is a labelled tree with vertices $\{i, i+1, \ldots, i+n\}$



The vertex i is the *stalk*, and $i, \ldots, i+n$ the *grapes*.

A *vine* $v = g_{i_1} \ldots g_{i_n}$ is a labelled rooted tree obtained by grafting bunches of grapes. Each stalk must be glued to the highest grape.



Define the *height* $h(g_{i_1} \ldots g_{i_k})$ to be k.

A solution corresponding to $\zeta(-1)$

If v is a vine, define a polynomial

$$x_v = \prod_{(i,j)\in E(v)} (x_j - x_i)$$

where $x_0 = 0$ and the product is over edges (i,j) with $i < j$. Define

$$\psi_{-1}^{(d)} = \sum_{v\in\mathcal{V}_d} \frac{(-1)^{h(v)+1}}{h(v)} \frac{1}{x_v x_d} \ ,$$

where the sum is over vines with d grapes. It has $\psi_{-1}^{(1)} = x_1^{-2}$.

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Theorem 9

The element $\psi_{-1} = (\psi_{-1}^{(\bullet)})$ is a solution to the double shuffle equations modulo products

The Lie algebra of polar associators

We obtain a graded Lie algebra

$$\mathcal{L} = \mathrm{Lie}_{\mathbb{Q}}\langle \psi_{-1}, \psi_3, \psi_5, \psi_7, \ldots \rangle$$

equipped with the Ihara bracket.

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equipped with the Ihara bracket. By adapting Racinet's theorem:

Theorem 10

Every $\xi \in \mathcal{L}$ is a solution to the double shuffle equations mod products.

Suppose that we have a rational combinations of brackets

$$\{\psi_{2n_1+1}, \{\psi_{2n_2+1}, \{\psi_{2n_3+1}, \ldots\}\} \cdots\} \in \mathcal{L}$$

such that all poles cancel. Such a combination will be in $\mathfrak{dmr}$. By Zagier's conjecture it will be in the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$. Conversely, every element in $\mathfrak{g}^{\mathfrak{m}}$ should have a 'Taylor expansion' in $\mathcal{L}$.

Anatomy of an associator

Anatomies of the canonical elements $\sigma_3, \sigma_5, \sigma_7, \sigma_9$:

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$$\begin{aligned}
\sigma_3 &\equiv \psi_3 \\
\sigma_5 &\equiv \psi_5 - \frac{1}{60}\{\psi_{-1},\{\psi_{-1},\psi_7\}\} - \frac{1}{5}\{\psi_3,\{\psi_3,\psi_{-1}\}\} \\
\sigma_7 &\equiv \psi_7 - \frac{1}{112}\{\psi_{-1},\{\psi_{-1},\psi_9\}\} - \frac{1}{14}\{\psi_5,\{\psi_3,\psi_{-1}\}\} \\
&\quad -\frac{29}{224}\{\psi_3,\{\psi_5,\psi_{-1}\}\} \; + \{\text{terms of depth } \geq 5\} \\
\sigma_9 &\equiv \psi_9 - \frac{1}{180}\{\psi_{-1},\{\psi_{-1},\psi_{11}\}\} - \frac{7}{180}\{\psi_7,\{\psi_3,\psi_{-1}\}\} \\
&\quad -\frac{113}{180}\{\psi_3,\{\psi_7,\psi_{-1}\}\} - \frac{1}{16}\{\psi_5,\{\psi_5,\psi_{-1}\}\} + \{\text{dpth } \geq 5\}
\end{aligned}$$

Subtraction of counter-terms

Idea is to construct expansion of σ_{2n+1} depth by depth.

Theorem 11

Let $\xi \in \mathcal{L}$ of weight ≥ 0. Then $\xi^{(r)}$ has at most simple poles.

Theorem 12

Let $\xi \in \mathcal{L}$ such that $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(d-1)}$ have no poles. Then $\xi^{(d)} \in \mathcal{O}_d$ only has poles along 'consecutive diagonals'

$$x_1 = 0, \quad x_2 = x_1, \quad , \dots , \quad x_{n-1} = x_n, \quad x_n = 0$$

What is the conjecture which guarantees the existence of this expansion?

Main conjecture (first version)

Definition

Let $\mathfrak{pls}_d$ be the space of solutions $f(x_1, \ldots, x_d) \in \mathcal{O}_d$ to the *linearized double shuffle equations in depth d* such that

$$x_1(x_2 - x_1)\ldots(x_d - x_{d-1})x_d\, f \quad \in \mathbb{Q}[x_1, \ldots, x_d]$$

Then $\mathfrak{pls} = \bigoplus_d \mathfrak{pls}_d$ is a Lie algebra for the Ihara bracket.

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Conjecture 3

The Lie algebra $\mathfrak{pls}$ is generated by

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Theorem 13

Conjecture 3 implies that $\mathfrak{g}^{\mathfrak{m}} \subset \mathcal{L} = \mathrm{Lie}_{\mathbb{Q}}\langle \psi_{-1}, \psi_3, \psi_5, \ldots \rangle$.

Remarks

- If conjecture 3 is true, this construction 'rigidifies' the motivic Lie algebra enormously. A priori, σ_{11} was ambiguous up to a multiple of $\{\sigma_3, \{\sigma_3, \sigma_5\}\}$. The above construction fixes σ_{11} uniquely.

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- If conjecture 3 is true, this construction 'rigidifies' the motivic Lie algebra enormously. A priori, σ_{11} was ambiguous up to a multiple of $\{\sigma_3, \{\sigma_3, \sigma_5\}\}$. The above construction fixes σ_{11} uniquely.
- The anatomical decomposition is not unique. We shall see that the ambiguity is precisely controlled by modular forms.
- The anatomical decomposition is extremely compact. A priori σ_9 has 512 coefficients. They can be completely reconstructed from

$$\frac{1}{180}, -\frac{7}{180}, -\frac{113}{180}, -\frac{1}{16}$$

just **four numbers**! In weights ≤ 13, the MZV data mine contains $\sim 30,000$ numbers. It should reduce to about 40.

European Research Council

Question: If power-series correspond to words via

$$e_0^{a_0} e_1 e_0^{a_1} \ldots e_1 e_0^{a_r} \quad \Longleftrightarrow \quad x_0^{a_0} x_1^{a_1} \ldots x_r^{a_r} \ ;$$

then what is the meaning of power-series **with poles**?

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then what is the meaning of power-series **with poles**?

Answer: Pass from genus 0 to genus 1.

Eisenstein derivations

$\mathcal{E}\backslash\{0\}$ punctured elliptic curve, $\mathrm{Lie}^{\mathrm{gr}}(\pi_1^{un}(\mathcal{E}\backslash\{0\})) \cong \mathrm{Lie}_{\mathbb{Q}}\langle a, b\rangle$. Let $\mathfrak{d}$ be the set of derivations δ on $T(a\,\mathbb{Q} \oplus b\,\mathbb{Q})$ such that

$$\delta([a, b]) = 0$$

and $a^\vee(\delta(b)) = 0$. Then δ is uniquely determined by $\delta(a)$.

Definition Nakamura-Asada, ..., Hain-Matsumoto, Pollack

For all $n \geq 0$ there is a unique $\varepsilon_{2n} \in \mathfrak{d}$ such that $\varepsilon_{2n}(a) = \mathrm{ad}(a)^{2n} b$.

Let $\mathfrak{u}^\epsilon \subset \mathfrak{d}$ be the Lie algebra spanned by ε_{2n}, $n \geq 0$. It encodes in particular the universal monodromy of $\mathfrak{M}_{1,1}$ on universal elliptic curve.

Words and rational functions

A derivation $\delta \in \mathfrak{d}$ is determined by

$$\delta(a) \in T(a\,\mathbb{Q} \oplus b\,\mathbb{Q})$$

The 'depth' is the now the degree in b. As before, map to words

$$\begin{array}{ccc} \mathrm{gr}^{(r)}_{\mathfrak{D}} T(a\,\mathbb{Q} \oplus b\,\mathbb{Q}) & \longrightarrow & \mathbb{Q}[x_0, \ldots, x_r] \\ a^{k_0} b a^{k_1} \ldots b a^{k_r} & \mapsto & x_0^{k_0} x_1^{k_1} \ldots x_r^{k_r} \end{array}$$

but this time divide by $(x_0 - x_1)\ldots(x_r - x_{r-1})(x_r - x_0)$.
This gives a map from derivations to **rational functions**:

$$\mathfrak{d} \longrightarrow \frac{\mathbb{Q}[x_0, x_1, \ldots, x_r]}{(x_1 - x_0)(x_2 - x_1)\ldots(x_r - x_{r-1})(x_r - x_0)}$$

Back to the linearized double shuffle equations

Everything is translation-invariant, so we set $x_0 = 0$ as usual.

Example 14

The elements ε_{2n} satisfy $\varepsilon_{2n}(a) = ad(a)^{2n}(b)$. Therefore

$$\varepsilon_{2n} \mapsto \frac{(x_1 - x_0)^{2n}}{(x_1 - x_0)^2} \xrightarrow{x_0=0} x_1^{2n-2}$$

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Theorem 15

The map above gives a morphism of Lie algebras

$$\mathfrak{u}^{\epsilon} \hookrightarrow \mathfrak{pls}$$

The Eisenstein derivations satisfy the ordinary linearized double shuffle equations.

Main conjecture revisited

Since $\mathfrak{u}^{\epsilon}$ is spanned by $\varepsilon_{2n} \leftrightarrow x_1^{2n-2}$, for $n \geq 0$, we get

Conjecture 3 revisited

$$\mathfrak{u}^{\epsilon} = \mathfrak{pls}$$

This is an elliptic analogue of Zagier's conjecture $\mathfrak{g}^{\mathfrak{m}} = \mathfrak{dmr}$.

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Theorem 16

The conjecture is true in depths ≤ 3.

In particular, Hain-Matsumoto-Pollack's period polynomial relations lift in depth 3, and these are the only relations.

Structures on $\mathfrak{pls}$

Recall that $\mathfrak{u}^{\epsilon} \subset \mathfrak{pls}$. We have:

1. An action of $\mathfrak{sl}_2$ on $\mathfrak{pls}$ which extends the action on $\mathfrak{u}^{\epsilon}$.
2. An increasing filtration $\mathfrak{R}_n\mathfrak{pls}$ which is preserved by the Lie bracket. It is concentrated in non-negative degrees. It satisfies

$$\mathfrak{ls} = \mathfrak{R}_0\mathfrak{pls}$$

   Therefore, $\mathfrak{dg}^{\mathfrak{m}} \subset \mathfrak{R}_0\mathfrak{pls}$.

3. So $\mathfrak{pls}$ is a triply-graded Lie algebra. It forms a bridge between the Broadhurst-Kreimer conjecture and the Lie algebra of Eisenstein derivations. It satisfies

$$\mathfrak{dg}^{\mathfrak{m}} \subset \mathfrak{pls} \supset \mathfrak{u}^{\epsilon}$$

Conclusion

- We defined an *explicit* Lie algebra

$$\mathcal{L} = \mathrm{Lie}_{\mathbb{Q}}\langle \psi_{-1}, \psi_3, \psi_5, \ldots \rangle$$

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- If conjecture true, then the motivic Lie algebra embeds

$$\mathfrak{g}^{\mathfrak{m}} \hookrightarrow \mathcal{L} \qquad \text{`Anatomy'}$$

  This is a theorem in depths $1, 2, 3, 4$. Also implies $\mathfrak{d}\mathfrak{g}^{\mathfrak{m}} = \mathfrak{R}_0 \mathfrak{u}^{\epsilon}$.

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- The elements ψ_{2n+1} should be viewed as a 'lifting' of the Eisenstein derivations ϵ_{2n+2} for $n \geq -1$. In other words $\mathcal{L} \longrightarrow \mathfrak{u}^{\epsilon}$
$$\psi_{2n+1} \mapsto \epsilon_{2n+2}$$

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- Similar story for associators $\Rightarrow$ Massive compression of MZV tables.

....Hold on!....

The double shuffle equations are in fact *vacuous* in depth 1.

We know that all solutions are necessarily even functions in depth 1, but what about solutions with poles...?

An exceptional solution

Return to double shuffle equations:

$$\begin{aligned} 0 &= \Phi^{(2)}(x_1,x_2)+\Phi^{(2)}(x_2,x_1)+\tfrac{\Phi^{(1)}(x_1)-\Phi^{(1)}(x_2)}{x_1-x_2} \\ 0 &= \Phi^{(2)}(x_1,x_1+x_2)+\Phi^{(2)}(x_2,x_1+x_2) \end{aligned} \tag{2}$$

Easy to see that the only solutions satisfy $\Phi^{(1)}(x_1)=\Phi^{(1)}(-x_1)$ when $\Phi^{(1)},\Phi^{(2)}$ are power series. When we allow poles, there is

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Exceptional solution

$$\Phi^{(1)}(x_1)=\frac{1}{x_1} \quad \text{and} \quad \Phi^{(2)}(x_1,x_2)=\frac{1}{3}\Big(\frac{2}{x_1x_2}+\frac{1}{x_1(x_1-x_2)}\Big) .$$

This element has weight 0: it corresponds to $\zeta(0)$.

A weight 0 element

Recall that we had an algebra of vines

$$\mathbb{Q}\langle g_1, g_2, \ldots \rangle$$

where g_n is in degree n. It is a Hopf algebra for the coproduct $\Delta(g_n) = \sum_{i+j=n} g_i \otimes g_j$. Define a series of elements

$$\mathsf{s}_n = S \star Y(g_n)$$

where $S \star Y$ is the Dynkin operator. Then

$$\mathsf{s}_1 = g_1 \qquad \mathsf{s}_2 = 2g_2 - g_1 g_1$$

We had a map x from vines to rational functions. Define

$$\psi_0^{(d)} = \binom{d+1}{2}^{-1} x_{\mathsf{s}_d}^{-1}$$

Theorem 17

$\psi_0 = (\psi_0^{(\bullet)})$ *solves the double shuffle equations mod products.*

Twisting with ψ_0

Definition of twisting

Let $0 \neq \alpha \in \mathcal{O}_d$ be a solution to the linearized double shuffle equations in depth d. Let $\widetilde{\alpha}^{(i)} = 0$ for $i < d$, and $\widetilde{\alpha}^{(d)} = \alpha$. Recursively define

$$\widetilde{\alpha}^{(d+k)} = \frac{1}{2k} \sum_{i=1}^{k} \{\psi_0^{(i)}, \widetilde{\alpha}^{(d+k-i)}\} \tag{3}$$

for $k \geq 1$. Let $\widetilde{\alpha} = (\widetilde{\alpha}^{(\bullet)})$. Its first non-zero component is α.

Theorem 18

$\widetilde{\alpha}$ is a solution to the double shuffle equations mod products.

Proof of theorem 18 in progress

Twisting with ψ_0

We could have saved ourselves a lot of trouble and defined

$$\chi_{2n+1} = \widetilde{x_1^{2n}} \in \mathfrak{pdmr}$$

We get solutions to double shuffle equations for free! But they are different from the ones defined earlier: $\chi_{2n+1} \neq \psi_{2n+1}$, and have bad poles.

Example: χ_{-1} starts to differ from ψ_{-1} starting from depth 5.

Twisting with ψ_0 gives an *unconditional* anatomical decomposition for the motivic Lie algebra in the space of solutions to the linearized double shuffle equations (with more general poles, this time).

An algebraic structure

What happens if we allow ψ_0 to twist with itself?
We get a copy of the Witt algebra.

Theorem 19

The elements $s_d = \binom{d+1}{2}\psi_0^{(d)}$ *satisfy*

$$\{s_m, s_n\} = (m-n)s_{m+n}$$

In fact, we get an action of the Witt algebra on the Hopf algebra of vines coming from the linearized Ihara action:

$$s_1 \underline{\circ}\, g_n = (n+1)g_{n+1} \qquad \text{and} \qquad s_2 \underline{\circ}\, g_n = (n+2)g_{n+2} - g_{n+1}g_1$$

which satisfies

$$a \underline{\circ} (b.c) = (a \underline{\circ}\, b)c + b(a \underline{\circ}\, c) - abc$$

What is the meaning of this algebraic structure?