AN EXACT SEQUENCE FOR THE BROADHURST-KREIMER CONJECTURE

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Don Zagier asked me whether the Broadhurst-Kreimer conjecture could be reformulated as a short exact sequence of spaces of polynomials in commutative variables. The purpose of this note is to describe just such a sequence.

1. Depth-graded double shuffle Hopf algebra

Recall the double shuffle equations from [5], Définition 1.3. Let $\Phi \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ be a non-commutative formal power series with coefficients in $\mathbb{Q}$. Let

$$\Delta_{\text{ш}} : \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \longrightarrow \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \widehat{\otimes}_{\mathbb{Q}} \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$$

denote the continuous comultiplication for which e_0, e_1 are primitive, such that $\Delta_{\text{ш}}$ is a homomorphism for the concatenation product. The coefficients of the element Φ *satisfy the shuffle equations* if $\Delta_{\text{ш}}\Phi = \Phi \otimes \Phi$. Now let $Y = \{y_1, y_2, \ldots, \}$ denote an alphabet in infinitely many elements y_i of degree i and let

$$\Delta_* : \mathbb{Q}\langle\langle Y\rangle\rangle \longrightarrow \mathbb{Q}\langle\langle Y\rangle\rangle \widehat{\otimes}_{\mathbb{Q}} \mathbb{Q}\langle\langle Y\rangle\rangle$$

denote the continuous comultiplication such that $\Delta_*(y_n) = \sum_{i+j=n} y_i \otimes y_j$, and such that Δ_* is a homomorphism for the concatenation product. The coefficients of an element $\Psi \in \mathbb{Q}\langle\langle Y\rangle\rangle$ *satisfy the stuffle equations* if $\Delta_*\Psi = \Psi \otimes \Psi$. Racinet's group of solutions to the *double shuffle equations* consists of elements $\Phi \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ such that

$$(1.1) \qquad \begin{aligned} \Delta_{\text{ш}}\Phi &= \Phi \otimes \Phi \\ \Delta_*\Phi_* &= \Phi_* \otimes \Phi_* \\ \Phi(e_0) = \Phi(e_1) = 0 \quad &, \quad \Phi(1) = 1 \end{aligned}$$

where $\Phi(w)$ is the coefficient of a word $w \in \{e_0, e_1\}^\times$ in Φ, and $\Phi_* \in \mathbb{Q}\langle\langle Y\rangle\rangle$ is obtained from Φ by a regularization procedure, which we shall not require here. We can view a solution to (1.1) either as an element $\Phi \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ or as an element $\Phi_* \in \mathbb{Q}\langle\langle Y\rangle\rangle$ since they determine each other uniquely.

1.1. **Depth-graded version.** Recall that the *depth filtration* is the decreasing filtration on $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ (and respectively $\mathbb{Q}\langle\langle Y\rangle\rangle$) with respect to the $\mathfrak{D}$-degree, where e_0 has $\mathfrak{D}$-degree 0, and e_1 has $\mathfrak{D}$-degree 1 (respectively y_n has $\mathfrak{D}$-degree 1). By passing to the associated weight and depth bigraded Hopf algebras, $\Delta_{\text{ш}}$ becomes the coproduct

$$(1.2) \qquad \Delta_{\text{ш}} : \mathbb{Q}\langle e_0, e_1\rangle \longrightarrow \mathbb{Q}\langle e_0, e_1\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle e_0, e_1\rangle$$

with respect to which e_0, e_1 are primitive (no change here), and Δ_* becomes

$$(1.3) \qquad \Delta^Y_{\text{ш}} : \mathbb{Q}\langle Y\rangle \longrightarrow \mathbb{Q}\langle Y\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle Y\rangle$$

for which y_n is primitive for all $n \geq 1$. Now define a map

$$(1.4) \qquad \begin{aligned} \alpha : \mathbb{Q}\langle e_0, e_1\rangle \quad &\longrightarrow \quad \mathbb{Q}\langle Y\rangle \\ \alpha(e_1 e_0^{n_1} \ldots e_1 e_0^{n_r}) \quad &= \quad y_{n_1+1} \ldots y_{n_r+1} \end{aligned}$$

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and such that α sends all words beginning in e_0 to zero. A section of this map is given by the map $\beta : \mathbb{Q}\langle Y\rangle \rightarrow \mathbb{Q}\langle e_0, e_1\rangle$ which sends y_n to $e_1 e_0^{n-1}$ and is a homomorphism for the concatenation products.

Definition 1.1. An element $\Psi \in \mathbb{Q}\langle Y\rangle$ is a solution to the *depth-graded double shuffle equations* if it equalizes the two coproducts (1.2) and (1.3) and is even in depth one:

$$\begin{aligned} \Delta_{\text{Ш}}\, \beta(\Psi) &= (\beta\otimes\beta)(\Delta^Y_{\text{Ш}}\, \Psi) \\ \Psi(y_n) &= 0 \text{ if } n \text{ is even or } n \text{ is } 1 \end{aligned} \tag{1.5}$$

The reason for the second condition in (1.5) is explained in [1]: the double shuffle equations, restricted to depth one, are vacuous. Nonetheless the full equations (in particular in depth ≤ 2) imply evenness in depth one, so this condition must be added back artificially to the depth-graded versions of the equations.

Definition 1.2. Let $\widetilde{D} \subset \mathbb{Q}\langle Y\rangle$ denote the largest bigraded subspace of the vector space of solutions to (1.5) which is a coalgebra for $\Delta^Y_{\text{Ш}}$.

For simplicity, we shall denote the coproduct by

$$\Delta : \widetilde{D} \longrightarrow \widetilde{D} \otimes_{\mathbb{Q}} \widetilde{D}\ .$$

Recall that the *linearized double shuffle equations* [1] are defined by the bigraded vector space $\mathfrak{ls}$ of elements $\Phi \in \mathbb{Q}\langle e_0, e_1\rangle$ which satisfy the equations:

$$\begin{aligned} \Delta_{\text{Ш}}\, \Phi &= 1\otimes\Phi + \Phi\otimes 1 \\ \Delta^Y_{\text{Ш}}\, \alpha(\Phi) &= 1\otimes\alpha(\Phi) + \alpha(\Phi)\otimes 1 \\ \Phi(e_0^i e_1) &= 0 \text{ if } i \text{ is odd} \end{aligned} \tag{1.6}$$

and satisfy $\Phi(e_0) = 0$ and $\Phi(e_1) = 0$. Note that, compared to [1] §7, we have added the condition that $\Phi(e_1)$ vanish to exclude the trivial solution to these equations.

Lemma 1.3. *The primitive elements in $\widetilde{D}$ are exactly given by $\mathfrak{ls}$.*

Proof. Suppose that $\Psi \in \widetilde{D} \subset \mathbb{Q}\langle Y\rangle$ is primitive for $\Delta^Y_{\text{Ш}}$ and therefore by (1.5), $\beta(\Psi)$ is primitive for $\Delta_{\text{Ш}}$. Since every word in $\{e_0, e_1\}^\times$ can be uniquely written as a linear combination of shuffles of e_0^n with words beginning in e_1, we can uniquely extend $\beta(\Psi)$ to an element $\Phi \in \mathbb{Q}\langle e_0, e_1\rangle$ which is primitive for $\Delta_{\text{Ш}}$ and satisfies $\Phi(e_0) = 0$ and $\alpha(\Phi) = \Psi$. The element Φ satisfies the linearized double shuffle equations (1.6). □

1.2. **The Ihara action.** The action on the pro-unipotent fundamental groupoid of $\mathbb{P}^1\backslash\{0, 1, \infty\}$ by automorphisms gives rise to a continuous $\mathbb{Q}$-linear map

$$\circ : {}_0\Pi_1(\mathbb{Q})\, \widehat{\otimes}\, {}_0\Pi_1(\mathbb{Q}) \longrightarrow {}_0\Pi_1(\mathbb{Q})\ ,$$

known as the *Ihara action*, where ${}_0\Pi_1(\mathbb{Q}) \subset \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ consists of invertible power series Φ which are group-like for $\Delta_{\text{Ш}}$. Concretely, $\circ$ is defined on power series by

$$F(e_0, e_1), G(e_0, e_1) \mapsto G(e_0, F(e_0, e_1)e_1F(e_0, e_1)^{-1})F(e_0, e_1)$$

One of the main results of Racinet's thesis [5] is the following.

Theorem 1.4. *(Racinet) The solutions to the double shuffle equations* (1.1) *are preserved by the Ihara action.*

In [2], we defined a variant of the Ihara action, called the *linearized Ihara action.*

Definition 1.5. The linearized Ihara action is the $\mathbb{Q}$-bilinear map

$$\underline{\circ} : \mathbb{Q}\langle e_0, e_1\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle e_0, e_1\rangle \to \mathbb{Q}\langle e_0, e_1\rangle$$

defined inductively as follows. For words a, w in e_0, e_1, and for any integer $n \geq 0$, let

$$a \,\underline{\circ}\, (e_0^n e_1 w) = e_0^n a e_1 w + e_0^n e_1 a^* w + e_0^n e_1 (a \,\underline{\circ}\, w) \tag{1.7}$$

where $a \,\underline{\circ}\, e_0^n = e_0^n\, a$, and for any $a_i \in \{e_0, e_1\}^\times$, $(a_1 \dots a_n)^* = (-1)^n a_n \dots a_1$.

The action $\underline{\circ}$ is not associative but satisfies

$$f_1 \,\underline{\circ}\, (f_2 \,\underline{\circ}\, g) - f_2 \,\underline{\circ}\, (f_1 \,\underline{\circ}\, g) = (f_1 \,\underline{\circ}\, f_2) \,\underline{\circ}\, g - (f_2 \,\underline{\circ}\, f_1) \,\underline{\circ}\, g \tag{1.8}$$

for all $f_1, f_2, g \in \mathbb{Q}\langle e_0, e_1\rangle$. A variant of Racinet's theorem is the following:

Corollary 1.6. *The linearized Ihara action defines a map*

$$\underline{\circ} : \mathfrak{ls} \otimes_{\mathbb{Q}} \widetilde{D} \longrightarrow \widetilde{D} \tag{1.9}$$

1.3. **A variant of the Milnor-Moore theorem.** Suppose that we have a coalgebra A over $\mathbb{Q}$, with coproduct

$$\Delta : A \longrightarrow A \otimes_{\mathbb{Q}} A$$

which is graded, connected, and cocommutative. Let $\mathfrak{a}$ denote the set of primitive elements of A. Suppose now that we have a $\mathbb{Q}$-bilinear map

$$\mathfrak{a} \circ A \longrightarrow A \tag{1.10}$$

which is graded, and such that $\Delta(f \circ g) = \Delta(f) \circ \Delta(g)$ for all $f \in \mathfrak{a}, g \in A$. Denote the antisymmetrization $\wedge^2 \mathfrak{a} \to \mathfrak{a}$ by $\{f, g\} = f \circ g - g \circ f$. Suppose furthermore that $\circ$ satisfies the pre-Lie identity:

$$f_1 \circ (f_2 \circ g) - f_2 \circ (f_1 \circ g) = \{f_1, f_2\} \circ g$$

for all $f_1, f_2 \in \mathfrak{a}, g \in A$. Then in particular, $\mathfrak{a}$ is a Lie algebra with respect to $\{,\}$ and the map (1.10) defines a map $\mathcal{U}\mathfrak{a} \to A$, where $\mathcal{U}\mathfrak{a}$ is the universal enveloping algebra of $\mathfrak{a}$.

Proposition 1.7. *With these assumptions, $\mathcal{U}\mathfrak{a} \cong A$.*

Proof. For any connected graded Hopf algebra H, let $\Delta^{(2)} = \Delta(x) - x \otimes 1 - 1 \otimes x$ denote the reduced coproduct, and define $\Delta^{(r)} : H \to H^{\otimes r}$ to be the iterated reduced coproduct for $r \geq 2$, and the identity for $r = 1$. For every $x \in H$ there is a smallest r such that $\Delta^{(r)}(x) = 0$. This defines an increasing (coradical) filtration R on H. In our situation, A is cocommutative and so we obtain a map

$$\Delta^{(r-1)} : \mathrm{gr}_r^R A \longrightarrow \mathrm{Sym}^{r-1} \mathfrak{a}\ .$$

By iterating (1.10) and symmetrizing, we obtain a map $m : \mathrm{Sym}^{r-1}\mathfrak{a} \to R_r A$, which is an inverse to $\Delta^{(r-1)}$ by the compatibility between Δ and $\circ$. Thus

$$\mathrm{gr}_r^R A \cong \mathrm{Sym}^{r-1} \mathfrak{a}\ .$$

But $\mathrm{Sym}^{r-1}\mathfrak{a}$ is isomorphic to $\mathrm{gr}_r^R \mathcal{U}\mathfrak{a}$ by the Poincaré-Birkhoff-Witt theorem. Therefore $\mathcal{U}\mathfrak{a} \to A$ is an isomorphism (of coalgebras). □

By applying the previous proposition to $\widetilde{D}$, we deduce from lemma 1.3 that the coalgebra $\widetilde{D}$ is isomorphic to the universal enveloping algebra of $\mathfrak{ls}$:

$$\mathcal{U}\mathfrak{ls} \cong \widetilde{D}\ . \tag{1.11}$$

2. Equations for polynomials in commuting variables

In [1], §3-5, it is explained how to translate Hopf algebraic properties of series Φ as described above, into functional equations for power series in commuting variables. The basic remark is that there is an isomorphism of graded vector spaces

$$\begin{array}{rcl} \mathrm{gr}^r_{\mathfrak{D}} \mathbb{Q}\langle Y \rangle & \longrightarrow & \mathbb{Q}[x_1, \ldots, x_r] \\ y_{i_1} \ldots y_{i_r} & \mapsto & x_1^{i_1} \ldots x_r^{i_r} \end{array}$$

where the weight of a polynomial in $\mathbb{Q}[x_1, \ldots, x_r]$ is defined to be the degree plus the number of variables. The (p,q)-th shuffle equations are defined to be the $(p,q)^{\mathrm{th}}$ component of $\Delta_{\text{ш}} \Psi$. If the depth $p+q$-component of Ψ is the element f, it is written

$$f^\sharp(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q})$$

where, using the notation from [4], we define

$$g^\sharp(x_1, \ldots, x_n) = g(x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_n) \ ,$$

and ш is the shuffle product acting formally on the arguments of the function f; thus $f(x_i u \,\text{ш}\, x_j v) = f(x_i, u \,\text{ш}\, x_j v) + f(x_j, x_i u \,\text{ш}\, v)$. Likewise, the (p,q)-th stuffle equation is defined to be the $(p,q)^{\mathrm{th}}$ component of $\Delta^Y_{\text{ш}} \Psi$. It is written

$$f(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q})$$

Corresponding to the $(i,j)^{\mathrm{th}}$ component of β, let us define a map $\beta_{i,j}$:

$$\beta_{i,j} f(x_1, \ldots, x_{i+j}) = f(x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_i, x_{i+1}, x_{i+1} + x_{i+2}, \ldots, x_i + \ldots + x_{i+j}) \tag{2.1}$$

Lemma 2.1. *The defining equations for $\widetilde{D}_n$, where $n \geq 2$ correspond to:*

$$\begin{array}{rcl} f^\sharp(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q}) & = & \beta_{p,q} f(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q}) \\ f(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q}) & \in & \widetilde{D}_p \otimes_{\mathbb{Q}} \widetilde{D}_q \end{array} \tag{2.2}$$

for all $1 \leq p \leq q$ where $p+q = n$. In the second line of these equations we identify $\mathbb{Q}[x_1, \ldots x_p] \otimes_{\mathbb{Q}} \mathbb{Q}[x_1, \ldots, x_q]$ with $\mathbb{Q}[x_1, \ldots, x_{p+q}]$.

For comparison, the defining equations for $\mathfrak{ls}_n$, where $n \geq 2$ correspond to

$$\begin{array}{ll} f^\sharp(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q}) = 0 & \text{for all } 1 \leq p \leq q \quad , \quad p+q = n \\ f(x_1 \ldots x_p \,\text{ш}\, x_{p+1} \ldots x_{p+q}) = 0 & \text{for all } 1 \leq p \leq q \quad , \quad p+q = n \end{array} \tag{2.3}$$

The defining equations for $\widetilde{D}_n$ and $\mathfrak{ls}_n$ in depth $n=1$ are simply $f(0) = 0$, $f(x_1)$ even, by the second lines of equations (1.5) and (1.6), giving

$$\widetilde{D}_1 \quad = \quad \mathfrak{ls}_1 \quad \cong \quad x_1^2 \mathbb{Q}[x_1^2] \tag{2.4}$$

Of course, $\widetilde{D}_0 = \mathbb{Q}$, by definition.

2.1. Linearized Ihara action for polynomials.

In [2] and [1] we wrote down the following explicit formula for the linearized Ihara action:

$$\underline{\circ} : \mathbb{Q}[x_1, \ldots, x_r] \otimes_{\mathbb{Q}} \mathbb{Q}[x_1, \ldots, x_s] \longrightarrow \mathbb{Q}[x_1, \ldots, x_{r+s}]$$

which is given explicitly by

$$f \,\underline{\circ}\, g\,(x_1,\dots,x_{r+s}) = \sum_{i=0}^{s} f(x_{i+1}-x_i,\dots,x_{i+r}-x_i)g(x_1,\dots,x_i,x_{i+r+1},\dots,x_{r+s})$$
$$-(-1)^{\deg f + r}\sum_{i=1}^{s} f(x_{i+r-1}-x_{i+r},\dots,x_i-x_{i+r})g(x_1,\dots,x_{i-1},x_{i+r},\dots,x_{r+s})$$

Specializing to the case when $r=1$, the previous formula reduces to

$$\begin{array}{rcl} \mathbb{Q}[x_1^2]\otimes_{\mathbb{Q}}\mathbb{Q}[x_1,\dots,x_{s-1}] & \longrightarrow & \mathbb{Q}[x_1,\dots,x_s] \\ x_1^{2n}\,\underline{\circ}\, g(x_1,\dots,x_{s-1}) & = & \displaystyle\sum_{i=1}^{s}\big((x_i-x_{i-1})^{2n}-(x_i-x_{i+1})^{2n}\big)g(x_1,\dots,\widehat{x_i},\dots,x_s) \end{array}$$

where $x_0 = 0$ and $x_{s+1} = x_s$ (i.e., the term $(x_s - x_{s+1})^{2n}$ is discarded).

2.2. **Examples in depths 2 and 3.**

2.2.1. *Depth 2.* The space $\widetilde{D}_2$ is defined by the equations

$$\begin{array}{rcl} f^\sharp(x_1 ш x_2) & = & f(x_1 ш x_2) \\ f(x_1 ш x_2) & \in & \widetilde{D}_1\otimes_{\mathbb{Q}}\widetilde{D}_1 \end{array} \tag{2.5}$$

Concretely, this is the pair of equations

$$\begin{array}{rcl} f(x_1,x_1+x_2)+f(x_2,x_1+x_2) & = & f(x_1,x_2)+f(x_2,x_1) \\ f(x_1,x_2)+f(x_2,x_1) & \in & x_1^2x_2^2\mathbb{Q}[x_1^2,x_2^2] \end{array} \tag{2.6}$$

Compare the space $\mathfrak{ls}_2$ of linearized double shuffle equations in depth 2, given by

$$\begin{array}{rcl} f(x_1,x_1+x_2)+f(x_2,x_1+x_2) & = & 0 \\ f(x_1,x_2)+f(x_2,x_1) & = & 0 \end{array} \tag{2.7}$$

The map $\mathfrak{ls}_1\otimes\mathfrak{ls}_1 \longrightarrow \widetilde{D}_2$ is given by

$$x_1^{2m}\,\underline{\circ}\, x_1^{2n} = x_1^{2m}x_2^{2n} + (x_2-x_1)^{2m}x_1^{2n} - (x_2-x_1)^{2m}x_2^{2n} \tag{2.8}$$

2.2.2. *Depth 3.* The space $\widetilde{D}_3$ is defined by the equations

$$\begin{array}{rcl} f^\sharp(x_1 ш x_2x_3) & = & \beta_{1,2}f(x_1 ш x_2x_3) \\ f(x_1 ш x_2x_3) & \in & \widetilde{D}_1\otimes_{\mathbb{Q}}\widetilde{D}_2 \end{array} \tag{2.9}$$

Concretely, this is the pair of equations

$$\begin{aligned} & f(x_1,x_{12},x_{123})+f(x_2,x_{12},x_{123})+f(x_2,x_{23},x_{123}) \\ &\qquad = f(x_1,x_2,x_{23})+f(x_2,x_1,x_{23})+f(x_2,x_{23},x_1) \end{aligned} \tag{2.10}$$

$$f(x_1,x_2,x_3)+f(x_2,x_1,x_3)+f(x_2,x_3,x_1)\in x_1^2\mathbb{Q}[x_1^2]\otimes_{\mathbb{Q}}\widetilde{D}_2$$

where we write x_{ab} for x_a+x_b, and x_{abc} for $x_a+x_b+x_c$.

Compare the space $\mathfrak{ls}_3$ of linearized double shuffle equations in depth 2, given by

$$\begin{array}{rcl} f(x_1,x_{12},x_{123})+f(x_2,x_{12},x_{123})+f(x_2,x_{23},x_{123}) & = & 0 \\ f(x_1,x_2,x_3)+f(x_2,x_1,x_3)+f(x_2,x_3,x_1) & = & 0 \end{array} \tag{2.11}$$

The map $\mathfrak{ls}_1\otimes_{\mathbb{Q}}\widetilde{D}_2 \longrightarrow \widetilde{D}_3$ is given by

$$\begin{aligned} & x_1^{2m}\,\underline{\circ}\, f(x_1,x_2) = x_1^{2m}f(x_2,x_3) + \\ &\quad (x_2-x_1)^{2m}(f(x_1,x_3)-f(x_2,x_3)) + (x_3-x_2)^{2m}(f(x_1,x_2)-f(x_1,x_3)) \end{aligned} \tag{2.12}$$

3. Relations and exceptional cuspidal elements

3.1. Period polynomials.

Definition 3.1. Let $n \geq 1$ and let $W^e_{2n} \subset \mathbb{Q}[X,Y]$ denote the vector space of homogeneous polynomials $P(X,Y)$ of degree $2n-2$ satisfying

$$P(X,Y) + P(Y,X) = 0 \quad , \quad P(\pm X, \pm Y) = P(X,Y) \tag{3.1}$$

$$P(X,Y) + P(X-Y,X) + P(-Y,X-Y) = 0 \ . \tag{3.2}$$

The space W^e_{2n} contains the polynomial $p_{2n} = X^{2n-2} - Y^{2n-2}$, and is a direct sum

$$W^e_{2n} \cong W^{e,0}_{2n} \oplus \mathbb{Q}\, p_{2n}$$

where $W^{e,0}_{2n}$ is the subspace of polynomials which vanish at $(X,Y) = (1,0)$. We write $W^{e,0} = \bigoplus_n W^{e,0}_{2n}$. By the Eichler-Shimura theorem and classical results on the space of modular forms, one knows that

$$\sum_{n\geq 1} \dim W^{e,0}_{2n} s^{2n} = \frac{s^{12}}{(1-s^4)(1-s^6)} \ . \tag{3.3}$$

3.2. Relations in depth 2.

The Ihara bracket gives a map

$$\{.,.\} : \mathfrak{ls}_1 \wedge \mathfrak{ls}_1 \longrightarrow \mathfrak{ls}_2 \ . \tag{3.4}$$

It follows immediately from formula (2.8) for $\underline{\circ}$ and the definition of $W^{e,0}$ that

$$W^{e,0} = \ker(\mathfrak{ls}_1 \wedge \mathfrak{ls}_1 \longrightarrow \mathfrak{ls}_2) \tag{3.5}$$

It is easy to show that the following sequence is exact

$$0 \longrightarrow W^{e,0} \longrightarrow \mathfrak{ls}_1 \wedge \mathfrak{ls}_1 \overset{\{,\}}{\longrightarrow} \mathfrak{ls}_2 \longrightarrow 0 \ . \tag{3.6}$$

and hence by lemma 1.3, the following sequence is also exact:

$$0 \longrightarrow W^{e,0} \longrightarrow \mathfrak{ls}_1 \otimes_{\mathbb{Q}} \mathfrak{ls}_1 \overset{\underline{\circ}}{\longrightarrow} \widetilde{D}^2 \longrightarrow 0 \ .$$

3.3. Exceptional elements in depth 4.

Let $f \in W^{e,0}_{2n+2}$ be an even period polynomial of degree $2n$ which vanishes at $y = 0$. It follows from (3.1) and (3.2) that it vanishes along $x = 0$ and $x - y = 0$. Therefore we can write

$$f = xy(x-y)f_0$$

where $f_0 \in \mathbb{Q}[x,y]$ is symmetric of homogeneous degree $2n-3$. Let us also write $f_1 = (x-y)f_0$. We have $f_1(-x,y) = f_1(x,-y) = -f_1(x,y)$.

Definition 3.2. Let $f \in \mathbb{Q}[x,y]$ be an even period polynomial as above. The following element was defined in [2]:

$$\begin{aligned} \mathbf{e}_f \quad &\in \quad \mathbb{Q}[y_0,y_1,y_2,y_3,y_4] \\ \mathbf{e}_f \quad &= \quad \sum_{\mathbb{Z}/\mathbb{Z}5} f_1(y_4-y_3, y_2-y_1) + (y_0-y_1)f_0(y_2-y_3, y_4-y_3) \ , \end{aligned} \tag{3.7}$$

where the sum is over cyclic permutations $(y_0,y_1,y_2,y_3,y_4) \mapsto (y_1,y_2,y_3,y_4,y_0)$. Its reduction $\overline{\mathbf{e}}_f \in \mathbb{Q}[x_1,\ldots,x_4]$ is obtained by setting $y_0 = 0, y_i = x_i$, for $i = 1,\ldots,4$.

Theorem 3.3. [2] *The reduced polynomial $\overline{\mathbf{e}}_f$ obtained from* (3.7) *satisfies the linearized double shuffle relations. In particular, we get an injective linear map*

$$\overline{\mathbf{e}} : W^{e,0} \longrightarrow \mathfrak{ls}_4$$

Definition 3.4. Let $\mathcal{E} \subset \mathfrak{ls}_4$ be the image of the map $\overline{\mathbf{e}}$.

By the previous theorem, $\mathcal{E} \cong W^{e,0}$. There is an explicit map $\mathcal{E} \to W^{e,0}$ given by $f(x_1, x_2, x_3, x_4) \mapsto x_1 x_2 f(x_1, x_2, 0, 0)$.

4. A three-term complex of vector spaces

Consider the following complex, where $n \geq 1$:

$$0 \longrightarrow W^{e,0} \otimes_{\mathbb{Q}} \widetilde{D}^{n-2} \longrightarrow (\mathfrak{ls}_1 \otimes_{\mathbb{Q}} \widetilde{D}^{n-1}) \oplus (\mathcal{E} \otimes_{\mathbb{Q}} \widetilde{D}^{n-4}) \longrightarrow \widetilde{D}^n \longrightarrow 0 \tag{4.1}$$

where the first map is the composite (identifying $\mathfrak{ls}_1 \otimes_{\mathbb{Q}} \mathfrak{ls}_1 \cong x_1^2 x_2^2 \mathbb{Q}[x_1, x_2]$),

$$W^{e,0} \otimes_{\mathbb{Q}} \widetilde{D}^{n-2} \quad \subset \quad \mathfrak{ls}_1 \otimes_{\mathbb{Q}} \mathfrak{ls}_1 \otimes_{\mathbb{Q}} \widetilde{D}^{n-2} \overset{id \otimes \underline{\circ}}{\longrightarrow} \mathfrak{ls}_1 \otimes_{\mathbb{Q}} \widetilde{D}^{n-1} ,$$

and the maps in the middle are given by the Ihara bracket (recall $\mathfrak{ls}_1 \cong \widetilde{D}^1$)

$$\mathfrak{ls}_1 \otimes_{\mathbb{Q}} \widetilde{D}^{n-1} \overset{\underline{\circ}}{\longrightarrow} \widetilde{D}^n \qquad , \qquad \mathcal{E} \otimes_{\mathbb{Q}} \widetilde{D}^{n-4} \overset{\underline{\circ}}{\longrightarrow} \widetilde{D}^n$$

The sequence (4.1) is a complex, by (3.5) and (1.8).

Conjecture 1. The complex (4.1) is an exact sequence.

If we use the notation

$$\mathbb{O}(s) = \frac{s^3}{1 - s^2} \quad , \quad \mathbb{S}(s) = \frac{s^{12}}{(1 - s^4)(1 - s^6)} \ . \tag{4.2}$$

then clearly the exactness of the sequence (4.1) implies that

$$\sum_{N, d \geq 0} (\dim_{\mathbb{Q}} \widetilde{D}^d_N) \, s^N t^d = \frac{1}{1 - \mathbb{O}(s)t + \mathbb{S}(s)t^2 - \mathbb{S}(s)t^4} \ . \tag{4.3}$$

where $\widetilde{D}^d_N$ is the part of $\widetilde{D}^d$ of weight N. By the arguments given in [2], this in turn implies the usual Broadhurst-Kreimer conjecture for motivic multiple zeta values (and much more besides).

4.1. **General remark on Lie algebras with split quadratic homology.** Let $\mathfrak{g}$ be a graded Lie algebra over a field k whose graded pieces are finite dimensional. Recall that the Chevalley-Eilenberg complex is given by

$$\longrightarrow \wedge^2 \mathfrak{g} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g} \longrightarrow k \longrightarrow 0$$

and is exact. Now suppose that $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{r} \subset \wedge^2 \mathfrak{h}$, such that the sequence

$$0 \longrightarrow \mathfrak{r} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathfrak{h} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g} \longrightarrow k \longrightarrow 0 \tag{4.4}$$

is exact. Then since this is a resolution of k, we immediately deduce (by tensoring with k, viewed as a $\mathcal{U}\mathfrak{g}$-module for the augmentation map) that

$$\begin{aligned} H_1(\mathfrak{g}; k) &\cong \mathfrak{h} \\ H_2(\mathfrak{g}; k) &\cong \mathfrak{r} \\ H_i(\mathfrak{g}; k) &= 0 \qquad \text{for all } i \geq 3 \end{aligned} \tag{4.5}$$

Conversely, suppose that (4.5) is true, where $\mathfrak{h} \subset \mathfrak{g}$, and $\mathfrak{r} \subset \ker(\wedge^2 \mathfrak{h} \to \mathfrak{g})$. The first line implies that $\mathfrak{g}$, and hence $\mathcal{U}\mathfrak{g}$ are generated by $\mathfrak{h}$. Thus there is a surjective map $\mathfrak{h} \otimes_k \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}_{>0}$, and we have

$$\mathfrak{r} \otimes_k \mathcal{U}\mathfrak{g} \subset \ker(\mathfrak{h} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g}_{>0}) \tag{4.6}$$

Standard arguments imply that the Poincaré series of $\mathcal{U}\mathfrak{g}$ is related to the Poincaré series of the homology of $\mathfrak{g}$ via $\chi_{\mathcal{U}\mathfrak{g}}(t) = (1 - \chi_{H_1(\mathfrak{g};k)}(t) + \chi_{H_2(\mathfrak{g};k)}(t))^{-1}$. This implies equality in (4.6) and hence the sequence

$$0 \longrightarrow \mathfrak{r} \otimes_k \mathcal{U}\mathfrak{g} \longrightarrow \mathfrak{h} \otimes_k \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}_{>0} \longrightarrow 0$$

is exact. This is equivalent to the exactness of (4.4).

4.2. **Closing remark.**

Theorem 4.1. *The exactness of sequence* (4.1) *(conjecture* 1*) is equivalent to the strong Broadhurst-Kreimer conjecture (conjecture 3 in* [2]*), which states that*

$$\begin{aligned} H_1(\mathfrak{ls};\mathbb{Q}) &\cong \mathfrak{ls}_1 \oplus \mathcal{E} \\ H_2(\mathfrak{ls};\mathbb{Q}) &\cong W^{e,0} \\ H_i(\mathfrak{ls};\mathbb{Q}) &= 0 \qquad \text{for all } i \geq 3 \end{aligned} \tag{4.7}$$

Proof. Apply the previous remarks to $\mathfrak{g} = \mathfrak{ls}$, and use the fact (lemma 1.3) that $\mathcal{U}\mathfrak{g} \cong \widetilde{D}$, together with (3.5). □

Question: Does there exist a natural splitting $\widetilde{D}^n \longrightarrow \mathcal{E} \otimes_{\mathbb{Q}} \widetilde{D}^{n-4}$ which is zero on the image of $\widetilde{D}^1 \otimes_{\mathbb{Q}} \widetilde{D}^{n-1}$? I.e., can one think of (4.1) as a (split) 4-term sequence?

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