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The block filtration for MZV's.

0.1. **Recap.** Let $\mathbb{Q}\langle e_0, e_1\rangle$ denote the tensor coalgebra on $\mathbb{Q}e_0 \oplus \mathbb{Q}e_1$. Its underlying $\mathbb{Q}$-vector space is generated by words in e_0, e_1. Recall that a convergent word is one which begins in e_1 and ends in e_0. To every convergent word $w = e_{i_1} \dots e_{i_n}$ we can associate the convergent multiple zeta value

$$\zeta(w) := \int_0^1 \omega_{i_1} \dots \omega_{i_n}$$

where we integrate from left to right, and $\omega_0 = \frac{dx}{x}$, $\omega_1 = \frac{dx}{1-x}$. We have $\zeta(\emptyset) = 1$.

0.2. **Alternating block decomposition.** Define a word in $\{e_0, e_1\}$ to be *alternating* if it is non-empty and has no repeats, i.e., contains no subsequence of the form e_0e_0 or e_1e_1. There are exactly two alternating words of any given length. The set of alternating words, ordered by length are

$$A = \{e_0 \ , \ e_1 \ , \ e_0e_1 \ , \ e_1e_0 \ , \ e_0e_1e_0 \ , \ e_1e_0e_1 \ , \ \dots\}$$

Clearly, every non-empty word w in e_0, e_1 can be written as a word in the alphabet A, i.e., a concatenation of alternating words. Define a *block decomposition* [3] of w to be a shortest such representation. For example,

$$e_0e_1e_1e_0e_1e_0e_0 = (e_0e_1)(e_1e_0e_1e_0)(e_0)$$

Note that the last letter in each block (alternating word) is equal to the first letter in the next one.

Lemma 0.1. *(Charlton [3]) Every word in $\{e_0, e_1\}$ has a unique block decomposition.*

Define the block degree $\deg_B$ to be the number of alternating words in the block decomposition minus 1. It counts the number of repeats in w:

$$\deg_B(w) = \text{number of subsequences of } w \text{ of the form } e_0e_0 \text{ or } e_1e_1 \ .$$

Let us define the block degree of the empty word to be $\deg_B(\emptyset) = 0$.

Lemma 0.2. *(Charlton [3]) A word w is convergent if and only if it begins in e_1 and*

$$\deg_B(w) - |w| \equiv 0 \pmod 2$$

where $|w|$ denotes the length of a word.

This is reminiscent of the depth-parity theorem. Charlton also observes that the block degree is preserved under the duality relation for multiple zeta values (induced by $t \mapsto 1 - t$ in the integral representation). Charlton applies the block decomposition not to the argument w of the integral for $\zeta(w)$ above, but to the word e_0we_1.

0.3. **Block filtration.** The block degree defines a grading on $\mathbb{Q}\langle e_0, e_1\rangle$. The block filtration is the associated increasing filtration.

Definition 0.3. Let us define the *block filtration*

$$B_n\mathbb{Q}\langle e_0, e_1\rangle \subset \mathbb{Q}\langle e_0, e_1\rangle$$

to be the subspace spanned by words whose block decomposition is of length $\leq n + 1$. Equivalently, it is the subspace spanned by words of length $\leq n + 1$ in A.

Note that the block degree of a convergent word w is the same as that of the word e_0we_1, so we obtain the same filtration whether we include the endpoints of integration or not (at least for convergent words). The block filtration induces a filtration on the ring of motivic multiple zeta values, and hence on multiple zeta values:

$$B_n\mathcal{Z}^{\mathfrak{m}} = \langle \zeta^{\mathfrak{m}}(w) : w \text{ convergent such that } \deg_B w \leq n\rangle_{\mathbb{Q}}$$
$$B_n\mathcal{Z} = \langle \zeta(w) : w \text{ convergent such that } \deg_B w \leq n\rangle_{\mathbb{Q}} \ .$$

Observe that

$$B_0\mathcal{Z}^{\mathfrak{m}} = \bigoplus_{n\geq 0} \zeta^{\mathfrak{m}}(\{2\}^n)\,\mathbb{Q} \ .$$

We know that $\zeta^{\mathfrak{m}}(\{2\}^n) \in \mathbb{Q}\zeta^{\mathfrak{m}}(2)^n$. Therefore $B_0\mathcal{Z} \cong \mathbb{Q}[\pi^2]$.

Lemma 0.4. *The block filtration induces the level filtration on the subspace spanned by the set of Hoffman motivic multiple zeta values $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ with $n_i \in \{2,3\}$, where the level is the number of indices n_i equal to* 3.

Proof. The word corresponding to $(n_1,\ldots,n_r)$ with $n_i \in \{2,3\}$ of level k has exactly k occurrences of the subsequence e_0e_0 and no occurences of the subsequence e_1e_1. Therefore its block degree is exactly $k+1$. □

Since the level is motivic, this motivates the following proposition. Let $G^{dR}_{\mathcal{MT}(\mathbb{Z})}$ denote the de Rham motivic Galois group of the category $\mathcal{MT}(\mathbb{Z})$ and $U^{dR}_{\mathcal{MT}(\mathbb{Z})}$ its unipotent radical. They both act upon $\mathcal{Z}^{\mathfrak{m}}$.

Proposition 0.5. *The block filtration is motivic: i.e., B_n is stable under the action of $G^{dR}_{\mathcal{MT}(\mathbb{Z})}$. Furthermore, the subgroup $U^{dR}_{\mathcal{MT}(\mathbb{Z})}$ acts trivially on $\mathrm{gr}^B_{\bullet}\mathcal{Z}^{\mathfrak{m}}$.*

Equivalently, if Δ denotes the (left) motivic coaction dual to the action of $U^{dR}_{\mathcal{MT}(\mathbb{Z})}$,

$$\Delta(B_n\mathcal{Z}^{\mathfrak{m}}) \subset \mathcal{O}(U^{dR}_{\mathcal{MT}(\mathbb{Z})}) \otimes_{\mathbb{Q}} B_{n-1}\mathcal{Z}^{\mathfrak{m}}$$

An immediate corollary of the main theorem of [2] is :

Theorem 0.6. *Every element in $B_n\mathcal{Z}^{\mathfrak{m}}$ and weight N can be written uniquely as a $\mathbb{Q}$-linear combination of motivic Hoffman elements of weight N and level $\leq n$.*

It follows that the associated graded for the block filtration has the following Poincaré series (where W is the MZV-weight):

$$\sum_{p,n\geq 0} \big(\dim_{\mathbb{Q}} \mathrm{gr}^B_p \mathrm{gr}^W_n \mathcal{Z}^{\mathfrak{m}}\big) s^p t^n = \frac{1}{1-t^2-st^3} \ .$$

Corollary 0.7. *Taking the period of the previous theorem: every multiple zeta value of block degree $\leq n$ and weight N can be written as a $\mathbb{Q}$-linear combination of Hoffman multiple zeta values of weight N with at most n threes.*

Corollary 0.8. *The block filtration on $\mathcal{Z}^{\mathfrak{m}}$ is the coradical filtration.*

Proof. If C denotes the coradical filtration, then the proposition implies $B_n \subset C_n$. The converse follows from lemma 0.4, and the fact [2], that the coradical filtration on the Hoffman basis for motivic multiple zeta values is given by the level filtration. □

The previous corollary is equivalent to the statement that the block filtration is given by the length filtration in any f-alphabet decomposition [1].

It follows from the last corollary that $B_0\mathcal{Z}^{\mathfrak{m}} = \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$ and

$$B_1\mathcal{Z}^{\mathfrak{m}} = \bigoplus_{n\geq 1} \zeta^{\mathfrak{m}}(2n+1)\mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$$

i.e., the space of polynomials in single motivic zeta values which are of degree at most one in the odd motivic zeta values. Taking the period implies that:

$$B_1\mathcal{Z} = \langle \zeta(2n+1)\pi^{2k-2}, \text{ for } k,n \geq 1\rangle_{\mathbb{Q}} \ .$$

0.4. **Proof of the proposition.** The motivic coaction factors through the action of operators D_{2r+1} defined in [2], (3.4). They act via the following formula, where $a_i \in \{0,1\}$:

$$(0.1) \quad D_n I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) =$$

$$\sum_{p=0}^{N-n} I^{u}(a_p; a_{p+1}, \ldots, a_{p+n}; a_{p+n+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+n+1}, a_{p+n+2}, \ldots, a_N; a_{N+1}) \ .$$

where $I^{\mathfrak{m}}$ is the motivic iterated integral, and $I^{\mathfrak{u}}$ the unipotent de Rham iterated integral. It is enough to show that

$$D_{2r+1} B_k \mathcal{Z}^{\mathfrak{m}} \quad \subset \quad \mathcal{O}(U^{dR}_{\mathcal{MT}(\mathbb{Z})}) \otimes B_{k-1}\mathcal{Z}^{\mathfrak{m}} \ .$$

Suppose that $a_0 = 0$, $a_{N+1} = 1$, and $a_1 \ldots a_N$ is convergent of B-degree k. It can be written as a concatenation

$$w = v_1 \ldots v_k$$

of k alternating words $v_i \in A$. In the formula for D_n, all terms with $a_p = a_{p+n+1}$ vanish by [2] **I0**. Therefore we can assume $a_p \neq a_{p+n+1}$, which proves that every non-zero term on the right-hand side is convergent and of B-degree $\leq k$. Furthermore, this implies that

$$a_p a_{p+1} \ldots a_{p+n+1} \notin A \ ,$$

since any alternating word of odd length necessarily begins and ends in the same letter. This implies that: if $p = 0$ then a_{p+n+1} is not a letter of v_1; if $p = N-n$ then a_p is not a letter of v_k; and in all other cases, a_p and a_{p+n+1} are not letters in the same word v_i. This implies that the right-hand side of (0.1) has B-degree $< k$: every non-zero term on the right-hand side is of the form $v_1 \ldots v_{i-1} w v_{i+j+1} \ldots v_k$, for some $w \in A$ and $j \geq 1$.

References

[1] F. Brown: *On the decomposition of motivic multiple zeta values*, Adv. Stud. Pure Math., 63, Math. Soc. Japan, Tokyo, (2012), 31-58.

[2] F. Brown: *Mixed Tate motives over* $\mathbb{Z}$, Ann. of Math. (2) 175 (2012), no. 2, 949-976.

[3] S. Charlton : *Identities arising from coproducts on Multiple Zeta Values and Multiple Polylogarithms*, Ph. D thesis, Durham University, September 2016.