DEDEKIND ZETA MOTIVES FOR TOTALLY REAL NUMBER FIELDS

FRANCIS C.S. BROWN

ABSTRACT. Let k be a totally real number field. For every odd $n \geq 3$, we construct an element in the category $\mathrm{MT}(k)$ of mixed Tate motives over k out of the quotient of a product of hyperbolic spaces by an arithmetic group. By a volume calculation, we prove that its period is a rational multiple of $\pi^{n[k:\mathbb{Q}]}\zeta_k^*(1-n)$, where $\zeta_k^*(1-n)$ denotes the special value of the Dedekind zeta function of k. We deduce that the group $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ is generated by the cohomology of a quadric relative to hyperplanes, and that $\zeta_k^*(1-n)$ is a determinant of volumes of geodesic hyperbolic simplices defined over k.

1. INTRODUCTION

1.1. **Background.** This paper was motivated by a desire to reconcile two important and complementary results due to Zagier and Goncharov. Firstly, in 1986, Zagier obtained a formula for the value $\zeta_K(2)$, where K is any number field, as a linear combination of products of values of the dilogarithm function at algebraic points [37]. The idea is to compute the volume of an arithmetic hyperbolic manifold in two different ways. Suppose that K is quadratic imaginary, and let Γ be a torsion-free subgroup of finite index of the Bianchi group $\mathsf{PSL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. Then Γ acts on hyperbolic 3-space $\mathbb{H}^3$, and a classical theorem due to Humbert states that

$$\mathrm{vol}(\mathbb{H}^3/\Gamma) = \frac{|d|^{3/2}\, r}{4\pi^2}\zeta_K(2) \ ,$$

where d is the discriminant of K, and r is the index of Γ in $\mathsf{PSL}_2(\mathcal{O}_K)$. On the other hand, such a hyperbolic 3-manifold can be triangulated using geodesic simplices, whose volume, by a calculation originally due to Milnor, can be expressed in terms of the Bloch-Wigner dilogarithm function

$$D(z) = \mathrm{Im}\left(\mathrm{Li}_2(z) + \log|z|\log(1-z)\right) \ , \ \text{ where } \mathrm{Li}_2(z) = \sum_{k \geq 1}\frac{z^k}{k^2} \ .$$

The volume of $\mathbb{H}^3/\Gamma$ is therefore a rational linear combination of $D(z)$ with algebraic arguments. Equating the two volume calculations yields Zagier's formula for $\zeta_K(2)$, which in turn motivated his conjectures relating polylogarithms to special values of zeta and L-functions [15]. He also obtained a formula for $\zeta_K(2)$ for any number field by considering groups acting on products of hyperbolic 3-space in [37]. However, this geometric argument is insufficient to prove the final form of Zagier's conjecture for two reasons: one is that the dilogarithms are evaluated over some finite extension of K, rather than K itself, and secondly there is no way to prove that the sums of products of dilogarithms are a determinant, as the conjecture demands.

Zagier's results were subsequently reinterpreted in terms of algebraic K-theory by many authors. See, for example, [6, 15, 22, 23, 33] and the references therein.

Essentially, the set of simplices in a triangulation of $\mathbb{H}^3/\Gamma$ gives a well-defined element in the Bloch group $B(\overline{\mathbb{Q}})$ via their gluing equations, and in turn defines an element in the K-theory group $K_3(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$. The dilogarithm can be interpreted as the Borel regulator map.

Secondly, in [17], Goncharov generalized these constructions for any discrete torsion-free group Γ of finite covolume acting on hyperbolic n-space $\mathbb{H}^n$, for $n = 2m - 1$ odd. For any such manifold $M = \mathbb{H}^n/\Gamma$, he constructed an element

$$\xi_M \in K_{2m-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q} \ ,$$

such that the volume of M is given by the regulator map on ξ_M. In fact, he gave two such constructions, but it is the motivic version which is of interest here. The main idea is that a finite geodesic simplex in hyperbolic space (with algebraic vertices) defines a mixed Tate motive in the sense of [13]. By triangulating the manifold M, and combining the motives of each simplex, one obtains a total motive for M, which is non-trivial if n is odd. He then proves that the motive of M is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(m)$, and identifies the group of such extensions with $K_{2m-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ (see below). The proof of this fact is analytic: it uses the fact that (solid) angles around faces in a triangulation sum to multiples of 2π, and uses the full faithfullness of the Hodge realization for mixed Tate motives over $\overline{\mathbb{Q}}$ ([13], §2.13).

In this paper, we give a common generalization of both Zagier and Goncharov's results. First we list the arithmetic groups which act on products of hyperbolic spaces $\mathbb{H}^{n_1} \times \ldots \times \mathbb{H}^{n_N}$, and give their covolumes, up to a rational multiple, in terms of zeta functions using well-known Tamagawa number arguments [24]. Next, we show that any such complete 'product-hyperbolic' manifold M of finite volume admits a triangulation using products of geodesic simplices, and use this to construct a well-defined mixed Tate motive $\mathsf{mot}(M)$. In the case when $N = 1$, and M is non-compact, our construction differs from Goncharov's. We then prove that

$$(1.1) \qquad \mathsf{mot}(M) \in \mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0), \mathbb{Q}(n_1)) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0), \mathbb{Q}(n_N))$$

using a geometric argument. Thus the construction is completely motivic. Now let k be a totally real number field, and n an odd integer ≥ 3. The Dedekind zeta motive is obtained in the case where Γ is a torsion-free subgroup of the restriction of scalars $R_{k/\mathbb{Q}}\mathsf{SO}(2n - 1, 1)$, and its period is directly related to $\zeta_k(n)$. Using Borel's calculation of the rank of algebraic K-groups, we show that $\mathsf{mot}(M)$ is a determinant of $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(k) \otimes \mathbb{Q}$. As a corollary, we obtain generators for $K_{2n-1}(k) \otimes \mathbb{Q}$ in terms of hyperbolic geometry and deduce that $\zeta_k(n)$ is a determinant of volumes of hyperbolic simplices.

1.2. **Explicit categories of mixed Tate motives and zeta values.** Let k be a number field, and let $\mathrm{MT}(k)$ be the abelian tensor category of mixed Tate motives over k [13]. Its simple objects are the Tate motives $\mathbb{Q}(n)$, for $n \in \mathbb{Z}$, and its structure is determined by its relation to algebraic K-theory:

$$(1.2) \qquad \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} 0 & \text{if } n \leq 0 \ , \\ K_{2n-1}(k) \otimes \mathbb{Q} & \text{if } n \geq 1 \ , \end{cases}$$

and the fact that all higher extension groups vanish. One has $K_1(k) = k^\times$, and a theorem due to Borel [7], states that for $n > 1$,

$$(1.3) \qquad \dim_{\mathbb{Q}}(K_{2n-1}(k) \otimes \mathbb{Q}) = \begin{cases} n_+ = r_1 + r_2, & \text{if } n \text{ is odd}; \\ n_- = r_2, & \text{if } n \text{ is even}, \end{cases}$$

where r_1 is the number of real places of k, and r_2 is the number of complex places of k. The Hodge realisation functor on $\mathrm{MT}(k)$ gives rise to a regulator map

$$(1.4) \qquad r_H : \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathbb{R}^{n\pm} \qquad (n > 1) \ .$$

Its covolume should be related to the values of the Dedekind zeta function $\zeta_k(s)$ via the Beilinson and Bloch-Kato conjectures.

It is an important open problem to construct $\mathrm{MT}(k)$ explicitly out of simple geometric building blocks. Concretely, one writes down a category $C \subseteq \mathrm{MT}(k)$ constructed from certain diagrams of varieties with Tate cohomology, and one wants to show that $C = \mathrm{MT}(k)$. To do this, firstly one must show that the groups $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ are spanned by elements of C (generation), and secondly, one must show that C is closed under extensions (freeness). Via such a construction one hopes to compute the periods of $\mathrm{MT}(k)$, and these include the numbers $\zeta_k(n)$ for $n \geq 2$, in particular. Natural candidates for C come from smooth hypersurfaces with Tate cohomology (or their blow-ups along linear subspaces), as follows.

Hyperplanes. The approach of [5] uses arrangements of hyperplanes in projective space. Consider a set of $2n+2$ hyperplanes $L_0, \ldots, L_n, M_0, \ldots, M_n \subset \mathbb{P}^n$ in general position, and write $L = \cup_{i=0}^n L_i$, $M = \cup_{i=0}^n M_i$. Let k be a number field. If L, M are defined over k, the pair (L, M) defines a mixed Tate motive:

$$(1.5) \qquad H^n\big(\mathbb{P}^n \backslash M, L \backslash (L \cap M)\big) \in \mathrm{MT}(k) \ .$$

Let $C^h(k) \subset \mathrm{MT}(k)$ denote the full subcategory spanned by the elements (1.5) and closed under tensor products and subquotients. The first part of the problem described above is to show that the natural map

$$(1.6) \qquad \mathrm{Ext}^1_{C^h(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$$

is surjective. Stefan Müller-Stach informed us that this follows from the work of Gerdes [16] (conjecture 4.7, remark 4.11) along with the proof of the rank conjecture for algebraic K-theory [36]. A variant of the above construction would be to consider blow-ups of degenerate configurations of hyperplanes. The motivic fundamental group of the punctured line minus roots of unity [13] is a special case of this latter construction in the particular case when k is abelian.

Quadrics and hyperplanes [17]. Let Q denote a smooth quadric in $\mathbb{P}^{2n-1}$, and let $L_1, \ldots, L_{2n}$ denote $2n$ hyperplanes in general position with respect to Q. Let $L = \cup_{i=1}^{2n} L_i$. If Q and L are defined over $\overline{\mathbb{Q}}$, this defines a *quadric motive*:

$$(1.7) \qquad H^{2n-1}\big(\mathbb{P}^{2n-1} \backslash Q, L \backslash (Q \cap L)\big) \in \mathrm{MT}(\overline{\mathbb{Q}}) \ .$$

As above, let $C^q(\overline{\mathbb{Q}}) \subset \mathrm{MT}(\overline{\mathbb{Q}})$ denote the full subcategory spanned by the elements (1.7), and let $C^q(k) \subset \mathrm{MT}(k)$ be the subcategory which is invariant under $\mathrm{Gal}(\overline{k}/k)$. The generation part of the problem is now to show that

$$(1.8) \qquad \mathrm{Ext}^1_{C^q(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$$

is surjective. In this paper, we shall prove this when k is totally real.

A much more precise statement should follow from Beilinson and Deligne's reformulation [4] of Zagier's conjecture [15]. For every $z \in k \backslash \{0, 1\}$, and $n \geq 1$, one

can construct a polylogarithm motive $P_n(z) \in \mathrm{MT}(k)$ as a subquotient of a (degenerate) hyperplane motive (1.5). If K_z denotes the Kummer motive which is the extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ given by z, then $P_n(z)$ is part of an exact sequence

$$0 \longrightarrow \left(\mathrm{Sym}^{n-1}K_z\right)(1) \longrightarrow P_n(z) \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

and satisfies $P_n(z)/W_{-4}P_n(z) \cong K_{1-z}$. The objects $P_n(z)$ span a strict subcategory $C^p(k) \subset \mathrm{MT}(k)$ and the classical polylogarithm $\mathrm{Li}_n(z) = \sum_{k \geq 1}\frac{z^k}{k^n}$ is a period of $P_n(z)$. A version of Zagier's conjecture would state that

$$(1.9) \qquad \phi^p : \mathrm{Ext}^1_{C^p(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ .$$

is surjective, i.e., (1.2) should be generated by polylogarithm motives. This is known only for $n \leq 3$ [18] but is supported by extensive numerical evidence [15].

1.2.1. *Zeta values.* The surjectivity of (1.6), (1.8), or (1.9) would imply a statement about values of zeta functions, as follows. Let $\zeta_k(s)$ denote the Dedekind zeta function of k, and let $\zeta_k^*(1-n)$ denote the leading coefficient in the Taylor expansion of $\zeta_k(s)$ at $s = 1-n$, for $n \geq 2$. The image of the Borel regulator

$$(1.10) \qquad r_B : K_{2n-1}(k) \otimes \mathbb{Q} \longrightarrow \mathbb{R}^{n\pm} \qquad (n > 1) \ ,$$

is a $\mathbb{Q}$-lattice $\Lambda_n(k)$ whose covolume is well-defined up to multiplication by an element in $\mathbb{Q}^\times$. Using the isomorphism of rational K-theory with the stable cohomology of the linear group, Borel proved by an analytic argument in [8] that

$$(1.11) \qquad \zeta_k^*(1-n) = \alpha \operatorname{covol}(\Lambda_n(k)) \quad \text{ for some } \alpha \in \mathbb{Q}^\times \ .$$

If one combines this result with the isomorphism (1.2), then the surjectivity of one of the maps $\phi^\bullet$ above yields a conjectural formula for $\zeta_k^*(1-n)$ modulo rationals. In the case of the map ϕ^h, this would express $\zeta_k^*(1-n)$ as a determinant of Aomoto polylogarithms [5], and in the case of the map ϕ^p, this would express $\zeta_k^*(1-n)$ as a determinant of classical polylogarithms [15].

1.3. **Main Results.** Let k be a totally real number field. If n is even, the group (1.2) vanishes. For every odd $n \geq 3$, we construct a canonical Dedekind zeta motive

$$(1.12) \qquad \mathsf{mot}_n(k) \in \mathrm{MT}(k) \ ,$$

using arithmetic subgroups of $\mathsf{SO}(2n-1, 1)$ (see below). The motive $\mathsf{mot}_n(k)$ is a subquotient of a sum of products of quadric motives (1.7).

Theorem 1.1. *The element* $\mathsf{mot}_n(k)$ *satisfies*

$$\mathsf{mot}_n(k) \in \det \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ ,$$

and its image under the regulator (1.4) *is a non-zero rational multiple of* $\zeta_k^*(1-n)$.

Theorem 1.1 is a motivic analogue of Borel's theorem (1.11) in the totally real case. However, the proof of theorem 1.1 is entirely different, as it uses r_H rather than r_B, and works completely inside the category $\mathrm{MT}(k)$. This opens up the possibility of studying other realisations of $\mathsf{mot}_n(k)$.

Theorem 1.2. *If k is totally real, the map ϕ^q of* (1.8) *is surjective.*

Thus every element in $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$, for k a totally real number field, is a subquotient of a direct sum of quadric motives. Theorems 1.1 and 1.2 also hold for certain quadratic extensions of k. A similar construction when $n = 2$ holds in fact for any number field.

The element $\mathsf{mot}_n(k)$ is defined in the following way. Let $\mathbb{H}^m$ denote hyperbolic space of dimension m, and let $\mathcal{O}_k$ denote the ring of integers of k. Then any torsion-free subgroup Γ of finite index of $\mathsf{SO}(2n-1, 1)(\mathcal{O}_k)$ acts properly discontinuously on $r = [k : \mathbb{Q}]$ copies of $\mathbb{H}^{2n-1}$. The quotient is a *product-hyperbolic* manifold

$$M = \mathbb{H}^{2n-1} \times \ldots \times \mathbb{H}^{2n-1}/\Gamma \ .$$

Then, using a trick due to Zagier, one can show that M can be triangulated using products of hyperbolic geodesic simplices defined over k. By an idea due to Goncharov, a hyperbolic geodesic simplex in the Klein model for $\mathbb{H}^{2n-1}$ is a Euclidean simplex L inside a smooth quadric Q, and so defines a quadric motive (1.7). The total motive $\mathsf{mot}(M)$ is defined to be a certain subquotient of the direct sum of tensor products of quadric motives corresponding to all the simplices in the triangulation of M, and is well-defined. Technically, this subquotient is extracted using framed equivalence classes (§5.1). The gluing relations between simplices imply that $\mathsf{mot}_n(k) \in \bigotimes_{[k:\mathbb{Q}]} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$, generalizing a result due to Goncharov for ordinary hyperbolic manifolds. Theorem 1.1 is then proved using the fact that

$$\mathrm{vol}(M) = \alpha \, \pi^{nr} \zeta_k^*(1-n) \ ,$$

for some $\alpha \in \mathbb{Q}^\times$, which follows from a Tamagawa number argument. This in turn implies theorem 1.2, using the fact that the rank of $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ is exactly r, which follows from (1.3). There are two corollaries.

Corollary 1.3. *Let $M \in \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$. The periods of M under $\pi^n r_H$ are $\mathbb{Q}$-linear combinations of volumes of hyperbolic $(2n-1)$-simplices defined over k.*

Corollary 1.4. *The special value $\zeta_k^*(1-n)$ is a rational multiple times a determinant of (π^{-n}-times) volumes of hyperbolic $(2n-1)$-simplices defined over k.*

This result is in the spirit of, but weaker than, Zagier's conjecture. The proof that the zeta value is a determinant uses the rank calculation (1.3) in an essential way, and does not seem to follow directly from the geometry of the triangulation. A similar argument to the above also works for $n = 2$, and holds for all number fields L. It is not hard to show that quadric motives in $\mathbb{P}^3$ are dilogarithm motives $P_2(z)$, thereby giving another proof of Zagier's conjecture on zeta values for $n = 2$.

1.4. **Plan.** In §2 we introduce notations and define product-hyperbolic manifolds. In §3, we list the non-exceptional arithmetic product-hyperbolic manifolds, and state their covolumes up to a rational multiple. In §4, we explain how to decompose a product-hyperbolic manifold M into a finite union of products of hyperbolic geodesic simplices $\Delta_1^{(i)} \times \ldots \times \Delta_N^{(i)}$, $1 \le i \le R$, where each simplex $\Delta_j^{(i)}$ has at most one vertex at infinity. In §5, we recall properties of framed mixed Tate motives and study the framed mixed Tate motive defined by such a geodesic simplex and its periods, with examples in §5.3. The total (framed) motive $\mathsf{mot}(M)$ is defined by

$$(1.13) \qquad \mathsf{mot}(M) = \sum_{i=1}^{R} \mathsf{mot}(\Delta_1^{(i)}) \otimes \ldots \otimes \mathsf{mot}(\Delta_N^{(i)}) \ .$$

and is well-defined and non-zero if M is modelled on products of hyperbolic spaces with odd-dimensional components only. In §5.2.5, we prove (1.1) from a geometric argument which uses only the fact that M has no boundary. Theorem 5.16 covers the case when M is defined arithmetically. In §6, these results are combined to prove the main theorems. Analogous results in the exceptional case $n = 2$ are given in §6.3, and some open questions are discussed in §6.4. Finally, in §7 we compute an explicit example of the motive of an Artin L-value at 3 over the field $\mathbb{Q}(\sqrt{5})$, which is based on a remarkable computation due to Bugaenko.

This work was written during my thesis at the University of Bordeaux, in 2004-2005. Many thanks to C. Soulé for his comments on a (very) early version of this paper, and also to H. Gangl and M. Belolipepsky for helpful discussions.

2. Product-hyperbolic manifolds.

2.1. **Euclidean and hyperbolic spaces.** We use the following notations. For $n \geq 1$, Euclidean space $\mathbb{E}^n$ is $\mathbb{R}^n$ equipped with the scalar product

$$(x, y) = x_1 y_1 + \ldots + x_n y_n .$$

Let $n \geq 2$, and let $\mathbb{R}^{n,1}$ denote $\mathbb{R}^{n+1}$ equipped with the inner product

$$(x, y) = -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n .$$

Hyperbolic space $\mathbb{H}^n$ is defined to be the half-hyperboloid:

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n,1} : (x, x) = -1, \ x_0 > 0 \} .$$

Let $\mathsf{SO}(n, 1)(\mathbb{R})$ denote the group of matrices preserving this scalar product. It has two components. Let $\mathsf{SO}^+(n, 1)(\mathbb{R})$ denote the connected component of the identity, which is the group of orientation-preserving symmetries of $\mathbb{H}^n$. The invariant metric on $\mathbb{H}^n$ is given by $ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_n^2$.

We will consider complete manifolds M which are locally modelled on products of the above, *i.e.*, Riemannian products of the form

$$(2.1) \qquad\qquad \mathbb{X}^{\mathfrak{n}} = \prod_{1 \leq i \leq N} \mathbb{H}^{n_i} \times \prod_{N+1 \leq i \leq N'} \mathbb{E}^{n_i} ,$$

where $\mathfrak{n} = (n_1, \ldots, n_{N'})$. A complete, orientable manifold modelled on $\mathbb{X}^{\mathfrak{n}}$ of finite volume is of the form $M = \mathbb{X}^{\mathfrak{n}}/\Gamma$, where Γ is a discrete torsion-free subgroup of the group of motions of $\mathbb{X}^{\mathfrak{n}}$. In this case, we say that M is a *flat-hyperbolic* manifold. If it is modelled only on products of hyperbolic spaces ($N' = N$), it will be called *product-hyperbolic*. It follows from Margulis' theorem on the arithmeticity of lattices of rank > 1 [21] that any irreducible discrete group of finite covolume Γ acting on $\mathbb{X}^{\mathfrak{n}}$ is arithmetic, or else $\mathbb{X}^{\mathfrak{n}} = \mathbb{H}^n$ and $\mathbb{H}^n/\Gamma$ is an ordinary hyperbolic manifold.

2.2. **Models of hyperbolic spaces.** We also need to consider the following models for hyperbolic space ([1], I,§2) and the absolute, which we denote by $\partial \mathbb{H}^n$.

2.2.1. *The Klein (projective) model.* The map $(x_0, \ldots, x_n) \mapsto (y_1, \ldots, y_n)$, where $y_i = x_i/x_0$, gives an isomorphism of the hyperboloid model with the Klein model:

$$(2.2) \qquad\qquad \mathbb{K}^n = \{ (y_1, \ldots, y_n) \in \mathbb{R}^n : r^2 = \sum_{i=1}^n y_i^2 < 1 \}$$

equipped with the metric $ds^2 = (1-r^2)^{-2}[(1-r^2)\sum_{i=1}^{n} dy_i^2 + (\sum_{i=1}^{n} y_i dy_i)^2]$. The action of $\mathsf{SO}^+(n,1)$ on $\mathbb{K}^n$ is by projective transformations, and extends continuously to the boundary $\partial \mathbb{K}^n$, which is the unit sphere in $\mathbb{R}^n$. Geodesics in this model are Euclidean lines, and in particular, Euclidean hyperplanes are totally geodesic, but note that hyperbolic and Euclidean angles do not agree.

2.2.2. *The Poincaré upper-half space model.* Let

$$(2.3) \qquad \mathbb{U}^n = \{(z_1, \ldots, z_{n-1}, t) : (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}, \, t > 0\} \ ,$$

equipped with the metric $t^{-2}(\sum_{i=1}^{n-1} dz_i^2 + dt^2)$. The absolute $\partial \mathbb{H}^n$ is identified with the Euclidean plane $\partial \mathbb{U}^n = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ at height $t = 0$, compactified by adding the single point at infinity ∞. Geodesics in this model are vertical line segments (which go to ∞) or segments of circles which meet the absolute at right angles. In this model, hyperbolic angles coincide with Euclidean angles.

3. Volumes of product-hyperbolic manifolds and L-functions

3.1. **Arithmetic groups acting on product-hyperbolic space.** There are four basic types of arithmetic groups which act on products of hyperbolic spaces.

Type (I) : Let $k/\mathbb{Q}$ be a totally real number field of degree r, and let $\mathcal{O}$ denote an order in k. Let $1 \leq t \leq r$. Consider a non-degenerate quadratic form

$$(3.1) \qquad q(x_0, \ldots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j \quad \text{where} \quad a_{ij} \in k \ , \ a_{ij} = a_{ji} \ ,$$

and suppose that $^{\sigma}q = \sum_{i,j=0}^{n} {}^{\sigma}a_{ij} \, x_i x_j$ has signature $(n,1)$ for t infinite places $\sigma \in \{\sigma_1, \ldots, \sigma_t\}$ of k, and is positive definite for the remaining $r - t$ infinite places of k. Let $\mathsf{SO}^+(q, \mathcal{O})$ be the group of linear transformations with coefficients in $\mathcal{O}$ preserving q, which map each connected component of $\{x \in \mathbb{R}^{n+1} : {}^{\sigma_i}q(x) < 0\}$ to itself for $1 \leq i \leq t$. Let Γ be a torsion-free subgroup of $\mathsf{SO}^+(q, \mathcal{O})$ of finite index. It acts properly discontinuously on $\prod_{i=1}^{t} \mathbb{H}^n$ via the map

$$\begin{aligned} \Gamma & \hookrightarrow \prod_{i=1}^{t} \mathsf{SO}^+(n,1)(\mathbb{R}) \\ A & \mapsto ({}^{\sigma_1}A, \ldots, {}^{\sigma_t}A) \ . \end{aligned}$$

Type (II) : Suppose that $n = 2m - 1$ is odd. Let $k/\mathbb{Q}$ be a totally real number field of degree r, and let $1 \leq t \leq r$. Consider a quaternion algebra D over k such that $D_\sigma = D \otimes_{k,\sigma} \mathbb{R}$ is isomorphic to $M_{2 \times 2}(\mathbb{R})$ for all embeddings σ of k. Let

$$(3.2) \qquad Q(x,y) = \sum_{i,j=1}^{m} \overline{x}_i a_{ij} y_j \quad \text{where} \quad a_{ij} \in D, \, a_{ij} = -\overline{a}_{ji} \ ,$$

be a non-degenerate skew-Hermitian form on D^m, where $x \mapsto \overline{x}$ denotes the conjugation map on D, which is an anti-homomorphism. For each embedding σ of k, the signature of $^{\sigma}Q$, where $^{\sigma}Q(x,y) = \sum_{i,j=1}^{m} \overline{x}_i \, {}^{\sigma}a_{ij} y_j$, is defined as follows. Let $D_\sigma \cong M_{2 \times 2}(\mathbb{R}) = \mathbb{R} \oplus \mathsf{i}\mathbb{R} \oplus \mathsf{j}\mathbb{R} \oplus \mathsf{k}\mathbb{R}$, where $\mathsf{i}^2 = \mathsf{j}^2 = 1$ and $\mathsf{ij} = -\mathsf{ji} = \mathsf{k}$. The endomorphism of D_σ given by right multiplication by i is of order two and has eigenvalues ± 1. Let

$$D_{\sigma, \pm} = \{x \in D : x\mathsf{i} = \pm x\} \ .$$

It follows that $D_\sigma^m = D_{\sigma,+}^m \oplus D_{\sigma,-}^m$, and because one has $D_{\sigma,-}^m = D_{\sigma,+}^m \mathsf{j}$, the dimension of $D_{\sigma,+}^m$ is $2m$. Writing $x = x_+ + x_-$, where $x_+\mathsf{i} = x_+$ and $x_-\mathsf{i} = -x_-$, one verifies that ${}^\sigma Q(x_+, y_+) = {}^\sigma Q(x_+\mathsf{i}, y_+) = -\mathsf{i}\,{}^\sigma Q(x_+, y_+)$, and ${}^\sigma Q(x_+, y_+) = {}^\sigma Q(x_+, y_+\mathsf{i}) = {}^\sigma Q(x_+, y_+)\mathsf{i}$. It follows that

$$
{}^\sigma Q(x_+, y_+) = f_\sigma(x_+, y_+)(\mathsf{j} - \mathsf{k}), \quad \text{where } f_\sigma(x_+, y_+) \in k .
$$

Since ${}^\sigma Q$ is skew-hermitian, f_σ is a non-degenerate symmetric bilinear form on $D_{\sigma,+}^m$. The signature of ${}^\sigma Q$ is defined to be the signature of f_σ. Let ϕ_σ denote the map which to Q associates the bilinear form f_σ. The form f_σ uniquely determines ${}^\sigma Q$, for instance, via the formula $-\mathsf{j}\,{}^\sigma Q(x_-, y_+) = {}^\sigma Q(x_-\mathsf{j}, y_+) = f_\sigma(x_-\mathsf{j}, y_+)(\mathsf{j} - \mathsf{k})$.

Suppose, therefore, that ${}^\sigma Q$ has signature $(2m - 1, 1)$ for t embeddings $\sigma \in \{\sigma_1, \ldots, \sigma_t\}$ of k, and is positive definite for the remaining $r - t$ embeddings. Let $\mathcal{O}$ be an order in D, and let $\mathsf{U}^+(Q, \mathcal{O})$ denote the group of $\mathcal{O}$-valued points of the unitary group which preserves the connected components of $\{x \in D_{\sigma_i,+}^m : f_{\sigma_i}(x, x) < 0\}$ for $1 \leq i \leq t$. If Γ is a torsion-free subgroup of $\mathsf{U}^+(Q, \mathcal{O})$, then it acts properly discontinuously on $\prod_{i=1}^t \mathbb{H}^n$ via the map $(\phi_{\sigma_1}, \ldots, \phi_{\sigma_t}) : \Gamma \hookrightarrow \prod_{i=1}^t \mathsf{SO}^+(n, 1)(\mathbb{R})$.

Type (III) : Let $L/\mathbb{Q}$ denote a number field of degree n with r_1 real places and $r_2 \geq 1$ complex places, and let $0 \leq t \leq r_1$. Let B denote a quaternion algebra over L which is unramified at t real places, and ramified at the other $r_1 - t$ real places of L. Then $\prod_{v|\infty}(B \otimes_L L_v)^* = (\mathfrak{H}^*)^{r_1 - t} \times \mathsf{GL}_2(\mathbb{R})^t \times \mathsf{GL}_2(\mathbb{C})^{r_2}$, where $\mathfrak{H}$ denotes Hamilton's quaternions. Let $\mathcal{O}$ denote an order in B and let Γ be a torsion-free subgroup of finite index of the group of elements in B of reduced norm 1. Then Γ defines a discrete subgroup of $\mathsf{PSL}_2(\mathbb{R})^t \times \mathsf{PSL}_2(\mathbb{C})^{r_2}$ and acts properly discontinuously on $(\mathbb{H}^2)^t \times (\mathbb{H}^3)^{r_2}$ [12].

Type (IV) : There are further exceptional cases for $\mathbb{H}^7$ which are related to Cayley's octonions. These will not be considered here.

Remark 3.1. When n is odd, every arithmetic group of type (I) can also be expressed as an arithmetic group of type (II). So in (I) we can assume n is even.

Our main results only use the trivial case $q = -x_0^2 + x_1^2 + \ldots + x_n^2$. Then we obtain the standard orthogonal group $\mathsf{SO}(n, 1)$, and Γ is any torsion-free subgroup of finite index of $R_{k/\mathbb{Q}}\mathsf{SO}^+(n, 1)(\mathbb{Z}) = \mathsf{SO}^+(n, 1)(\mathcal{O}_k)$. In the case $n = 2$, the main example is given by $B = M_2(L)$, where L is a number field as in (III). Then Γ is a torsion-free subgroup of finite index of the Bianchi group $\mathsf{PSL}_2(\mathcal{O}_L)$.

3.2. **Covolumes of arithmetic product-hyperbolic manifolds.** For k a number field, let $\zeta_k(s)$ denote the Dedekind zeta function of k, and let d_k denote the absolute value of the discriminant of k. If χ is the non-trivial character of a quadratic extension L/k, let $L(\chi, s) = \zeta_L(s)/\zeta_k(s)$ denote its Artin L-function, and let $d_{L/k} = d_L/d_k^2$. For any $\alpha, \alpha' \in \mathbb{R}$, we write $\alpha \sim_{\mathbb{Q}^\times} \alpha'$ if $\alpha = \beta\alpha'$ for some $\beta \in \mathbb{Q}^\times$.

Theorem 3.2. *The covolumes of arithmetic groups of types $(I) - (III)$ acting on products of hyperbolic spaces are as follows.*

(1) *Let $n = 2m$, let k be a totally real field of degree r, and let Γ be of type (I) acting on $\prod_{i=1}^t \mathbb{H}^n$, where $1 \leq t \leq r$. Let $M = (\prod_{i=1}^t \mathbb{H}^n)/\Gamma$. Then*

$$
\mathrm{vol}(M) \sim_{\mathbb{Q}^\times} |d_k|^{n(n+1)/4} \, \pi^{-m^2 r + m(r-t)} \, \zeta_k(2) \ldots \zeta_k(2m) ,
$$

(2) *Let $n = 2m - 1$ be odd and let k denote a totally real field of degree r. Let Γ be of type (II) acting on $\prod_{i=1}^{t} \mathbb{H}^n$ for some $1 \le t \le r$, defined in terms of a skew-Hermitian form Q over a quaternion algebra D. Let $d \in k^{\times}/k^{\times 2}$ denote the reduced norm of the discriminant of Q, and let χ denote the non-trivial character of the quadratic extension $L = k(\sqrt{d})$ of k. Let $L(\chi, m) = \zeta_L(m)\zeta_k(m)^{-1}$. If $M = (\prod_{i=1}^{t} \mathbb{H}^n)/\Gamma$, then*

$$\text{vol}(M) \sim_{\mathbb{Q}^\times} \begin{cases} |d_{L/k}|^{1/2} |d_k|^{n(n+1)/4} \pi^{-m^2 r + mt} \zeta_k(2) \ldots \zeta_k(2m-2)\, L(\chi, m), & \text{if } [L:K] = 2 , \\ |d_k|^{n(n+1)/4} \pi^{-m^2 r + mt} \zeta_k(2) \ldots \zeta_k(2m-2)\, \zeta_k(m), & \text{if } L = K , \end{cases}$$

In the special case where Γ is of type (I), the same formula holds, where d is the discriminant of the quadratic form q of (3.1).

(3) *Let L be a number field with r_1 real places and $r_2 \ge 1$ complex places. Let Γ denote an arithmetic group of type (III) acting on $\mathbb{X} = \prod_{i=1}^{t} \mathbb{H}^2 \times \prod_{j=1}^{r_2} \mathbb{H}^3$ where $1 \le t \le r_1$. If $M = \mathbb{X}/\Gamma$, then*

$$\text{vol}(M) \sim_{\mathbb{Q}^\times} |d_L|^{3/2} \pi^{t - 2r_1 - 2r_2} \zeta_L(2) .$$

In order to reduce the length of the paper, we have omitted the proof. More generally, let G be a connected semisimple algebraic group defined over a number field k, such that for at least one infinite place v of k, $G(k_v)$ is not compact. Let K denote a maximal compact subgroup of G, and let $X_v = K(k_v)\backslash G(k_v)$ be the corresponding symmetric space for each infinite place v of k. Set $X = \prod_{v|\infty} X_v$, and let Γ denote a torsion-free subgroup of finite index of the group of $\mathcal{O}_k$-valued points of G, which acts properly discontinuously on X. In [24] it was shown that

$$\text{vol}(X/\Gamma) \sim_{\mathbb{Q}^\times} |d_k|^{\dim G/2} \Delta_G \left(\prod_{v|\infty} \text{vol}(K_v)^{-1} \right) \prod_{v < \infty} \frac{q_v^{\dim G}}{\#G(\mathbb{F}_v)} ,$$

for the standard invariant volume forms on X and K_v (which are canonical up to a sign). Here, $\mathbb{F}_v$ denotes the residue field of k at a finite place v, and $q_v = \#\mathbb{F}_v$. In the cases we are interested in, K_v is an orthogonal group, whose volume is easily computed and yields a power of π. The infinite product on the right converges ([25], appendix II), and the point counts of G over finite fields are well-known [24]. More precisely, in case (I) of §3.1, G is of type B_n and we have

$$\#G(\mathbb{F}_v) = q^{m^2} \prod_{k=1}^{m} (q^{2k} - 1) .$$

In case (II), G is of type D_m and we have

$$\#G(\mathbb{F}_v) = \begin{cases} q_v^{m(m-1)}(q_v^m - 1) \prod_{k=1}^{m-1}(q_v^{2k} - 1) & \text{if } d \in \mathbb{F}_v^2 , \\ q_v^{m(m-1)}(q_v^m + 1) \prod_{k=1}^{m-1}(q_v^{2k} - 1) & \text{if } d \notin \mathbb{F}_v^2 . \end{cases}$$

Finally, in case (III), G is of type A_1 and $\#G(\mathbb{F}_v) = q(q^2 - 1)$. This yields the product of L-values in the volume formula. Finally, the invariant Δ_G is in $\mathbb{Q}^\times$ when G is an inner form of $SO(n+1)$ (cases (I), (III) and case (II) when $d \in k^2$) or $d_{L/k}^{1/2} \mathbb{Q}^\times$ if it is a non-trivial outer form (case (III) when $d \notin k^2$).

3.3. **Summary of volume computations.** One can show by a non-compact version of the Gauss-Bonnet formula [19], or by using Poincaré's formula for the volumes of even-dimensional simplices and a geodesic triangulation (e.g. [28] §11.3), that if M is a product-hyperbolic manifold of finite volume modelled on products of even-dimensional spaces, i.e., $M = \prod_{i \in I} \mathbb{H}^{2n_i}/\Gamma$, then we have:

$$(3.3) \qquad \qquad \mathrm{vol}(M) \sim_{\mathbb{Q}^\times} \pi^{\sum_{i \in I} n_i}.$$

This implies the following theorem due to Siegel and Klingen.

Corollary 3.3. *If k is a totally real field, $\zeta_k(1 - 2m) \in \mathbb{Q}^\times$ for all integers $m \geq 1$.*

Proof. Let $n = 2m$, and let $\Gamma \leq \mathsf{SO}^+(n, 1)(\mathcal{O}_k)$ be a torsion-free subgroup of finite index, where k is totally real of degree r. Applying (3.3) to $M = (\mathbb{H}^n)^r/\Gamma$ gives $\mathrm{vol}(M) \in \pi^{mr}\mathbb{Q}^\times$. Theorem 3.2 with $t = r$ implies that

$$\mathrm{vol}(M) \sim_{\mathbb{Q}^\times} |d_k|^{n(n+1)/4}\pi^{-rm^2}\zeta_k(2)\dots\zeta_k(2m) \ .$$

The functional equation for $\zeta_k(s)$ gives $\zeta_k(1 - 2m) \sim_{\mathbb{Q}^\times} d_k^{(2m-1)/2}\pi^{-2mr}\zeta_k(2m)$, and hence $\zeta_k(-1)\,\zeta_k(-3)\dots\zeta_k(1 - 2m) \in \mathbb{Q}^\times$, for all $m \geq 1$. $\qquad \square$

Corollary 3.4. *If an arithmetic product-hyperbolic manifold is modelled on even-dimensional spaces, its volume is a rational multiple of a power of π. In the case where $n = 2m - 1$ is odd, or $n = 2$, the formulae of theorem 3.2 simplify to*

$$\pi^{mt}\sqrt{|d_k|}\left(\frac{\zeta_k(m)}{\pi^{m[k:\mathbb{Q}]}}\right) \ , \quad \pi^{mt}\sqrt{|d_{L/k}d_k|}\left(\frac{L(\chi, m)}{\pi^{m[k:\mathbb{Q}]}}\right) \ , \quad \pi^{t+2r_2}\sqrt{|d_L|}\left(\frac{\zeta_L(2)}{\pi^{2[L:\mathbb{Q}]}}\right) \ .$$

By the corresponding functional equations, these are, respectively

$$(3.4) \qquad \pi^{mt}\zeta_k^*(1 - m) \ , \quad \pi^{mt}L^*(\chi, 1 - m) \ , \quad \pi^{t+2r_2}\zeta_L^*(-1) \quad \mathrm{mod}\ \mathbb{Q}^\times \ .$$

4. Triangulation of Product-hyperbolic manifolds.

In order to define the motive of a product-hyperbolic manifold, we must decompose it into compact and cuspidal parts, and triangulating each part with products of simplices using an inclusion-exclusion argument due to Zagier.

4.1. **Decomposition of product-hyperbolic manifolds into cusp sectors.** The cusps of a product-hyperbolic manifold are best described using the upper-half space model $\mathbb{U}^n \cong \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0}\}$ (§2). For all $r > 0$, let $B_n(r) \subset \mathbb{H}^n$ denote the closed horoball near the point at infinity:

$$B_n(r) = \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : t \geq r\}.$$

Its boundary, the horosphere, is isometric to a Euclidean plane $\mathbb{E}^{n-1}$ at height r. Now let $\mathbb{X}^{\mathfrak{n}} = \prod_{i \in I} \mathbb{H}^{n_i}$. For any subset $\emptyset \neq S \subset I$, and any set of parameters $\underline{r} = \{r_i > 0\}_{i \in S}$ indexed by S, we define the corresponding product-horoball to be

$$B_S(\underline{r}) = \prod_{i \in S} B_{n_i}(r_i) \times \prod_{i \in I \setminus S} \mathbb{H}^{n_i} \subset \mathbb{X}^{\mathfrak{n}}.$$

Its horosphere is $\prod_{i \in S} \partial B_{n_i}(r_i) \times \prod_{i \in I \setminus S} \mathbb{H}^{n_i}$ which is diffeomorphic to the space $\prod_{i \in S} \mathbb{E}^{n_i-1} \times \prod_{i \in I \setminus S} \mathbb{H}^{n_i}$. The cusps of a product-hyperbolic manifold are diffeomorphic to a product $F \times \mathbb{R}_{>0}^{|S|}$ where F is a compact flat-hyperbolic manifold.

Theorem 4.1. *Any complete product-hyperbolic manifold M of finite volume has a decomposition into finitely many disjoint pieces*

$$M = \bigcup_{\ell=0}^{p} M_{(\ell)}$$

where $M_{(0)}$ is compact, and each $M_{(\ell)}$ for $\ell \geq 1$ is isometric to a cusp, i.e., $M_{(\ell)} \cong F_\ell \times \mathbb{R}_{>0}^{k_\ell}$, where $k_\ell \geq 1$ and F_ℓ is a compact flat-hyperbolic manifold with boundary.

The proof of the theorem is different in the case when M is arithmetic or non-arithmetic. By Margulis' theorem, an irreducible non-arithmetic lattice is necessarily of rank one and acts on a single hyperbolic space. In this case, M is an ordinary hyperbolic manifold of finite volume and the decomposition is well-known ([2], III, §10 and appendix). In the case of arithmetically-defined groups, the decomposition follows from [20], Proposition 4.6, or the main result of [30].

4.2. **Geodesic simplices and rational points.**

4.2.1. *Rational points on product-hyperbolic manifolds.* Let k be a subfield of $\mathbb{R}$, and let $\mathbb{H}_k^n$ (resp. $\mathbb{E}_k^n$) denote the set of points with k-rational coordinates in the hyperboloid model for hyperbolic space (resp. Euclidean space). The former coincides with the set of k-rational points in the upper-half model, but is strictly contained in the set of k-points in the Klein model. Define $\partial \mathbb{H}_k^n$ to be the set of k-rational points on the absolute $k^{n-1} \times \{0\} \subset \partial \mathbb{U}^n = \mathbb{R}^n \times \{0\}$ (§2.2.2), and write $\overline{\mathbb{H}}_k^n = \mathbb{H}_k^n \cup \partial \mathbb{H}_k^n$. The sets $\mathbb{H}_k^n$ and $\partial \mathbb{H}_k^n$ are preserved by $\mathsf{SO}^+(n,1)(k)$ and are dense in $\mathbb{H}^n$ and $\partial \mathbb{H}^n$ respectively. Now consider products of hyperbolic spaces $\mathbb{X}^{\mathfrak{n}} = \prod_{i=1}^{N} \mathbb{H}^{n_i}$. Let $S = (k_1, \ldots, k_N)$ denote a tuple of fields, with $k_i \subset \mathbb{R}$. We define the set of S-rational points on $\mathbb{X}^{\mathfrak{n}}$ to be $\mathbb{X}_S^{\mathfrak{n}} = \prod_{i=1}^{N} \mathbb{H}_{k_i}^{n_i}$. We say that a product-hyperbolic manifold $M = \mathbb{X}^{\mathfrak{n}}/\Gamma$ is *defined over S* if Γ is conjugate to Γ' which satisfies

$$\Gamma' \leq \prod_{1 \leq i \leq N} \mathsf{SO}^+(n_i, 1)(k_i) \, .$$

Remark 4.2. In dimension $n = 3$, there is an exceptional isomorphism between $\mathsf{SO}^+(3,1)(\mathbb{R})$ and $\mathsf{PSL}_2(\mathbb{C})$. For any field $L \subset \mathbb{C}$, we define the set of L-points $\mathbb{H}_L^3$ of $\mathbb{H}^3$ to be the orbit of a fixed rational point of $\mathbb{H}^3$ under $\mathsf{PSL}_2(L)$. From now on, we assume for simplicity that $\mathbb{X}$ has no hyperbolic components of dimension 3 and all fields k_i are real. The translation to the exceptional case $n = 3$ is straightforward.

Theorem 4.3. *Let M be a complete product-hyperbolic manifold of finite volume. Then M is defined over $S = (k_1, \ldots, k_r)$ where k_i are (embedded) number fields.*

Proof. For arithmetic groups this follows from the definition. For non-arithmetic groups, it suffices to consider discrete subgroups of $\mathsf{SO}^+(n,1)$, by Margulis' arithmeticity theorem. By Weil's rigidity theorem, Γ is conjugate to a subgroup Γ' of $\mathsf{SO}^+(n,1)(\overline{\mathbb{Q}})$. Since Γ' is finitely generated, its entries lie in a field $k \subset \mathbb{R}$ which is finite over $\mathbb{Q}$. $\square$

We define the set of S-rational points of such an M defined over S to be

(4.1) $$M_S = \mathbb{X}_S^{\mathfrak{n}}/\Gamma' \, .$$

The set of S-rational points M_S is dense in M.

Lemma 4.4. *The fixed point $z \in \partial\mathbb{H}^n$ of any parabolic motion $\gamma \in \mathsf{SO}^+(n,1)(k)$ is defined over k, i.e., $z \in \partial\mathbb{H}_k^n$.*

Proof. In the Klein model for hyperbolic space, the group $\mathsf{SO}^+(n,1)(k)$ acts by projective transformations on the absolute $\partial\mathbb{H}^n = \{z_1, \ldots, z_n : \sum_i z_i^2 = 1\}$. It follows that a point $z \in \partial\mathbb{H}^n$ which is stabilised by $\gamma \in \mathsf{SO}^+(n,1)(k)$ satisfies an equation $\gamma z = z$, which is algebraic in the coordinates of z, and has coefficients in k. If γ is parabolic, then this equation has a unique solution on the absolute. By uniqueness, z coincides with its conjugates under $\mathrm{Gal}(\overline{k}/k)$, and hence $z \in \partial\mathbb{H}_k^n$. $\quad\square$

4.2.2. *The action of the symmetric group.* Now let $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$ denote a set of distinct real embeddings of a fixed field k as above. Set $k_i = \sigma_i k$, and let $S = (k_1, \ldots, k_N)$. For each pair of indices $1 \le i, j \le N$, there is a bijection $\sigma_i \sigma_j^{-1} : \overline{\mathbb{H}}_{k_j}^n \xrightarrow{\sim} \overline{\mathbb{H}}_{k_i}^n$. Let $\mathfrak{S}_N$ denote the symmetric group on N letters $\{1, \ldots, N\}$.

Definition 4.5. If the dimensions of all hyperbolic components n_i are equal to n, the symmetric group $\mathfrak{S}_N$ acts on $\overline{\mathbb{X}}_S^{\mathfrak{n}} = \prod_{i=1}^N \overline{\mathbb{H}}_{k_i}^n$ as follows:

$$\pi(x_1, \ldots, x_N) = (\sigma_1 \sigma_{\pi(1)}^{-1} x_{\pi(1)}, \ldots, \sigma_N \sigma_{\pi(N)}^{-1} x_{\pi(N)}), \quad \text{where } \pi \in \mathfrak{S}_N .$$

We define the *equivariant points* of $\overline{\mathbb{X}}_S^{\mathfrak{n}}$ to be the fixed points under this action.

Definition 4.6. A product-hyperbolic manifold $M = \prod_{i=1}^N \mathbb{H}^{n_i}/\Gamma$, with all n_i equal to n, is *equivariant* with respect to $\mathfrak{S}_N$ if Γ lies in the image of

$$
\begin{aligned}
e : \mathsf{SO}(n,1)(k) &\longrightarrow \prod_{1 \le i \le N} \mathsf{SO}(n,1)(\sigma_i(k)) \\
A &\mapsto (\sigma_1(A), \ldots, \sigma_N(A)) .
\end{aligned}
$$

Note that the fixed point of a parabolic motion on an equivariant product-hyperbolic manifold is necessarily equivariant.

4.2.3. *Geodesic simplices and the action of $\mathfrak{S}_N$.* Let $\mathbb{X}$ be $\mathbb{E}^n$ or $\mathbb{H}^n$, for $n \ge 2$. If $x_0, \ldots, x_n$ are $n+1$ distinct points in $\mathbb{X}$, let $\Delta(x_0, \ldots, x_n)$ denote the geodesic simplex whose vertices are $x_0, \ldots, x_n$. This is defined to be the convex hull of the points $\{x_0, \ldots, x_n\}$. If the points $x_0, \ldots, x_n$ lie in a proper geodesic subspace, then $\Delta(x_0, \ldots, x_n)$ will be degenerate. The boundary faces of $\Delta(x_0, \ldots, x_n)$ are the convex hulls of nonempty strict subsets of the points $\{x_0, \ldots, x_n\}$. When $\mathbb{X} = \mathbb{H}^n$, we can allow some or all of the vertices x_i to lie on the boundary $\partial\mathbb{H}^n$. One can show that such a simplex $\Delta(x_0, \ldots, x_n)$ always has finite, and in fact bounded, volume. The simplex $\Delta(x_0, \ldots, x_n)$ is said to be *defined over a field $k \subset \mathbb{R}$*, if $x_0, \ldots, x_n \in \overline{\mathbb{X}}_k$. Suppose that we are given a map of fields $\sigma : k \hookrightarrow k'$. It induces an action on the set of geodesic simplices defined over k:

$$(4.2) \qquad \sigma\Delta(x_0, \ldots, x_n) = \Delta(\sigma x_0, \ldots, \sigma x_n) \subset \overline{\mathbb{X}}_{k'} .$$

In a product of spaces of type (2.1), we consider products of the form

$$(4.3) \qquad \Delta = \Delta_1 \times \ldots \times \Delta_N ,$$

where Δ_i are geodesic simplices in each component. We call this a *geodesic product-simplex*. It is *defined over the fields $S = (k_1, \ldots, k_N)$* if Δ_i is defined over k_i for all $1 \le i \le N$. In the equivariant case $S = (\sigma_1 k, \ldots, \sigma_N k)$, and all n_i are equal, the symmetric group $\mathfrak{S}_N$ acts on geodesic product simplices defined over S as follows:

$$(4.4) \quad \pi(\Delta_1 \times \ldots \times \Delta_N) = \sigma_1 \sigma_{\pi(1)}^{-1} \Delta_{\pi(1)} \times \ldots \times \sigma_1 \sigma_{\pi(N)}^{-1} \Delta_{\pi(N)} \text{ for any } \pi \in \mathfrak{S}_N .$$

4.2.4. *Cones over Euclidean simplices.* Let $\Delta(x_1, \ldots, x_n) \subset \mathbb{E}^{n-1}$ be a Euclidean geodesic simplex. Let us identify the horosphere in $\mathbb{U}^n$ at height $R > 0$ with $\mathbb{E}^{n-1} \times \{R\} \subset \mathbb{U}^n$. If ∞ denotes the point at infinity, let us write $\Delta_\infty = \Delta(x_1, \ldots, x_n, \infty)$ for the geodesic hyperbolic simplex $\Delta \times [R, \infty)$ which is the cone over Δ.

4.3. **Generalities on virtual triangulations.** Let M denote a flat-hyperbolic manifold. In order to decompose M into products of geodesic simplices, we must consider geodesic polytopes in products of Euclidean and hyperbolic spaces, which may have vertices at infinity. Let P denote an n-dimensional convex polytope in Euclidean or hyperbolic space $\overline{\mathbb{H}}^n$. A *faceting* $F(P)$ of P ([28], §11.1) is a finite collection of closed, $(n-1)$-dimensional convex polytopes F_i, called *facets*, which are contained in the boundary ∂P of P such that:

(1) Each face of P is a union of facets.
(2) Any two facets are either disjoint or meet along their common boundaries.

The set of all codimension 1 faces of P, for example, defines a faceting of P. A *product-polytope* in $\overline{\mathbb{X}}^n$ is a product of convex geodesic polytopes, and one defines a faceting in a similar manner. For any product-polytope P, let P^f denote the polytope P with all infinite components removed. Now let R denote a finite set of geodesic product-polytopes in $\overline{\mathbb{X}}^n$, and suppose we are given a faceting for each product-polytope in R. A set of gluing relations for R is a way to identify all facets of all polytopes $P \in R$ in pairs which are isometric. Let

$$T = \coprod_{P \in R} P^f / \sim$$

denote the topological space obtained by identifying glued facets. A *product-tiling* of M is then an isometry

$$(4.5) \qquad\qquad f : T \longrightarrow M .$$

Since facets may be strictly contained in the faces of each geodesic simplex, the tiling is not always a triangulation. See also [28], §11 for a much more detailed treatment of tilings and triangulations in a general context.

We will also need to consider *virtual* tilings. To define a virtual tiling, consider a continuous surjective map $f : T \longrightarrow M$, of finite degree which is not necessarily étale. We assume that the restriction of f to the interior of each product polytope P in T is an isometry. We define the local multiplicity of f on P to be $+1$ if $f|_P$ is orientation-preserving, and -1 if $f|_P$ is orientation-reversing. The condition that f be a virtual tiling is that the total multiplicity of f is almost everywhere equal to 1. In the case when M is defined over the fields $S = (k_1, \ldots, k_N)$, we will say that the (virtual) product tiling is *defined over S* if the geodesic product simplices which occur in T are defined over S.

4.4. **Tiling of product-hyperbolic manifolds.** One can construct a product-tiling of any complete, orientable, finite-volume flat-hyperbolic manifold M, using a variant of an argument due to Zagier [37].

Lemma 4.7. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ denote any sets, and let $\epsilon \in \{0, 1\}^n$. We write $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, and let*

$$X_i(\epsilon) = \begin{cases} X_i \cap Y_i & \text{if} \quad \epsilon_i = 0 , \\ X_i \backslash (X_i \cap Y_i) & \text{if} \quad \epsilon_i = 1 , \end{cases}$$

and define $Y_i(\epsilon)$ similarly. Any union of products $(X_1 \times \ldots \times X_n) \cup (Y_1 \times \ldots \times Y_n)$ can be written

$$(X_1 \cap Y_1) \times \ldots \times (X_n \cap Y_n) \cup \bigcup_{0 \neq \epsilon \in \{0,1\}^n} X_1(\epsilon) \times \ldots \times X_n(\epsilon) \cup \bigcup_{0 \neq \epsilon \in \{0,1\}^n} Y_1(\epsilon) \times \ldots \times Y_n(\epsilon),$$

where all unions are disjoint.

We apply the lemma to a pair of geodesic product-simplices in flat-hyperbolic space. Let $\Delta = \prod_{i \in I} \Delta_i$ and $\Delta' = \prod_{i \in I} \Delta'_i$ where Δ_i, Δ'_i are geodesic simplices in $\mathbb{E}^{n_i}$ or $\overline{\mathbb{H}}^{n_i}$ for each $i \in I$. By the lemma, $\Delta \cup \Delta'$ can be decomposed as a disjoint union of products of $\Delta_i \cap \Delta'_i$, $\Delta_i \backslash (\Delta_i \cap \Delta'_i)$, or $\Delta'_i \backslash (\Delta_i \cap \Delta'_i)$, for $i \in I$. In each case, the intersection of two geodesic simplices (or its complement) is a union of geodesic polytopes. Every such polytope can be decomposed as a disjoint union of geodesic simplices by subdividing and triangulating. It follows that we can triangulate any overlapping union $\Delta \cup \Delta'$ with geodesic product-simplices.

Corollary 4.8. *Any finite union of product-simplices can be obtained by gluing finitely many geodesic product-simplices along pairs of common facets.*

We can now prove the main result of this section, following [37].

Proposition 4.9. *Every product-hyperbolic manifold M of finite volume admits a finite tiling by products of geodesic simplices. If M is defined over a tuple of fields S as in §4.2, then we can assume the product simplices have S-rational vertices.*

Proof. By theorem 4.1, there is a finite decomposition $M = \bigcup_\ell M_{(\ell)}$, where $M_{(0)}$ is compact and for $\ell \geq 1$, $M_{(\ell)}$ is a cone over F_ℓ, a compact flat-hyperbolic manifold with boundary. We first tile each $M_{(\ell)}$ for $\ell \geq 1$ by constructing a tiling of F_ℓ and taking the cone at infinity over this tiling using §4.2.4. We then obtain a tiling of M on taking the union of all the geodesic product-simplices involved in the tiling of each piece $M_{(\ell)}$ and excising the overlaps using corollary 4.8.

Let $M = \mathbb{X}^\mathbf{n}/\Gamma$ and let $\pi : \mathbb{X}^\mathbf{n} \to M$ denote the covering map. Let $\mathcal{F} \subset \mathbb{X}^\mathbf{n}$ be a fundamental set for M, with a decomposition $\mathcal{F} = \bigcup_\ell \mathcal{F}_\ell$, where $\mathcal{F}_0$ is compact, and each $\mathcal{F}_\ell$ for $\ell \geq 1$ is diffeomorphic to $\mathbb{R}_{>0}^{k_\ell} \times D_\ell$, where D_ℓ is a compact domain in flat-hyperbolic space $\prod_{i \in S} \mathbb{E}^{n_i - 1} \times \prod_{j \in I \backslash S} \mathbb{H}^{n_j}$ (and hence $k_\ell = |S|$). Cover D_ℓ with geodesic product-simplices as follows. Since D_ℓ is compact, choose compact polyhedra $K_i \subset \mathbb{E}^{n_i - 1}$, $K_j \subset \mathbb{H}^{n_j}$, where $i \in S, j \in I \backslash S$, such that $D_\ell \subset \prod_{i \in S} K_i \times \prod_{j \in I \backslash S} K_j$. We denote the restriction of the covering map $D_\ell \to F_\ell$ by π also. Each set K_i can be triangulated by finitely many oriented geodesic simplices $\Delta_{a_i}^{(i)}$ which are sufficiently small such that any product $\Delta_{\mathfrak{a}} = \prod_{i \in S} \Delta_{a_i}^{(i)} \times \prod_{j \in I \backslash S} \Delta_{a_j}^{(j)}$, where $\mathfrak{a} = (a_i)_{i \in I}$, is mapped isometrically onto $\pi(\Delta_{\mathfrak{a}})$. Let $\mathfrak{a} \neq \mathfrak{b}$ be indices such that the pair of simplices $\pi(\Delta_{\mathfrak{a}})$, $\pi(\Delta_{\mathfrak{b}})$ have non-empty overlap. It follows that there is a $\gamma \in \Gamma$ such that $\gamma \Delta_{\mathfrak{a}} \cap \Delta_{\mathfrak{b}} \neq \emptyset$. Applying the previous corollary to the union $\gamma \Delta_{\mathfrak{a}} \cup \Delta_{\mathfrak{b}}$, we can replace $\Delta_{\mathfrak{a}} \cup \Delta_{\mathfrak{b}}$ with a union of product-simplices whose interiors are disjoint after projection down to F_ℓ. We can repeatedly excise the overlap between simplices using the previous lemma and its corollary to obtain the required tiling of D_ℓ with product-simplices. To show that this process terminates after finitely many steps, let $d(x) \in \mathbb{N}$ denote the multiplicity of the tiling at each point $x \in D_\ell$. Since $\mathcal{F}$ is a fundamental set, there is an integer N such that $d(x) \leq N$ for all $x \in D_\ell$. Every time an excision is applied, the local multiplicities $d(x)$ for all x in some open subset of D_ℓ decrease by 1. Since D_ℓ is compact,

this process terminates when $d(x) = 1$ almost everywhere. This gives the required tiling of D_ℓ. Each Euclidean component of D_ℓ can be identified with a suitable horoball neighbourhood of infinity in $\mathbb{U}^n$. By the construction of §4.2.4, replace every product of flat-hyperbolic geodesic simplices that occurs in the tiling of D_ℓ:

$$\prod_{i \in S} \Delta^{(i)} \times \prod_{j \in I \backslash S} \Delta^{(j)} \in \prod_{i \in S} \mathbb{E}^{n_i - 1} \times \prod_{j \in I \backslash S} \mathbb{H}^{n_j} ,$$

with its cone at infinity (§4.2.4):

$$\prod_{i \in S} \Delta_\infty^{(i)} \times \prod_{j \in I \backslash S} \Delta^{(j)} \in \prod_{i \in S} \overline{\mathbb{H}}^{n_i} \times \prod_{j \in I \backslash S} \mathbb{H}^{n_j} .$$

This gives a covering of each end $\mathcal{F}_\ell$, for $\ell \geq 1$, with product-simplices which maps via π to a tiling of $M_{(\ell)}$. Finally, the compact part $\mathcal{F}_0$ of $\mathcal{F}$ can be covered by a large compact set K_0 which can be triangulated with geodesic product-simplices as before. All together, we have a covering of $\mathcal{F}$ with finitely many geodesic product-simplices which maps to a tiling in each cusp $M_{(\ell)}$. By applying corollary 4.8 as above, we can excise the overlaps between these simplices all over again. Since the $K_0 \cap \mathcal{F}_i$ are compact, we obtain a tiling of M after finitely many steps.

For the last statement, suppose that M is defined over a tuple of fields S, as in §4.2. Since the set of S-rational points is dense in M, and since by lemma 4.9, the coordinates at infinity of the cusps of M are S-rational, we can ensure in the previous argument that all product-simplices have vertices defined over S. $\qquad \square$

Note that the geodesic simplices which occur have at most one vertex at infinity, a fact which will be used later. It does not matter if degenerate simplices occur.

4.5. **Equivariance.** Now suppose that the product-hyperbolic manifold M is equivariant (§4.2.2). By proposition 4.9, there exists a product-tiling of M which is defined over S, where $S = (\sigma_1 k, \ldots, \sigma_N k)$. Let T denote the set of hyperbolic product-simplices in this tiling. The symmetry group $\mathfrak{S}_N$ acts on the product-simplices, and preserves the subdivisions of a face into its facets F_i: if F is tiled by facets F_i for $1 \leq i \leq m$, then ${}^\sigma F$ is tiled by facets ${}^\sigma F_i$, for all $\sigma \in \mathfrak{S}_N$. Note that some of the simplices ${}^\sigma F_i$ may be oriented negatively, so the latter tiling is in fact a virtual tiling of ${}^\sigma F$. Let ${}^\sigma T$ denote the set of images of the elements of T under $\sigma \in \mathfrak{S}_N$. We can glue the simplices in ${}^\sigma T$ back together according to the same gluing pattern to form a virtual tiling of ${}^\sigma M$. Since M was assumed to be equivariant, this gives a new tiling of M, which is also defined over S.

Lemma 4.10. *Let M denote an equivariant product-hyperbolic manifold. If T is a product-tiling of M defined over S, then ${}^\sigma T$ is also a virtual product-tiling of M defined over S, for all $\sigma \in \mathfrak{S}_N$.*

Note that, given a finite virtual tiling of M, one can obtain a genuine tiling by subdividing and excising any overlaps using a variant of corollary 4.8.

5. THE MOTIVE OF A PRODUCT-HYPERBOLIC MANIFOLD

5.1. **Framed Mixed Tate motives and their periods.**

5.1.1. *Mixed Tate motives.* Let k be a number field, and let $\mathrm{MT}(k)$ denote the abelian tensor category of mixed Tate motives over k [13]. Its simple objects are the pure Tate motives $\mathbb{Q}(n)$, where $n \in \mathbb{Z}$, and its structure is determined by:

$$(5.1) \qquad \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} 0 & \text{if } n \leq 0 , \\ K_{2n-1}(k) \otimes \mathbb{Q} , & \text{if } n \geq 1 , \end{cases}$$

and the fact that $\mathrm{Ext}^2_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for all $n \in \mathbb{Z}$. Recall that $K_1(k) \cong k^\times$, and Borel proved in [7] that for $n > 1$,

$$(5.2) \qquad \dim_{\mathbb{Q}}(K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}) = \begin{cases} r_1 + r_2, & \text{if } n \text{ is odd} ; \\ r_2, & \text{if } n \text{ is even} , \end{cases}$$

where r_1 is the number of real places of k, and r_2 is the number of complex places of k. Although we do not explicitly require (5.2) to prove theorem 1.1, it is used implicitly in the construction of $\mathrm{MT}(k)$.

Every mixed Tate motive $M \in \mathrm{MT}(k)$ has a canonical weight filtration, which is increasing, finite, and indexed by even integers. For each $n \in \mathbb{Z}$, its graded piece of weight $-2n$ is denoted by $\mathrm{gr}^W_{-2n} M = W_{-2n} M / W_{-2n-1} M$, and is isomorphic to a finite sum of copies of $\mathbb{Q}(n)$. Following [13], one sets

$$\omega_n(M) = \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n} M) .$$

The de Rham realisation of M is defined to be the graded vector space $M_{DR} = \omega(M) \otimes_{\mathbb{Q}} k$, where $\omega(M) = \oplus_n \omega_n(M)$. For every embedding $\sigma : k \hookrightarrow C$ into an algebraic closure C of k, there is a Betti realisation M_σ, which is a finite dimensional vector space over $\mathbb{Q}$ equipped with a weight filtration such as above [13], §2.11. There is a comparison isomorphism which respects the weight filtrations

$$(5.3) \qquad \mathrm{comp}_{\sigma,DR} : M_{DR} \otimes_{k,\sigma} C \xrightarrow{\sim} M_\sigma \otimes_{\mathbb{Q}} C ,$$

and is functorial with respect to σ. For each $\sigma : k \hookrightarrow C$, the data

$$(5.4) \qquad H_\sigma = (M_{DR}, M_\sigma, \mathrm{comp}_{\sigma,DR})$$

is called a mixed Hodge-Tate structure [13], §2.13. The Hodge realisation functor:

$$M \mapsto (M_{DR}, M_\sigma, \mathrm{comp}_{\sigma,DR})_{\sigma:k\hookrightarrow C}$$

to the category of systems of mixed Hodge-Tate structures is fully faithful, although we shall not require this fact. Note that $\omega_n(M) \subset W_{-2n} M_{DR}$ can be retrieved from the data (5.4) (*loc. cit.*).

5.1.2. *Framed objects in* $\mathrm{MT}(k)$. Let $M \in \mathrm{MT}(k)$, and let $n \geq 0$.

Definition 5.1. An *n-framing* of M [5, 18] consists of non-zero morphisms:

$$\begin{aligned} v_0 &\in \omega_0(M) = \mathrm{Hom}(\mathbb{Q}(0), \mathrm{gr}^W_0 M) , \\ f_n &\in \omega_n(M)^\vee = \mathrm{Hom}(\mathrm{gr}^W_{-2n} M, \mathbb{Q}(n)) . \end{aligned}$$

A morphism from (M, v_0, f_n) to (M, v_0', f_n') is a morphism $\phi : M \to M'$ such that $\phi(v_0) = v_0'$ and $f_n'(\phi) = f_n$. Morphisms generate an equivalence relation on the set of n-framed objects. The equivalence class of (M, v_0, f_n) is written $[M, v_0, f_n]$.

Let $\mathfrak{A}_n(k)$ denote the set of equivalence classes of n-framed objects in $\mathrm{MT}(k)$. One shows that $\mathfrak{A}_n(k)$ is a $\mathbb{Q}$-vector space with respect to the addition rule:

$$(5.5) \qquad [M, v_0, f_n] + [M', v_0', f_n'] = [M \oplus M', v_0 \oplus v_n', f_n + f_n'] ,$$

and scalar multiplication $\alpha[M, v_0, f_n] = [M, \alpha v_0, f_n] = [M, v_0, \alpha f_n]$ for all $\alpha \in \mathbb{Q}^\times$. The zero element is given by the equivalence class of $\mathbb{Q}(0) \oplus \mathbb{Q}(n)$ with trivial framings, and $\mathfrak{A}_0(k) \cong \mathbb{Q}$. Consider the graded $\mathbb{Q}$-vector space:

$$(5.6) \qquad \mathfrak{A}(k) = \bigoplus_{n \geq 0} \mathfrak{A}_n(k) \ .$$

It is equipped with a coproduct $\Delta : \mathfrak{A}(k) \to \mathfrak{A}(k) \otimes_\mathbb{Q} \mathfrak{A}(k)$, whose components

$$\Delta_{r,n-r} : \mathfrak{A}_n(k) \to \mathfrak{A}_r(k) \otimes_\mathbb{Q} \mathfrak{A}_{n-r}(k)$$

can be computed as follows. For $M \in \mathrm{MT}(k)$, let $\{e_i\}_{1 \leq i \leq N}$ be any basis of $\omega_r(M)$, and let $\{e_i^\vee\}_{1 \leq i \leq N}$ denote the dual basis in $\omega_r(M)^\vee$. Define

$$(5.7) \qquad \Delta_{r,n-r}[M, v_0, f_n] = \sum_{i=1}^N [M, v_0, e_i^\vee] \otimes [M, e_i, f_n](-r) \ ,$$

where $[M, e_i, f_n](-r)$ is the Tate-twisted object $M(-r)$ with corresponding framings. Let $\Delta_n = \bigoplus_{0 \leq r \leq n} \Delta_{r,n-r}$, and set $\Delta = \bigoplus_{n \geq 0} \Delta_n$. One verifies that the above constructions are well-defined and that $\mathfrak{A}(k)$ can be made into a graded commutative Hopf algebra. Define the reduced coproduct $\widetilde{\Delta}$ on $\mathfrak{A}(k)$ by $\widetilde{\Delta}(X) = \Delta(X) - 1 \otimes X - X \otimes 1$. Its kernel is the set of primitive elements in $\mathfrak{A}(k)$.

Proposition 5.2. *(see e.g., [17]) There is an isomorphism:*

$$(5.8) \quad \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \ker \left(\widetilde{\Delta}_n : \mathfrak{A}_n(k) \longrightarrow \bigoplus_{1 \leq r \leq n-1} \mathfrak{A}_r(k) \otimes_\mathbb{Q} \mathfrak{A}_{n-r}(k) \right) \ .$$

This gives a strategy for defining elements in rational algebraic K-theory via (5.1). The essential idea [4] is to construct a framed mixed Tate motive out of simple algebraic varieties in such a way that the reduced coproduct vanishes.

5.1.3. *Real periods.* Let $M \in \mathrm{MT}(k)$, and $\sigma : k \hookrightarrow \mathbb{C}$. Consider the isomorphism:

$$P_\sigma : \omega(M) \otimes_\mathbb{Q} \mathbb{C} \overset{\sim}{\longrightarrow} M_{DR} \otimes_{k,\sigma} \mathbb{C} \overset{\mathrm{comp}_{\sigma, DR}}{\longrightarrow} M_\sigma \otimes_{k,\sigma} \mathbb{C} \ ,$$

and let $P_\sigma^* = (P_\sigma^\vee)^{-1}$ be the inverse dual map from $\omega(M)^\vee \otimes_\mathbb{Q} \mathbb{C}$ to $M_\sigma^\vee \otimes_{k,\sigma} \mathbb{C}$. Denote the natural pairing $\left(M_\sigma \otimes_{k,\sigma} \mathbb{C} \right) \otimes \left(M_\sigma \otimes_{k,\sigma} \mathbb{C} \right)^\vee \to \mathbb{C}$ by $\langle \cdot, \cdot \rangle$.

Definition 5.3. Let $M \in \mathrm{MT}(k)$, with n-framings $v_0 \in \omega(M)$, $f_n \in \omega(M)^\vee$. For every $\sigma : k \hookrightarrow \mathbb{C}$ its real period ([17], §4) is defined by

$$\langle \mathrm{Im} \left((2i\pi)^{-n} P_\sigma(v_0 \otimes 1) \right), \mathrm{Re} \left(P_\sigma^*(f_n \otimes 1) \right) \rangle \in \mathbb{R} \ .$$

This only depends on the equivalence class $[M, v_0, f_n]$, yielding a map:

$$(5.9) \qquad \mathcal{R}_\sigma : \mathfrak{A}_n(k) \longrightarrow \mathbb{R} \ ,$$

which is called the real period. Note that its normalization varies in the literature.

To calculate the real period, choose a graded k-basis of M_{DR} and a $\mathbb{Q}$-basis of M_σ which is compatible with the weight filtration. The matrix $\mathrm{comp}_{\sigma, DR}$ in these bases can be computed from the integration pairing

$$M_{DR} \otimes_\mathbb{Q} M_\sigma^\vee \to \mathbb{C} \ .$$

We can assume that $\mathrm{comp}_{\sigma, DR}$ on $\mathrm{gr}^W_{-2n} M_{DR} \otimes_{k,\sigma} \mathbb{C} \overset{\sim}{\to} \mathrm{gr}^W_{-2n} M_\sigma \otimes_{k,\sigma} \mathbb{C}$ is $(2\pi i)^n$ times the identity. Then our k-basis of M_{DR} is a $\mathbb{Q}$-basis of $\omega(M)$ ([13], (2.11.3)) and our matrix represents P_σ. It is well-defined up to change of weight-filtered

$\mathbb{Q}$-basis of M_σ, i.e., multiplication by a triangular unipotent matrix with entries in $\mathbb{Q}$. By (5.8), we have $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \ker \widetilde{\Delta}_n \subset \mathfrak{A}_n(k)$, and hence a map:

$$(5.10) \qquad R_\sigma : \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathbb{R} \ .$$

To compute this map, we can represent an element $\xi \in \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ by an extension $0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0$. For every $\sigma : k \hookrightarrow \mathbb{C}$, the prescription above yields a period matrix for M which is of the form

$$P_\sigma(M) = \begin{pmatrix} (2\pi i)^n & 0 \\ \alpha & 1 \end{pmatrix} \text{ for some } \alpha \in \mathbb{C}.$$

Therefore its real period, for the obvious framings, is $R_\sigma(\xi) = \mathrm{Im}\left(\frac{\alpha}{(2\pi i)^n}\right)$.

5.1.4. *Regulators.* Let k be a number field, and let $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$ denote any set of distinct embeddings of k into $\mathbb{C}$. Writing $k_i = \sigma_i(k)$, there is an isomorphism:

$$(5.11) \qquad \rho : \mathfrak{A}_n(k) \otimes_\mathbb{Q} \ldots \otimes_\mathbb{Q} \mathfrak{A}_n(k) \xrightarrow{\sim} \mathfrak{A}_n(k_1) \otimes_\mathbb{Q} \ldots \otimes_\mathbb{Q} \mathfrak{A}_n(k_N) \ .$$

Let $\mathfrak{A}_n(k)^{\otimes N}$ denote the left-hand side of (5.11). The symmetric group $\mathfrak{S}_N$ acts upon it by permuting the factors. Let $\mathfrak{A}_n(k)^{\otimes \Sigma}$ denote the right-hand side of (5.11), with the induced $\mathfrak{S}_N$-action. If $\xi_i \in \mathfrak{A}_n(k_i)$, for $1 \le i \le N$, $\mathfrak{S}_N$ acts by:

$$\pi(\xi_1 \otimes \ldots \otimes \xi_N) = \sigma_1 \sigma_{\pi(1)}^{-1}(\xi_{\pi(1)}) \otimes \ldots \otimes \sigma_N \sigma_{\pi(N)}^{-1}(\xi_{\pi(N)}) \quad \text{for } \pi \in \mathfrak{S}_N \ .$$

For any subspace $V \subset \mathfrak{A}_n(k)$ write $\bigwedge^N V \subset V^{\otimes N}$ for the subspace of elements which are alternating with respect to the action of $\mathfrak{S}_N$. Then by (5.8):

$$(5.12) \qquad \begin{array}{ccc} \bigwedge^N \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) & \subset \bigwedge^N \mathfrak{A}_n(k) & \subset \mathfrak{A}_n(k)^{\otimes N} \\ \wr\downarrow & \wr\downarrow & \wr\downarrow \\ \bigwedge^\Sigma \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) & \subset \bigwedge^\Sigma \mathfrak{A}_n(k) & \subset \mathfrak{A}_n(k)^{\otimes \Sigma} \end{array}$$

where a superscript Σ in the second line denotes the subspace of alternating elements with respect to the action of $\mathfrak{S}_N$ on $\mathfrak{A}_n(k)^{\otimes \Sigma}$. Our construction of Dedekind zeta motives will naturally lie in the second line of (5.12).

Definition 5.4. Suppose that k has r_1 distinct real embeddings $\sigma_1, \ldots, \sigma_{r_1}$ and $2r_2$ distinct complex embeddings $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma}_{r_1+1}, \ldots, \overline{\sigma}_{r_1+r_2}$. Let

$$\Sigma_n = \begin{cases} \{\sigma_1, \ldots, \sigma_{r_1+r_2}\}, & \text{if } n \text{ is odd} , \\ \{\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}\}, & \text{if } n \text{ is even} . \end{cases}$$

Let $n > 1$. By (5.1) and (5.2), $\dim_\mathbb{Q}\left(\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))\right) = |\Sigma_n|$. In this situation $\bigwedge^{|\Sigma_n|} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ is a $\mathbb{Q}$-vector space of dimension 1. Let

$$R_{\Sigma_n} = \prod_{\sigma \in \Sigma_n} R_\sigma : \mathfrak{A}_n(k) \otimes_\mathbb{Q} \ldots \otimes_\mathbb{Q} \mathfrak{A}_n(k) \longrightarrow \mathbb{R} \ ,$$

be the product of the real periods, and define the *Hodge regulator map* :

$$(5.13) \qquad R_{\Sigma_n} : \det \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathbb{R} \qquad \text{for } n > 1 \ .$$

It factors through (5.12). Its image is a one-dimensional $\mathbb{Q}$-lattice in $\mathbb{R}$, whose covolume is a well-defined element $R_n(k) \in \mathbb{R}^\times / \mathbb{Q}^\times \cup \{0\}$.

The Hodge regulator $r_H : \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \mathbb{R}^{n\pm}$ of (1.4) is the map $r_H = \bigoplus_{\sigma \in \Sigma_n} R_\sigma$. Its image is a $\mathbb{Q}$-lattice whose covolume is $R_n(k) \mod \mathbb{Q}^\times$.

5.2. **Construction of the motive of a product-hyperbolic manifold.** We begin by considering the motive of a single hyperbolic geodesic simplex; first in the generic case following [17] §3.3, and then with one vertex at infinity.

5.2.1. *The motive of a finite geodesic simplex.* Let $m \geq 1$ and let $\Delta \subset \mathbb{H}^m$ be a hyperbolic geodesic simplex with no vertices at infinity. In the Klein model for $\mathbb{H}^m$, Δ is a Euclidean simplex inside the unit sphere $\partial \mathbb{H}^m \subset \mathbb{R}^m$. This data can be represented by a smooth quadric $Q \subset \mathbb{P}^m$, and a set of hyperplanes $\{L_0, \ldots, L_m\} \subset \mathbb{P}^m$, defined over $\mathbb{R}$, such that for some affine open $\mathbb{A}^m \subset \mathbb{P}^m$, $Q(\mathbb{R}) \cap \mathbb{A}^m$ is the unit sphere in $\mathbb{R}^m$, and the boundary faces of Δ are contained in $\bigcup_i L_i(\mathbb{R}) \cap \mathbb{A}^m$.

Let $L = \bigcup_i L_i$. If Q, L are defined over $\overline{\mathbb{Q}}$, this data defines a mixed Tate motive

$$(5.14) \qquad h(\Delta) = H^m(\mathbb{P}^m \backslash Q, L \backslash (L \cap Q)) \,,$$

which only depends on Δ. Note that $L \cup Q$ is a normal crossing divisor. To determine the structure of $h(\Delta)$, observe that for any smooth quadric $Q \subset \mathbb{P}^m$,

$$H^m(\mathbb{P}^m \backslash Q) = \left\{ \begin{array}{ll} \mathbb{Q}(-n) \,, & \text{if } m = 2n - 1 \,, \\ 0 \,, & \text{if } m \text{ is even} \,. \end{array} \right.$$

Whenever $m = 2n - 1$ is odd, one verifies that $\mathrm{gr}_{2n}^W h(\Delta) \cong H^m(\mathbb{P}^m \backslash Q)$, and hence we obtain a class (see §5.2.3 below) which we denote by

$$(5.15) \qquad \omega_Q : \mathbb{Q}(-n) \xrightarrow{\sim} \mathrm{gr}_{2n}^W h(\Delta) \,.$$

Write $L_I = \bigcap_{i \in I} L_i$ for any subset $I \subset \{0, \ldots, m\}$, and let $Q_I = Q \cap L_I$. Every face F of Δ is equal to $\Delta \cap L_I$ for some such I. Since F is a geodesic simplex in $\mathbb{H}^{m-|I|}$, it defines a motive $h(F)$, corresponding to the quadric $Q_I \subset L_I$ relative to the hyperplanes $L_I \cap L_j$, for $j \notin I$. For each face F of odd dimension $2i - 1$, the inclusion $F \hookrightarrow \Delta$ induces a map $i_F : h(F) \to h(\Delta)$ and we set

$$(5.16) \qquad e_F = i_F \circ \omega_{Q_F} : \mathbb{Q}(-i) \longrightarrow \mathrm{gr}_{2i}^W h(\Delta) \,.$$

Proposition 5.5. *Let $m = 2n - 1 \geq 1$. Then $\mathrm{gr}_0^W h(\Delta) = \mathbb{Q}(0)$ and*

$$(5.17) \qquad \mathrm{gr}_{2(n-r)}^W h(\Delta) = \bigoplus_{|I| = 2r} \mathbb{Q}(r - n) \quad \text{for} \quad 0 \leq r < n \,.$$

A basis is given by the classes $e_F(1)$ for all odd-dimensional faces F.

This result will follow from our proof of proposition 5.6 below.

5.2.2. *The motive of a simplex with a vertex at infinity.* Let Δ be a hyperbolic geodesic simplex with a single vertex x on the boundary $\partial \mathbb{H}^m$. As above, Δ defines a set of hyperplanes $L_0, \ldots, L_m$ and a smooth quadric Q in $\mathbb{P}^m$, which do not cross normally at x. Number the hyperplanes so that $L_1, \ldots, L_m$ intersect at x, and L_0 is the hyperplane which does not contain x. Let $\widetilde{\mathbb{P}}^m$ denote the blow-up of $\mathbb{P}^m$ at x, let $\widetilde{L}_{-1}$ denote the exceptional divisor, and let $\widetilde{Q}, \widetilde{L}_i$ be the strict transforms of Q, L_i respectively. When Q, L_i are defined over $\overline{\mathbb{Q}}$, we define a mixed Tate motive:

$$(5.18) \qquad h(\Delta) = H^m(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}, \widetilde{L} \backslash (\widetilde{L} \cap \widetilde{Q})) \,.$$

It has an identical structure to the motive of a finite simplex (5.14), except in graded weight 2. This corresponds to the fact that the one-dimensional faces of Δ which meet x have infinite length. We construct a basis of $\mathrm{gr}_\bullet^W h(\Delta)$ in the case $m = 2n - 1$ is odd as follows. As in the case of a finite simplex, there is a map

$$(5.19) \qquad \omega_{\widetilde{Q}} : \mathbb{Q}(-n) \xrightarrow{\sim} \mathrm{gr}_{2n}^W h(\Delta) \cong H^{2n-1}(\widetilde{\mathbb{P}}^{2n-1} \backslash \widetilde{Q}) \,.$$
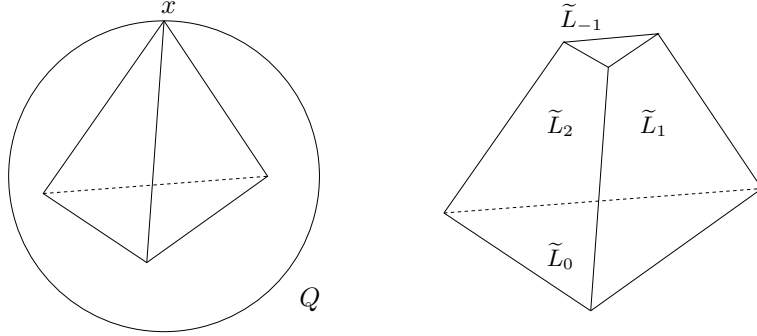
FIGURE 1. A hyperbolic 3-simplex Δ with one vertex at infinity x in the Klein model. After blowing up x, the exceptional divisor $\widetilde{L}_{-1}$ meets the faces of its inverse image $\widetilde{\Delta}$ in a Euclidean triangle.

Similarly, for each face F of odd dimension $2i - 1 \geq 3$, the inclusion of the face $F \hookrightarrow \Delta$ defines a map $i_{\widetilde{F}} : h(F) \to h(\Delta)$ and gives rise to a map:

$$(5.20) \qquad e_{\widetilde{F}} : i_{\widetilde{F}} \circ \omega_{\widetilde{Q} \cap \widetilde{F}} : \mathbb{Q}(-i) \xrightarrow{\sim} \mathrm{gr}_{2i}^W h(F) \longrightarrow \mathrm{gr}_{2i}^W h(\Delta) .$$

The same holds for one-dimensional faces which do not contain x. However, for every one-dimensional face F containing x, we have $\mathrm{gr}_2^W h(F) = 0$ since its strict transform meets $\widetilde{Q}$ in a single point, and $H^1(\mathbb{P}^1 \backslash \{1 \text{ point}\}) = 0$.

What happens instead is the following. Let G be a 2-dimensional face of Δ which meets x. Its strict transform corresponds to the complement of the blow-up of a smooth quadric in the projective plane $\widetilde{\mathbb{P}}^2 \backslash \widetilde{Q}_G$. There is a class

$$(5.21) \qquad \eta_{\widetilde{Q}_G} : \mathbb{Q}(-1) \xrightarrow{\sim} H^2(\widetilde{\mathbb{P}}^2 \backslash \widetilde{Q}_G) \cong \mathrm{gr}_2^W h(G) ,$$

which we shall define below. As previously, the inclusion of the face $G \hookrightarrow \Delta$ induces a map $i_{\widetilde{G}} : h(G) \to h(\Delta)$, and we define:

$$(5.22) \qquad \alpha_{\widetilde{G}} = i_{\widetilde{G}} \circ \eta_{\widetilde{Q}_G} : \mathbb{Q}(-1) \longrightarrow \mathrm{gr}_2^W h(\Delta) .$$

Now consider the set of hyperplanes $L_1, \ldots, L_m$ which contain x. Their strict transforms $\widetilde{L}_1, \ldots, \widetilde{L}_m$ intersect $\widetilde{L}_{-1}$ in an $m - 1$ simplex Δ_∞. Let $F_k(\Delta_\infty)$ denote the set of faces of Δ_∞ of dimension k. From (5.22) we deduce a map

$$(5.23) \qquad \alpha : \mathbb{Q}(-1)^{F_1(\Delta_\infty)} \longrightarrow \mathrm{gr}_2^W h(\Delta)$$

since every 1-dimensional face of Δ_∞ corresponds to a 2-dimensional face of Δ containing x. Choose an orientation on Δ_∞ and set

$$(5.24) \qquad V_x = \mathrm{coker}(\mathbb{Q}^{F_2(\Delta_\infty)} \xrightarrow{\partial} \mathbb{Q}^{F_1(\Delta_\infty)})$$

where ∂ is the boundary map. We shall show that (5.23) factors through

$$\alpha : V_x \otimes_{\mathbb{Q}} \mathbb{Q}(-1) \longrightarrow \mathrm{gr}_2^W h(\Delta) .$$

Note that V_x is of dimension $m - 1$ and isomorphic to the space

$$(5.25) \qquad V_x \cong \ker(\mathbb{Q}^{F_0(\Delta_\infty)} \longrightarrow \mathbb{Q}) ,$$

spanned by linear combinations of 1-dimensional faces of Δ which meet x such that the sum of coefficients is 0. This corresponds to the fact that there is a relation between the internal angles of the Euclidean simplex at infinity Δ_∞ (see §5.2.7).

Proposition 5.6. *Let $m = 2n - 1 \geq 1$ be odd. Then $\mathrm{gr}_0^W h(\Delta) \cong \mathbb{Q}(0)$ and*

$$(5.26) \qquad \mathrm{gr}_{2(n-r)}^W h(\Delta) = \bigoplus_{|I|=2r} \mathbb{Q}(r-n) \quad for \quad 0 \leq r \leq n-2$$

in weights ≥ 4. A basis is given by $e_{\widetilde{F}}(1)$ for all odd-dimensional faces F of Δ of dimension ≥ 3. In graded weight 2, we have

$$(5.27) \qquad \mathrm{gr}_2^W h(\Delta) \cong \mathbb{Q}(-1)^{m(m+1)/2-1} \ ,$$

which is spanned by the $e_{\widetilde{F}}(1)$ for the $m(m-1)/2$ one-dimensional faces F of Δ which do not contain x, and $\alpha_{\widetilde{G}}(1)$ for all two-dimensional faces G of Δ which contain x, modulo boundaries of three-dimensional faces of Δ which meet x.

Before proving proposition 5.6 it is useful to state the following lemma.

Lemma 5.7. *Let $Q \subset \mathbb{P}^m$ be a smooth quadric, and x a point in Q. Let $\widetilde{\mathbb{P}}^m$ denote the blow-up of $\mathbb{P}^m$ at x, and $\widetilde{Q}$ the strict transform of Q. Then*

$$(5.28) \qquad H^i(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}) = \begin{cases} \mathbb{Q}(-n) \ , & if\ i = m\ is\ odd\ and\ equal\ to\ 2n-1 \ ; \\ \mathbb{Q}(-1) \ , & if\ i = 2 \ ; \\ \mathbb{Q}(0) \ , & if\ i = 0 \ ; \\ 0 \ , & otherwise\ \ . \end{cases}$$

Proof. Recall that the cohomology of a smooth quadric Q of dimension ℓ vanishes in odd degrees, and satisfies $H^{2i}(Q) = \mathbb{Q}(-i)$ for $1 \leq i \leq \ell$, unless $\ell = 2k$ is even, in which case $H^{2k}(Q) = \mathbb{Q}(-k) \oplus \mathbb{Q}(-k)$ in middle degree. When $\ell > 1$, the strict transform $\widetilde{Q}$ is just the blow-up of Q in the point x. From the Gysin sequence:

$$\ldots \longrightarrow H^{i-2}(\widetilde{Q})(-1) \longrightarrow H^i(\widetilde{\mathbb{P}}^m) \longrightarrow H^i(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}) \longrightarrow H^{i-1}(\widetilde{Q})(-1) \longrightarrow \ldots \ ,$$

and the formula for the cohomology of a blow-up of a smooth variety in a point, we obtain the statement. $\qquad \square$

Proof of proposition 5.6. The motive (5.18) is the hypercohomology of the complex of sheaves

$$(5.29) \qquad \mathbb{Q}_{\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}} \to \bigoplus_{|I|=1} \mathbb{Q}_{\widetilde{L}_I \backslash \widetilde{Q}_I} \to \cdots \to \bigoplus_{|I|=m} \mathbb{Q}_{\widetilde{L}_I \backslash \widetilde{Q}_I}$$

where $I \subset \{-1, 0, \ldots, m\}$. It defines a spectral sequence with

$$(5.30) \qquad E_1^{p,q} = \bigoplus_{|I|=p} H^q(\widetilde{L}_I \backslash \widetilde{Q}_I) \quad for\ p \geq 1 \ , \quad and \quad E_1^{0,q} = H^q(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}) \ ,$$

which converges to $H^{p+q}(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}, \widetilde{L} \backslash (\widetilde{L} \cap \widetilde{Q}))$. If $-1 \in I$ then $\widetilde{L}_{-1} \backslash \widetilde{Q}_{-1}$ is isomorphic to affine space, and so $H^q(\widetilde{L}_I \backslash \widetilde{Q}_I) = 0$ for all $q \geq 1$. If $0 \in I$ then $\widetilde{L}_I \backslash \widetilde{Q}_I \cong L_I \backslash Q_I$ is the complement of a smooth quadric in $\mathbb{P}^{m-|I|}$ and $H^q(L_I \backslash Q_I)$ vanishes when $q > 0$, unless $|I| = 2r$ is even, and $q = m - 2r$. For all other I, $H^q(\widetilde{L}_I \backslash \widetilde{Q}_I)$ is given by (5.28). The spectral sequence (5.30) degenerates at E_2, and one deduces that

$$(5.31) \qquad \mathrm{gr}_{2(n-r)}^W h(\Delta) = \bigoplus_{-1 \notin I, |I|=2r} H^{2(n-r)-1}(\widetilde{L}_I \backslash \widetilde{Q}_I) \cong \bigoplus_{-1 \notin I, |I|=2r} \mathbb{Q}(r-n) \ ,$$

for $0 \leq r \leq n-2$, and where we write $\widetilde{L}_\emptyset \backslash \widetilde{Q}_\emptyset$ for $\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}$. Now by (5.28)

$$E_1^{p,2} = \bigoplus_{I \subset \{1,\ldots,m\}, \, |I|=p} \mathbb{Q}(-1) \cong \mathbb{Q}(-1)^{\binom{m}{p}} \qquad \text{for } 0 \leq p \leq m-2 \, .$$

The complex $0 \to E_1^{0,2} \to E_1^{1,2} \to \ldots \to E_1^{m-2,2}$ is precisely the simplicial complex corresponding to Δ_∞:

$$0 \to \mathbb{Q}^{F_{m-1}(\Delta_\infty)} \to \ldots \to \mathbb{Q}^{F_2(\Delta_\infty)} \to \mathbb{Q}^{F_1(\Delta_\infty)}$$

tensored with $\mathbb{Q}(-1)$, which is exact except in the last position. By (5.24),

$$E_2^{m-2,2} \cong V_x \otimes_\mathbb{Q} \mathbb{Q}(-1) \cong \mathbb{Q}(-1)^{m-1} \, ,$$

and $E_2^{p,2} = 0$ for $p < m-2$. Finally, we consider the one-dimensional edges. If $|I| = m-1$, then $\widetilde{L}_I \cong \mathbb{P}^1$. It meets $\widetilde{Q}_I$ in two points if $0 \in I$, and in exactly one point if $0 \notin I$. In the latter case $H^1(\widetilde{L}_I \backslash \widetilde{Q}_I) = 0$, so we have

$$E_2^{m-1,1} = E_1^{m-1,1} = \bigoplus_{0 \in I, \, |I|=m-1} H^1(\widetilde{L}_I \backslash \widetilde{Q}_I) = \mathbb{Q}(-1)^{m(m-1)/2} \, .$$

The total contributions in graded weight 2 are therefore:

$$\mathrm{gr}_2^W h(\Delta) \cong E_2^{m-2,2} \oplus E_2^{m-1,1} \cong \mathbb{Q}(-1)^{m(m+1)/2-1} \, .$$

Finally, $E_1^{p,0} = \bigoplus_{|I|=m-p} \mathbb{Q}(0)$ and $\ldots \to E_1^{0,p} \to E_1^{0,p+1} \to \ldots$ is the simplicial complex of the set of hyperplanes $\widetilde{L}$, which has the homology of a sphere. Thus

$$(5.32) \qquad \mathrm{gr}_0^W h(\Delta) = E_2^{m,0} = \mathrm{coker}\Big(\bigoplus_{|I|=m-1} \mathbb{Q}(0) \to \bigoplus_{|I|=m} \mathbb{Q}(0) \Big) \cong \mathbb{Q}(0) \, .$$

$\square$

5.2.3. *Framings.* In both cases (whether Δ has a vertex at infinity or not), we have

$$\mathrm{gr}_{2n}^W h(\Delta) \cong H^{2n-1}(\mathbb{P}^{2n-1} \backslash Q) \cong \mathbb{Q}(-n) \quad \text{and} \quad \mathrm{gr}_0^W h(\Delta) \cong \mathbb{Q}(0) \, ,$$

by (5.31) and (5.32). In the first isomorphism, a generator is given by an element $\pm(1,-1)$ in $H^{2n}(Q)(-1) \cong \mathbb{Q}(-n) \oplus \mathbb{Q}(-n)$, and is well-defined up to a sign. In the second isomorphism, a generator is given by the set of vertices of L (or the vertices of the total transform polytope $\widetilde{L}$ by (5.32)). In the Hodge realisation, we can write these generators explicitly. Let $x_0, \ldots, x_{2n-1}$ denote coordinates on $\mathbb{P}^{2n-1}$, and let $q = \sum_{ij} a_{ij} x_i x_j$ be a quadratic form defining Q. Then

$$(5.33) \qquad \omega_Q = \pm i^n \sqrt{\det q} \, \frac{\sum_{j=0}^{2n-1} (-1)^j x_j \, dx_0 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{2n-1}}{q^n(x)} \, ,$$

defines a class $[\omega_Q]$ which generates $H_{DR}^{2n-1}(\mathbb{P}^{2n-1} \backslash Q)$. It does not depend on the choice of coordinates. The framing on $\mathrm{gr}_0^W h(\Delta)^\vee$ is given by the relative homology class of the real simplex Δ (or its inverse image $\widetilde{\Delta}$ in the case when it has a vertex at infinity), and defines a class $[\Delta] \in \mathrm{gr}_0^W H_{B,2n-1}(\mathbb{P}^{2n-1}, L) \cong \mathrm{gr}_0^W H_B^0(\mathbb{P}^{2n-1}, L)^\vee$ in both cases. In practice, we shall consider many simplices Δ and one fixed quadric Q. Thus we fix a sign of ω_Q at the outset, and all signs of $[\Delta]$ are determined by an orientation of Δ: its sign is normalised so that,

$$\mathrm{vol}(\Delta) = i^{1-n} \int_\Delta \omega_Q \in \mathbb{R}$$

is positive if Δ is positively oriented, and negative otherwise.

Definition 5.8. Let Δ be an oriented hyperbolic geodesic simplex in $\mathbb{H}^{2n-1}$, with at most one vertex at infinity. Suppose it is defined over $\overline{\mathbb{Q}}$. Let $h(\Delta)$ be defined by (5.14) in the case where Δ is finite, and by (5.18) in the case where Δ has a vertex at infinity. Writing (n) for the Tate twist, we set

$$(5.34) \qquad \mathsf{mot}(\Delta) = \big[h(\Delta), [\omega_Q], [\Delta]\big](n) \in \mathfrak{A}_n(\overline{\mathbb{Q}}) \ .$$

Note that it is convenient to write the framings in terms of the de Rham and Betti classes, although they can be defined without reference to the Hodge realisation.

In the case of an oriented hyperbolic geodesic triangle Δ in $\mathbb{H}^2$ defined over k with one vertex at infinity, let $\eta_{\widetilde{Q}_\Delta}$ denote the class (5.21), (5.28) and set:

$$(5.35) \qquad \mathsf{mot}(\Delta) = \big[h(\Delta), [\eta_{\widetilde{Q}_\Delta}], [\widetilde{\Delta}]\big](1) \in \mathfrak{A}_1(\overline{\mathbb{Q}}) \ ,$$

where $\widetilde{\Delta}$ is the total inverse image of Δ in $\widetilde{\mathbb{P}}^2$.

Remark 5.9. One can extend the definition of $h(\Delta)$ to the case where any number of vertices of Δ lie on the absolute in much the same way. When the simplex Δ is degenerate, the resulting framed object is equivalent to zero.

5.2.4. *Definition of the framed motive of a product-hyperbolic manifold.* We first require the following subdivison lemma. Let m be odd and ≥ 1.

Lemma 5.10. *Let $x_0, \ldots, x_m$ be distinct points in $\overline{\mathbb{H}}^m_{\overline{\mathbb{Q}}}$ in general position except that some may lie on the absolute, and let $\Delta(x_0, \ldots, x_m)$ denote the geodesic simplex whose vertices are given by the x_i. Given any finite point $y \in \mathbb{H}^m_{\overline{\mathbb{Q}}}$,*

$$(5.36) \qquad \mathsf{mot}(\Delta(x_0, \ldots, x_m)) = \sum_{i=0}^{m} \mathsf{mot}(\Delta(x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m)) \ .$$

Proof. Let J (resp. I) be the set of m-element subsets of $\{x_0, \ldots, x_m\}$ (resp. $\{x_0, \ldots, x_m, y\}$), and let L_i for $i \in I$ denote the hyperplane passing through m points. Let Q denote a smooth quadric in $\mathbb{P}^m$ corresponding to the absolute of $\mathbb{H}^m$. Let $\widetilde{\mathbb{P}}^m$ denote the blow-up of $\mathbb{P}^m$ at all intersections of hyperplanes which do not cross normally in increasing order of dimension so that the strict transforms $\widetilde{L}_i$ of L_i, the strict transform $\widetilde{Q}$ of Q, and the exceptional loci L_h for h in some indexing set H, are normal crossing. The hypercohomology of the complex of sheaves

$$(5.37) \qquad \mathbb{Q}_{\widetilde{\mathbb{P}}^m \setminus \widetilde{Q}} \to \bigoplus_{i \in I \cup H} \mathbb{Q}_{\widetilde{L}_i \setminus (\widetilde{L}_i \cap \widetilde{Q})} \to \bigoplus_{i,j \in I \cup H} \mathbb{Q}_{\widetilde{L}_{ij} \setminus (\widetilde{L}_{ij} \cap \widetilde{Q})} \to \cdots$$

defines a mixed Tate motive. There are $m + 1$ similar complexes defined relative to the vertices $\{x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m\}$ for each i, which map to (5.37). Via these maps, there are unique framings on the motive of (5.37) which make it equivalent to $\sum_{i=0}^{m} \mathsf{mot}(\Delta(x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m))$. Likewise, the complex

$$(5.38) \qquad \mathbb{Q}_{\widetilde{\mathbb{P}}^m \setminus \widetilde{Q}} \to \bigoplus_{i \in J \cup H} \mathbb{Q}_{\widetilde{L}_i \setminus (\widetilde{L}_i \cap \widetilde{Q})} \to \bigoplus_{i,j \in J \cup H} \mathbb{Q}_{\widetilde{L}_{ij} \setminus (\widetilde{L}_{ij} \cap \widetilde{Q})} \to \cdots$$

also defines a mixed Tate motive, which can be given unique framings making it equivalent to $\mathsf{mot}(\Delta(x_0, \ldots, x_m))$ (after blowing-down superfluous divisors). The natural map from (5.38) to (5.37) is an equivalence of framed motives, and corresponds to the fact that the relative homology class $[\Delta(x_0, \ldots, x_m))]$ is equal to the sum of the relative homology classes $\sum_{i=0}^{m} [\Delta(x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m))]$. $\qquad \square$

Definition 5.11. Let M be a finite-volume product-hyperbolic manifold modelled on $\mathbb{X}^{\mathfrak{n}} = \mathbb{H}^{2n_1-1} \times \ldots \times \mathbb{H}^{2n_N-1}$ and defined over the fields $S = (k_1, \ldots, k_N)$. By proposition 4.9, M admits a product-tiling with geodesic-product simplices $\Delta_1^{(i)} \times \ldots \times \Delta_N^{(i)}$, for $1 \leq i \leq R$, where $\Delta_j^{(i)}$ are oriented geodesic simplices in $\overline{\mathbb{H}}^{n_j}$, defined over k_j, with at most one vertex at infinity.

The (framed) motive of M is defined to be the element

$$(5.39) \quad \mathsf{mot}(M) = \sum_{i=1}^{R} \mathsf{mot}(\Delta_1^{(i)}) \otimes \ldots \otimes \mathsf{mot}(\Delta_N^{(i)}) \in \mathfrak{A}_{n_1}(\overline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathfrak{A}_{n_N}(\overline{\mathbb{Q}}) \ .$$

It follows from lemma 5.10 that $\mathsf{mot}(M)$ is well-defined, since one can construct a common subdivision of any two distinct product-tilings of M.

5.2.5. *Vanishing of the reduced coproduct.* We prove that $\mathsf{mot}(M)$ is a product of extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n_i)$ by showing that it lies in the kernel of the reduced coproduct. Let Δ be a geodesic hyperbolic simplex defined over $\overline{\mathbb{Q}}$ with at most one vertex at infinity. The main point is that the reduced coproduct of $\mathsf{mot}(\Delta)$ only depends on the boundary of Δ. By definition (5.7)

$$(5.40) \qquad \Delta_{n-r,r}\mathsf{mot}(\Delta) = \sum_i [h(\Delta), [\omega_Q], e_i^{\vee}](n) \otimes [h(\Delta), e_i, [\Delta]](r)$$

where $\{e_i\}$ is a basis for $\mathrm{gr}_{2r}^W h(\Delta)$. By proposition 5.6, such a basis is provided by the classes $e_{\widetilde{F}}$, where F ranges over all faces of Δ of dimension $2r-1$, or, in the case when Δ has a vertex at infinity and $r=1$, by linear combinations of classes $\alpha_{\widetilde{G}}$ where G is a face of Δ of dimension two. For any such face F, we have

$$(5.41) \qquad\qquad \mathsf{mot}(F) = [h(\Delta), e_i, [\Delta]](r) \ ,$$

if e_i is $e_{\widetilde{F}}$ or $\alpha_{\widetilde{F}}$. Thus we can write the reduced coproduct:

$$(5.42) \qquad\qquad \widetilde{\Delta}(\mathsf{mot}(\Delta)) = \sum_F c_F \otimes \mathsf{mot}(F) \ ,$$

where the sum is over the boundary faces of Δ. We say that c_F is the coefficient of the face F. It is a framed motive that we will compute in §5.2.7.

Theorem 5.12. *Let M be a product-hyperbolic manifold defined over the number fields $(k_1, \ldots, k_N)$ as above. Then $\mathsf{mot}(M) \in \mathfrak{A}_{n_1}(\overline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathfrak{A}_{n_r}(\overline{\mathbb{Q}})$ and defines an element in $\mathrm{Ext}^1_{\mathrm{MT}(k_1)}(\mathbb{Q}(0), \mathbb{Q}(n_1)) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathrm{Ext}^1_{\mathrm{MT}(k_N)}(\mathbb{Q}(0), \mathbb{Q}(n_N))$.*

Proof. Let $\mathbb{X}^{\mathfrak{n}} = \prod_{i=1}^{N} \mathbb{H}^{2n_i-1}$, and $M = \mathbb{X}^{\mathfrak{n}}/\Gamma$ (notations from §2). By proposition 4.9, M admits a tiling with product simplices $\Delta_1^{(i)} \times \ldots \times \Delta_N^{(i)}$ for $1 \leq i \leq R$ defined over $(k_1, \ldots, k_N)$ with at most one vertex at infinity. It suffices to show that

$$(5.43) \qquad \mathsf{mot}(M) \in \ker \left(\mathrm{id}_1 \otimes \ldots \otimes \mathrm{id}_{j-1} \otimes \Delta_{r,n_j-r} \otimes \mathrm{id}_{j+1} \otimes \ldots \otimes \mathrm{id}_N \right) \ ,$$

for each $1 \leq j \leq N$ and all $1 \leq r \leq n_j - 1$. This would imply that

$$\mathsf{mot}(M) \in \ker \widetilde{\Delta}_{n_1} \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \ker \widetilde{\Delta}_{n_N} \ ,$$

and it follows by corollary 5.2 that

$$\mathsf{mot}(M) \in \mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0), \mathbb{Q}(n_1)) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0), \mathbb{Q}(n_N)) \ .$$

Since the tiling of M is defined over $(k_1, \ldots, k_N)$, it is invariant under the natural action of $\mathrm{Gal}(\overline{k}_1/k_1) \times \ldots \times \mathrm{Gal}(\overline{k}_N/k_N)$ on the product-simplices. It follows from

Galois descent for K-theory, or more precisely, [13], (2.16.2), that $\mathsf{mot}(M)$ lies in the subspace $\mathrm{Ext}^1_{\mathrm{MT}(k_1)}(\mathbb{Q}(0), \mathbb{Q}(n_1)) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathrm{Ext}^1_{\mathrm{MT}(k_N)}(\mathbb{Q}(0), \mathbb{Q}(n_N))$.

To prove (5.43), let $1 \leq r \leq n_j - 1$ and let

$$V \subset \bigotimes_{i=1}^{j-1} \mathfrak{A}_{n_i}(\overline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \left(\mathfrak{A}_r(\overline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathfrak{A}_{n_j-r}(\overline{\mathbb{Q}})\right) \otimes_{\mathbb{Q}} \bigotimes_{i=j+1}^{N} \mathfrak{A}_{n_i}(\overline{\mathbb{Q}})$$

denote the $\mathbb{Q}$-vector space spanned by all the terms

$$(5.44) \qquad \bigotimes_{i=1}^{j-1} \mathsf{mot}(\Delta_i^{(k)}) \otimes (c_F \otimes \mathsf{mot}(F)) \otimes \bigotimes_{i=j+1}^{N} \mathsf{mot}(\Delta_i^{(k)})$$

which can occur in $\mathrm{id}_1 \otimes \ldots \otimes \mathrm{id}_{j-1} \otimes \Delta_{r,n_j-r} \otimes \mathrm{id}_{j+1} \otimes \ldots \otimes \mathrm{id}_N\big(\mathsf{mot}(M)\big)$ by (5.42). Since the triangulation is finite, V is finite dimensional and only depends on the set of faces of codimension ≥ 1. Let us set

$$v = \mathrm{id}_1 \otimes \ldots \otimes \mathrm{id}_{j-1} \otimes \Delta_{r,n_j-r} \otimes \mathrm{id}_{j+1} \otimes \ldots \otimes \mathrm{id}_N(\mathsf{mot}(M)) \in V .$$

We shall show that v is zero. For simplicity, let us assume that M is compact and that all simplices $\Delta_j^{(i)}$ have no vertices at infinity. The tiling of M lifts to a Γ-equivariant tiling of the whole of $\mathbb{X}^{\mathfrak{n}}$. Let $D = \bigcup_{i=1}^{R} \Delta_1^{(i)} \times \ldots \times \Delta_N^{(i)} \subset \mathbb{X}^{\mathfrak{n}}$ be a lift of the tiling of M. Now there exists an $s > 0$, such that for all $r \gg 0$ sufficiently large, there are elements $\gamma_1, \ldots, \gamma_{N_r}$ such that

$$B_r \subset \bigcup_{1 \leq i \leq N_r} \gamma_i(D) \subset B_{r+s}$$

where $B_r \subset B_{r+s} \subset \mathbb{X}^{\mathfrak{n}}$ are balls of radius r and $r + s$, and the translates of D are distinct and tile B_r. Since γ_i is an isometry, $\mathsf{mot}(\bigcup_i \gamma_i(M)) = N_r \, \mathsf{mot}(M)$, and so

$$\mathrm{id}_1 \otimes \ldots \otimes \mathrm{id}_{j-1} \otimes \Delta_{r,n_j-r} \otimes \mathrm{id}_{j+1} \otimes \ldots \otimes \mathrm{id}_N(\mathsf{mot}(\bigcup_i \gamma_i(D))) = N_r \, v$$

where N_r is of order $r^{\dim \mathbb{X}^{\mathfrak{n}}}$, by volume considerations. On the other hand, by a version of lemma 5.10 (which easily generalizes to hold for polytopes), and the fact that the reduced coproduct of a simplex only depends on its boundary, contributions from internal faces cancel and we can rewrite

$$\mathrm{id}_1 \otimes \ldots \otimes \mathrm{id}_{j-1} \otimes \Delta_{r,n_j-r} \otimes \mathrm{id}_{j+1} \otimes \ldots \otimes \mathrm{id}_N(\mathsf{mot}(\bigcup_i \gamma_i(D)))$$

as a sum of terms in V of the form (5.44) where all faces are contained in $B_{r+s} \backslash B_r$. By volume considerations, the number of such faces is of order $r^{(\dim \mathbb{X}^{\mathfrak{n}}-1)}$. Since V is finite-dimensional, choose a norm $||.||$ on V. The above argument shows that $||N_r v||$ is of order $r^{(\dim \mathbb{X}^{\mathfrak{n}}-1)}$. Since N_r is of order $r^{\dim \mathbb{X}^{\mathfrak{n}}}$, and r is arbitrarily large, we have $||v|| = 0$ and hence $v = 0$ as required. The proof in the case when the simplices have vertices at infinity is entirely similar. $\square$

5.2.6. *The symmetric group action.* It remains to show, in the case when M is equivariant, that its framed motive $\mathsf{mot}(M)$ is a determinant. Assume that all hyperbolic components are of equal odd dimension $2n - 1$, *i.e.*, $n_1 = \ldots = n_N = n$,

and let k be a totally real number field with a set of embeddings $\Sigma = \{\sigma_1, \dots, \sigma_N\}$ into $\mathbb{R}$.[1] Let $k_i = \sigma_i(k)$ and suppose that M is equivariant with respect to Σ.

Theorem 5.13. *Let M be an equivariant product-hyperbolic manifold as above. Then, in the notation of §5.1.4, the framed motive of M is a determinant:*

$$\mathsf{mot}(M) \in \bigwedge{}^{\Sigma}\mathfrak{A}_n(k) \ .$$

Proof. Let $\Delta_1^{(i)} \times \dots \times \Delta_N^{(i)}$, for $1 \le i \le R$ be a product-tiling for M, where $\Delta_j^{(i)}$ is defined over k_j, and has at most one vertex at infinity. Since the cusps of M are equivariant the image of this tiling under the twisted action of the symmetric group $\mathfrak{S}_N$ is another tiling for $\pi(M) = M$ (lemma 4.10):

$$M = \bigcup_{i=1}^{R} (\sigma_1 \sigma_{\pi(1)}^{-1} \Delta_{\pi(1)}^{(i)}) \times \dots \times (\sigma_N \sigma_{\pi(N)}^{-1} \Delta_N^{(i)}) \ , \ \text{ for all } \pi \in \mathfrak{S}_N \ ,$$

It follows that $\mathsf{mot}(M)$ and $\mathsf{mot}(\pi(M))$ are equivalent up to a sign, determined by the action of π on the framings. First, π preserves the framing in $\mathrm{gr}_{-2n}^W \mathsf{mot}(M)^\vee$ corresponding to the fundamental class of M, because M is equivariant. Let Q_i denote the quadric in $\mathbb{P}^{2n_i - 1}$ which corresponds to $\partial \mathbb{H}^{2n_i - 1}$, for $1 \le i \le N$. The framing in $\mathrm{gr}_0^W \mathsf{mot}(M)$ corresponds to the volume form on $\mathbb{X}^n$, which is alternating:

$$\pi[\omega_{Q_1} \wedge \dots \wedge \omega_{Q_N}] = \varepsilon(\pi)[\omega_{Q_1} \wedge \dots \wedge \omega_{Q_N}] \ , \quad \text{for all } \pi \in \mathfrak{S}_N \ ,$$

since each ω_{Q_i} is of odd degree. Here, ε is the sign of a permutation. Thus,

$$\mathsf{mot}(M) = \varepsilon(\pi) \, \pi(\mathsf{mot}(M)) \quad \text{for all } \pi \in \mathfrak{S}_N \ ,$$

and $\mathsf{mot}(M) \in \mathfrak{A}_n(\sigma_1(k)) \otimes_{\mathbb{Q}} \dots \otimes_{\mathbb{Q}} \mathfrak{A}_n(\sigma_N(k))$ is a determinant (§5.1.4). $\qquad\square$

5.2.7. *Quotient motives and spherical angles.* Let $m = 2n - 1$, and let $\Delta \subset \overline{\mathbb{H}_{\mathbb{Q}}^m}$ denote a geodesic simplex with one vertex at infinity (the case when Δ is finite is simpler and left to the reader). We wish to compute the coefficient motives c_F of any two or odd-dimensional face F of Δ. Using the notations of §5.2.2, the motive $h(\Delta)$ is the hypercohomology of the complex of sheaves (5.29):

$$(5.45) \qquad \mathbb{Q}_{\widetilde{\mathbb{P}}^m \setminus \widetilde{Q}} \to \bigoplus_{|I|=1} \mathbb{Q}_{\widetilde{L}_I \setminus \widetilde{Q}_I} \to \dots \to \bigoplus_{|I|=m} \mathbb{Q}_{\widetilde{L}_I \setminus \widetilde{Q}_I}$$

where $I \subset \{-1, 0, \dots, m\}$. First let F be a face of Δ of odd dimension $2r - 1$. We assume that F does not contain the vertex at infinity if F is one-dimensional. It corresponds to a hyperplane L_J where $J \subset \{0, \dots, m\}$, so $F = \Delta \cap L_J$, and $\mathsf{mot}(F)$ is the hypercohomology of the following subcomplex of sheaves:

$$(5.46) \qquad \mathbb{Q}_{\widetilde{L}_J \setminus \widetilde{Q}} \to \bigoplus_{|I|=|J|+1, J \subset I} \mathbb{Q}_{\widetilde{L}_I \setminus \widetilde{Q}_I} \to \dots \to \bigoplus_{|I|=m, J \subset I} \mathbb{Q}_{\widetilde{L}_I \setminus \widetilde{Q}_I}$$

where $I \subset \{-1, 0, \dots, m\}$. Consider the complex

$$(5.47) \qquad \mathbb{Q}_{\widetilde{\mathbb{P}}^m \setminus \widetilde{Q}} \to \bigoplus_{|I|=1, I \subset J} \mathbb{Q}_{\widetilde{L}_I \setminus \widetilde{Q}_I} \to \dots \to \mathbb{Q}_{\widetilde{L}_J \setminus \widetilde{Q}_J} \ .$$

Its hypercohomology in degree m is $H^m(\widetilde{\mathbb{P}}^m \setminus \widetilde{Q}, \widetilde{L}^J \setminus \widetilde{Q}^J)$, where $L^J = \cup_{j \in J} L_j$, $Q^J = L^J \cap Q$, and $\widetilde{L}^J, \widetilde{Q}^J$ are their strict transforms. There is a map from (5.45)

---

[1]In the exceptional case $n = 2$, if we identify $\mathsf{SO}^+(3,1)$ with $\mathsf{PSL}_2(\mathbb{C})$ we can allow k be to be any number field L and σ_i to be complex places of L.

to (5.47). The divisor $Q \cup L^J$ is normal crossing, so we can blow down once again to obtain an isomorphism of $H^m(\widetilde{\mathbb{P}}^m \backslash \widetilde{Q}, \widetilde{L}^J \backslash \widetilde{Q}^J)$ with a motive we call

$$(5.48) \qquad h(\Delta_F) = H^m(\mathbb{P}^m \backslash Q, L^J \backslash Q^J) \ .$$

There are framings on $h(\Delta_F)$ given by $\omega_Q : \mathbb{Q}(-n) \xrightarrow{\sim} \mathrm{gr}_{2n}^W H^m(\mathbb{P}^m \backslash Q) \cong \mathrm{gr}_{2n}^W h(\Delta_F)$, and a class $T_F : \mathbb{Q}(-r) \xrightarrow{\sim} \mathrm{gr}_{2r}^W h(\Delta_F)^\vee$, which we define as follows. Since Q is smooth, the residue map onto Q gives a morphism

$$(5.49) \qquad h(\Delta_F) \longrightarrow H^{m-1}(Q, Q \cap L^J)(-1) \ ,$$

and gives an identification $\mathbb{Q}(-r) \cong \mathrm{gr}_{2r}^W h(\Delta_F) \cong \mathrm{gr}_{2(r-1)}^W H^{m-1}(Q, Q \cap L^J)$ which is dual to $H_{m-1}(Q, Q \cap L^J)$. An element in the Betti realization of this group is given by the relative homology class of the spherical simplex $S_F \subset Q(\mathbb{R})$ cut out by the hyperplanes $(L_j \cap Q)_{j \in J}$ on the set of real points of the quadric Q (which can be identified as the boundary $\partial \mathbb{H}^m$ in the Klein model), and T_F is defined to be the corresponding framing on $\mathrm{gr}_{2r}^W h(\Delta_F)$. It can be represented by a tubular neighbourhood N_F of S_F. The sign is determined by the orientations, and one can verify that T_F is dual to the class induced by e_F. Finally, the residue of ω_Q along Q is a certain rational multiple of the volume form on Q.

Lemma 5.14. *Let Δ be as above. For every strict odd-dimensional subface F of Δ, which is not a one-dimensional face which contains the vertex at infinity, the coefficient c_F of $\mathsf{mot}(F)$ in (5.42) is equivalent to the framed mixed Tate motive*

$$[h(\Delta_F), [\omega_{\widetilde{Q}}], T_F](n) \ .$$

Its period is computed by integrating $\omega_{\widetilde{Q}}$ over the tubular neighbourhood N_F. By the residue formula and (5.33), this is $2\pi i^n$ times a certain rational multiple of the the spherical volume of S_F. In particular, it lies in $i^n \mathbb{R}$.

By proposition 5.6, the coproduct (5.42) also contains contributions (5.27) from the two-dimensional faces G of Δ which contain the point at infinity. Such a face G defines a framed submotive $\mathsf{mot}(G)$ as before. To obtain the coefficients, consider two distinct such faces G_i and G_j. They intersect along a line $G_i \cap G_j$ which contains the point at infinity, but nonetheless the complex (5.47) still defines a framed motive $h(\Delta_{G_i \cap G_j})$. Inspection of the spectral sequence which computes the hypercohomology of (5.47) shows that the framing $T_{G_i \cap G_j}$ corresponds via (5.49) to the dual of $\pm(\alpha_{\widetilde{G}_i} - \alpha_{\widetilde{G}_j})$. Thus the set of classes $T_{G_i \cap G_j}$ satisfy a single relation and define a dual basis to $\alpha(V_x \otimes_{\mathbb{Q}} \mathbb{Q}(-1))$ by (5.25). The period, as before, is proportional to the dihedral angle subtended by G_i and G_j. The relation between the $T_{G_i \cap G_j}$ corresponds to the analytic fact that there is a single relation between the angles of the even-dimensional Euclidean simplex at infinity Δ_∞.

In all cases, the periods of the coefficient motives c_F are πi^n times a rational multiple of the spherical volumes of the lunes S_F (see also [17] §4.7).

5.2.8. *Volume and the main theorem.* The volume of a hyperbolic simplex with at most one vertex at infinity is determined by its real period. The proof is identical to Goncharov's proof of the same result for finite simplices [17].

Corollary 5.15. *Let Δ denote an oriented hyperbolic geodesic simplex in $\overline{\mathbb{H}}_{\mathbb{Q}}^{2n-1}$ with at most one vertex at infinity. Then there is a number field k and $\sigma : k \to \mathbb{R}$*

such that $\mathsf{mot}(\Delta) \in \mathrm{MT}(k)$ *and*

$$\mathrm{vol}(\Delta) = (2\pi)^n R_\sigma(\mathsf{mot}(\Delta)) \ .$$

Proof. Let P_σ be a period matrix for $\mathsf{mot}(\Delta)$. Its first column is given by integrating the form $\omega_{\widetilde{Q}}$ over a basis for the homology of $h(\Delta)$. The previous discussion proves that, in the basis given by the classes T_F, all entries lie in $i^n\mathbb{R}$ except for the first, which is given by $i^{n-1}\mathrm{vol}(\Delta)$. Thus, up to signs, $\mathrm{Im}\left((2i\pi)^{-n}P_\sigma v_0\right)$ is the column vector $((2\pi)^{-n}\mathrm{vol}(\Delta), 0, \ldots, 0)$, and it follows from the definition and the choice of signs on the framings that $R_\sigma[h(\Delta), \omega_{\widetilde{Q}}, \Delta] = \langle \mathrm{Im}\left((2i\pi)^{-n}P_\sigma[\omega_{\widetilde{Q}}]\right), \mathrm{Re}\left(P_\sigma^*[\Delta]\right)\rangle$ which is exactly $(2\pi)^{-n}\mathrm{vol}(\Delta) \times 1$. $\qquad\square$

Putting the various elements together, we obtain the following theorem.

Theorem 5.16. *Let M be a product-hyperbolic manifold modelled on $\mathbb{H}^{2n_1-1} \times \ldots \times \mathbb{H}^{2n_N-1}$, and defined over the fields $(k_1, \ldots, k_N)$. Then the framed motive of M is a well-defined element:*

$$\mathsf{mot}(M) \in \mathrm{Ext}^1_{\mathrm{MT}(k_1)}(\mathbb{Q}(0), \mathbb{Q}(n_1)) \otimes_\mathbb{Q} \ldots \otimes_\mathbb{Q} \mathrm{Ext}^1_{\mathrm{MT}(k_N)}(\mathbb{Q}(0), \mathbb{Q}(n_N)) \ ,$$

such that the volume of M is given by the Hodge regulator:

$$\mathrm{vol}(M) = (2\pi)^{n_1+\cdots+n_N} R_\Sigma(\mathsf{mot}(M)) \ .$$

If M is equivariant with respect to $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$, and $n_i = n$ for all i, then

$$\mathsf{mot}(M) \in \bigwedge\nolimits^\Sigma \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ .$$

5.3. **Examples.**

5.3.1. *A hyperbolic line element.* First consider the simplest case of a hyperbolic line segment L in $\mathbb{H}^1 \cong \mathbb{R}$. It corresponds to a pair of points $\{x_0, x_1\} \in \mathbb{P}^1$, and a quadric $Q \subset \mathbb{P}^1$ which consists of a pair of points $\{q_0, q_1\} \in \mathbb{P}^1$. Then

(5.50) $$h(L) = H^1(\mathbb{P}^1 \backslash \{q_0, q_1\}, \{x_0, x_1\}) \ ,$$

is a Kummer motive, *i.e.*, $\mathrm{gr}^W_\bullet h(L) \cong \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$. The period defined by integrating the form ω_Q over the interval $[x_0, x_1]$ is

$$\int_{x_0}^{x_1} \frac{1}{2}\Big(\frac{dx}{x-q_0} - \frac{dx}{x-q_1}\Big) = \frac{1}{2}\log\Big(\frac{(x_1-q_0)(x_0-q_1)}{(x_0-q_0)(x_1-q_1)}\Big) = \frac{1}{2}\log\left([x_1\,x_0|q_0\,q_1]\right) \ .$$

The real period of $h(L)$ is half the real part of the logarithm $\log\big|[x_1\,x_0|q_0, q_1]\big|$, which, up to a sign, is the hyperbolic length of the oriented line segment $\{x_0, x_1\}$.

5.3.2. *Hyperbolic triangles in the hyperbolic plane.* Now consider a finite triangle T in $\mathbb{H}^2$. It defines a set of 3 lines L_0, L_1, L_2 and a smooth quadric Q in $\mathbb{P}^2$. Then

$$h(T) = H^2\big(\mathbb{P}^2\backslash Q, \bigcup_{0 \le i \le 2} L_i\backslash(L_i \cap Q)\big) \ .$$

From (5.17), we have $\mathrm{gr}^W_2 h(T) = \mathbb{Q}(-1)^3$, and $\mathrm{gr}^W_0 h(T) = \mathbb{Q}(0)$. Each side of T defines a motive (5.50) whose period is its hyperbolic length. It follows that $h(T)$ splits as a direct sum of Kummer motives, one corresponding to each side. The periods of $h(T)$ are $2\ell_0, 2\ell_1, 2\ell_2$ where ℓ_i is the hyperbolic length of each side. Note that the triple $\{\ell_0, \ell_1, \ell_2\}$ is a complete isometry invariant for T. Since $\mathrm{gr}^W_4 h(T) = 0$, there is no interesting framed motive to speak of in this case.

Now consider what happens when one vertex $x = L_1 \cap L_2$ of T is at infinity. After blowing up the point x, we obtain a smooth quadric Q in $\widetilde{\mathbb{P}^2} \cong \mathbb{P}^1 \times \mathbb{P}^1$, and a configuration of four lines $\widetilde{L}_{-1}, \widetilde{L}_0, \widetilde{L}_1, \widetilde{L}_2$ as shown below (left). Now,

$$h(T) = H^2(\widetilde{\mathbb{P}}^2 \backslash \widetilde{Q}, \bigcup_{-1 \le i \le 2} \widetilde{L}_i \backslash (\widetilde{L}_i \cap \widetilde{Q})) \,,$$

and (5.27) gives $\mathrm{gr}_2^W h(T) \cong \mathbb{Q}(-1)^2$, $\mathrm{gr}_0^W h(T) = \mathbb{Q}(0)$. It is therefore the direct sum of two Kummer motives. One of them comes from the inclusion of the side L_0 of finite length, whose period is its hyperbolic length. To compute the period of the other, consider the following diagram. Note that the motive $h(T)$ is uniquely
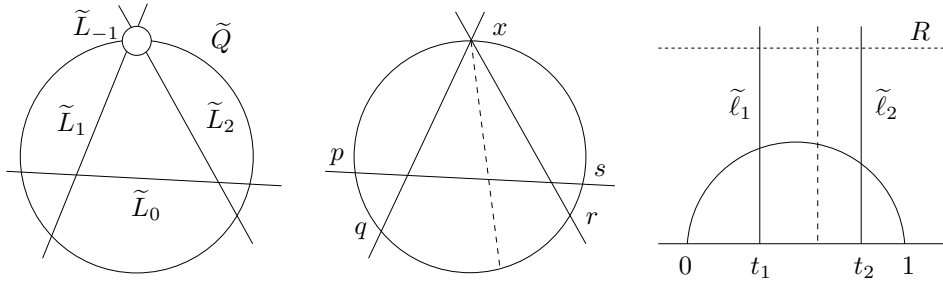
determined by the five points $x, p, q, r, s \in \partial \mathbb{H}^2 \cong \mathbb{P}^1(\mathbb{R})$.

Lemma 5.17. *The periods of $h(T)$ are given by the two quantities*

$$(5.51) \quad \ell_x = \frac{1}{2} \log \left(\frac{(x-q)^2(p-r)(r-s)}{(x-r)^2(p-q)(q-s)} \right) \quad and \quad \ell_0 = \frac{1}{2} \log \left(\frac{(p-r)(q-s)}{(p-q)(r-s)} \right) \,.$$

Proof. The periods of $h(T)$ define a Kummer variation on the configuration space of 5 distinct points in $\mathbb{P}^1(\mathbb{R})$ modulo the action of $\mathsf{PSL}_2(\mathbb{R})$. This is the moduli space $\mathfrak{M}_{0,5}(\mathbb{R})$ of genus 0 curves with 5 marked points. By projective transformation, set $p = 0, q = t_1, r = t_2, s = 1, x = \infty$. The space of logarithms on $\mathfrak{M}_{0,5}$ is spanned by $\log(t_1)$, $\log(t_2)$, $\log(1-t_1)$, $\log(1-t_2)$, $\log(t_2-t_1)$. The periods of $h(T)$ are additive with respect to subdivision (the dotted line above). A simple calculation shows that the vector space of additive functions is spanned by the two functions:

$$2\,\ell_x = \log \left(\frac{t_2(1-t_2)}{t_1(1-t_1)} \right) \quad and \quad 2\,\ell_0 = \log \left(\frac{t_2(1-t_1)}{t_1(1-t_2)} \right) \,.$$

Rewriting $t_1, t_2, 1-t_1, 1-t_2$ as cross-ratios, we obtain formula (5.51). $\square$

The quantity ℓ_x (resp. ℓ_0) is anti-invariant (resp. invariant) under the transformation $(p,q) \leftrightarrow (s,r)$. One checks that ℓ_0 is the hyperbolic length of the face defined by L_0, and ℓ_x is the difference of the regularised lengths $\widetilde{\ell}_1 - \widetilde{\ell}_2$ of the sides L_1, L_2. Here $\widetilde{\ell}_i$ is defined to be the length of the truncated line segment of side L_i up to a horoball neighbourhood of x (depicted by a horizontal dotted line at height R in the figure above (right)). The quantity $\widetilde{\ell}_1 - \widetilde{\ell}_2$ is independent of R.

5.3.3. *A finite simplex in hyperbolic 3 space.* Consider a finite hyperbolic geodesic simplex Δ in $\mathbb{H}^3$, which is given by four hyperplanes $L_0, \ldots, L_3$ in general position relative to a smooth quadric Q in $\mathbb{P}^3$. From (5.17), we have

$$\mathrm{gr}_4^W h(\Delta) \cong \mathbb{Q}(-2) , \quad \mathrm{gr}_2^W h(\Delta) \cong \mathbb{Q}(-1)^6 , \quad \mathrm{gr}_0^W h(\Delta) \cong \mathbb{Q}(0) .$$

To compute the periods, choose an edge L_{ij}. It defines a complex of sheaves

$$(5.52) \qquad\qquad \mathbb{Q}_{L_{ij}\backslash Q_{ij}} \to \bigoplus_{k\in\{0,1,2,3\}\backslash\{i,j\}} \mathbb{Q}_{L_{ijk}} ,$$

whose hypercohomology is the Kummer submotive of the line element L_{ij}, relative to two points, whose real period is its hyperbolic length. Now consider the analogue of (5.47) obtained from the set of faces containing the edge L_{ij}:

$$(5.53) \qquad\qquad \mathbb{Q}_{\mathbb{P}^3\backslash Q} \to \mathbb{Q}_{L_i\backslash Q_i} \oplus \mathbb{Q}_{L_j\backslash Q_j} \to \mathbb{Q}_{L_{ij}\backslash Q_{ij}} .$$

It defines a Kummer motive $H^3(\mathbb{P}^3\backslash Q, (L_i \cup L_j)\backslash Q)$ which maps to

$$H^3(\mathbb{P}^3\backslash Q, (L_i \cup L_j)\backslash Q) \longrightarrow H^2(Q, (L_i \cup L_j)\cap Q)(-1) ,$$

via the residue. The period of the latter is computed as follows. The two hyperplanes L_i, L_j cut out a spherical lune on the quadric $Q(\mathbb{R})$ (see figure below).
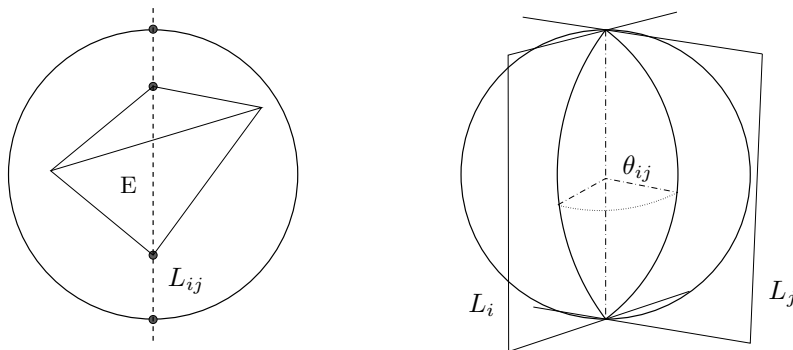

FIGURE 2. Left: the complex (5.52) corresponds to the edge E. Right: L_i and L_j define a spherical lune on $Q(\mathbb{R})$ whose real period is twice the dihedral angle θ_{ij}.

The period is easily computed in spherical coordinates. Suppose that Q is given by the equation $x^2 + y^2 + z^2 = 1$. Then $\omega_Q = i(x^2 + y^2 + z^2 - 1)^{-2}dx\,dy\,dz$ is just $i\rho^2(1 - \rho^2)^{-2}d\rho\,\sin(\phi)d\phi\,d\theta$. Its residue at $\rho = 1$ is $i4^{-1}\sin(\phi)d\phi\,d\theta$, which is $1/4i$ times the volume form on the sphere. Thus the period obtained by integrating over the spherical lune is $\frac{1}{2i}\theta_{ij}$, half the dihedral angle between the hyperplanes L_i, L_j. If X_{ij} is a tubular neighbourhood around Q of the lune whose boundary is contained in $L_i\cup L_j$, the relative homology classes $[X_{ij}]$ form a basis for $\mathrm{gr}_2^W h(\Delta)^\vee$. In conclusion, we can write a (dual) period matrix for $\mathsf{mot}(\Delta)$ as follows:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 2\ell_{01} & 2i\pi & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 2\ell_{23} & 0 & \cdots & 2i\pi & 0 \\ i\mathrm{vol}(\Delta) & 2\pi\theta_{01} & \cdots & 2\pi\theta_{23} & (2i\pi)^2 \end{pmatrix}$$

where ℓ_{ij} is the hyperbolic length of the edge L_{ij}, for $0 \leq i < j \leq 3$, and θ_{ij} is the dihedral angle subtended at that edge. The reduced coproduct map (5.7) on the level of period matrices can therefore be written as a Dehn invariant:

$$\widetilde{\Delta}(\mathsf{mot}(\Delta)) = \sum_{0 \leq i < j \leq 3} \begin{pmatrix} 1 & 0 \\ 2\ell_{ij} & 2i\pi \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ i\,\theta_{ij} & 2i\pi \end{pmatrix} (-1) \ .$$

5.3.4. *The case of a simplex in hyperbolic 3 space with a vertex at infinity.* Now consider the case when Δ has a single vertex at infinity $x = L_{123}$. After blowing-up this point, we obtain a new hyperplane $\widetilde{L}_{-1}$ which is the exceptional locus, and set

$$h(\Delta) = H^3(\widetilde{\mathbb{P}} \backslash \widetilde{Q}, \bigcup_{-1 \leq i \leq 3} \widetilde{L}_i \backslash (\widetilde{Q} \cap \widetilde{L}_i)) \ .$$

From (5.27), $\mathrm{gr}_4^W h(\Delta) = \mathbb{Q}(-2)$, $\mathrm{gr}_2^W h(\Delta) = \mathbb{Q}(-1)^5$ and $\mathrm{gr}_0^W h(\Delta) = \mathbb{Q}(0)$. The graded weight 2 part is spanned by the Kummer submotives coming from each of the three finite-length edges L_{01}, L_{02}, L_{03}, whose periods are their hyperbolic lengths, and a further 3 classes $e_{\widetilde{L}_{ij}} = \alpha_{L_i} - \alpha_{L_j}$, for $1 \leq i < j \leq 3$, where

$$\alpha_{L_i} \in H^2(\widetilde{L}_i \backslash \widetilde{Q}_i, \bigcup_{j \neq i} \widetilde{L}_{ij} \backslash \widetilde{Q}_{ij}) \ ,$$

is the class whose period is the quantity defined in (5.51), on the left. The remaining periods are the dihedral angles θ_{ij} subtended at the edge L_{ij} in all cases, exactly as in the case where Δ is finite. There is a single relation between the classes $e_{\widetilde{L}_{ij}}$, and correspondingly, the angles subtended at infinity $\theta_{13}, \theta_{12}, \theta_{23}$ sum to π.

6. Applications

Let M be a complete product-hyperbolic manifold of finite volume, which is modelled on a product of odd-dimensional hyperbolic spaces. Then we can write $M = \mathbb{X}/\Gamma$, where $\mathbb{X} = \prod_{1 \leq i \leq N} \mathbb{H}^{2n_i - 1}$, and Γ is a discrete torsion-free subgroup of the group of automorphisms of $\mathbb{X}$. By theorem 4.3, Γ is defined over number fields $(k_1, \ldots, k_N)$. In §4 and §5, we constructed a framed motive

$$\mathsf{mot}(M) \in \mathfrak{A}_{n_1}(k_1) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \mathfrak{A}_{n_N}(k_N) \ .$$

Now let Γ' denote another discrete torsion-free group acting on $\mathbb{X}$, which is commensurable with Γ, *i.e.*, $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'. Then if $M' = \mathbb{X}/\Gamma'$,

$$(6.1) \qquad \mathsf{mot}(M') = \mathsf{mot}(M) \frac{[\Gamma : \Gamma \cap \Gamma']}{[\Gamma' : \Gamma \cap \Gamma']} \ .$$

This is clear from the construction: a tiling for $\mathbb{X}/\Gamma \cap \Gamma'$ can be obtained by taking $[\Gamma : \Gamma \cap \Gamma']$, or $[\Gamma' : \Gamma \cap \Gamma']$, copies of a tiling for M, or M' respectively. In this way, one can define the motive of a product-hyperbolic orbifold. Let Γ denote any discrete subgroup of automorphisms of $\mathbb{X}$ which is not necessarily torsion-free. After choosing a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index, define

$$(6.2) \qquad \mathsf{mot}(\mathbb{X}/\Gamma) = [\Gamma : \Gamma_0]^{-1} \mathsf{mot}(\mathbb{X}/\Gamma_0) \ .$$

If M is isometric to a product $M_1 \times M_2$, then $\mathsf{mot}(M) = \mathsf{mot}(M_1) \otimes \mathsf{mot}(M_2)$. We compute the motives for the cases when M is an arithmetic manifold of type (II), (III) and omit the exceptional case (IV).

6.1. **Dedekind Zeta motives for totally real number fields.** Let k denote a totally real number field of degree r, and let $m = 2n - 1 \geq 3$ be an odd integer. Let D be a quaternion algebra over k satisfying the conditions of (II), and let Q be a skew-Hermitian form over D. Let d be the reduced norm of its discriminant and set $L = k(\sqrt{d})$. Suppose that Q has signature $(n, 1)$ for t places of k, where $t \geq 1$, and is positive definite for $r - t$ places. As in §3, let

$$\Gamma \leq \mathsf{U}^+(Q, \mathcal{O}) \quad \text{and} \quad M_\Gamma = \Big(\prod_{i=1}^{t} \mathbb{H}^n \Big) / \Gamma \ ,$$

where Γ is any subgroup of finite index (not necessarily torsion-free). By remark 3.1, the corresponding cases $\Gamma \leq \mathsf{SO}^+(q, \mathcal{O})$ of type (I), n odd, are subsumed in this construction. Let $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ denote the set of real embeddings of k.

There are two cases to consider. If $L = k$, then the signature of $^\sigma Q$ must be the same for all embeddings $\sigma \in \Sigma$, and hence we must have $t = r$. Otherwise, $[L : k] = 2$, and L has exactly $2t$ real embeddings, and $r - t$ pairs of complex conjugate embeddings. Let $\tau \in \mathrm{Gal}(L/k)$ be a generator. Then the action of τ gives an eigenspace decomposition:

$$(6.3) \qquad \qquad \mathrm{Ext}^1_{\mathrm{MT}(L)}\big(\mathbb{Q}(0), \mathbb{Q}(n) \big) = E^+ \oplus E^- \ ,$$

where $E^+ \cong \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$. Let χ denote the non-trivial character of $\mathrm{Gal}(L/k)$, and let $L(\chi, s) = \zeta_L(s) \zeta_k(s)^{-1}$ denote the corresponding Artin L-function.

Theorem 6.1. *Let Γ, M_Γ be as above. Let $\mathsf{mot}(M_\Gamma)$ denote the framed motive corresponding to M_Γ as defined by (6.2). If $k = L$ then*

$$(6.4) \qquad \qquad \mathsf{mot}(M_\Gamma) \in \bigwedge\nolimits^{\Sigma} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ ,$$

and there exists a non-zero rational number α such that $(2\pi)^{-mr} \mathrm{vol}(M_\Gamma)$ is

$$(6.5) \qquad \qquad R_\Sigma(\mathsf{mot}(M_\Gamma)) = \alpha \, \zeta_k^*(1 - m) \ .$$

Otherwise, in the case where $[L : k] = 2$,

$$(6.6) \qquad \mathsf{mot}(M_\Gamma) \in \bigwedge\nolimits^{\Sigma} E^- = \bigwedge\nolimits^{\Sigma} (\tau - 1) \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ ,$$

and there exists a non-zero rational number α such that $(2\pi)^{-mt} \mathrm{vol}(M_\Gamma)$ is

$$(6.7) \qquad \qquad R_\Sigma(\mathsf{mot}(M_\Gamma)) = \alpha \, L^*(\chi, 1 - m) \ .$$

Proof. In the first case when $k = L$, the manifold M_Γ is defined over $(k_1, \ldots, k_r)$, where $k_i = \sigma_i k$ for $1 \leq i \leq r$, and is equivariant by definition. Then (6.4) follows from theorem 5.16, and (6.5) follows from corollary 3.4.

In the second case, when $[L : k] = 2$, the manifold M_Γ is defined over $(L_1, \ldots, L_r)$, where $L_i = \sigma_i k(\sqrt{d})$, and is also equivariant with respect to Σ. Let τ be a generator of $\mathrm{Gal}(L/k)$. By construction, it maps a tiling of M_Γ to another tiling of M_Γ, and therefore preserves the framing given by the relative homology classes of the simplices defining $\mathsf{mot}\, M$. However τ clearly acts with sign -1 on the volume form (5.33), and therefore the framing in $\mathrm{gr}_0^W(\mathsf{mot}(M_\Gamma))$ is anti-invariant under τ. We deduce that $\tau(\mathsf{mot}(M_\Gamma)) = -\mathsf{mot}(M_\Gamma)$. Now (6.6) and (6.7) follow from theorem 5.16 and corollary 3.4 as before. $\qquad \square$

Note that since $\mathrm{vol}(M_\Gamma)$ is non-vanishing, theorem 6.1 implies, "independently" of Borel's theorem (5.2), that $\dim_{\mathbb{Q}} E^+ \geq r$ and $\dim_{\mathbb{Q}} E^- \geq t$.

Definition 6.2. Many different discrete groups Γ give rise to the same framed motive, up to multiplication by a rational number. For number-theoretic applications, we can take simple representatives for Γ as follows. Let $d \in k$, and set

$$q(x_0, \ldots, x_{2n}) = -dx_0^2 + x_1^2 + \ldots + x_{2n}^2 \ .$$

We define $\mathsf{mot}(k, d, n) = \mathsf{mot}(M_\Gamma)$, where $\Gamma = \mathsf{SO}^+(q, \mathcal{O}_k)$.

Corollary 6.3. *As above, let k be a totally real number field, let $n > 1$ be odd, let $L = k(\sqrt{d})$, where $[L : k] = 2$ and $d \in k$ is positive for at least one embedding of k. Let Σ_k denote the set of real embeddings of k. Then*

$$\mathcal{L}_k = \bigwedge\nolimits^{\Sigma_k} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ , \quad \mathcal{L}'_L = \bigwedge\nolimits^{\Sigma_k} (1 - \tau) \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(n)) \ ,$$

are 1-dimensional $\mathbb{Q}$-vector spaces, and $\mathsf{mot}(k, 1, n) \in \mathcal{L}_k$ and $\mathsf{mot}(k, d, n) \in \mathcal{L}'_L$ are generators. Up to a rational factor, the Hodge regulator on each element is, respectively, $\zeta_k^*(1 - n)$, $L^*(\chi, 1 - n)$, where χ is a non-trivial character of $Gal(L/k)$.

Proof. By (5.2), we have $\dim_\mathbb{Q} E^+ = r$, and $\dim_\mathbb{Q} E^- = t$. $\qquad\square$

Corollary 6.4. *The special value $\pi^{nr} \zeta_k^*(1-n)$ is a determinant of sums of volumes of hyperbolic simplices defined over k with at most one vertex at infinity.*

This corollary uses Borel's bound for the rank of algebraic K-groups and does not follow directly from a decomposition of M_Γ into simplices.

6.2. Quadric motives and generators for $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$. Let k be a totally real number field, and let $L = k(\sqrt{d})$ where $d \in k^\times$ is positive for at least one embedding of k. Let us fix a smooth quadric Q_d in $\mathbb{P}^{2n-1}$ by:

$$Q_d = \{-dx_1^2 + x_2^2 + \ldots + x_{2n}^2 = 0\} \ .$$

Definition 6.5. Let $L_1, \ldots, L_{2n}$ denote a set of hyperplanes in general position and defined over k. Define a finite quadric motive over k to be:

$$(6.8) \qquad m(Q_d, L) = H^{2n-1}(\mathbb{P}^{2n-1} \backslash Q_d, \bigcup_{1 \le i \le 2n} L_i \backslash (L_i \cap Q_d)) \in \mathrm{MT}(\overline{\mathbb{Q}}) \ ,$$

with its framing as defined in §5. In the case where $Q_d, L_1, \ldots, L_{2n}$ do not cross normally, we blow up the points $x_i = L_1 \cap \ldots \cap \widehat{L_i} \cap \ldots \cap L_{2n}$ which meet Q_d. Let $\widetilde{\mathbb{P}}^{2n-1}$ denote the blow-up of $\mathbb{P}^{2n-1}$ in $\{x_i : 1 \le i \le 2n | x_i \in Q_d\}$, and let $\widetilde{L}_i, \widetilde{Q}_d$ denote the strict transforms of L_i, Q_d. Define a quadric motive in this case to be:

$$(6.9) \qquad m(Q_d, L) = H^{2n-1}(\widetilde{\mathbb{P}}^{2n-1} \backslash \widetilde{Q}_d, \bigcup_{1 \le i \le 2n} \widetilde{L}_i \backslash (\widetilde{L}_i \cap \widetilde{Q}_d)) \in \mathrm{MT}(\overline{\mathbb{Q}}) \ ,$$

with its framing as given in §5.

For each embedding σ of k into $\mathbb{R}$ for which $\sigma(d)$ is positive, the points $\sigma(x_i)$ define a hyperbolic geodesic simplex in the Klein model which has finite volume.

Theorem 6.6. *Let d, k, n be as above. Every $M \in \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(n))$, viewed as an element of $\mathfrak{A}_n(L)$, is equal to a linear combination of quadric motives:*

$$(6.10) \qquad\qquad M = \sum_{i=1}^R m(Q_d, L_i) \ ,$$

and its real periods are $(2\pi)^{-n}$ times the corresponding sum of hyperbolic volumes.

Proof. Let Σ be the set of places of k and assume that $[L:k]=1$, i.e., $d \in k^{\times 2}$. Let $\overline{V} \subset \mathfrak{A}_n(\overline{\mathbb{Q}})$ denote the $\mathbb{Q}$-vector space spanned by all linear combinations of quadric motives (6.9) in $\mathbb{P}^{2n-1}$. Let $V = \overline{V} \cap \mathfrak{A}_n(k)$ and let

$$V_0 \subset \ker\left(\widetilde{\Delta}_n : \mathfrak{A}_n(k) \longrightarrow \bigoplus_{1 \leq r \leq n-1} \mathfrak{A}_r(k) \otimes_{\mathbb{Q}} \mathfrak{A}_{n-r}(k)\right)$$

denote the subspace of V on which the reduced coproduct vanishes. The right-hand side of the previous line is isomorphic to $E^+ = \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$, and we identify V_0 with its image in E^+. We have constructed an element

$$\mathsf{mot}(k,1,n) \in \textstyle\bigwedge^\Sigma V_0 \subset \bigwedge^\Sigma E^+ \; ,$$

which is non-zero since its regulator does not vanish, by corollary 6.3. It follows that $\dim_{\mathbb{Q}} V_0 \geq r$, and by Borel's theorem, $\dim_{\mathbb{Q}} E^+ = r$. Therefore $V_0 = E^+$, and every element in $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$ is a linear combination of quadric motives over k. Now if $L = k(\sqrt{d})$ and $[L:k]=2$, the same argument applied to $\mathsf{mot}(k,d,n)$ shows that every element in E^- is a linear combination of motives (6.9) with $d \notin k^{\times 2}$. The theorem follows from the fact that $\mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong E^+ \oplus E^-$. $\qquad\square$

The proof of theorem 5.16 only requires hyperbolic simplices with at most one vertex at infinity. We can therefore assume that all quadric motives which occur have at most one vertex x_i lying on Q_d. This proves theorem 1.2.

One can show using a theorem due to Sah [29] that it suffices to consider finite simplices only, since a hyperbolic geodesic simplex with vertices at infinity is stably scissors-congruent to a sum of finite ones.

6.3. **The case $n = 2$ and Zagier's conjecture.** The case of hyperbolic 3-space is different because of the exceptional isomorphism $\mathsf{SO}^+(3,1)(\mathbb{R}) \cong \mathsf{PSL}_2(\mathbb{C})$, and as a result, the analogues of the previous theorems hold for all number fields, not just totally real ones. We can also use ideal triangulations in this case [22].

Let us identify the boundary of hyperbolic 3-space with the complex projective line: $\partial \mathbb{H}^3 \cong \mathbb{P}^1(\mathbb{C})$, with the action of $\mathsf{PSL}_2(\mathbb{C})$ by Möbius transformations. An ideal hyperbolic 3-simplex is given by 4 distinct points on $\partial \mathbb{H}^3$, and by projective transformation, we can assume that 3 of them are at $0, 1$ and ∞, and denote the last point by $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. If z lies in a number field $L' \subset \mathbb{C}$, then $\Delta(0, 1, \infty, z)$ defines a framed mixed Tate motive by (6.9) with graded pieces $\mathbb{Q}(0), \mathbb{Q}(-1), \mathbb{Q}(-2)$. One can verify that it is defined over the field L'.

Now consider an arithmetic group Γ of type (III). So let L be a number field of degree r with r_1 real places and r_2 complex places, where $r_2 \geq 1$, and let $0 \leq t \leq r_1$. Let B denote a quaternion algebra over L which is unramified at t real places, and ramified at $r_1 - t$ real places. For any order $\mathcal{O}$ in B, let Γ denote a subgroup of finite index of the elements of $\mathcal{O}$ of reduced norm 1. Let Σ denote the set of complex places of L. By triangulating over a suitable splitting field using ideal hyperbolic 3-simplices (see [22]), we obtain a framed motive $\mathsf{mot}(M_\Gamma)$ as before.

Theorem 6.7. *The element* $\mathsf{mot}(M_\Gamma) \in \bigwedge^\Sigma \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(2))$ *satisfies*

$$R_\Sigma\big(\mathsf{mot}(M_\Gamma)\big) = \alpha\, \zeta_L^*(-1) \quad \text{for some } \alpha \in \mathbb{Q}^\times \; .$$

Borel's theorem (5.2) implies that the rank of $\bigwedge^\Sigma \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(2))$ is exactly r_2, and hence it is generated by the class $\mathsf{mot}(M_\Gamma)$. More simply, one can take Γ

to be precisely $\mathsf{PSL}_2(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of L. Then Γ acts on $(\mathbb{H}^3)^{r_2}$ and the quotient is a Bianchi orbifold. We denote its motive by $\mathsf{mot}(L,2)$.

Corollary 6.8. *The element $\mathsf{mot}(L,2)$ is a generator of $\bigwedge^{\Sigma} \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0),\mathbb{Q}(2))$.*

The analogue of theorem 6.6 is the following.

Corollary 6.9. *Every element $M \in \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0),\mathbb{Q}(2))$ is framed equivalent to a linear combination of motives of hyperbolic simplices $\Delta(0,1,\infty,z)$ where $z \in L$.*

A closer analysis of the gluing equations between hyperbolic 3-simplices (or, by looking at the coproduct on the corresponding framed motives, which can be written explicitly), one verifies that every such element is a linear combination $\sum_{i=1}^{R} n_i \, \mathsf{mot}(\Delta(0,1,\infty,z_i))$ where $n_i \in \mathbb{Z}$, and the elements $z_i \in L \backslash \{0,1\}$ satisfy:

$$(6.11) \qquad \sum_{i=1}^{R} n_i z_i \wedge (1-z_i) = 0 \in \bigwedge^2 L^{\times} \ .$$

Using the well-known fact (see below) that the volume of $\Delta(0,1,\infty,z)$ is given by the Bloch-Wigner dilogarithm $D(z) = \mathrm{Im}\,(\mathrm{Li}_2(z) + \log|z|\log(1-z))$, the analogue of theorem 6.4 is precisely Zagier's conjecture for $n=2$.

Corollary 6.10. *There exist formal linear combinations $\xi_j = \sum n_{i,j}[z_{i,j}]$ for $1 \leq i \leq r$ satisfying (6.11), and a non-zero rational number α such that:*

$$\zeta_L^*(-1) = \alpha \det(D(\sigma_i(\xi_j))) \ .$$

Remark 6.11. This corollary is well-known by the work of Bloch, Suslin and many others, by relating the Borel regulator on $K_3(L) \otimes \mathbb{Q}$ to the Bloch-Wigner dilogarithm D on the Bloch group of L. The relation with the zeta-value comes from Borel's theorem (1.11), which is a separate argument. Note that in the proof outlined above, Zagier's conjecture and theorem 6.7 are proved by the same argument, since quadric motives in dimension 3 are equivalent to dilogarithmic motives $P_2(z)$.

6.3.1. *The volume of an ideal hyperbolic 3-simplex.* The moduli space of ideal 3-simplices is $\mathfrak{M}_{0,4} \cong \mathbb{P}^1 \backslash \{0,1,\infty\}$, parameterized by a single coordinate z. The volume is a function of z we denote by $v(z)$. By definition (6.9), each ideal 3-simplex defines a mixed Tate motive with graded pieces $\mathbb{Q}(0)$, $\mathbb{Q}(-1)$ and $\mathbb{Q}(-2)$ only. It therefore defines a unipotent variation of mixed Hodge structure on $\mathfrak{M}_{0,4}$. In particular, the function $v(z)$ satisfies the following properties:

  (1) It is a unipotent function on $\mathfrak{M}_{0,4}$ of weight 2.
  (2) It is single-valued and extends continuously to $\overline{\mathfrak{M}}_{0,4} \cong \mathbb{P}^1$.
  (3) It vanishes at the points $0,1$ and ∞.

The algebra of all single-valued unipotent functions on $\mathfrak{M}_{0,4}$ was explicitly constructed in [9]. A basis for the vector-space of such functions is $\mathcal{L}_w(z)$, where w is a word in the alphabet on two letters x_0, x_1. It follows that an arbitrary single-valued unipotent function F of weight 2 is of the form:

$$F(z) = a_{x_0^2}\,\mathcal{L}_{x_0^2}(z) + a_{x_0 x_1}\,\mathcal{L}_{x_0 x_1}(z) + a_{x_1 x_0}\,\mathcal{L}_{x_1 x_0}(z) + a_{x_1^2}\,\mathcal{L}_{x_1^2}(z) \qquad a_w \in \mathbb{R} \ ,$$

where $\mathcal{L}_{x_0^n}(z) = \frac{1}{n}\log^n |z|^2$, $\mathcal{L}_{x_1^n}(z) = \frac{1}{2}\log^n |1-z|^2$, and, for example,

$$\mathcal{L}_{x_0 x_1}(z) = 2i\,\mathrm{Im}\,(\mathrm{Li}_2(z) + \log|z|\log(1-z)) - 2\log|z|\log|1-z| \ .$$

One also has the shuffle relation $\mathcal{L}_{x_0}(z)\mathcal{L}_{x_1}(z) = \mathcal{L}_{x_0 x_1}(z) + \mathcal{L}_{x_1 x_0}(z)$, and many other properties [9]. From property (3), we conclude that $v(z) = a(\mathcal{L}_{x_0 x_1}(z) - \mathcal{L}_{x_1 x_0}(z))$. The constant $a = (4i)^{-1}$ is easily determined from a special case, giving

$$v(z) = \mathrm{Im}\left(\mathrm{Li}_2(z) + \log|z|\log(1-z)\right),$$

which is none other than the Bloch-Wigner dilogarithm. The same result has been proved many times over by a wide variety of methods.

6.3.2. *Volumes of higher-dimensional hyperbolic simplices.*

A similar variational argument allows one to show that the volumes of hyperbolic simplices in $\mathbb{H}^{2n-1}$ are expressible in terms of (logarithms and products of) multiple polylogarithms

$$\mathrm{Li}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}},$$

of weight $n = n_1 + \ldots + n_r$. It is possible to construct single-valued versions of these functions in the same way as for the multiple polylogarithms in one variable [9]. By corollary 6.4, we deduce that the special value $\zeta_k^*(1-n)$, where k is a totally real number field, is expressible in terms of powers of π and a determinant of values of multiple polylogarithms evaluated in possibly an extension of k.

In the case $n = 3$, it is relatively easy to show that every multiple polylogarithm of weight 3 can be expressed in terms of the trilogarithm $\mathrm{Li}_3(z)$. From this fact, one could presumably obtain a proof of Zagier's conjecture for $n = 3$ for totally real number fields. The case $n = 4$ is the first interesting case, since every multiple polylogarithm of weight 4 can be expressed in terms of two functions:

$$\mathrm{Li}_4(z) \qquad \text{and} \qquad \mathrm{Li}_{2,2}(x,y)$$

since $\mathrm{Li}_4(z)$ does not suffice on its own. The expectation is that if M is an arithmetic product-hyperbolic manifold modelled on products of $\mathbb{H}^7$, then the vanishing of the reduced coproduct for $\mathsf{mot}(M)$ (i.e., the gluing equations for the simplices) should imply that the sum of the volume terms of the form $\mathrm{Li}_{2,2}(x,y)$ over all simplices in the triangulation should be expressible in terms of $\mathrm{Li}_4(z)$. In short, M should define a functional equation for combinations of $\mathrm{Li}_{2,2}(x,y)$ in terms of $\mathrm{Li}_4(z)$.

6.4. Some open questions.

(1) Since our proof of theorem 1.1 is motivic, it is natural to ask what happens in other realisations, and in particular the p-adic case.

(2) We have only considered arithmetic product-hyperbolic manifolds, but there exists an abundance of non-arithmetic manifolds $\mathbb{H}^n/\Gamma$, which define elements in $\mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0),\mathbb{Q}(n))$. Does there exist a volume formula for non-arithmetic hyperbolic manifolds $M = \mathbb{H}^n/\Gamma$ as a Dirichlet-type series? To the author's knowledge, there does not seem to exist a formula, conjectured or otherwise, for the regulator on single elements of $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0),\mathbb{Q}(n))$ in the case where k is a non-abelian Galois extension of $\mathbb{Q}$.

(3) Does the construction of $\mathsf{mot}_n(k)$ generalise to other symmetric spaces? The case of the special linear group would give a result for values of Dedekind zeta functions for all number fields, not just the totally real ones.

(4) Can one construct non-trivial iterated extensions in the category $MT(k)$ by considering manifolds with boundary? One can dream of proving the freeness conjecture (§1.2) by defining nested families of manifolds with boundary which yield elements of $\mathfrak{A}_n(k)$ which are coindependent by construction. The simplest example would be a $\zeta(3,5)$-manifold whose boundary is related to the manifolds (3) for $\zeta(3)$ and $\zeta(5)$.

7. Example: A Coxeter motive for $L(\chi, 3)$

The following example of a fundamental domain, for an arithmetic reflection group acting on $\mathbb{H}^5$, is due to Bugaenko [11]. Only exceptionally few examples can hope to have such a simple and explicit description. Let $\phi = \frac{1+\sqrt{5}}{2}$, and let $k = \mathbb{Q}(\sqrt{5})$. Its ring of integers $\mathcal{O}_k$ is $\mathbb{Z}[\phi]$. Consider the quadratic form:

$$q(x_0, \ldots, x_5) = -\phi\, x_0^2 + x_1^2 + \ldots + x_5^2 \ ,$$

and let $\Gamma = \mathsf{SO}^+(\mathcal{O}_k, q)$ be the group of $\mathcal{O}_k$-valued matrices which preserve q and which map each connected component of $\{x : q(x) < 0\}$ to itself. Then Γ is a discrete group of automorphisms of $\mathbb{H}^5$ of type (I) as defined in §2. Note, however, that Γ has torsion. Consider the seven hyperplanes in $\mathbb{P}^5$:

$$
\begin{aligned}
L_1 &: & x_2 - x_1 &= 0 \ , \\
L_2 &: & x_3 - x_2 &= 0 \ , \\
L_3 &: & x_4 - x_3 &= 0 \ , \\
L_4 &: & x_5 - x_4 &= 0 \ , \\
L_5 &: & x_5 &= 0 \ , \\
L_6 &: & (\phi - 1)\, x_0 + \phi\, x_1 &= 0 \ , \\
L_7 &: & (1 + \phi)\, x_0 + \phi\, (x_1 + x_2 + x_3 + x_4 + x_5) &= 0 \ .
\end{aligned}
$$

These hyperplanes bound a convex polytope P. If $Q = \{x \in \mathbb{P}^5 : q(x) = 0\}$ denotes the quadric defined by q, the set of real points of $\mathbb{P}^5 \backslash Q$ (more precisely $\{x \in \mathbb{P}^5(\mathbb{R}) : q(x) < 0\}$) is a projective model for $\mathbb{H}^5$. One proves [11], that the group generated by hyperbolic reflections in the L_i, for $1 \leq i \leq 7$ generates Γ, and therefore that the interior of P is an open fundamental domain for Γ. The Coxeter diagram for Γ is the following:
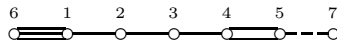
The polytope P has the combinatorial structure of a prism, *i.e.*, the product of a 5-simplex with an interval, and has no non-trivial symmetries.

Now consider the motive

$$h(\Gamma) = H^3\big(\mathbb{P}^5 \backslash Q, \textstyle\bigcup_{i=1}^7 L_i \backslash (L_i \cap Q)\big) \in \mathrm{MT}(\overline{\mathbb{Q}}) \ .$$

It has canonical framings $[P] \in \mathrm{gr}_6^W H_3(\mathbb{P}^5, \bigcup_{i=1}^7 L_i) \cong (\mathrm{gr}_0^W h(\Gamma))^\vee$ given by the class of the polytope P, and by the class of the volume form

$$\omega_Q = \sqrt{\phi}\, \frac{\sum_{i=0}^5 (-1)^i x_i dx_0 \ldots \widehat{dx_i} \ldots dx_5}{q(x_0, \ldots, x_n)^3} \ , \quad \text{and} \quad [\omega_Q] \in \mathrm{gr}_6^W H^3(\mathbb{P}^5 \backslash Q) \cong \mathrm{gr}_6^W h(\Gamma).$$

Let Γ' denote a torsion-free subgroup of Γ of finite index. We can construct a fundamental domain for Γ' by gluing $[\Gamma : \Gamma']$ polytopes P together. By §4, we know

that the framed equivalence class of the corresponding motive, which is an integral multiple of $[h(\Gamma), [P], [\omega_Q]]$, has vanishing coproduct. It follows that:

$$\mathsf{mot}(\Gamma) = [h(\Gamma), [P], [\omega_Q]] \in \mathrm{Ext}^1_{\mathrm{MT}(\overline{\mathbb{Q}})}(\mathbb{Q}(0), \mathbb{Q}(3)) \ .$$

The motive $h(\Gamma)$ is clearly defined over k, and its framing is defined over $L = k(\sqrt{\phi})$. It is clear, therefore, that $\mathsf{mot}(\Gamma) \in \mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(3))$, and furthermore, that it is anti-invariant under the action of τ, the non-trivial generator of $\mathrm{Gal}(L/k)$. This is because τ fixes Q, the L_i and $[P]$, but sends $[\omega_Q]$ to $-[\omega_Q]$. Thus

$$(7.1) \qquad\qquad \mathsf{mot}(\Gamma) \in (1 - \tau)\,\mathrm{Ext}^1_{\mathrm{MT}(L)}(\mathbb{Q}(0), \mathbb{Q}(3)) \ .$$

By (5.1) and (5.2), the right-hand side is a $\mathbb{Q}$-vector space of dimension 1, and the framed motive $\mathsf{mot}(\Gamma)$ is therefore a generator.

Next, by the volume computations given in §3, we deduce that

$$(7.2) \qquad \mathrm{vol}(\mathbb{H}^5/\Gamma') \sim_{\mathbb{Q}^\times} \sqrt{|d_{L/k}d_k|}\frac{L(\chi, 3)}{\pi^3} \sim_{\mathbb{Q}^\times} \pi^3 L^*(\chi, -2) \ ,$$

where χ is the non-trivial quadratic character of $\mathrm{Gal}(L/k)$. We conclude using the fact that $\mathrm{vol}(\mathbb{H}^5/\Gamma) = \int_{[P]}[\omega]$ is the real period of $\mathsf{mot}(\Gamma)$.

Corollary 7.1. *We have explicitly constructed* $\mathsf{mot}(\Gamma) \in \mathrm{Ext}^1_{\mathrm{MT}(L)}\big(\mathbb{Q}(0), \mathbb{Q}(3)\big)$ *such that* $\tau(\mathsf{mot}(\Gamma)) = -\mathsf{mot}(\Gamma)$ *and* $L^*(\chi, -2) = R_\tau\,\mathsf{mot}(\Gamma)$, *where* R_τ *is the real period map (Hodge regulator).*

Note that it is possible in principle to compute the exact volume of P, using the methods of [27]. It is also known that the volume of a hyperbolic 5-simplex can be written as a linear combination of values of (single-valued) trilogarithms (§6.3.2).

Remark 7.2. For the main result of the paper we need instead to take the group of $\mathcal{O}_k$-automorphisms of the quadratic form $q(x) = -x_0^2 + x_1^2 + \ldots + x_5^2$ whose automorphism group now acts on the pair of hyperbolic spaces $\mathbb{H}^5 \times \mathbb{H}^5$. Finding an explicit fundamental domain in this case should be possible using the methods of Epstein and Penner [14], but is complicated in practice.

The above example has a very simple fundamental domain precisely because it is a reflection group. Reflection groups are known only to exist in hyperbolic spaces $\mathbb{H}^n$ for bounded n and for number fields of bounded discriminant (when $n \geq 4$).

References

[1] **D. Alekseevskii, E. Vinberg, A. Soldovnikov**: *Geometry of spaces of constant curvature*, in Encyc. of Math. Sciences **29**, Springer (1993).

[2] **W. Ballmann, M. Gromov, V. Schroeder**: *Manifolds of non-positive curvature*, Prog. Math **61**, Birkhäuser, Boston (1985)

[3] **A. Beilinson**: *Higher regulators and values of L-functions*, J. Soviet Math **30**, (1985), 2036-2070.

[4] **A. Beilinson, P. Deligne**: *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs*, Proc. Symp. Pure Math. **55**, Pt. 2, (1994), 97-121.

[5] **A. Beilinson, A. Goncharov, V. Schechtman, A. Varchenko**: *Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of a pair of triangles in the plane*, The Grothendieck Festschrift, Birkhauser, (1990), 131-172.

[6] **S. Bloch**: *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*, Proc. Int. Symp. on algebraic geometry, Kyoto 1977, (1977), 103-114.

[7] **A. Borel**: *Stable real cohomology of arithmetic groups*, Annales Ecole Normale Sup. **7**, No. 4, (1974), 235-272.

[8] **A. Borel**: *Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers*, Annali della Scuola Norm. di Pisa, (1976), 613-635, + erratum.

[9] **F. C. S. Brown**: *Single-valued multiple polylogarithms in one variable*, C.R. Acad. Sci. Paris, Ser. I **338** (2004), 527-532.

[10] **J. I. Burgos Gil**: *The regulators of Beilinson and Borel*, CRM Monographs **15**, American Math. Soc., (2002)

[11] **V. O. Bugaenko**: *Groups of Automorphisms of unimodular hyperbolic quadratic forms over the ring* $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]$, Vestnik Mosk. Univ. Matematika, Vol. 39, No.5, (1984), 6-12.

[12] **T. Chinberg**: *Volumes of hyperbolic manifolds*, J. Differ. Geom. **18**, (1983), 783-789.

[13] **P. Deligne, A. B. Goncharov**: *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. Ecole Norm. Sup., Sér. IV **38**, No. 1, (2005), 1-56.

[14] **D.B.A. Epstein, R.C. Penner**: *Euclidean decompositions of noncompact hyperbolic manifolds*, J. Diff. Geom **27** (1988), 67-80.

[15] **H. Gangl, D. Zagier**: *Classical and elliptic polylogarithms and special values of L-series*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 561–615, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[16] **W. Gerdes**: *The linearization of higher Chow cycles of dimension one*, Duke Math. Journal, Vol. 62, No.1, (1991), 105-129.

[17] **A.B. Goncharov**: *Volumes of hyperbolic manifolds and mixed Tate motives*, Journal AMS, **12**, No. 2, (1999), pp. 569-618.

[18] **A.B. Goncharov**: *Geometry of configurations, polylogarithms, and motivic cohomology* Adv. Math. 114 (1995), no. 2, 197–318

[19] **G. Harder**: *A Gauss-Bonnet formula for discrete arithmetically defined groups*, Ann. Scient. Ec. Norm. Sup. ,4ième série, (1971), 409-455.

[20] **J. Li, R. MacPherson**: *Geometry of compactifications of locally symmetric spaces*, Ann. Inst. Fourier, Grenoble **52**, 2, (2002), 457-559.

[21] **G. A. Margulis**, *Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than* 1, Invent. Math. 76 (1984), no. 1, 93–120.

[22] **W. Neumann, J. Yang**: *Bloch invariants of hyperbolic 3-manifolds*, Duke Math. Journal **96** (1999), 29-59.

[23] **W. Neumann, D. Zagier**: *Volumes of hyperbolic three-manifolds*, Topology **24**, 307332 (1985)

[24] **T. Ono**: *On algebraic groups and discontinuous groups*, Nagoya Math. Journal, **27**, (1966), 279-322. Annals of Math., **84**, 279-322 (1965)

[25] **T. Ono**, *On the relative theory of Tamagawa numbers*, Annals of Math., **82**, (1965), 88-111.

[26] **G. Prasad**: *Volumes of S-arithmetic quotients of semi-simple groups*, Publ. Math. IHES **69**, 91-114 (1989).

[27] **J. Ratcliffe**: *Foundations of hyperbolic manifolds*, Graduate Texts in Math. **149**, Springer, (1994)

[28] **C. H. Sah**: *Scissors Congruences, I*, Math. Scand. **49**, 181-210 (1981).

[29] **L. Saper**: *Tilings and finite energy retractions of locally symmetric spaces*, Comment. Math. Helv. **72** (1997), 167-202

[30] **C. Siegel**, *The volume of the fundamental domain for some infinite groups*, Trans. Amer. Math. Soc 39 (1936), 209-218.

[31] **C. Siegel**, *Zur Bestimmung des Volumens des Fundamentalbereichs der unimodularen Gruppe*, Math. Ann. 137, 427-432 (1959).

[32] **W. Thurston**, *The geometry and topology of 3-manifolds*, Princeton lecture notes (19781981).

[33] **E. Vinberg, O. Shvartsman**: *Discrete groups of motions of spaces of constant curvature*, in Encyc. of Math. Sciences **29**, Springer (1993).

[34] **Weil, A.**: *Adèles and algebraic groups*, Progress in Math. **23**, Birkhäuser, Boston, (1961).

[35] **Yang, J.** : *On the real cohomology of arithmetic groups and the rank conjecture for number fields*, Annales Sci. de l'ENS, 25, no. 3, (1992), 287-306.

[36] **D. Zagier**: *Hyperbolic manifolds and special values of Dedekind zeta-functions*, Invent. Math. **83** (1986), 285-301.

CNRS, IMJ

E-mail address: brown@math.jussieu.fr