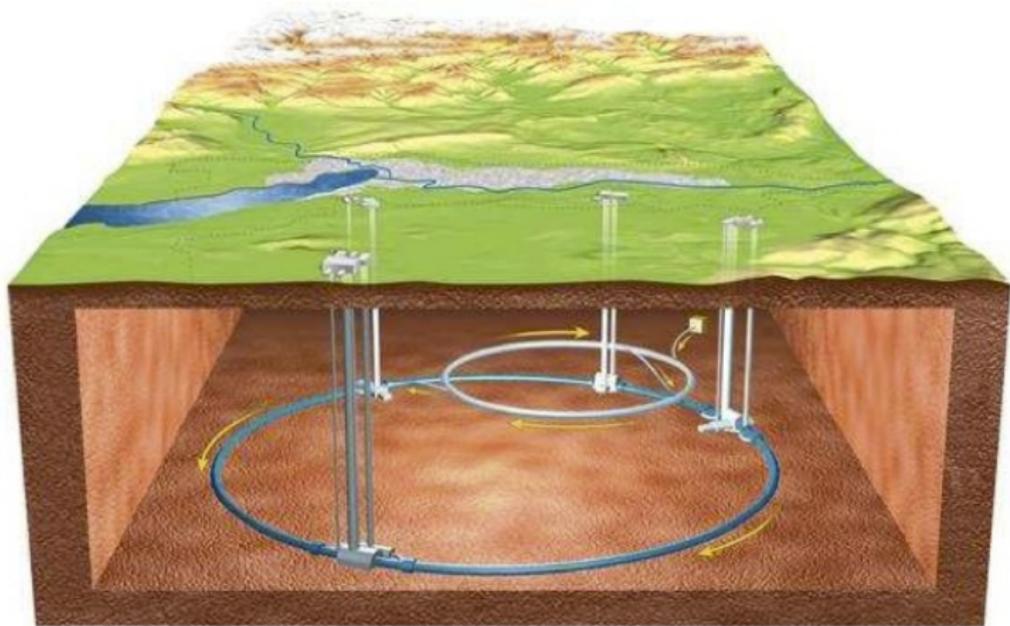


Periods, Galois theory and particle physics

Francis Brown
All Souls College, Oxford

Gergen Lectures,
21st-24th March 2016

Particle Physics

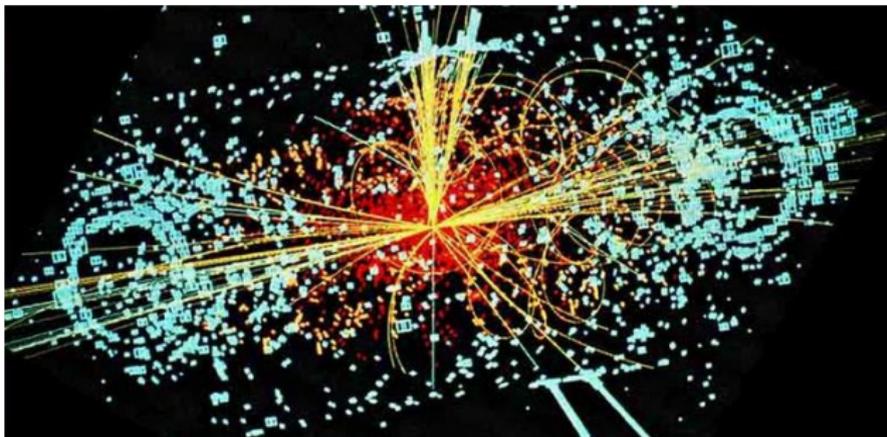


Large Hadron Collider (LHC) at CERN. 27 km.

Goal is to study the forces and fundamental particles of nature.

Collision of beams

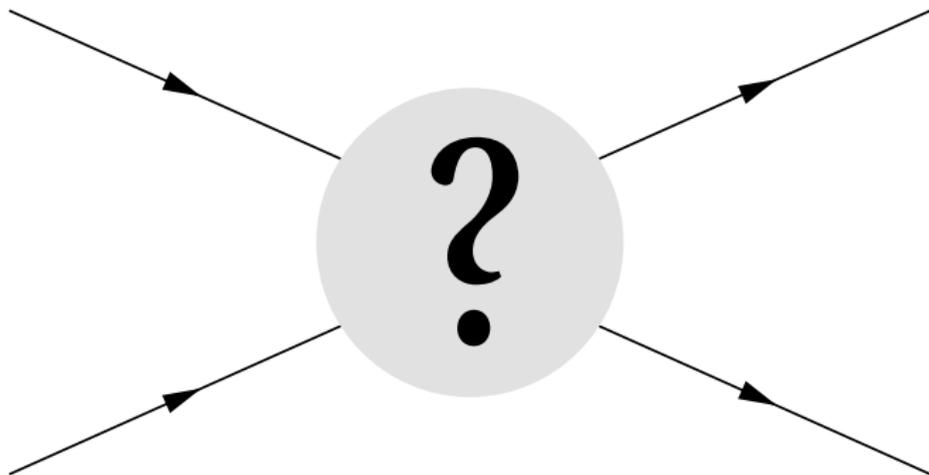
Two beams of particles moving in opposite directions around the beam pipe are brought together at very high energy resulting in a cascade of new particles (via $E = mc^2$).



By analysing the particle tracks the fundamental laws of physics can be tested.

Perturbative Quantum Field theory

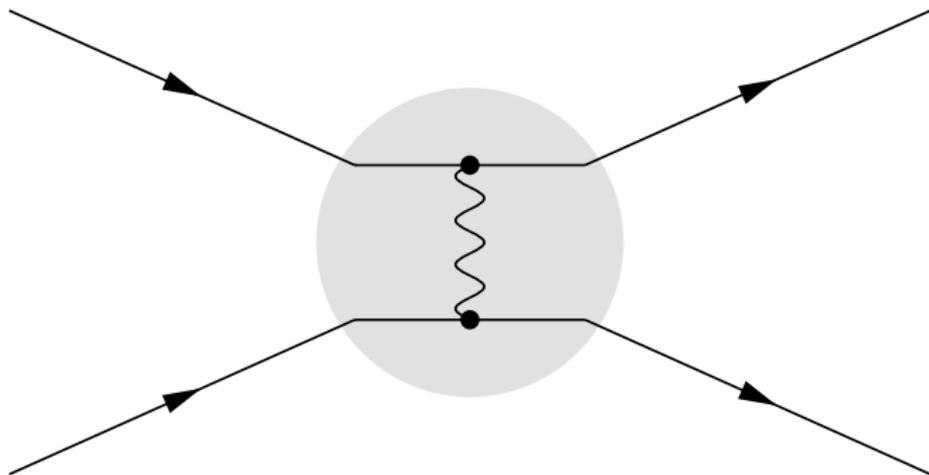
Quantum Field theory is the general framework describing fundamental forces and particles. Scattering of particles are represented by Feynman graphs:



Every Feynman graph represents a possible particle interaction. Only the incoming and outgoing particles can be observed. To every Feynman graph G one assigns a complex probability amplitude called the *Feynman amplitude* I_G .

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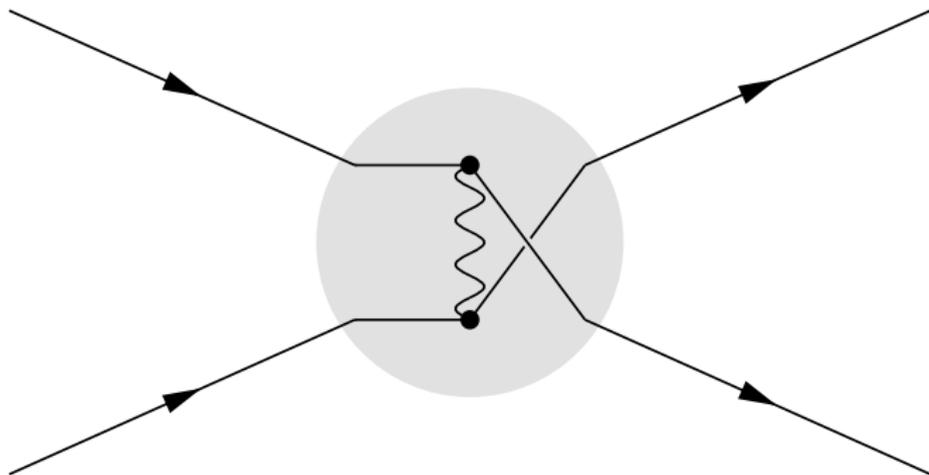
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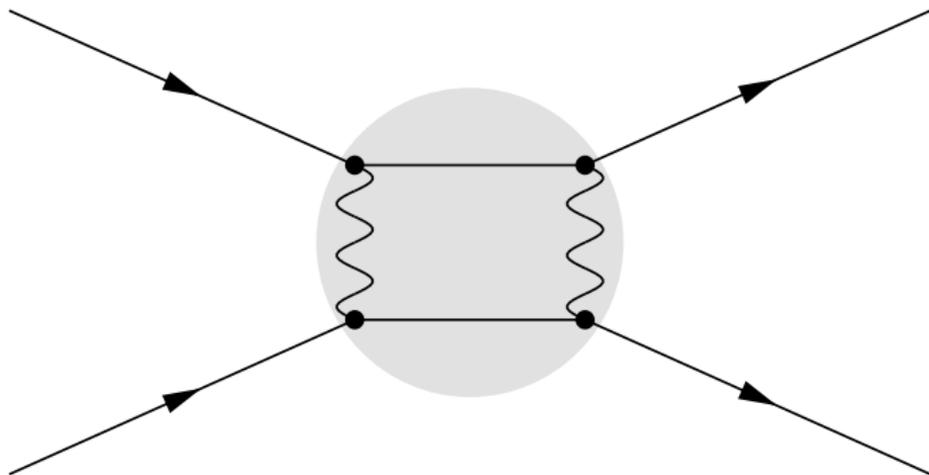
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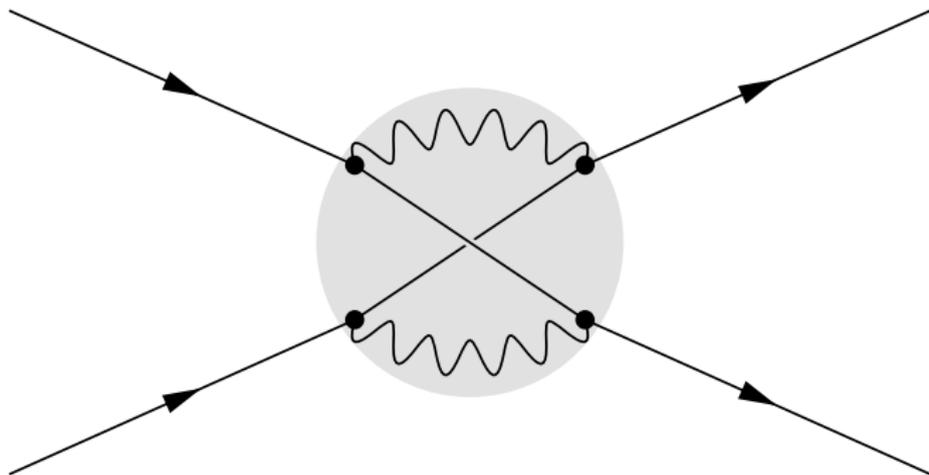
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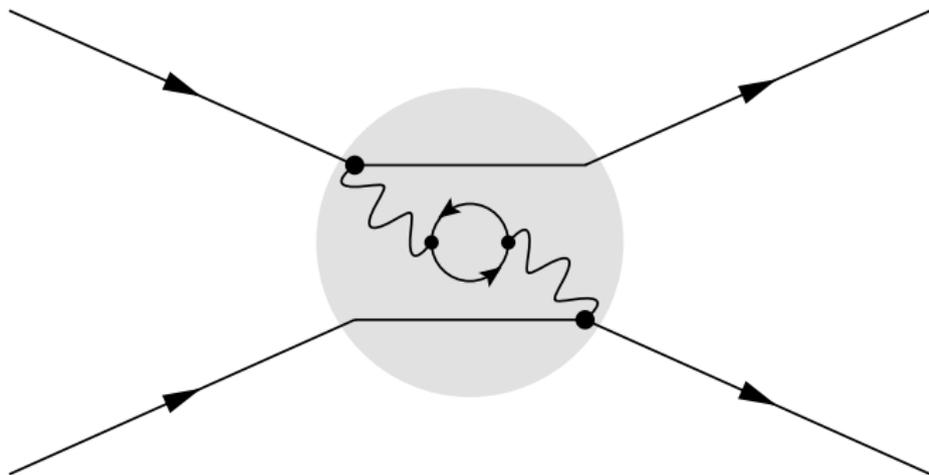
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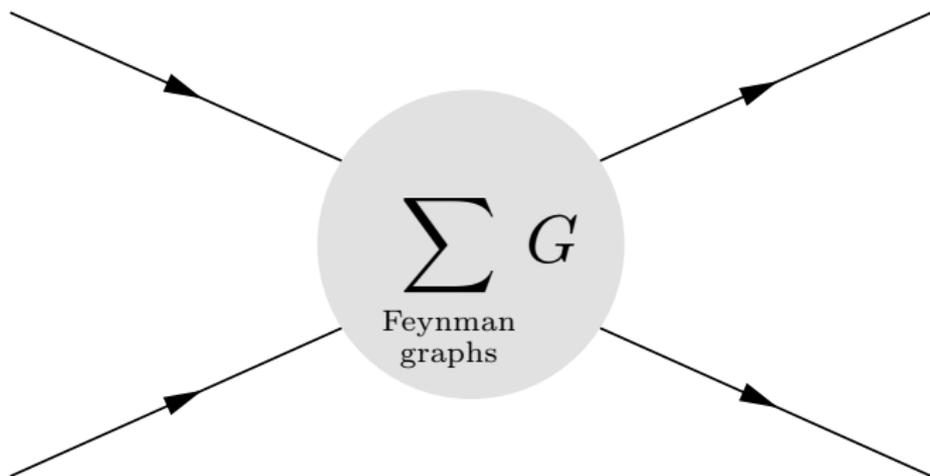
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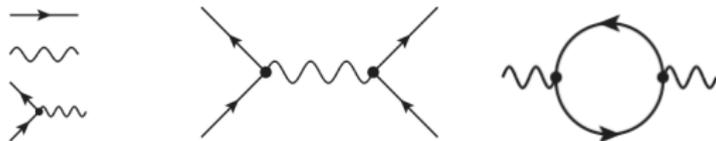


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Perturbative Quantum Field Theory

The probability of observing a given outcome is obtained by summing the amplitudes I_G over all graphs and taking the norm squared. This is a divergent series, and it is hoped that it is resumable. Nobody worries about this in practice.

Mathematically, we can represent a quantum field theory by a collection of graph elements (types of edges, and vertices), and build out of it a certain class of graphs.



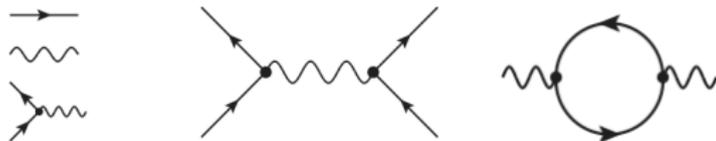
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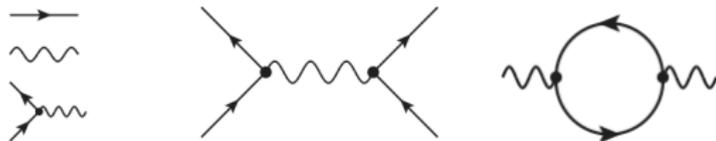
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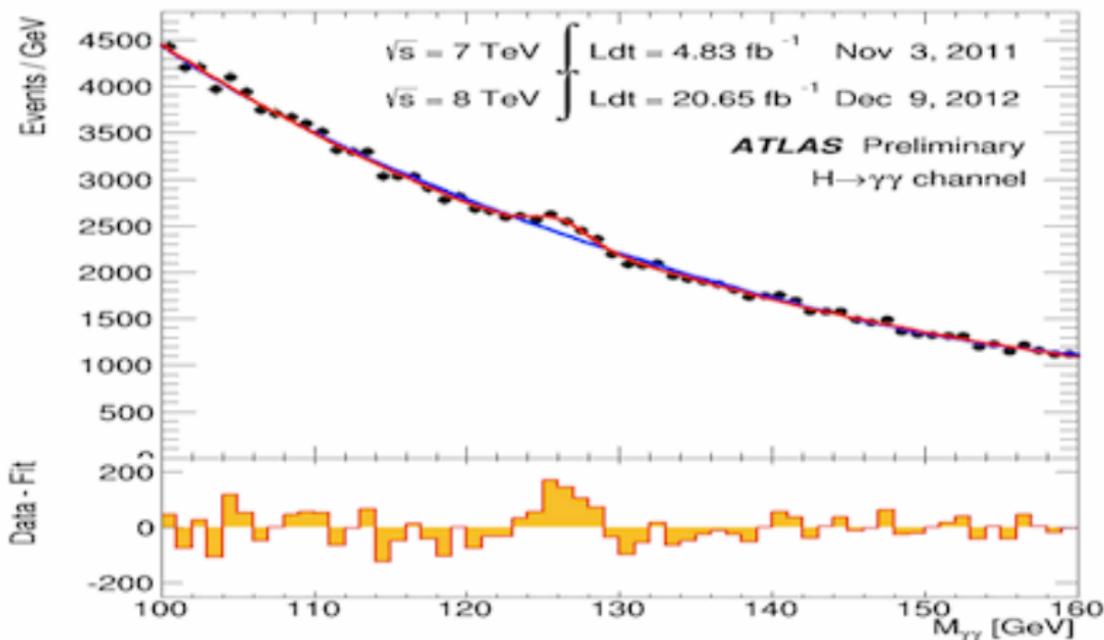
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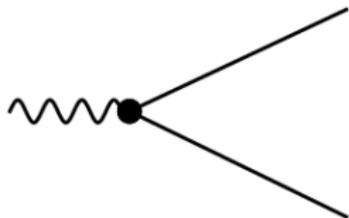
Application: discovery of Higgs boson



The blue line (background) requires calculating a huge number of Feynman amplitudes.

Motivation II

Anomalous magnetic moment of the electron. One of the physical quantities that can be measured the most accurately.



The best experimental value for this quantity is

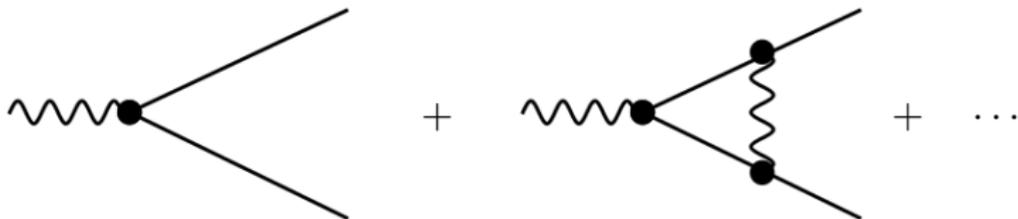
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The prediction from quantum field theory (Feynman diagrams) is

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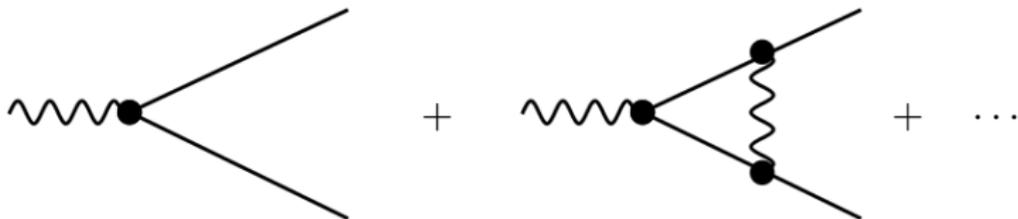
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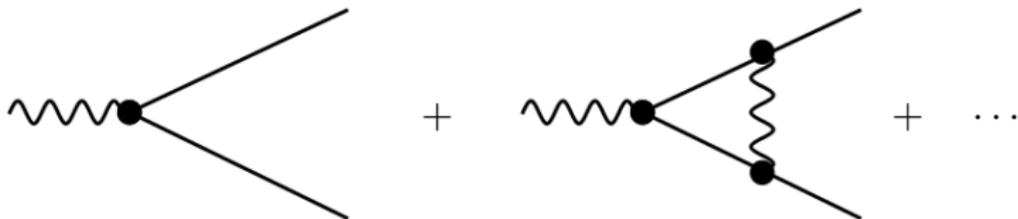
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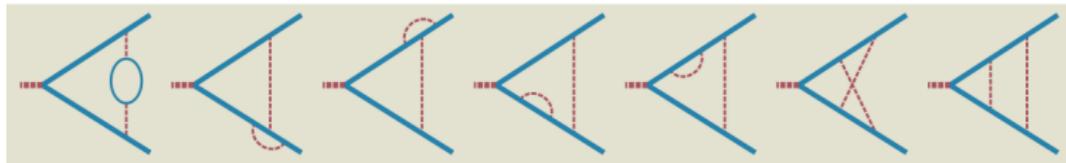
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The 2-loop Feynman diagrams were computed in 1957/8:



The sum of the Feynman amplitudes for these 7 graphs is

$$\frac{197}{144} + \frac{1}{2}\zeta(2) - 3\zeta(2)\log 2 + \frac{3}{4}\zeta(3)$$

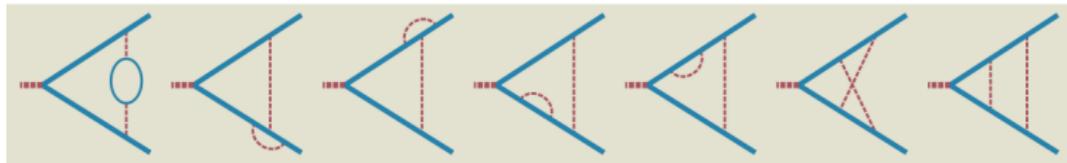
There are 72 diagrams with 3-loops, and the answer is also known analytically as from 1996 (Laporta, Remiddi).

The 4-loop amplitude is only known *numerically* (891 diagrams).

The 5-loop QED contributions known numerically using supercomputers (2012, $\sim 12,000$ diagrams, T. Kinoshita et al.)

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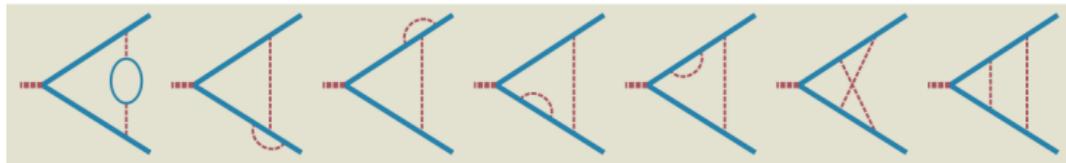
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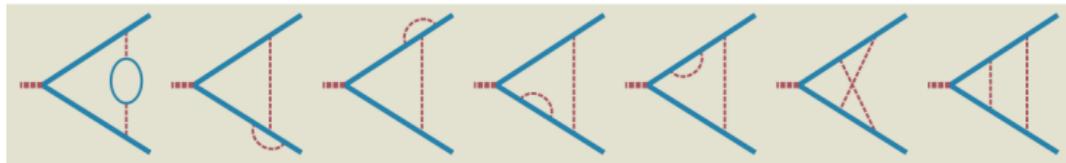
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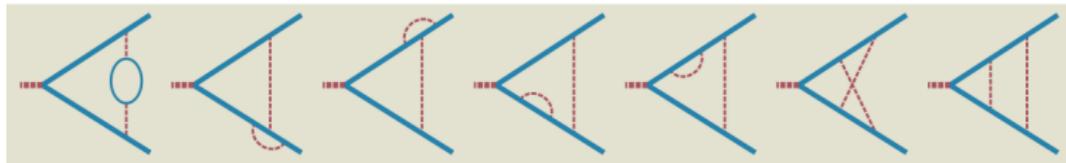
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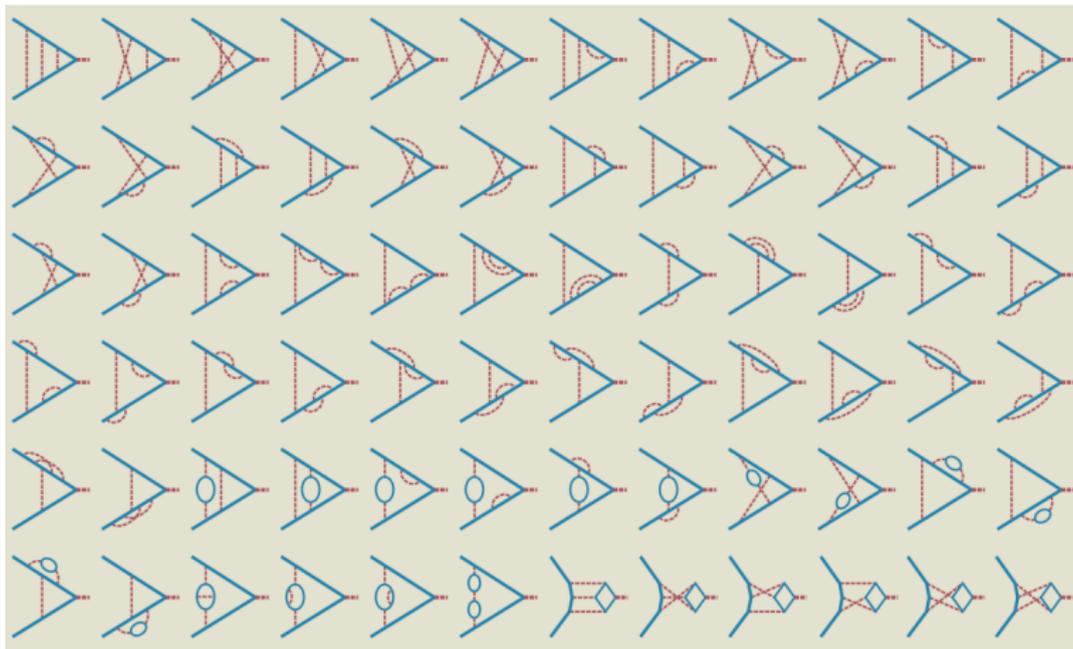
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Three-loop diagrams

Here are the 72 diagrams with three loops.

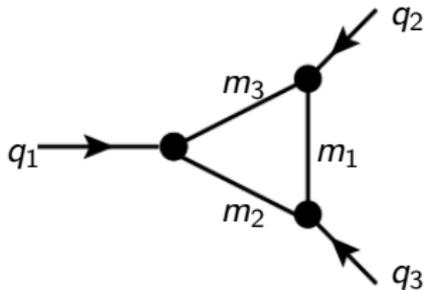


The latest results also require taking into account QCD diagrams.

Scalar Feynman graphs

For simplicity, consider graphs with straight edges only. A *Feynman graph* is a graph $G = (V_G, E_G, E_G^{ext})$ with vertices V_G , internal edges $E_G \subset \text{Sym}^2 V_G$, external edges E_G^{ext} , and data:

- a momentum $q_e \in \mathbb{R}^D$ for every external edge e ,
- a mass $m_e \geq 0$ for every internal edge e .



These are subject to momentum conservation:

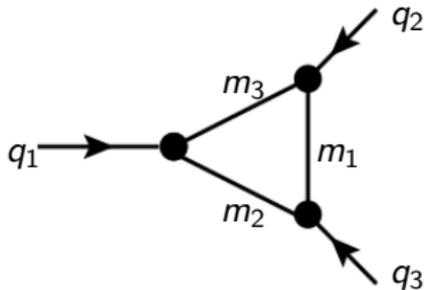
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The integer D is the dimension of space-time and will be even (typically 4). Use Euclidean norm $q^2 = \sum_{1 \leq i \leq D} q_i^2$.

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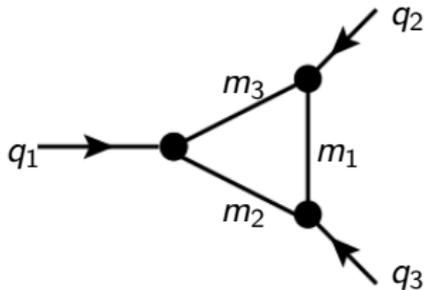
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Graph polynomials

Let G be a connected Feynman graph. The amplitude of G will be constructed out of two polynomials called Symanzik polynomials, in α_e , e edge of G . The first polynomial

$$\Psi_G \in \mathbb{Z}[\alpha_e, e \in E_G]$$

was defined by Kirchhoff in 1847 while studying electrical circuits.

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e \quad T \text{ spanning tree}$$

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e \quad T_1 \cup T_2 \text{ spanning 2-tree}$$

where q^{T_1} is the total momentum entering T_1 .

A subgraph $T \subset G$ is a *spanning k tree* if it is a forest (it is simply connected), with k connected components and if it spans G . This means that it meets every vertex of G , or $V(T) = V(G)$.

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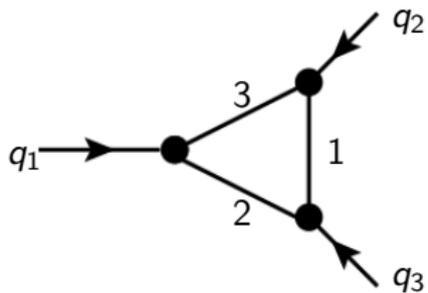
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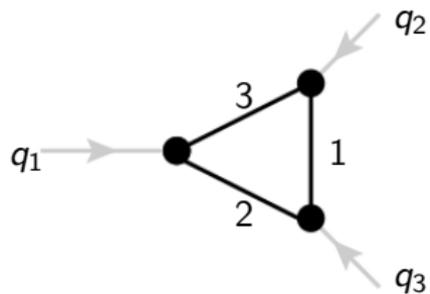
Amplitudes in parametric form

Example:



Amplitudes in parametric form

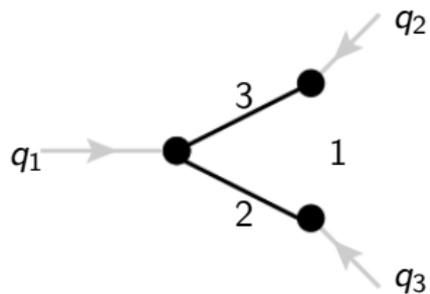
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$$\Psi_G = ?$$

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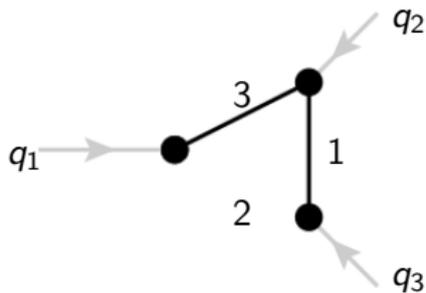
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$$\Psi_G = \alpha_1$$

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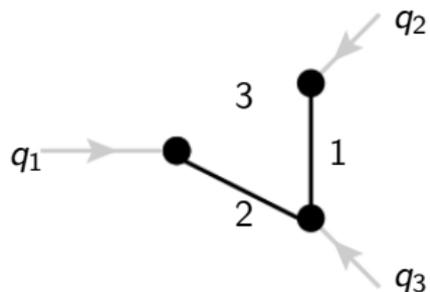
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$$\Psi_G = \alpha_1 + \alpha_2$$

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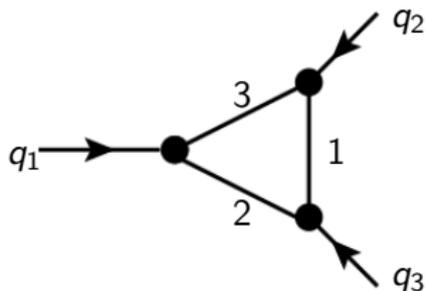
Example:



$$\Psi_G = \alpha_1 + \alpha_2 + \alpha_3$$

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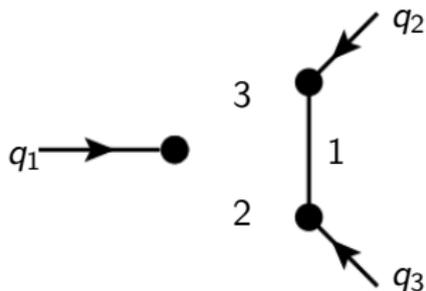


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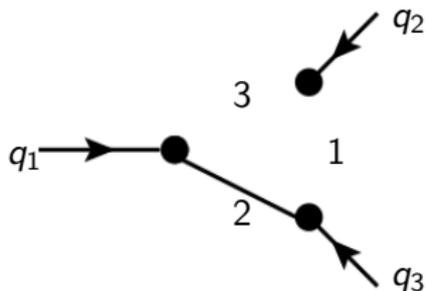


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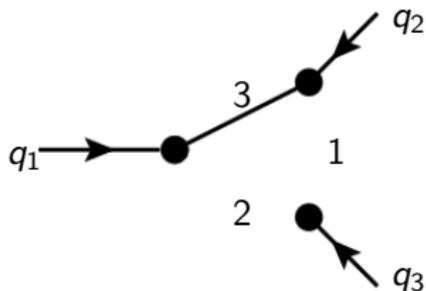


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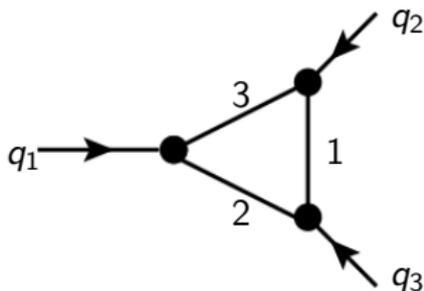


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Let m_e denote the particle mass of edge e , set

$$\Xi_G(m, q) = \Phi_G(q) + \left(\sum_e m_e^2 \alpha_e \right) \Psi_G$$

Amplitudes in parametric form

In general, Ψ_G is homogeneous of degree h_G , the first Betti number of G (also known as the 'loop number' of G), and Φ_G is homogeneous in the α_e of degree $h_G + 1$. Let $N_G = \#E(G)$.

The amplitude in parametric form is (up to some pre-factors)

$$I_G(q, m) = \int_{\sigma} \frac{1}{(\Psi_G)^{2h_G}} \left(\frac{\Psi_G}{\Xi_G(q, m)} \right)^{N_G - 2h_G} \Omega_G$$

where

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_{N_G}$$

and the domain of integration σ is the real coordinate simplex

$$\sigma = \{(\alpha_1 : \dots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$

The integrand is homogeneous of degree 0.

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$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_{N_G}$$

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The integrand is homogeneous of degree 0.

Amplitudes in parametric form

In general, Ψ_G is homogeneous of degree h_G , the first Betti number of G (also known as the 'loop number' of G), and Φ_G is homogeneous in the α_e of degree $h_G + 1$. Let $N_G = \#E(G)$.

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A class of periods

Key observation

Amplitudes are (families of) periods

- The integrals often diverge. The theory of *renormalisation* provides a way to remove ultra-violet singularities consistently.
- Gauge theory amplitudes (e.g. QED) have polynomials in the numerator. The types of numbers which occur only depend on the *denominator* and not the numerator.

A huge effort goes into the computation of Feynman amplitudes. The general 1-loop amplitude involves the dilogarithm

$$\text{Li}_2(z) = \sum_{k \geq 1} \frac{z^k}{k^2} = \int_{0 \leq t_1 \leq t_2 \leq z} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$$

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What is the class of periods which occur amplitudes in QFT?

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Some single-scale examples

The simplest situation occurs when $N_G = 2h_G$, and we can set

$$I_G = \int_{0 < \alpha_e < \infty} \frac{\Omega_G}{\Psi_G^2}$$

No dependence on m, q . Only involves the first Symanzik polynomial. In order to guarantee convergence we shall assume:

- G is *overall log-divergent*: $N_G = 2h_G$
- G is *primitive*: $N_\gamma > 2h_\gamma$ for all $\gamma \subsetneq G$.

A graph is deemed to be 'physical' if all its vertices have degree at most 4, written

$$G \in \phi^4$$

The result is that we get a map from graphs to numbers:

$$I : \{\text{Primitive, log-divergent graphs in } \phi^4\} \longrightarrow \mathbb{R}$$

The problem is to try to understand this map.

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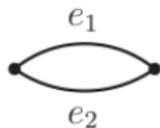
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Trivial example: consider the graph



It satisfies $\Psi_G = \alpha_1 + \alpha_2$.

We can compute the integral on the affine chart $\alpha_2 = 1$ in \mathbb{P}^1 , where it reduces to

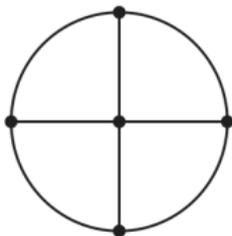
$$I_G = \int_0^\infty \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1$$

The Zoo

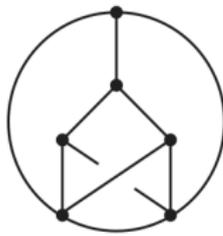
Some selected examples of primitive, log-divergent graphs in ϕ^4 theory, at 3, 4, 5 and 6 loops, and their amplitudes:



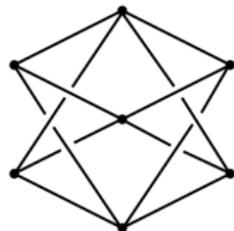
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The number $N_{3,5}$ is given by

$$N_{3,5} = \frac{27}{5}\zeta(5, 3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$$

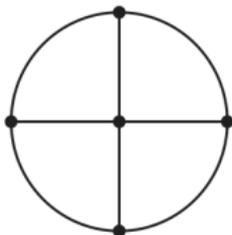
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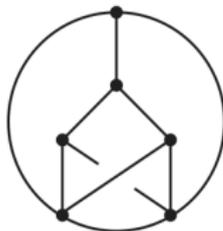
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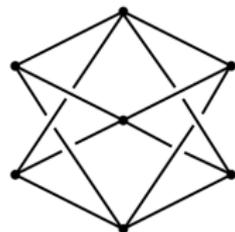
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Recap on Multiple Zeta Values

Let n_1, \dots, n_{r-1} be integers ≥ 1 , and let $n_r \geq 2$. Euler defined

$$\zeta(n_1, \dots, n_r) = \sum_{1 \leq k_1 < k_2 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \in \mathbb{R}$$

When $r = 1$, these are values of the Riemann zeta function $\zeta(n)$.

The *weight* is the quantity $n_1 + \dots + n_r$.

We saw an explicit formula expressing them as period integrals. They are obtained by integrating a sequence of the forms

$$\frac{dz}{z} \quad \text{and} \quad \frac{dz}{1-z}$$

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These are examples of *iterated integrals* on $\mathbb{C} \setminus \{0, 1\}$.

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Metempsychosis

The amplitude for the wheel with 3 spokes is $I_G = 6\zeta(3)$.

This same $\zeta(3)$ shows up in numerous 'real' quantum field theories, including QED and QCD, and also string perturbation theory and $N = 4$ Super-Yang Mills theory.

It can be viewed as an identity between periods in the ring P :

$$\int_{\alpha_i \geq 0} \frac{\Omega_6}{\underbrace{(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \dots)}_{16 \text{ terms}}}^2 = 6 \int_{0 \leq t_1 \leq t_2 \leq t_3 \leq 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3}$$

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Main folklore conjecture

In the 90's, Broadhurst and Kreimer made very extensive computations of I_G , and found that for all graphs G for which I_G can be computed (e.g. $h_G \leq 6$), it is *numerically* an MZV.

Folklore conjecture

The numbers I_G are \mathbb{Q} -linear combinations of multiple zeta values.

Inspired by analogies with 2-dimensional Quantum Field Theories, and deformation quantization.

The conjecture is true for infinitely many graphs.

Theorem (B. 2009)

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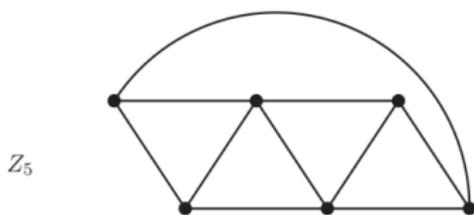
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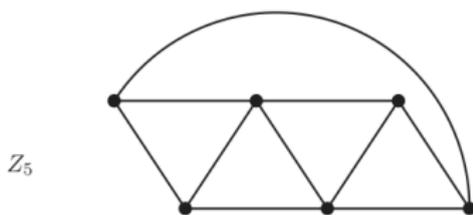
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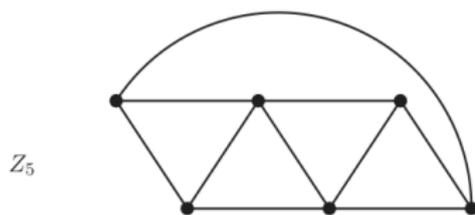
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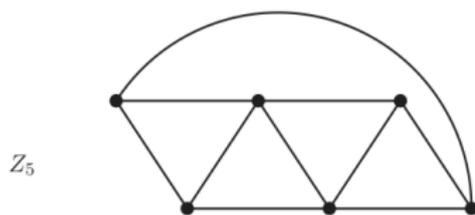
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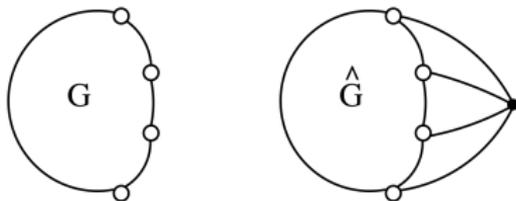
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Some Identities

- 1 (Partial multiplication law). When G_1 and G_2 each have two trivalent vertices connected by an edge, we can form the two-vertex join $G_1 : G_2$. Then $I_{G_1 : G_2} = I_{G_1} I_{G_2}$.



- 2 (Completion) Every non-trivial primitive log-divergent graph G in ϕ^4 theory has exactly 4 trivalent vertices. Let \hat{G} be the graph obtained by connecting them to a single new vertex.



If $\hat{G}_1 \cong \hat{G}_2$ then $I_{G_1} = I_{G_2}$.

Digression: graph Laplacians

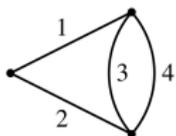
The *Laplacian matrix* of G is the symmetric $V_G \times V_G$ matrix $M_{i,j}$

$$M_{i,j} = \begin{cases} \sum_e x_e & \text{if } i=j, \text{ sum over edges meeting } i \\ -\sum_e x_e & \text{if } i \neq j, \text{ sum over } e \text{ joining } i \text{ and } j \end{cases}$$

The matrix M has rank $V_G - 1$. If we delete any row and column, the determinant of the resulting minor is

$$\pm \Psi_G(x_e^{-1}) \prod_{e \in E_G} x_e$$

Example: Let G be



$$M = \begin{pmatrix} x_1 + x_2 & -x_1 & -x_2 \\ -x_1 & x_1 + x_3 + x_4 & -x_3 - x_4 \\ -x_2 & -x_3 - x_4 & x_2 + x_3 + x_4 \end{pmatrix}$$

$\Psi_G = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_3\alpha_4$, where $\alpha_i = x_i^{-1}$.

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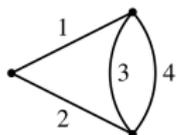
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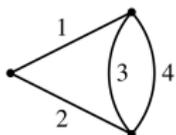
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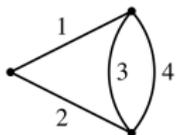
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The graph polynomials satisfy contraction-deletion identities.

If e is an (internal) edge of G then

$$\Psi_G = \alpha_e \Psi_{G \setminus e} + \Psi_{G // e}$$

where $G \setminus e$ is the graph obtained by removing e , and $G // e$ the graph obtained by contracting e and identifying its endpoints. The contraction of a self-edge is defined to be zero.

This can be used (inefficiently) to compute the graph polynomial in terms of smaller graphs.

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Factorization property of graph polynomials

The following property is very important. Let $\gamma \subset G$ be a subgraph. Let G/γ be the graph obtained by contracting γ .

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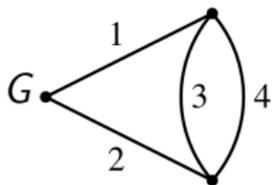
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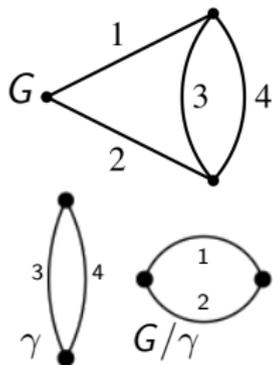
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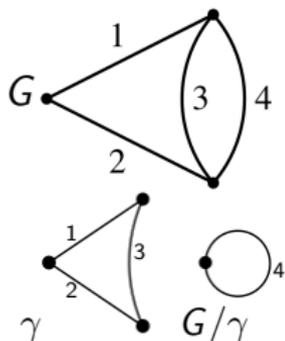
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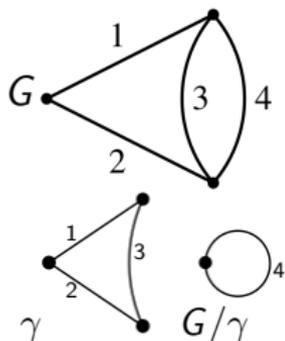
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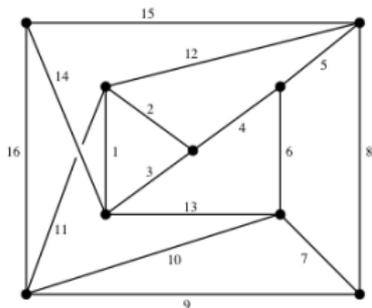
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This property uniquely determines the graph polynomial Ψ_G and is fundamental to the theory of renormalisation.

Modular counter-example

Let G_8 be the following graph with 8 loops. It is primitive log-divergent, in ϕ^4 theory (and has $vw(G) = 4$).

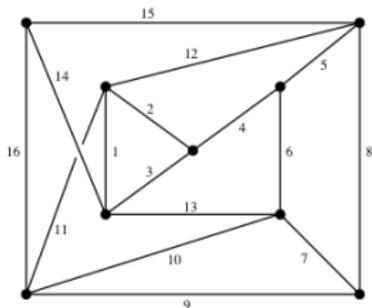


In joint work with O. Schnetz (2012), and D. Doryn (2014) we showed that the Feynman integral of this graph is ‘mixed modular’ and is not the period of a mixed Tate motive (recall that MZV’s are of the latter type).

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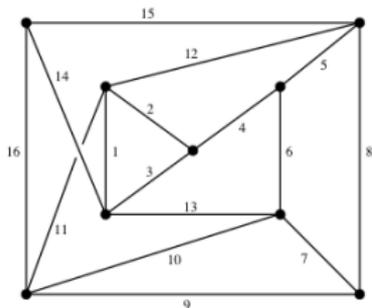


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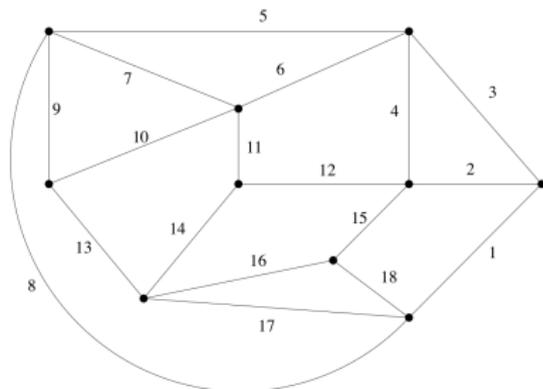
A planar counter-example

In particular, we showed that the number of solutions to

$$\Psi_G(\alpha_1, \dots, \alpha_{N_G}) = 0$$

where α_i are in the finite field \mathbb{F}_q with q elements, is given in terms of the Fourier coefficients of the modular form $(\eta(z)\eta(z^7))^3$ of weight 3 and level 7. Here $\eta(z) = z^{1/24} \prod_{n \geq 1} (1 - z^n)$ is equal to $\Delta(z)^{1/24}$ that made an appearance in the depth filtration for MZV's in the previous lecture.

There is even a planar graph with the same property:



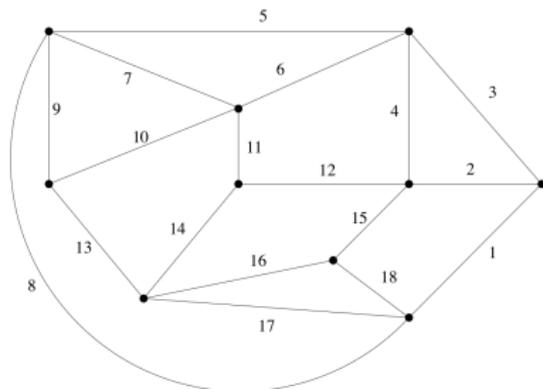
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Conclusion

One of the simplest natural families of periods in mathematics are the multiple zeta values. These are obtained by iteratively integrating the two differential forms

$$\frac{dz}{z} \quad \text{and} \quad \frac{dz}{1-z}$$

over a single space: $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We saw that they arise in many different mathematical contexts.

On the other hand, the simplest family of periods in quantum field theory are the amplitudes I_G . The folklore conjecture, based on a vast amount of evidence stated that

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All versions of the folklore conjecture are true for small graphs but completely false in general. This suggests that there is no simple answer to describe amplitudes at all loop orders: the situation is much more complex, and interesting, than anyone imagined.

This begs a mathematical question: what is the next class of periods we should study after iterated integrals on

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