Motivic Periods and $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

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Zeta values and Euler's theorem

Recall the Riemann zeta values

$$\zeta(n) = \sum_{k \geq 1} \frac{1}{k^n} \qquad \text{for } n \geq 2$$

Euler proved that $\zeta(2) = \frac{\pi^2}{6}$ and more generally

$$\zeta(2n) = -\frac{B_{2n}}{2} \frac{(2\pi i)^{2n}}{(2n)!} \qquad \text{for } n \geq 1$$

where B_m is the m^{th} Bernoulli number.

Folklore conjecture

The odd Riemann zeta values $\zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}[\pi]$.

Few known results: Lindemann, Apéry, Rivoal, Ball-Rivoal, Zudilin.

Multiple Zeta values

Whilst searching for quadratic relations between zeta values, Euler computed a product of two zeta values

$$\sum_{k\geq 1}\frac{1}{k^m}\sum_{\ell\geq 1}\frac{1}{\ell^n}=\Big(\sum_{k<\ell}+\sum_{\ell<k}+\sum_{k=\ell}\Big)\frac{1}{k^m\ell^n}$$

and was led to introduce *multiple zeta values* (MZV's):

$$\zeta(n_1,\dots,n_r)=\sum_{1\leq k_1<\dots<k_r}\frac{1}{k_1^{n_1}\dots k_r^{n_r}}$$

where $n_1,\dots,n_r\geq 1$ and $n_r\geq 2$ to ensure convergence. Its *weight* is the quantity $n_1+\dots+n_r$. The above equation can be written

$$\zeta(m)\zeta(n)=\zeta(m,n)+\zeta(n,m)+\zeta(m+n)$$

and is the first example of a huge family of algebraic relations.

Transcendence conjecture

Sudden re-appearance in mathematics and independently in physics around the 1990's: Ecalle, Hoffman, Zagier, Broadhurst-Kreimer

- *Amplitudes in quantum field theory, superstring theory, moduli spaces of curves, periods of mixed Tate motives,* $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\})$*, Grothendieck-Teichmuller theory, resurgence, conformal field theory, quantum groups, deformation quantization, knot invariants ...*

Conjecture (Zagier)

Let $\mathcal{Z}_n$ denote the $\mathbb{Q}$-vector space of MZV's of weight n. Then

$$\sum_{n \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_n t^n = \frac{1}{1 - t^2 - t^3}$$

The weight is a grading. If $\mathcal{Z} = \sum \mathcal{Z}_n$ then

$$\mathcal{Z} = \bigoplus_{n \geq 0} \mathcal{Z}_n$$

Periods: elementary definition

A period is a type of complex number. Its real and imaginary parts are integrals of rational differential forms, over domains defined by polynomial inequalities (all with rational coefficients). Kontsevich, Zagier

Examples

$$\log(2) = \int_{1\leq z\leq 2} \frac{dz}{z} \quad , \quad \zeta(2) = \int_{0\leq t_1\leq t_2\leq 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

Multiple zeta values are periods (later).

General philosophy of motives (Grothendieck) suggests that there should be a Galois theory of periods: a motivic Galois group (pro-algebraic group) which acts on the space of periods.

Algebraic numbers are periods - contains usual Galois theory.

Periods and cohomology

Classical periods: Let X be a smooth algebraic variety over $\mathbb{Q}$. To it we associate its Betti or singular cohomology

$$H^n_B(X) := H^n(X(\mathbb{C}); \mathbb{Q})$$

It also has algebraic de Rham cohomology (Grothendieck)

$$H^n_{dR}(X) := \mathbb{H}^n(X; \Omega^\bullet_{X/\mathbb{Q}})$$

Both are vector spaces over $\mathbb{Q}$. There is a comparison isomorphism

$$\mathrm{comp}_{B,dR} : H^n_{dR}(X) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X) \otimes \mathbb{C}$$

The comparison map boils down to integration of differential forms over singular chains since $H^n_B(X) = H_n(X(\mathbb{C}); \mathbb{Q})^\vee$.

Generalisation: use relative cohomology.

Framework for motivic periods

Let $\mathcal{T}$ be a Tannakian category over $\mathbb{Q}$, with two fiber functors

$$\omega_B, \omega_{dR} : \mathcal{T} \longrightarrow \mathrm{Vec}_{\mathbb{Q}}$$

and a functorial isomorphism $\omega_{dR}(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_B(M) \otimes \mathbb{C}$. $\mathcal{T}$ is equivalent to the category of representations of an affine group scheme G^{dR} defined over $\mathbb{Q}$.

1. There is a ring of motivic periods $P^{\mathfrak{m}}_{\mathcal{T}}$ generated by symbols

$$[M, \omega, \sigma]^{\mathfrak{m}} \qquad M \in \mathcal{T}\ ,\ \omega \in \omega_{dR}(M)\ ,\ \sigma \in \omega_B(M)^{\vee}$$

   modulo a certain equivalence relation.
2. The Galois group acts on motivic periods:

$$G^{dR} \times P^{\mathfrak{m}}_{\mathcal{T}} \longrightarrow P^{\mathfrak{m}}_{\mathcal{T}}\ .$$

3. There is a period homomorphism to numbers:

$$\mathrm{per}\ : P^{\mathfrak{m}}_{\mathcal{T}} \longrightarrow \mathbb{C}$$

What to take for $\mathcal{T}$?

Would like to take $\mathcal{T}$ a category of 'mixed motives'. Not currently available. We have at least three options:

- Category of mixed Tate motives over a number field.[1] Let $\mathcal{MT}(\mathbb{Z})$ category of unramified mixed Tate motives/$\mathbb{Q}$.
- Nori's Tannakian category of mixed motives $\mathcal{MM}$.
- A category of realisations H. Objects are pairs:

$$(M_B, M_{dR}) \qquad \text{where } M_B, M_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$$

  with an isomorphism $M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} M_B \otimes \mathbb{C}$, and various filtrations so that M_B is a $\mathbb{Q}$ mixed Hodge structure.

We have a homorphism of motivic periods

$$P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})} \longrightarrow P^{\mathfrak{m}}_{H}$$

The main theorem actually takes place in $P^{\mathfrak{m}}_{H}$!

[1] Deligne-Goncharov, Levine, Voevodsky, Hanamura, Bloch, Beilinson, Borel, ...

Example: motivic version of $2\pi i$

Let $X = \mathbb{P}^1 \backslash \{0, \infty\}$, $X(\mathbb{C}) = \mathbb{C}^\times$. Let γ be a loop around 0.

$$H^1_B(X) \cong \mathbb{Q}[\gamma]^\vee \qquad \text{and} \qquad H^1_{dR}(X) \cong \mathbb{Q}[\tfrac{dz}{z}]$$

are $\mathbb{Q}$-vector spaces of dimension 1. The Tannaka group G^{dR} acts by linear automorphisms of H^1_{dR}, i.e., multiplication by a scalar. Let

$$\mathbb{L}^{\mathfrak{m}} = [H^1(X), [\tfrac{dz}{z}], [\gamma]]^{\mathfrak{m}}$$

It is the motivic version of $2\pi i$, its period is

$$\mathrm{per}\,(\mathbb{L}^{\mathfrak{m}}) = \int_\gamma \frac{dz}{z} = 2\pi i \ .$$

Then $G^{dR}(\mathbb{Q})$ acts via the multiplicative group $\mathbb{Q}^\times$ and transforms

$$\mathbb{L}^{\mathfrak{m}} \mapsto \lambda\, \mathbb{L}^{\mathfrak{m}} \qquad \lambda \in \mathbb{Q}^\times$$

We should think of $2\pi i$ not as a number, but as a function!

More examples

- Let p prime. Then $\log^{\mathfrak{m}}(p)$ transforms like

$$\log^{\mathfrak{m}}(p) \;\mapsto\; \lambda \log^{\mathfrak{m}}(p) + \nu_p$$

  where $\lambda \in \mathbb{Q}^{\times}$ the same λ as before and $\nu_p \in \mathbb{Q}$. This is an avatar of the fact that $\log(z)$ is a multivalued function.

- The *odd zeta values* transform like

$$\zeta^{\mathfrak{m}}(2n+1) \;\mapsto\; \lambda^{2n+1}\zeta^{\mathfrak{m}}(2n+1) + \mu_{2n+1}$$

  where λ as above, and $\mu_{2n+1} \in \mathbb{Q}$.

- The *even zeta values* transform trivially

$$\zeta^{\mathfrak{m}}(2n) \;\mapsto\; \lambda^{2n}\zeta^{\mathfrak{m}}(2n)$$

  by a version of Euler's theorem : $\zeta^{\mathfrak{m}}(2n) \in \mathbb{Q}^{\times}(\mathbb{L}^{\mathfrak{m}})^{2n}$.

The power of λ defines a weight grading. The motivic Galois group can move each $\zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \ldots$ *independently*.

Bounds on dimensions

It follows from deep theorems due to Borel that $\mathrm{Lie}^{gr}\, G^{dR}_{\mathcal{MT}(\mathbb{Z})}$ is isomorphic to the free graded Lie algebra on generators

$$\sigma_3, \sigma_5, \sigma_7, \ldots$$

of degree $-3, -5, -7, \ldots$. They act by $\sigma_{2n+1}\zeta^{\mathfrak{m}}(2m+1) = \delta_{m,n}$. One can define motivic multiple zeta values

$$\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \quad \in \quad \mathcal{P} \subset P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}$$

in the subspace $\mathcal{P}$ of real, geometric motivic periods of $\mathcal{MT}(\mathbb{Z})$. An easy counting argument gives the enumeration

$$\text{If } d_n = \dim_{\mathbb{Q}} \mathcal{P}_n \qquad \text{then} \qquad \sum_{n \geq 1} d_n t^n = \frac{1}{1 - t^2 - t^3}$$

The period map $\mathrm{per} : \mathcal{P}_n \to \mathcal{Z}_n$ is a surjection. We get:

Theorem (Terasoma, Goncharov) : $\dim_{\mathbb{Q}} \mathcal{Z}_n \leq d_n$

Statement of the main theorem for MZV's

Main Theorem (2011)

The following motivic multiple zeta values are linearly independent:

$$(*) \qquad \zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \quad \text{where} \quad n_i \in \{2, 3\}$$

We deduce that $(*)$ is a *basis* for $\mathcal{P}$.

Corollary (Goncharov conjecture)

The periods of every mixed Tate motive over $\mathbb{Z}$ are in $\mathcal{Z}[(2i\pi)^{-1}]$.

Corollary (Hoffman conjecture)

Every MZV is a linear comb. of $\zeta(n_1, \ldots, n_r)$, $n_i \in \{2, 3\}$.

Corollary: The Deligne-Ihara conjecture is true

Also deduce existence of canonical generators σ_{2n+1}; existence of canonical rational associators; lower bounds for $\mathfrak{grt}$; ...

Grothendieck's period conjecture

Version of Grothendieck's period conjecture for $\mathcal{MT}(\mathbb{Z})$: Y. André

Conjecture

The homomorphism $\mathrm{per} : P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})} \longrightarrow \mathbb{C}$ is injective.

It says that there is a relation between MZV's if and only if the corresponding relation holds for motivic MZV's. It implies Zagier's transcendence conjecture for MZV's, and the folklore conjecture for zeta values.

It suggests that the action of $G^{dR}_{\mathcal{MT}(\mathbb{Z})}$ on motivic multiple zeta values should also give a well-defined action on actual MZV's.

A philosophy for motivic periods

The main theorem is a statement about *independence, or transcendence, of motivic periods*. Recall there is a map

$$P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})} \longrightarrow P^{\mathfrak{m}}_{H}$$

If the images of elements in $P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}$ are independent in $P^{\mathfrak{m}}_{H}$, then *a fortiori* they are independent in $P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}$. So we can drop the motives and work in the elementary category H.

- Write down interesting arithmetic examples of periods
- Compute the Galois action on their motivic versions in $P^{\mathfrak{m}}_{H}$
- Try to prove independence theorems.

Other approaches to periods emphasise the *relations* between periods. This is the opposite point of view.

Part II: The projective line minus 3 points

Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, and $x \in X(\mathbb{Q})$. The profinite completion of $\pi_1(X(\mathbb{C}), x)$ admits an action by $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$.

Theorem (Belyi 1979):

$$\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\widehat{\pi}_1(X, x)) \qquad \text{is injective.}$$

This theorem initiated Grothendieck's 'Esquisse d'un programme'.

Can replace profinite completion with prounipotent completion (Deligne, Drinfeld, Ihara $\sim$ 1990). I want to explain

Theorem (Deligne-Ihara conjecture)

$$G^{dR}_{\mathcal{MT}(\mathbb{Z})} \longrightarrow \mathrm{Aut}(\pi_1^{\mathrm{dR}}(X, \overrightarrow{1}_0)) \qquad \text{is injective.}$$

Difference: Belyi's theorem is *geometric*, this is *arithmetic*.

Fundamental groupoid of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

Let $x, y \in X(\mathbb{Q})$. Realisations of $\pi_1^{\mathrm{mot}}(X, x, y)$ are:

- A scheme $\pi_1^B(X, x, y)$ over $\mathbb{Q}$ equipped with morphisms

$$\pi_1^B(X, x, y) \times \pi_1^B(X, y, z) \longrightarrow \pi_1^B(X, x, z)$$

  The topological π_1 maps to its $\mathbb{Q}$-points:

$$i^B : \pi_1(X(\mathbb{C}), x, y) \longrightarrow \pi_1^B(X, x, y)(\mathbb{Q}) \ .$$

- An affine group scheme $\pi_1^{dR}(X, x, y)$ over $\mathbb{Q}$.
- A comparison isomorphism (of schemes)

$$\pi_1^B(X, x, y) \times \mathbb{C} \xrightarrow{\sim} \pi_1^{dR}(X, x, y) \times \mathbb{C}$$

Deligne-Goncharov: if $x = \vec{1}_0$, $y = -\vec{1}_1$ unit tangent vectors at 0 and 1, then the motivic fundamental groupoid is in $\mathcal{MT}(\mathbb{Z})$.

Periods of $\pi_1^{\mathrm{mot}}(X, \vec{1}_0, -\vec{1}_1)$

1. Domain of integration: take the straight path γ from 0 to 1.
2. Differential forms: take any element in

$$\mathcal{O}(\pi_1^{dR}) = T(H^1_{dR}(X)) \cong \bigoplus_{n \geq 1} \left(\mathbb{Q}\omega_0 \oplus \mathbb{Q}\omega_1\right)^{\otimes n}$$

where $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{1-z}$.

3. The comparison map is given by the iterated integral:

$$\begin{aligned} \int_\gamma \underbrace{\omega_1\omega_0^{n_1-1}\ldots\omega_1\omega_0^{n_r-1}}_{\omega_{n_1,\ldots,n_r}} \quad &:= \quad (-1)^r \int_{0\leq t_1\leq\cdots\leq t_n\leq 1} \frac{dt_1}{t_1-\epsilon_1}\cdots\frac{dt_n}{t_n-\epsilon_n} \\ &= \quad \zeta(n_1,\ldots,n_r) \end{aligned}$$

where $(\epsilon_1,\ldots,\epsilon_n) = 10^{n_1-1}\ldots 10^{n_r-1}$.

Motivic multiple zeta values are thus motivic periods:

$$\zeta^{\mathfrak{m}}(n_1,\ldots,n_r) = [\mathcal{O}(\pi_1^{\mathrm{mot}}), \omega_{n_1,\ldots,n_r}, i^B\gamma]^{\mathfrak{m}} \quad \in \quad P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}$$

How to compute the Galois action?

In fact, we can't in general. One can only show that the Galois action factorizes through a certain group which respects the symmetries of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ (Ihara, Goncharov,...). Astonishingly, this actually gives some useful information.

Upshot: there is a coaction

$$\Delta : P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})} \to \left(P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}/\zeta^{\mathfrak{m}}(2)\right) \otimes P^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z})}$$

which can be computed on motivic MZV's. Example:

$$\begin{aligned}\Delta\zeta^{\mathfrak{m}}(4,3,3) = \zeta^{\mathfrak{u}}(3) \otimes \zeta^{\mathfrak{m}}(4,3) + 10\zeta^{\mathfrak{u}}(5) \otimes \zeta^{\mathfrak{m}}(2,3) \\ + \zeta^{\mathfrak{u}}(3,3) \otimes \zeta^{\mathfrak{m}}(4) - (2\zeta^{\mathfrak{u}}(4,3) + 4\zeta^{\mathfrak{u}}(3,4)) \otimes \zeta^{\mathfrak{m}}(3) \\ + (6\zeta^{\mathfrak{u}}(4,4) + 2\zeta^{\mathfrak{u}}(3,5) + 6\zeta^{\mathfrak{u}}(5,3)) \otimes \zeta^{\mathfrak{m}}(2)\end{aligned}$$

where $\zeta^{\mathfrak{u}}$ denotes $\zeta^{\mathfrak{m}}$ taken modulo $\zeta^{\mathfrak{m}}(2)$. It is *very* complicated.

Note! The motivic MZV's satisfy very complex relations which are not fully understood. The coaction respects these relations.

Proof of Deligne-Ihara conjecture

We have to wrestle with this coaction formula. It is compounded by the fact that there are 2^N multiple zeta values in weight N, but they span a vector space of dimension $\sim (\frac{4}{3})^N$. There are a huge number of relations, which are not understood.

A key input: a formula due to Zagier for

$$\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$$

It is proved by a clever use of generating function methods, and *analytic methods*. Subsequent work by Terasoma and Li.

The idea is to prove independence of $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$ where $n_i \in \{2, 3\}$ by induction on the weight and number of 3's, by using the coaction formula.

Generalization: Roots of unity

Where to go from here? Deligne has previously studied

$$X = \mathbb{P}^1 \backslash \{0, \mu_N, \infty\}$$

where μ_N denotes N^{th} roots of unity (motivation from Broadhurst's calculations in quantum field theory).

- In exceptional cases $N = 2, 3, 4, 6, 8$, Deligne proved analogous results (faithfulness of Galois action on motivic fundamental group). It uses the *depth filtration*. (The depth filtration is pathological for $N = 1$ and related to modular forms).
- If N prime ≥ 11, Goncharov showed that $\pi_1^{\mathrm{mot}}(X)$ *cannot* generate the corresponding category of mixed Tate motives. There are *missing periods* already in weight 2.

I want to suggest a completely different approach.

A programme

1. Replace $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ with $\Gamma \backslash\backslash \mathfrak{H}$ where $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is any subgroup of finite index and $\mathfrak{H}$ is the upper-half plane.
2. Instead of unipotent completion, we must now take *relative* unipotent completion, with respect to $\Gamma \to \mathrm{SL}_2(\mathbb{Q})$. It defines a mixed Hodge structure (Hain).
3. Examples of periods are iterated integrals of modular forms on $\mathfrak{H}$ between cusps (Eichler, Shimura, Manin). Building blocks are values of L-functions of modular forms.
4. By (2), we can define the corresponding motivic periods in $P_H^{\mathfrak{m}}$. There is a general formula for the Galois (co)action.

We can apply the whole philosophy of motivic periods in this set up. It gives information about a huge conjectural Tannakian category of mixed modular motives (containing the motives of all algebraic curves by Belyi's theorem).

Towards modularity of mixed Tate motives?

- Main theorem for $\mathbb{P}^1\backslash\{0,1,\infty\}$ was *arithmetic*. There is no clue from the geometry as to the structure of the motivic Galois group. Equivalently, there are too many periods on $\mathbb{P}^1\backslash\{0,1,\infty\}$, with many complicated relations. We had to work around this with tricky combinatorics.
- On the other hand, for $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathcal{H}$, we have exactly one Eisenstein series for every generator σ_{2n+1} of $\mathrm{Lie}\, G^{dR}_{\mathcal{MT}(\mathbb{Z})}$.

$$E_4, E_6, E_8, \ldots \qquad \leftrightarrow \qquad \sigma_3, \sigma_5, \sigma_7, \ldots$$

This suggests there is a *geometric* way to obtain similar results but using modular forms. The price to pay is that we must wander outside the category of mixed Tate motives.

Can we construct (motivic periods of), e.g., $\mathcal{MT}(\mathbb{Q})$ from congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$?