Coaction structure for Feynman amplitudes and a small graphs principle

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New geometric structures in scattering amplitudes,
Oxford University, 23rd September 2014

Overview and goals

The main goals:

1. Formulate O. Schnetz' coaction conjecture for scalar massless amplitudes. Explain its remarkable predictive power for high-loop amplitudes.
2. Define motivic amplitudes. This a vast generalisation of the notion of 'symbol', but contains more information.
3. Prove a version of the coaction conjecture. The small graphs principle allows one to deduce *all-order results in perturbation theory* from a finite computation.

Point (3) states that there is a hidden recursive structure in the amplitudes of quantum field theories: information about low-loop amplitudes propagates to all higher loop orders.

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A simple analogy

An analogy is Erastosthenes' sieve. Suppose that we have a set S of natural numbers with the following property:

- If $n \in S$, and m is a divisor of n, then $m \in S$.

Write the natural numbers in a table:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Now suppose that we have some low-order information:

- $2 \notin S$. Cross off all multiples of 2
- $3 \notin S$. Cross off all multiples of 3

The fact that S has few low-order elements means that S is full of holes at all orders.

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What happens for amplitudes?

Let P be the vector space of amplitudes of, e.g. massless ϕ^4. The coaction conjecture predicts the following property for amplitudes.

- If $\xi \in P$, and ξ' is a *Galois conjugate* of ξ, then $\xi' \in P$.

At low loop orders, the amplitudes are multiple zeta values. Write a basis for multiple zeta values in a table.

1	$\zeta(2)$	$\zeta(3)$	$\zeta(2)^2$	$\zeta(5)$	$\zeta(3)^2$	$\zeta(7)$	$\zeta(3,5)$
				$\zeta(3)\zeta(2)$	$\zeta(2)^3$	$\zeta(5)\zeta(2)$	$\zeta(3)^2\zeta(2)$
						$\zeta(3)\zeta(2)^2$	$\vdots$

Now look at amplitudes of small graphs (with ≤ 4 loops). There are very few of them. We see that:

- $\zeta(2) \notin P$. Cross off all linear terms in $\zeta(2)$
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A finite calculation leads to constraints at *all higher loop orders*.

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Amplitudes in parametric form

General form of Feynman amplitude:

$$\frac{I_G(q,m)}{\Gamma(N_G - h_G d/2)} = \int_{[0,\infty]^{N_G}} \frac{\Psi_G^{N_G-(h_G+1)d/2}}{(\Psi_G \sum_e m_e^2 \alpha_e - \Phi_G(q))^{N_G - h_G d/2}} \,\delta(\sum_e \alpha_e - 1)$$

for a graph G with N_G edges, h_G loops in $d \in 2\mathbb{Z}$ space-time dimensions, internal masses m_e. Symanzik polynomials:

$$\Psi_G = \sum_{T\subset G} \prod_{e\notin E_T} \alpha_e$$

$$\Phi_G = \sum_{T_1\cup T_2} \prod_{e\notin T_1\cup T_2} \alpha_e (q^{T_1})^2$$

where the first sum is over spanning trees of G, the second over spanning 2-trees, and q^{T_1} is momentum flow through T_1.

Almost everything that follows is valid for such integrals. I will focus on the *massless, single-scale* case.

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Massless single-scale amplitudes

Suppose $d = 4$. Assume

- G is *overall log-divergent*: $N_G = 2h_G$
- G is *primitive*: $N_\gamma > 2h_\gamma$ for all $\gamma \subsetneq G$.

The Feynman amplitude reduces to the convergent integral

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2} \in \mathbb{R}$$

It is the coefficient of ε^{-1} in dim. reg. Here

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots d\alpha_{N_G}$$

and the domain of integration σ is the real coordinate simplex

$$\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$

If subdivergences: either *renormalize under integral* (B. - Kreimer) or work in dim. reg. (Panzer) to get convergent period integrals.

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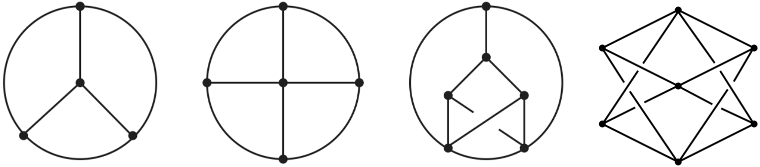
Examples in massless ϕ^4

Examples of primitive, log-divergent graphs in ϕ^4 theory, at $3, 4, 5$ and 6 loops, and their amplitudes (Broadhurst-Kreimer):



I_G : $6\zeta(3)$ $20\zeta(5)$ $36\zeta(3)^2$ $N_{3,5}$

where $N_{3,5} = \frac{27}{5}\zeta(5,3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$. Multiple Zeta Values are defined for integers $n_1, \ldots, n_{r-1} \geq 1$, and $n_r \geq 2$ by

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \quad \in \mathbb{R}$$

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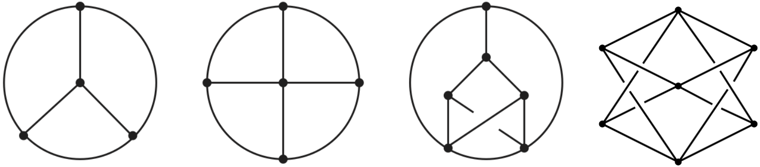

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Known results

1. *Calculus* of weights. Combinatorial criteria for graphs to have maximal weight or weight-drop (B.-K. Yeats, B.-Doryn).
2. *Sufficient* combinatorial conditions for graphs to be multiple zeta values (B.).
3. *Modular counter-examples.* There exist graphs with ≥ 8 loops whose amplitudes are periods of a mixed modular motive (expected *not to be an MZV*) (B.-Schnetz).
4. *Polylogarithms at roots of unity.* Amplitudes at ≥ 7 loops which are analogues of MZV's but with $2nd$ or $6th$ roots of unity in numerator (Panzer and Schnetz).
5. *Effective algorithms* for the symbolic computation of amplitudes at high loop orders (Panzer, Bogner-B. for linearly-reducible graphs; Schnetz, using graphical functions).
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Motivic multiple zeta values

Algebra of *motivic multiple zeta values* $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

It is equipped with a *period homomorphism*

$$\mathrm{per} : \mathcal{H} \longrightarrow \mathbb{R}$$

which sends $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$ to $\zeta(n_1, \ldots, n_r)$. We gain an action of a motivic Galois group on $\mathcal{H}$. This is equivalent to a *coaction*

$$\Delta : \mathcal{H} \longrightarrow \mathcal{A} \otimes \mathcal{H}$$

where $\mathcal{A} = \mathcal{H}/\langle \zeta^{\mathfrak{m}}(2) \rangle$. It respects all algebraic relations between motivic MZV's, and is effectively computable (Goncharov, B.).

Motivic multiple zeta values

Algebra of *motivic multiple zeta values* $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

It is equipped with a *period homomorphism*

$$\mathrm{per} : \mathcal{H} \longrightarrow \mathbb{R}$$

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Structure of motivic multiple zeta values

We have a model for $\mathcal{H}$. Let

$$\mathcal{U}' = \mathbb{Q}\langle f_3, f_5, f_5, \ldots \rangle$$

denote the graded $\mathbb{Q}$-vector space spanned by words in f_{2i+1}, where f_{2i+1} has degree $2i+1$, with shuffle product. Set

$$\mathcal{U} = \mathcal{U}' \otimes \mathbb{Q}[f_2]$$

where f_2 has degree 2, and commutes with all f_{2i+1}. Coaction

$$\begin{aligned} \Delta : \mathcal{U} &\longrightarrow \mathcal{U}' \otimes \mathcal{U} \\ f_{i_1} \ldots f_{i_m} f_2^r &\mapsto \sum_{k=0}^{m} f_{i_1} \ldots f_{i_k} \otimes f_{i_{k+1}} \ldots f_{i_m} f_2^r \end{aligned}$$

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There is an isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{U}$, and $\mathcal{A} \xrightarrow{\sim} \mathcal{U}'$ of algebra comodules, i.e., respecting the coactions.

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The theorem says that to every motivic MZV, we can uniquely associate a linear combination of words in f_{2i+1}, f_2:

$$\begin{array}{ccc} \zeta^{\mathfrak{m}}(2n+1) & \leftrightarrow & f_{2n+1} \\ \zeta^{\mathfrak{m}}(2)^r & \leftrightarrow & f_2^r \end{array}$$

By shuffle product:

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) \leftrightarrow f_3 f_5 + f_5 f_3$$

A more complicated example:

$$\zeta^{\mathfrak{m}}(3,5) \quad \leftrightarrow -5 f_3 f_5 + \tfrac{1586}{4725} f_2^4$$

The (de Rham) *Galois conjugates* of a motivic MZV $\xi \in \mathcal{H}$ are elements of the comodule generated by ξ under Δ. They spanned by right factors of the corresponding elements in $\mathcal{U}$. Examples:

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Coaction conjecture

O. Schnetz' coaction conjecture states that the amplitudes I_G in ϕ^4 theory are closed under the coaction.

- Tested by Schnetz for ~ 250 amplitudes up to 11 loops.
- Recent work of Panzer and Schnetz gave first explicit computation of amplitudes in ϕ^4 which are not MZV's but polylogarithms at $2nd$ and $6th$ roots of unity. Deligne proved analogue of the structure theorem for such numbers. The coaction conjecture still holds true for such examples.
- Equivalent formulation: if P_{ϕ^4} is the algebra generated by the (motivic) amplitudes of ϕ^4 theory then it is *stable under the action of the motivic Galois group* G:

$$G \times P_{\phi^4} \longrightarrow P_{\phi^4}$$

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The coaction conjecture in action I

Look at all graphs with $1, 3, 4, 5, 6$ loops. By an earlier theorem, we know they are MZV's. The coaction conjecture unravels much of the structure of the possible amplitudes:

Loops	Weights	Possible MZV's
1	1	1
3	3	f_3
4	5	$f_5 \quad f_3 f_2$
5	7	$f_7 \quad f_5 f_2 \quad f_3 f_2^2$
wd	6	$f_3^2 \quad f_2^3$
6	9	$f_9 \quad f_7 f_2 \quad f_5 f_2^2 \quad f_3 f_2^3 \quad f_3^3$
wd	8	$f_3 f_5 \quad f_5 f_3 \quad f_3^2 f_2 \quad f_2^4$

No amplitudes of weights 2 and 4 $\Rightarrow$ no f_2, f_2^2.
We know which graphs have weight-drops (B.- Yeats) $\Rightarrow$ no f_2^3.

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The coaction conjecture in action II

The coaction conjecture imposes stronger and stronger constraints as we increase the loop order.

At 6 loops and weight 8, one expects to see $f_3 f_5, f_5 f_3$ and f_2^4 but because there are few graphs, only these combinations occur:

$$f_3 f_5 + f_5 f_3 \quad , \quad f_3 f_5 + \alpha f_2^4$$

At 7 loops: we expect a vector space of MZV's of dimension 9. In reality, we only have a vector space of dimension 4 of amplitudes. The terms $f_3 f_3 f_5$, $f_3 f_5 f_3$, $f_3 f_2^4$ must occur in the linear combination

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There are many more striking examples. At each loop order, there are new constraints ('holes' in the set of amplitudes) which in turn propagate to all higher loop orders.

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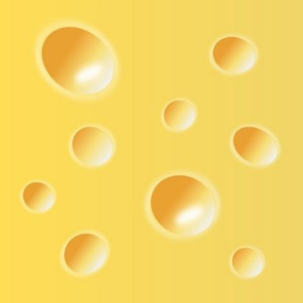
Swiss cheese

The amplitudes P_{ϕ^4} are stable under a group G_{ϕ^4} (coaction conjecture). But P_{ϕ^4} is full of holes (there are few small graphs).



Each hole engenders infinitely many more holes. [1]

[1] Of course, it is better to speak about which numbers actually *occur* rather than don't occur. There is a precise, but technical mathematical formulation to express this. For the exposition, I will keep talking about holes.

Part II: Plan

The previous picture is a conjectural prototype for the general structure of any perturbative quantum field theory. In order to turn it into a theory, we must modify the problem slightly. We must:

1. *Enlarge* the class of amplitudes considered.
2. *Define* 'motivic' versions of these amplitudes. With the right definition, there is automatically a coaction, and furthermore, the coaction conjecture is true for this class.
3. There is an underlying *operad* structure. It is the same structure which governs the renormalisation group equation.
4. Using the theory of weights in mixed Hodge theory, we reduce the calculation of the Galois conjugates to studying motivic amplitudes of small graphs.
5. Since there are very few small graphs, we get lots of holes.

Motivic periods

Let $\mathcal{T}$ be a Tannakian category over $\mathbb{Q}$ with two fiber functors:

$$\omega_B, \omega_{dR} : \mathcal{T} \longrightarrow \mathrm{Vec}_{\mathbb{Q}}$$

Suppose that there is a canonical isomorphism

$$\mathrm{comp}_{B,dR} : \omega_{dR}(M) \otimes \mathbb{C} \longrightarrow \omega_B(M) \otimes \mathbb{C}$$

for all $M \in \mathcal{T}$. Define the *ring of motivic periods* $P^{\mathfrak{m}}_{\mathcal{T}}$ of $\mathcal{T}$ to be the affine ring $\mathcal{O}(\mathrm{Isom}_{\mathcal{T}}(\omega_{dR}, \omega_B))$. The ring of *de Rham periods* is $P^{dR}_{\mathcal{T}} = \mathcal{O}(\mathrm{Aut}_{\mathcal{T}}(\omega_{dR}))$. There is a period homomorphism

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Motivic amplitudes

Various possibilities for $\mathcal{T}$. The weakest is to take a category of realisations H. Objects are pairs:

$$(M_B, M_{dR}) \qquad \text{where } M_B, M_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$$

with an isomorphism $M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} M_B \otimes \mathbb{C}$, and various filtrations so that M_B is a $\mathbb{Q}$-mixed Hodge structure.

For a Feynman graph G one can associate an object

$$M_G \in H$$

the 'graph mixed Hodge structure', and elements $\omega_G \in \omega_{dR}(M_G)$ and $\sigma \in \omega_B(M)^\vee$. We will obtain a *motivic amplitude*

$$[M, \omega_G, \sigma]^{\mathfrak{m}} \in P_H^{\mathfrak{m}}$$

It is the function $\phi :\mapsto \langle \phi(\omega_G), \sigma \rangle : \mathrm{Isom}(\omega_{dR}, \omega_B)(\mathbb{Q}) \to \mathbb{Q}$.

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Motivic amplitudes: what do we gain?

We gain:

1. A rigorous notion of *weight*. There is weight *filtration* on the ring $P^{\mathfrak{m}}_{H}$. The 'transcendental weight' can be a half-integer.
2. A coaction from the general formalism.
3. The motivic amplitude (in the case when there are external kinematics) knows everything about differential equations, monodromy equations, etc. Recover symbol from coaction.

The graph mixed Hodge structure is known explicitly in the following cases:

1. Subdivergence-free, massless amplitudes in ϕ^4 (Bloch-Esnault-Kreimer)
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3. The motivic amplitude (in the case when there are external kinematics) knows everything about differential equations, monodromy equations, etc. Recover symbol from coaction.

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Coaction Theorem

Let $P^{\mathfrak{m}}_{\phi^4}$ denote the space of the *specific* motivic amplitudes $I^{\mathfrak{m}}_G$ of sub-divergence free graphs in ϕ^4 (as considered above).

Coaction conjecture (Schnetz)

$P^{\mathfrak{m}}_{\phi^4}$ is stable under the coaction, $\Delta : P^{\mathfrak{m}}_{\phi^4} \longrightarrow P^{dR}_{H} \otimes P^{\mathfrak{m}}_{\phi^4}$

Idea: Enlarge the class of amplitudes. Let $P^{\mathfrak{m}}_{\widetilde{\phi}^4}$ denote the space of *all the* motivic amplitudes of the same class of graphs.

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Generalising the amplitudes: ϕ^4 versus $\widetilde{\phi}^4$

The amplitudes we considered in ϕ^4 are of the form

$$I_G = \int_\sigma \omega_G \qquad \text{where} \qquad \omega_G = \frac{\Omega_G}{\Psi_G^2}$$

where Ψ_G is the graph polynomial. They are periods of motivic amplitudes $[M_G, \omega_G, \sigma]^{\mathfrak{m}}$ in $P^{\mathfrak{m}}_{\phi^4}$.

The generalised *motivic amplitudes* we need are of the form

$$[M_G, \omega, \sigma]^{\mathfrak{m}} \in P^{\mathfrak{m}}_{\widetilde{\phi}^4}$$

where $\omega \in \omega_{dR}(M_G)$ is any differential form that can be integrated along σ. This includes convergent integrals of the form

$$I_G = \int_\sigma \omega \qquad \text{where} \qquad \omega = \frac{P(\alpha_e)\Omega_G}{\Psi_G^N}$$

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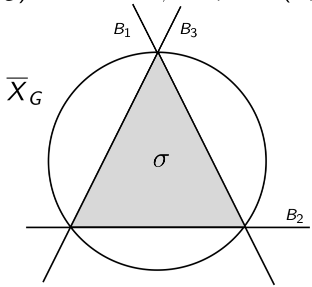
The graph MHS (Bloch-Esnault-Kreimer 2007)

Recall that

$$I_G = \int_\sigma \omega_G \qquad \text{where} \qquad \omega_G = \frac{\Omega_G}{\Psi_G^2}$$

How to interpret this as a period? Consider the graph hypersurface, and coordinate hyperplanes in projective space:

$$\overline{X}_G = V(\Psi_G) \subset \mathbb{P}^{N_G-1} \quad , \quad B_i = V(\alpha_i) \subset \mathbb{P}^{N_G-1}$$



$$\omega_G \in \Omega^{N_G-1}(\mathbb{P}^{N_G-1}\backslash\overline{X}_G) \qquad \text{and} \qquad \partial\sigma \subset B = \cup_i B_i \ .$$

The graph mixed Hodge structure (II)

The naive mixed Hodge structure is

$$H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash\overline{X}_G, B\backslash(B\cap\overline{X_G})$$

However, in reality, the domain of integration σ meets the singular locus $\overline{X}_G$ so we must do some blow-ups. B-E-K construct an explicit local resolution of singularities $\pi : P \to \mathbb{P}^{N_G-1}$ and define

$$M_G = H^{N_G-1}(P\backslash\widetilde{\overline{X}}_G, \widetilde{B}\backslash(\widetilde{B}\cap\widetilde{\overline{X}}_G))$$

Theorem (Bloch-Esnault-Kreimer 2007)

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Small graphs principle

The power of the method comes from two features: the coaction and the fact that there are missing periods (holes).

However, we added new periods to make the coaction conjecture into a theorem. Have we inadvertently filled in all the holes too?

The answer is no. But now we need much stronger results to prove that the holes are still there. We now need to understand the *amplitudes in* $\widetilde{\phi}_4$ up to a given weight.

This requires a detailed knowledge of the mixed Hodge structure. Involves: the theory of weights, relative cohomology spectral sequence, and some geometric properties of graph hypersurfaces

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Factorization property of graph polynomials

Let $\gamma \subset G$ be any subgraph. Let $G//\gamma$ be the quotient graph: it is obtained by contracting γ.

Key factorisation property:

$$\Psi_G = \Psi_\gamma \Psi_{G//\gamma} + R^1_{\gamma,G}$$

$$\Phi_G(q) = \Psi_\gamma \Phi_{G//\gamma}(q) + R^2_{\gamma,G}$$

The polynomials $R^i_{\gamma,G}$ are of higher degree in the γ-variables.

$$\underbrace{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}_{\Psi_G} = \underbrace{(\alpha_1 + \alpha_2)}_{\Psi_\gamma} \underbrace{\alpha_3}_{\Psi_{\Gamma//\gamma}} + \underbrace{\alpha_1\alpha_2}_{R^1_{\gamma,G}}$$

In the limit as the subgraph variables (here α_1, α_2) go to zero, the graph polynomials factorise

$$\Psi_G \sim \Psi_\gamma \Psi_{\Gamma//\gamma}$$

The small graphs principle

Geometrically, each boundary facet is a product of graph hyper-surfaces. Gives an operad structure on the cohomology.

Theorem (Small graphs principle)

The elements in the right-hand side of the coaction $\Delta[M_G, \omega, \sigma]^{\mathfrak{m}}$ can be expressed in the form

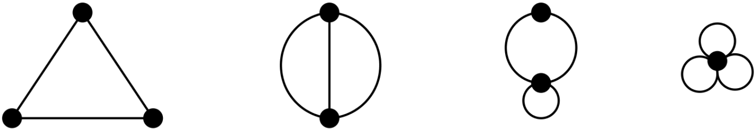
$$\prod_i [M_{\gamma_i}, \omega_i, \sigma]^{\mathfrak{m}}$$

where γ_i are sub and quotient graphs of G.

By general theorems on weights in mixed Hodge structures, the weight $\leq k$ part of the RHS of the coaction come from sub and quotient graphs with approx. $k+1$ edges in total.

Example: logarithms

Any $\log^{\mathfrak{m}}(p)$ occurring in the RHS of the coaction come from graphs with at most 3 edges. Write down all possibilities:



$\alpha_1 + \alpha_2 + \alpha_3$ $\quad$ $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$ $\quad$ $\alpha_1(\alpha_2 + \alpha_3)$ $\quad$ $\alpha_1\alpha_2\alpha_3$

The corresponding mixed Hodge structures are very simple. You can never get $\log(p)$ as an integral with these denominators.

Corollary

There is no $\log^{\mathfrak{m}}(p)$ in the right hand side of the coaction.

From these easy calculations + the theorems we actually deduce highly non-trivial constraints at all loop orders using the coaction.

Some immediate corollaries

Let $G \in \phi^4$ be primitive divergent.

Theorem

Suppose that $I_G^{\mathfrak{m}}$ is a motivic MZV at 2nd roots of unity. Then $\log^{\mathfrak{m}}(2)$ is not a Galois conjugate of $I_G^{\mathfrak{m}}$.

Let ζ_6 be a primitive 6th root of unity. Similarly, an inspection of 4-edge graphs immediately gives the following corollary.

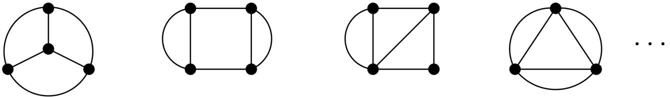
Theorem

Suppose that $I_G^{\mathfrak{m}}$ is a motivic MZV at 6th roots of unity. Then $\mathrm{Li}_2^{\mathfrak{m}}(\zeta_6)$ is not a Galois conjugate of $I_G^{\mathfrak{m}}$.

Recent examples ($P_{7,11}$, $P_{8,33}$, $P_{9,136}$, $P_{9,36}$, $P_{9,108}$) due to Panzer and Schnetz satisfy these conditions. We get strong *a priori* constraints on the possible amplitudes at 7, 8, 9 loops from a back-of-an envelope calculation.

Non-appearance of $\zeta^{\mathfrak{m}}(2)$

Expectation: There is no $\zeta^{\mathfrak{m}}(2)$ in $\widetilde{\phi}_4$. To prove this, it suffices to look at graphs γ with at most 6 edges:



and compute the mixed Hodge structures. One must show

$$\mathrm{gr}_4^W M_\gamma = 0$$

for every 6-edge graph γ. If so, then there is no $\zeta^{\mathfrak{m}}(2)$ in $\widetilde{\phi}_4$ and this propagates to an infinite number of constraints at all loop orders by the coaction theorem.

Remark: It appears that $P_{\phi^4} = P_{\widetilde{\phi}_4}$ in low weights. If it is true up a given weight, then Schnetz' coaction conjecture follows as a consequence, up to some loop order.

Generalizations

We can also look at processes depending on external parameters by replacing mixed Hodge structures with variations of MHS. Expect a coaction theorem and small graphs theorem.

Because there are very few small graphs, we expect to see many holes in the space of amplitudes.

Many known physical results should be interpretable as describing different pieces in the coaction (differential equations, monodromy, Cutcosky rules, etc). In the special case when we have variations of mixed Hodge-Tate structures (polylogarithms), then the symbol is obtained from the motivic amplitude by sending all constants to 0. The coaction reduces to the coproduct on the symbol.

Conclusion

- The theory of motivic periods gives an organising principle for much of the known structure of amplitudes.
- Surprising new structural features such as the coaction conjecture emerge. It gives extremely strong constraints on the possible numbers which can occur as amplitudes.
- By enlarging the space of amplitudes slightly, the coaction conjecture becomes a theorem.
- Programme: compute the mixed Hodge structures underlying the amplitudes of small graphs. This lead to constraints *to all orders in perturbation theory.*