Multiple Zeta Values in depth ≤ 3

Francis Brown
All Souls College, Oxford

70th Birthday conference in honour of Michel Waldschmidt
Leuca, 16th June 2016

Euler's theorem ($\sim$ 1740's)

Euler's theorem ($\sim$ 1740's)

$$\zeta(2n) = -\frac{(2\pi i)^{2n}}{2}\frac{B_{2n}}{(2n)!} \qquad n \geq 1$$

Euler's theorem ($\sim$ 1740's)

$$\zeta(2n) = -\frac{(2\pi i)^{2n}}{2}\frac{B_{2n}}{(2n)!} \qquad\qquad n \geq 1$$

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

Euler's theorem ($\sim$ 1740's)

$$\zeta(2n) = -\frac{(2\pi i)^{2n}}{2} \frac{B_{2n}}{(2n)!} \qquad n \geq 1$$

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

$$b(x) = \frac{1}{x} + \frac{1}{12}x + \frac{1}{720}x^3 + \ldots + \frac{B_{2n}}{(2n)!}x^{2n-1} + \ldots$$

Euler's theorem ($\sim$ 1740's)

$$\zeta(2n) = -\frac{(2\pi i)^{2n}}{2}\frac{B_{2n}}{(2n)!} \qquad n \geq 1$$

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

$$b(x) = \frac{1}{x} + \frac{1}{12}x + \frac{1}{720}x^3 + \ldots + \frac{B_{2n}}{(2n)!}x^{2n-1} + \ldots$$

Expect $\zeta(3), \zeta(5), \ldots$ algebraically independent over $\mathbb{Q}[\pi]$, so no such formula should exist for $\zeta(2n+1)$.

Reformulation

Consider the generating series

$$Z^{(1)} = \sum_{n \geq 2} \zeta(n) x^{n-1} \qquad \in \qquad \mathbb{R}[[x]]$$

Reformulation

Consider the generating series

$$Z^{(1)} = \sum_{n\geq 2} \zeta(n) x^{n-1} \qquad \in \qquad \mathbb{R}[[x]]$$

Let

$$\sigma_{2n+1} = \text{“coefficient of } \zeta(2n+1)\text{”}$$
$$\tau = \sum_{n\geq 0} \text{“coefficient of } (2\pi i)^{2n}\text{”}$$

Reformulation

Consider the generating series

$$Z^{(1)} = \sum_{n\geq 2} \zeta(n) x^{n-1} \qquad \in \qquad \mathbb{R}[[x]]$$

Let

$$\sigma_{2n+1} = \text{“coefficient of } \zeta(2n+1)\text{”}$$
$$\tau = \sum_{n\geq 0} \text{“coefficient of } (2\pi i)^{2n}\text{”}$$

$$\sigma_{2n+1}(Z^{(1)}) \quad = \quad x^{2n}$$

Reformulation

Consider the generating series

$$Z^{(1)} = \sum_{n\geq 2} \zeta(n) x^{n-1} \qquad \in \qquad \mathbb{R}[[x]]$$

Let

$$\sigma_{2n+1} = \text{“coefficient of } \zeta(2n+1)\text{”}$$

$$\tau = \sum_{n\geq 0} \text{“coefficient of } (2\pi i)^{2n}\text{”}$$

$$\begin{aligned} \sigma_{2n+1}(Z^{(1)}) &= x^{2n} \\ \tau(Z^{(1)}) &= \frac{1}{2x} - \frac{b(x)}{2} \end{aligned}$$

Reformulation

Improvement if we allow a simple pole in x:

$$\widetilde{Z}^{(1)} = \sum_{n \geq 0} \zeta(n) x^{n-1} \qquad \in \mathbb{R}[x^{-1}][[x]]$$

where we define

$$\zeta(1) = 0 \quad , \quad \zeta(0) = -1/2 \ .$$

Reformulation

Improvement if we allow a simple pole in x:

$$\widetilde{Z}^{(1)} = \sum_{n\geq 0} \zeta(n) x^{n-1} \qquad \in \mathbb{R}[x^{-1}][[x]]$$

where we define

$$\zeta(1) = 0 \quad , \quad \zeta(0) = -1/2 \; .$$

$$\begin{aligned} \sigma_{2n+1}(\widetilde{Z}^{(1)}) &= x^{2n} \\ \tau(\widetilde{Z}^{(1)}) &= -\frac{b(x)}{2} \end{aligned}$$

Goal

Define the generating series of depth r Multiple Zeta Values:

$$Z^{(r)} = \sum_{n_1,\ldots,n_r \geq 1} \zeta(n_1, \ldots, n_r)\, x_1^{n_1-1} \ldots x_r^{n_r-1}$$

where, for $n_r \geq 2$,

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}$$

Goal

Define the generating series of depth r Multiple Zeta Values:

$$Z^{(r)} = \sum_{n_1,\dots,n_r \geq 1} \zeta(n_1,\dots,n_r)\, x_1^{n_1-1} \dots x_r^{n_r-1}$$

where, for $n_r \geq 2$,

$$\zeta(n_1,\dots,n_r) = \sum_{1 \leq k_1 < k_2 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}$$

Ultimate goal

Write down homogeneous polynomials

$$\sigma^{(r)}_{2n+1} := \sigma_{2n+1}(Z^{(r)}) \qquad \in \qquad \mathbb{Q}[x_1,\dots,x_r]$$

of degree $2n+1-r$, and power series:

$$\tau^{(r)} := \tau(Z^{(r)}) \qquad \in \qquad \mathbb{Q}[[x_1,\dots,x_r]]$$

Problem - totally unreasonable!

Problem - totally unreasonable!

This makes no sense! The σ_{2n+1}, τ depend on a choice of basis for multiple zeta values, and inaccesible transcendence conjectures.

Problem - totally unreasonable!

This makes no sense! The σ_{2n+1}, τ depend on a choice of basis for multiple zeta values, and inaccesible transcendence conjectures. Nonetheless,

Theorem

There exists a canonical and explicit choice of elements

$$(\sigma^c_{2n+1})^{(r)} \in \mathbb{Q}[x_1, \ldots, x_r] \qquad \text{for } 1 \leq r \leq 4$$

$$(\tau^c)^{(r)} \in \mathbb{Q}[[x_1, \ldots, x_r]] \qquad \text{for } 1 \leq r \leq 3$$

with all the required properties.

Problem - totally unreasonable!

This makes no sense! The σ_{2n+1}, τ depend on a choice of basis for multiple zeta values, and inaccesible transcendence conjectures. Nonetheless,

Theorem

There exists a canonical and explicit choice of elements

$$(\sigma^c_{2n+1})^{(r)} \in \mathbb{Q}[x_1, \ldots, x_r] \qquad \text{for } 1 \leq r \leq 4$$

$$(\tau^c)^{(r)} \in \mathbb{Q}[[x_1, \ldots, x_r]] \qquad \text{for } 1 \leq r \leq 3$$

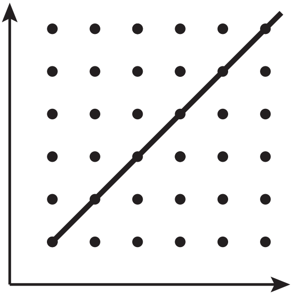
with all the required properties.

Modular forms, monodromy of π_1 of universal elliptic curve.

“Standard” relations for MZV’s

Stuffle equations

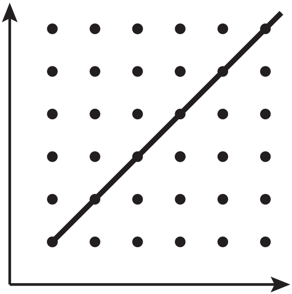
$$\sum_{k\geq 1}\frac{1}{k^m}\sum_{\ell\geq 1}\frac{1}{\ell^n} = \Bigg(\sum_{1\leq k<\ell} + \sum_{1\leq \ell<k} + \sum_{1\leq k=\ell}\Bigg)\frac{1}{k^m\ell^n}$$



$$\zeta(m)\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)\ .$$

Stuffle equations

$$\sum_{k\geq 1}\frac{1}{k^m}\sum_{\ell\geq 1}\frac{1}{\ell^n}=\Big(\sum_{1\leq k<\ell}+\sum_{1\leq \ell<k}+\sum_{1\leq k=\ell}\Big)\frac{1}{k^m\ell^n}$$



$$\zeta(m)\zeta(n)=\zeta(m,n)+\zeta(n,m)+\zeta(m+n)\ .$$

Stuffle equations in depth 3:

$$\zeta(n_1)\zeta(n_2,n_3)=\zeta(n_1+n_2,n_3)+\zeta(n_2,n_1+n_3)$$
$$+\zeta(n_1,n_2,n_3)+\zeta(n_2,n_1,n_3)+\zeta(n_2,n_3,n_1)$$

Shuffle relations

$$\zeta(2)\zeta(2) = \int_{0\leq t_1\leq t_2\leq 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \int_{0\leq s_1\leq s_2\leq 1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}$$

Shuffle relations

$$\zeta(2)\zeta(2) = \int_{0\leq t_1\leq t_2\leq 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \int_{0\leq s_1\leq s_2\leq 1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}$$

Decompose region of summation into six sets, including

$$\int_{0\leq t_1\leq s_1\leq s_2\leq t_2\leq 1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} = \zeta(1,3)\ .$$

Shuffle relations

$$\zeta(2)\zeta(2) = \int_{0\le t_1\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \int_{0\le s_1\le s_2\le 1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}$$

Decompose region of summation into six sets, including

$$\int_{0\le t_1\le s_1\le s_2\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} = \zeta(1,3)\ .$$

We deduce the equation

$$\zeta(2)\zeta(2) = 4\,\zeta(1,3) + 2\,\zeta(2,2)$$

Shuffle relations

$$\zeta(2)\zeta(2) = \int_{0\le t_1\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \int_{0\le s_1\le s_2\le 1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}$$

Decompose region of summation into six sets, including

$$\int_{0\le t_1\le s_1\le s_2\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} = \zeta(1,3)\ .$$

We deduce the equation

$$\zeta(2)\zeta(2) = 4\,\zeta(1,3) + 2\,\zeta(2,2)$$

Generalises to an infinite family of relations: *shuffle relations*.

Shuffle relations

$$\zeta(2)\zeta(2) = \int_{0\le t_1\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \int_{0\le s_1\le s_2\le 1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}$$

Decompose region of summation into six sets, including

$$\int_{0\le t_1\le s_1\le s_2\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} = \zeta(1,3)\ .$$

We deduce the equation

$$\zeta(2)\zeta(2) = 4\,\zeta(1,3) + 2\,\zeta(2,2)$$

Generalises to an infinite family of relations: *shuffle relations*.

Regularisation relation (Hoffman): Example $\zeta(1,2) = \zeta(3)$.
Involves making sense of $\zeta(n_1,\dots,n_r)$ for $n_r = 1$ (divergent case).

Example

In weight 4 there are four MZV's, $\zeta(4)$, $\zeta(1,3)$, $\zeta(2,2)$ and $\zeta(1,1,2)$. We have the equations:

$$\begin{aligned}
\zeta(2)^2 &= 2\zeta(2,2) + \zeta(4) && \text{(stuffle)} \\
\zeta(2)^2 &= 4\zeta(1,3) + 2\zeta(2,2) && \text{(shuffle)} \\
\zeta(1,3) + \zeta(4) &= 2\zeta(1,3) + \zeta(2,2) && \text{(reg.)} \\
2\zeta(1,1,2) + \zeta(2,2) + \zeta(1,4) &= 3\zeta(1,1,2) && \text{(reg.)}
\end{aligned}$$

Example

In weight 4 there are four MZV's, $\zeta(4)$, $\zeta(1,3)$, $\zeta(2,2)$ and $\zeta(1,1,2)$. We have the equations:

$$\begin{aligned}
\zeta(2)^2 &= 2\zeta(2,2)+\zeta(4) && \text{(stuffle)}\\
\zeta(2)^2 &= 4\zeta(1,3)+2\zeta(2,2) && \text{(shuffle)}\\
\zeta(1,3)+\zeta(4) &= 2\zeta(1,3)+\zeta(2,2) && \text{(reg.)}\\
2\zeta(1,1,2)+\zeta(2,2)+\zeta(1,4) &= 3\zeta(1,1,2) && \text{(reg.)}
\end{aligned}$$

Exercise: deduce that $\zeta(4)=\frac{2}{5}\zeta(2)^2$ (Euler).

Double shuffle ring

Define

$$\mathcal{Z}^f = \bigoplus_{n \geq 0} \mathcal{Z}^f_n$$

to be the graded $\mathbb{Q}$-algebra generated by symbols 1 and

$$\zeta^f(n_1, \ldots, n_r)$$

modulo the shuffle, stuffle and regularisation equations.

Double shuffle ring

Define

$$\mathcal{Z}^f = \bigoplus_{n \geq 0} \mathcal{Z}^f_n$$

to be the graded $\mathbb{Q}$-algebra generated by symbols 1 and

$$\zeta^f(n_1, \ldots, n_r)$$

modulo the shuffle, stuffle and regularisation equations. Grading is given by the weight $n_1 + \ldots + n_r$.

Double shuffle ring

Define

$$\mathcal{Z}^f = \bigoplus_{n \geq 0} \mathcal{Z}^f_n$$

to be the graded $\mathbb{Q}$-algebra generated by symbols 1 and

$$\zeta^f(n_1, \ldots, n_r)$$

modulo the shuffle, stuffle and regularisation equations. Grading is given by the weight $n_1 + \ldots + n_r$.

There is a canonical homomorphism $\zeta^f(n_1, \ldots, n_r) \mapsto \zeta(n_1, \ldots, n_r)$

$$\mathcal{Z}^f \longrightarrow \mathbb{R}$$

Double shuffle ring

Define

$$\mathcal{Z}^f = \bigoplus_{n \geq 0} \mathcal{Z}^f_n$$

to be the graded $\mathbb{Q}$-algebra generated by symbols 1 and

$$\zeta^f(n_1, \ldots, n_r)$$

modulo the shuffle, stuffle and regularisation equations. Grading is given by the weight $n_1 + \ldots + n_r$.

There is a canonical homomorphism $\zeta^f(n_1, \ldots, n_r) \mapsto \zeta(n_1, \ldots, n_r)$

$$\mathcal{Z}^f \longrightarrow \mathbb{R}$$

Zagier conjectured that this map is injective.

Two problems

Given a set of equations one wants to know: 1) how many solutions there are, and 2) what are the solutions?

Two problems

Given a set of equations one wants to know: 1) how many solutions there are, and 2) what are the solutions?

1. Compute the dimensions

$$\dim_{\mathbb{Q}} \mathcal{Z}_n^f$$

Two problems

Given a set of equations one wants to know: 1) how many solutions there are, and 2) what are the solutions?

1. Compute the dimensions

$$\dim_{\mathbb{Q}} \mathcal{Z}_n^f$$

2. Find the solutions over $\mathbb{Q}$ to the double shuffle equations.

Only discuss 2 here.

Reformulation of the second problem

Goal

Write down a homomorphism

$$\begin{array}{rcl} \tau : \mathcal{Z}^f & \longrightarrow & \mathbb{Q} \\ \zeta^f(2) & \mapsto & \frac{1}{24} \end{array}$$

and linear maps of weight $2n+1$:

$$\begin{array}{rcl} \sigma_{2n+1} : \dfrac{\mathcal{Z}^f_{>0}}{\mathcal{Z}^f_{>0}\mathcal{Z}^f_{>0}} & \longrightarrow & \mathbb{Q} \\ \zeta^f(2n+1) & \mapsto & 1 \end{array}$$

Reformulation of the second problem

Goal

Write down a homomorphism

$$\begin{array}{rcl} \tau : \mathcal{Z}^f & \longrightarrow & \mathbb{Q} \\ \zeta^f(2) & \mapsto & \frac{1}{24} \end{array}$$

and linear maps of weight $2n+1$:

$$\begin{array}{rcl} \sigma_{2n+1} : \dfrac{\mathcal{Z}^f_{>0}}{\mathcal{Z}^f_{>0}\mathcal{Z}^f_{>0}} & \longrightarrow & \mathbb{Q} \\ \zeta^f(2n+1) & \mapsto & 1 \end{array}$$

Furusho $\Rightarrow \tau$ defines a 'rational associator' (Drinfeld: τ exists).

Applications: knot invariants, deformation quantization, Kashiwara-Vergne problem, Mixed Tate motives over $\mathbb{Z}$.

Example

$$\sigma_5^{(1)} = x_1^4.$$

Example

$\sigma_5^{(1)} = x_1^4$. In depth 2,

$$\sigma_5^{(2)} = -3x_1^3 + \frac{9}{2}x_1^2x_2 - \frac{11}{2}x_1x_2^2 + 2x_2^3$$

Example

$\sigma_5^{(1)} = x_1^4$. In depth 2,

$$\sigma_5^{(2)} = -3x_1^3 + \frac{9}{2}x_1^2 x_2 - \frac{11}{2}x_1 x_2^2 + 2x_2^3$$

The coefficients encode the information:

$$\begin{aligned}
\zeta(3,2) &= \frac{9}{2}\zeta(5) - 2\zeta(3)\zeta(2) \\
\zeta(2,3) &= -\frac{11}{2}\zeta(5) + 3\zeta(3)\zeta(2) \\
\zeta(1,4) &= 2\zeta(5) - \zeta(3)\zeta(2)
\end{aligned}$$

Example

$\sigma_5^{(1)} = x_1^4$. In depth 2,

$$\sigma_5^{(2)} = -3x_1^3 + \frac{9}{2}x_1^2x_2 - \frac{11}{2}x_1x_2^2 + 2x_2^3$$

The coefficients encode the information:

$$\begin{aligned}
\zeta(3,2) &= \frac{9}{2}\zeta(5) - 2\zeta(3)\zeta(2) \\
\zeta(2,3) &= -\frac{11}{2}\zeta(5) + 3\zeta(3)\zeta(2) \\
\zeta(1,4) &= 2\zeta(5) - \zeta(3)\zeta(2)
\end{aligned}$$

These formulae were obtained by solving the double shuffle equations in weight 5.

Ill-definedness

The maps τ and σ_{2n+1} are ill-defined in general!

$$\zeta^f(3,5) = 5\zeta^f(3)\zeta^f(5) - \frac{5}{2}\zeta^f(6,2) + \frac{-7}{2764800}(2i\pi)^8$$

versus

$$\zeta^f(3,5) = -\zeta^f(3)\zeta^f(5) - \zeta^f(5,3) + \frac{1}{2419200}(2i\pi)^8$$

They depend on choice of generating family. *A priori* not a reasonable problem.

Main theorem

Theorem

There is an explicit homomorphism

$$\tau^c : D_3\mathcal{Z}^f \longrightarrow \mathbb{Q}$$

and linear maps of weight $2n+1$

$$\sigma_{2n+1} : \frac{D_4\mathcal{Z}^f}{\mathcal{Z}^f_{>0}\mathcal{Z}^f_{>0}} \longrightarrow \mathbb{Q}$$

defined only in terms of Bernoulli numbers.

Main theorem

Theorem

There is an explicit homomorphism

$$\tau^c : D_3\mathcal{Z}^f \longrightarrow \mathbb{Q}$$

and linear maps of weight $2n+1$

$$\sigma_{2n+1} : \frac{D_4\mathcal{Z}^f}{\mathcal{Z}^f_{>0}\mathcal{Z}^f_{>0}} \longrightarrow \mathbb{Q}$$

defined only in terms of Bernoulli numbers.

They should correspond to a canonical generating set of MZV's in depths $\leq 3, 4$. I have no idea what this is.

Challenge: extend to all depths.

Commutative generating series

Consider the ($\bullet$-regularized) generating series in depth r

$$Z_\bullet^{(r)}(x_1, \ldots, x_r) = \sum_{n_1, \ldots, n_r \geq 1} \zeta_\bullet(n_1, \ldots, n_r)\, x_1^{n_1 - 1} \ldots x_r^{n_r - 1}$$

Commutative generating series

Consider the ($\bullet$-regularized) generating series in depth r

$$Z_{\bullet}^{(r)}(x_1,\dots,x_r) = \sum_{n_1,\dots,n_r\geq 1} \zeta_{\bullet}(n_1,\dots,n_r)\, x_1^{n_1-1}\dots x_r^{n_r-1}$$

Relations between MZV's $\longleftrightarrow$ Functional equations for $Z^{(\bullet)}$

Double shuffle equations in depth 2

$$\begin{aligned} f_*^{(1)}(x_1)f_*^{(1)}(x_2) &= f_*^{(2)}(x_1,x_2) + f_*^{(2)}(x_2,x_1) + \frac{f_*^{(1)}(x_1) - f_*^{(1)}(x_2)}{x_1 - x_2} \\ f^{(1)}(x_1)f^{(1)}(x_2) &= f^{(2)}(x_1,x_1+x_2) + f^{(2)}(x_2,x_1+x_2) \end{aligned}$$

Where

$$f^{(1)} = f_*^{(1)} \qquad \text{and} \qquad f_*^{(2)} = f^{(2)} + \frac{1}{48}$$

Double shuffle equations in depth 2

$$
\begin{aligned}
f_*^{(1)}(x_1)f_*^{(1)}(x_2) &= f_*^{(2)}(x_1,x_2) + f_*^{(2)}(x_2,x_1) + \frac{f_*^{(1)}(x_1) - f_*^{(1)}(x_2)}{x_1 - x_2} \\
f^{(1)}(x_1)f^{(1)}(x_2) &= f^{(2)}(x_1,x_1+x_2) + f^{(2)}(x_2,x_1+x_2)
\end{aligned}
$$

Where

$$
f^{(1)} = f_*^{(1)} \qquad \text{and} \qquad f_*^{(2)} = f^{(2)} + \frac{1}{48}
$$

Examples of solutions:

$$
(f_\bullet^{(1)}, f_\bullet^{(2)}) = (Z_\bullet^{(1)}, Z_\bullet^{(2)}) \in \mathbb{R}[[x_1]] \times \mathbb{R}[[x_2]]
$$

Double shuffle equations in depth 2

$$\begin{aligned} f_*^{(1)}(x_1)f_*^{(1)}(x_2) &= f_*^{(2)}(x_1,x_2) + f_*^{(2)}(x_2,x_1) + \frac{f_*^{(1)}(x_1) - f_*^{(1)}(x_2)}{x_1 - x_2} \\ f^{(1)}(x_1)f^{(1)}(x_2) &= f^{(2)}(x_1,x_1+x_2) + f^{(2)}(x_2,x_1+x_2) \end{aligned}$$

Where

$$f^{(1)} = f_*^{(1)} \qquad \text{and} \qquad f_*^{(2)} = f^{(2)} + \frac{1}{48}$$

Examples of solutions:

$$(f_\bullet^{(1)}, f_\bullet^{(2)}) = (Z_\bullet^{(1)}, Z_\bullet^{(2)}) \in \mathbb{R}[[x_1]] \times \mathbb{R}[[x_2]]$$

$$(\tau^{(1)}, \tau^{(2)}) \in \mathbb{Q}[[x_1]] \times \mathbb{Q}[[x_1, x_2]]$$

The equations modulo products

$$
\begin{aligned}
0 &= f_*^{(2)}(x_1, x_2) + f_*^{(2)}(x_2, x_1) + \frac{f_*^{(1)}(x_1) - f_*^{(1)}(x_2)}{x_1 - x_2} \\
0 &= f^{(2)}(x_1, x_1 + x_2) + f^{(2)}(x_2, x_1 + x_2)
\end{aligned}
$$

Where

$$
f^{(1)} = f_*^{(1)} \qquad \text{and} \qquad f_*^{(2)} = f^{(2)} + \frac{1}{48}
$$

The equations modulo products

$$\begin{aligned} 0 &= f_*^{(2)}(x_1,x_2) + f_*^{(2)}(x_2,x_1) + \frac{f_*^{(1)}(x_1) - f_*^{(1)}(x_2)}{x_1 - x_2} \\ 0 &= f^{(2)}(x_1,x_1+x_2) + f^{(2)}(x_2,x_1+x_2) \end{aligned}$$

Where

$$f^{(1)} = f_*^{(1)} \qquad \text{and} \qquad f_*^{(2)} = f^{(2)} + \frac{1}{48}$$

Examples of solutions:

$$(\sigma_{2n+1}^{(1)}, \sigma_{2n+1}^{(2)}) \in \mathbb{Q}[x_1] \times \mathbb{Q}[x_1,x_2]$$

Strategy

1. Write down explicit solutions with *poles* in the x_i
2. Cancel out the poles.

Strategy

1. Write down explicit solutions with *poles* in the x_i
2. Cancel out the poles.

In (2) use group/torsor structure of solutions.

Related to Ecalle's ARI/GARI?

Group structure

A Lie algebra

Work in a graded ring of *rational functions*

$$Q = \bigoplus_{r \geq 1} \mathbb{Q}(x_1, \ldots, x_r)$$

The $\sigma^{(r)}_{2n+1}$ lie in subspace $\bigoplus_r \mathbb{Q}[x_1, \ldots, x_r]$. For τ, work in some completed version $\widehat{Q}$.

A Lie algebra

Work in a graded ring of *rational functions*

$$Q = \bigoplus_{r \geq 1} \mathbb{Q}(x_1, \ldots, x_r)$$

The $\sigma_{2n+1}^{(r)}$ lie in subspace $\bigoplus_r \mathbb{Q}[x_1, \ldots, x_r]$. For τ, work in some completed version $\widehat{Q}$. Define a Lie algebra stucture

$$\{\,,\} : Q \times Q \longrightarrow Q$$

A Lie algebra

Work in a graded ring of *rational functions*

$$Q = \bigoplus_{r \geq 1} \mathbb{Q}(x_1, \ldots, x_r)$$

The $\sigma_{2n+1}^{(r)}$ lie in subspace $\bigoplus_r \mathbb{Q}[x_1, \ldots, x_r]$. For τ, work in some completed version $\widehat{Q}$. Define a Lie algebra stucture

$$\{\,,\} : Q \times Q \longrightarrow Q$$

$\{f, g\} = f \,\underline{\circ}\, g - g \,\underline{\circ}\, f$ where

$$\underline{\circ} : Q \times Q \longrightarrow Q$$

$$(f \,\underline{\circ}\, g)^{(m)} = \sum_{i+j=m} f^{(i)} \,\underline{\circ}\, g^{(j)}$$

“Linearized” Ihara action

$$\underline{\circ} : \mathbb{Q}(x_1, \ldots, x_r) \times \mathbb{Q}(x_1, \ldots, x_s) \longrightarrow \mathbb{Q}(x_1, \ldots, x_{r+s})$$

“Linearized” Ihara action

$$\underline{\circ} : \mathbb{Q}(x_1, \ldots, x_r) \times \mathbb{Q}(x_1, \ldots, x_s) \longrightarrow \mathbb{Q}(x_1, \ldots, x_{r+s})$$

$$f \,\underline{\circ}\, g\,(x_1, \ldots, x_{r+s}) =$$

$$\sum_{i=0}^{s} f(x_{i+1} - x_i, \ldots, x_{i+r} - x_i) g(x_1, \ldots, x_i, x_{i+r+1}, \ldots, x_{r+s}) \qquad +$$

$$(-1)^{\deg f + r} \sum_{i=1}^{s} f(x_{i+r+1} - x_i, \ldots, x_{i+1} - x_i) g(x_1 \ldots, x_{i-1}, x_{i+r}, \ldots, x_{r+s})$$

“Linearized” Ihara action

$$\underline{\circ} : \mathbb{Q}(x_1, \ldots, x_r) \times \mathbb{Q}(x_1, \ldots, x_s) \longrightarrow \mathbb{Q}(x_1, \ldots, x_{r+s})$$

$$f \,\underline{\circ}\, g\,(x_1, \ldots, x_{r+s}) =$$

$$\sum_{i=0}^{s} f(x_{i+1}-x_i, \ldots, x_{i+r}-x_i) g(x_1, \ldots, x_i, x_{i+r+1}, \ldots, x_{r+s}) \qquad +$$

$$(-1)^{\deg f + r} \sum_{i=1}^{s} f(x_{i+r+1}-x_i, \ldots, x_{i+1}-x_i) g(x_1 \ldots, x_{i-1}, x_{i+r}, \ldots, x_{r+s})$$

Proposition: $(Q, \{\})$ is a graded Lie algebra.

Likewise, we get a left action:

$$\underline{\circ} : Q \times \widehat{Q} \longrightarrow \widehat{Q}$$

Variant of Racinet's theorem:

1. The solutions (e.g. σ_{2n+1}) of the double shuffle equations modulo products in Q form a Lie algebra under $\{,\}$.
2. The solutions (e.g. τ) of reg. double shuffle equations in $\widehat{Q}$ are stable under the the left action for $\underline{\circ}$, of *even* solutions to double shuffle modulo products.

Variant of Racinet's theorem:

1. The solutions (e.g. σ_{2n+1}) of the double shuffle equations modulo products in Q form a Lie algebra under $\{,\}$.
2. The solutions (e.g. τ) of reg. double shuffle equations in $\widehat{Q}$ are stable under the the left action for $\underline{\circ}$, of *even* solutions to double shuffle modulo products.

Idea: construct explicit solutions by bootstrapping using the operation $\underline{\circ}$.

Bootstrapping solutions

STEP I

Define

$$
\begin{aligned}
s^{(1)} &= \frac{1}{2\,x_1} \\
s^{(2)} &= \frac{1}{6}\left(\frac{1}{x_1 x_2} + \frac{1}{x_2(x_1 - x_2)}\right)
\end{aligned}
$$

STEP I

Define

$$
\begin{aligned}
s^{(1)} &= \frac{1}{2\,x_1} \\
s^{(2)} &= \frac{1}{6}\Big(\frac{1}{x_1 x_2} + \frac{1}{x_2(x_1 - x_2)}\Big)
\end{aligned}
$$

s solve double shuffle mod. products in depths ≤ 2.

s of weight 0, so 'corresponds' to $\zeta(0)$. Has no reason to be there.

STEP I

Define

$$
\begin{aligned}
s^{(1)} &= \frac{1}{2\,x_1} \\
s^{(2)} &= \frac{1}{6}\Big(\frac{1}{x_1 x_2} + \frac{1}{x_2(x_1 - x_2)}\Big)
\end{aligned}
$$

s solve double shuffle mod. products in depths ≤ 2.

s of weight 0, so 'corresponds' to $\zeta(0)$. Has no reason to be there.

It is surprising that there exists such a solution at all. Can be extended to all higher depths, but in infinitely many different ways!

STEP II

Recall $\sigma_{2n+1}^{(1)} = x_1^{2n}$.

STEP II

Recall $\sigma^{(1)}_{2n+1} = x_1^{2n}$. Set

$$
\begin{aligned}
\xi^{(1)}_{2n+1} &= x_1^{2n} \\
\xi^{(2)}_{2n+1} &= \{s^{(1)}, x_1^{2n}\} \\
\xi^{(3)}_{2n+1} &= \{s^{(2)}, x_1^{2n}\} + \frac{1}{2}\{s^{(1)}, \{s^{(1)}, x_1^{2n}\}\}
\end{aligned}
$$

STEP II

Recall $\sigma^{(1)}_{2n+1} = x_1^{2n}$. Set

$$\begin{aligned}
\xi^{(1)}_{2n+1} &= x_1^{2n} \\
\xi^{(2)}_{2n+1} &= \{s^{(1)}, x_1^{2n}\} \\
\xi^{(3)}_{2n+1} &= \{s^{(2)}, x_1^{2n}\} + \frac{1}{2}\{s^{(1)}, \{s^{(1)}, x_1^{2n}\}\}
\end{aligned}$$

ξ solve double shuffle mod. products with poles in depths ≤ 3.

STEP II

Recall $\sigma^{(1)}_{2n+1} = x_1^{2n}$. Set

$$\begin{aligned}
\xi^{(1)}_{2n+1} &= x_1^{2n} \\
\xi^{(2)}_{2n+1} &= \{s^{(1)}, x_1^{2n}\} \\
\xi^{(3)}_{2n+1} &= \{s^{(2)}, x_1^{2n}\} + \frac{1}{2}\{s^{(1)}, \{s^{(1)}, x_1^{2n}\}\}
\end{aligned}$$

ξ solve double shuffle mod. products with poles in depths ≤ 3.

ξ_{-1} of weight -1, so 'corresponds' to $\zeta(-1)$. There exist infinitely many possible generalisations to all higher depths.

STEP III

Heresy!

STEP III

Heresy! Rescale

$$\zeta(2n+1) \qquad \text{to} \qquad \frac{(2n)!}{B_{2n}}\zeta(2n+1)$$

STEP III

Heresy! Rescale

$$\zeta(2n+1) \qquad \text{to} \qquad \frac{(2n)!}{B_{2n}}\zeta(2n+1)$$

Define the heretical normalisations

$$\begin{aligned} \underline{\xi}_{-1} &= \frac{1}{12}\,\xi_{-1} \\ \underline{\xi}_{2n+1} &= \frac{B_{2n}}{(2n)!}\,\xi_{2n+1} \quad \text{for } n \geq 1 \end{aligned}$$

Canonical σ_{2n+1}'s

Canonical σ_{2n+1}'s

Define

$$\underline{\sigma}^c_{2n+1} = \underline{\xi}_{2n+1} + \sum_{a+b=n} \frac{1}{2b}\{\underline{\xi}_{2a+1}, \{\underline{\xi}_{2b+1}, \underline{\xi}_{-1}\}\}$$

Canonical σ_{2n+1}'s

Define

$$\underline{\sigma}^{c}_{2n+1} = \underline{\xi}_{2n+1} + \sum_{a+b=n} \frac{1}{2b}\{\underline{\xi}_{2a+1}, \{\underline{\xi}_{2b+1}, \underline{\xi}_{-1}\}\}$$

Theorem

The $\underline{\sigma}^{c}_{2n+1}$ have no poles in depths ≤ 4 (with the exception of σ^{c}_{3}), and are solutions to double shuffle mod. products .

Canonical σ_{2n+1}'s

Define

$$\underline{\sigma}^{c}_{2n+1} = \underline{\xi}_{2n+1} + \sum_{a+b=n} \frac{1}{2b} \{\underline{\xi}_{2a+1}, \{\underline{\xi}_{2b+1}, \underline{\xi}_{-1}\}\}$$

Theorem

The $\underline{\sigma}^{c}_{2n+1}$ have no poles in depths ≤ 4 (with the exception of σ^{c}_{3}), and are solutions to double shuffle mod. products .

The formula for $\underline{\sigma}^{c}_{2n+1}$ should extend to an infinite series in terms of Lie brackets of the $\underline{\xi}_{2n+1}$.

Challenge: guess the correct formula?

If we use canonical, instead of heretical normalisations, then the coefficients in σ^c_{2n+1} involve products

$$\binom{2a+2b}{2a} B_{2a} B_{2b}$$

which are coefficients in the regularised Eichler integral

$$\int_0^{i\infty} E_{2n+2}(\tau)(X+\tau Y)^{2n} d\tau$$

Bootstrapping for τ

Double Bernoulli series

Think of

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

as a deformation of $\frac{1}{x}$. Recall $s^{(1)} = \frac{1}{2x_1}$

Double Bernoulli series

Think of

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

as a deformation of $\frac{1}{x}$. Recall $s^{(1)} = \frac{1}{2x_1}$

Recall the element

$$s^{(2)} = \frac{1}{6}\left(\frac{1}{x_1 x_2} + \frac{1}{x_2(x_1 - x_2)}\right)$$

Double Bernoulli series

Think of

$$b(x) = \frac{1}{2} + \frac{1}{e^x - 1}$$

as a deformation of $\frac{1}{x}$. Recall $s^{(1)} = \frac{1}{2x_1}$

Recall the element

$$s^{(2)} = \frac{1}{6}\left(\frac{1}{x_1 x_2} + \frac{1}{x_2(x_1 - x_2)}\right)$$

It motivates the *double Bernoulli series*

$$b_2(x_1, x_2) = \frac{1}{3}\Big(b(x_1)b(x_2) + b(x_2)b(x_1 - x_2)\Big)$$

STEP IV

Set

$$\begin{aligned} 2\gamma^{(1)} &= -b_1 \\ 4\gamma^{(2)} &= -b_2 + \tfrac{1}{2} b_1 \underline{\circ}\, b_1 \\ 8\gamma^{(3)} &= b_2 \underline{\circ}\, b_1 - \tfrac{1}{6} b_1 \underline{\circ}\, (b_1 \underline{\circ}\, b_1) \end{aligned}$$

STEP IV

Set

$$
\begin{aligned}
2\gamma^{(1)} &= -b_1 \\
4\gamma^{(2)} &= -b_2 + \tfrac{1}{2} b_1 \underline{\circ}\, b_1 \\
8\gamma^{(3)} &= b_2 \underline{\circ}\, b_1 - \tfrac{1}{6} b_1 \underline{\circ}\, (b_1 \underline{\circ}\, b_1)
\end{aligned}
$$

The elements γ solve the 'semi-homogeneous' double shuffle equations. In depth 2 these are:

$$
\begin{aligned}
\gamma^{(2)}(x_1, x_1 + x_2) + \gamma^{(2)}(x_2, x_1 + x_2) &= \gamma^{(1)}(x_1)\gamma^{(1)}(x_2) \\
\gamma_*^{(2)}(x_1, x_2) + \gamma_*^{(2)}(x_2, x_1) &= \gamma^{(1)}(x_1)\gamma^{(1)}(x_2)
\end{aligned}
$$

where $\gamma_*^{(2)} = \gamma^{(2)} + \frac{1}{48}$.

STEP V

Set

$$
\begin{aligned}
\Theta^{(1)} &= \gamma^{(1)} \\
\Theta^{(2)} &= \gamma^{(2)} + s^{(1)} \underline{\circ}\, \gamma^{(1)} \\
\Theta^{(3)} &= \gamma^{(3)} + s^{(1)} \underline{\circ}\, \gamma^{(2)} + \tfrac{1}{2} s^{(2)} \underline{\circ}\, \gamma^{(1)} + \tfrac{1}{2} s^{(1)} \underline{\circ} \left(s^{(1)} \underline{\circ}\, \gamma^{(1)} \right)
\end{aligned}
$$

STEP V

Set

$$
\begin{aligned}
\Theta^{(1)} &= \gamma^{(1)} \\
\Theta^{(2)} &= \gamma^{(2)} + s^{(1)} \underline{\circ} \gamma^{(1)} \\
\Theta^{(3)} &= \gamma^{(3)} + s^{(1)} \underline{\circ} \gamma^{(2)} + \tfrac{1}{2} s^{(2)} \underline{\circ} \gamma^{(1)} + \tfrac{1}{2} s^{(1)} \underline{\circ} (s^{(1)} \underline{\circ} \gamma^{(1)})
\end{aligned}
$$

Remove garbage term of degree $-r$ in $\Theta^{(r)}$ to get a new element $\Phi^{(r)}$. They are solutions to the double shuffle equations with poles.

STEP V

Set

$$\begin{aligned}
\Theta^{(1)} &= \gamma^{(1)} \\
\Theta^{(2)} &= \gamma^{(2)} + s^{(1)} \,\underline{\circ}\, \gamma^{(1)} \\
\Theta^{(3)} &= \gamma^{(3)} + s^{(1)} \,\underline{\circ}\, \gamma^{(2)} + \tfrac{1}{2} s^{(2)} \,\underline{\circ}\, \gamma^{(1)} + \tfrac{1}{2} s^{(1)} \,\underline{\circ}\, (s^{(1)} \,\underline{\circ}\, \gamma^{(1)})
\end{aligned}$$

Remove garbage term of degree $-r$ in $\Theta^{(r)}$ to get a new element $\Phi^{(r)}$. They are solutions to the double shuffle equations with poles.

Now we have to remove the poles.

STEP VI

STEP VI

Set

$$C = \sum_{n \geq 1} \frac{1}{2n} \{\underline{\xi}_{-1}, \underline{\xi}_{2n+1}\}$$

STEP VI

Set

$$C = \sum_{n \geq 1} \frac{1}{2n} \{\underline{\xi}_{-1}, \underline{\xi}_{2n+1}\}$$

$$\begin{aligned}
\tau^{(1)} &= \Phi^{(1)} \\
\tau^{(2)} &= \Phi^{(2)} + C^{(2)} \\
\tau^{(3)} &= \Phi^{(3)} + C^{(2)} \underline{\circ} \Phi^{(1)} + C^{(3)}
\end{aligned}$$

STEP VI

Set

$$C = \sum_{n \geq 1} \frac{1}{2n} \{\underline{\xi}_{-1}, \underline{\xi}_{2n+1}\}$$

$$\begin{aligned} \tau^{(1)} &= \Phi^{(1)} \\ \tau^{(2)} &= \Phi^{(2)} + C^{(2)} \\ \tau^{(3)} &= \Phi^{(3)} + C^{(2)} \underline{\circ} \Phi^{(1)} + C^{(3)} \end{aligned}$$

Theorem

The $\tau^{(r)}$ is a solution to double shuffle equations with no poles.

Corollary

Goncharov computed the dimension of the space of solutions of double shuffle equations in depth 3 (hard!).

Corollary

Goncharov computed the dimension of the space of solutions of double shuffle equations in depth 3 (hard!).

Corollary

Every solution to the double shuffle equations in depths ≤ 3 can be constructed explicitly out of σ^c_{2n+1}, τ^c, and $\underline{\circ}$.

Corollary

Goncharov computed the dimension of the space of solutions of double shuffle equations in depth 3 (hard!).

Corollary

Every solution to the double shuffle equations in depths ≤ 3 can be constructed explicitly out of σ^c_{2n+1}, τ^c, and $\underline{\circ}$.

Corollary

The tables for multiple zeta values in depths ≤ 3 are redundant.

Corollary

Goncharov computed the dimension of the space of solutions of double shuffle equations in depth 3 (hard!).

Corollary

Every solution to the double shuffle equations in depths ≤ 3 can be constructed explicitly out of σ^c_{2n+1}, τ^c, and $\underline{\circ}$.

Corollary

The tables for multiple zeta values in depths ≤ 3 are redundant.

Example:

$$\zeta(5,2,2) = \frac{-3319}{72}\zeta(9) + \frac{2}{3}\zeta(3)^3 + 31\,\zeta(7)\zeta(2) - \zeta(5)\zeta(4) - \frac{25}{6}\zeta(3)\zeta(6)$$

All coefficients can be determined by the above recipe.

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.
2. Infinitesimal Tate curve $E^{\times}_{\partial/\partial q}$ (fiber of universal punctured elliptic curve at tangential base point at the cusp of $M_{1,1}$)

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.
2. Infinitesimal Tate curve $E^{\times}_{\partial/\partial q}$ (fiber of universal punctured elliptic curve at tangential base point at the cusp of $M_{1,1}$)
3. $\mathrm{Lie}\,\pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$ is free Lie algebra $\mathbb{L}(a, b)$.

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.
2. Infinitesimal Tate curve $E^{\times}_{\partial/\partial q}$ (fiber of universal punctured elliptic curve at tangential base point at the cusp of $M_{1,1}$)
3. $\mathrm{Lie}\,\pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$ is free Lie algebra $\mathbb{L}(a,b)$.
4. $\pi_1(M_{1,1}, \partial/\partial q) = \mathrm{SL}_2(\mathbb{Z})$. Relative completion $\mathrm{SL}_2(\mathbb{Z})^{rel}$ acts on $\mathbb{L}(a,b)$ via derivations (Nakamura) in $\mathrm{Der}\mathbb{L}(a,b)$

$$\varepsilon_{2n} : a \mapsto \mathrm{ad}(a)^{2n} b \qquad \text{and} \qquad [a,b] \mapsto 0$$

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.
2. Infinitesimal Tate curve $E^{\times}_{\partial/\partial q}$ (fiber of universal punctured elliptic curve at tangential base point at the cusp of $M_{1,1}$)
3. $\mathrm{Lie}\,\pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$ is free Lie algebra $\mathbb{L}(a,b)$.
4. $\pi_1(M_{1,1}, \partial/\partial q) = \mathrm{SL}_2(\mathbb{Z})$. Relative completion $\mathrm{SL}_2(\mathbb{Z})^{rel}$ acts on $\mathbb{L}(a,b)$ via derivations (Nakamura) in $\mathrm{Der}\mathbb{L}(a,b)$

$$\varepsilon_{2n} : a \mapsto \mathrm{ad}(a)^{2n} b \qquad \text{and} \qquad [a,b] \mapsto 0$$

5. Hain constructed a morphism of MHS (MT motives over $\mathbb{Z}$) $\pi_1^{un}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1) \to \pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$.

Geometric interpretation

1. The geometric meaning of the elements ξ_{2n+1}, in depths ≤ 3 is clear. I have no geometric interpretation for τ.
2. Infinitesimal Tate curve $E^{\times}_{\partial/\partial q}$ (fiber of universal punctured elliptic curve at tangential base point at the cusp of $M_{1,1}$)
3. $\mathrm{Lie}\,\pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$ is free Lie algebra $\mathbb{L}(a,b)$.
4. $\pi_1(M_{1,1}, \partial/\partial q) = \mathrm{SL}_2(\mathbb{Z})$. Relative completion $\mathrm{SL}_2(\mathbb{Z})^{rel}$ acts on $\mathbb{L}(a,b)$ via derivations (Nakamura) in $\mathrm{Der}\mathbb{L}(a,b)$
$$\varepsilon_{2n} : a \mapsto \mathrm{ad}(a)^{2n} b \qquad \text{and} \qquad [a,b] \mapsto 0$$
5. Hain constructed a morphism of MHS (MT motives over $\mathbb{Z}$) $\pi_1^{un}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1) \to \pi_1^{un}(E^{\times}_{\partial/\partial q}, \vec{1}_1)$.
6. Dictionary: $s \leftrightarrow$ Hain morphism, $\xi_{2n+1} \leftrightarrow \varepsilon_{2n+2}$, and $\xi_{-1} \leftrightarrow b\partial/\partial a$ in $\mathfrak{sl}_2$. Only holds in depths ≤ 3!??