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Avril 2006

IHES/P/06/21

AEI-2006-023
IHES/P/06/21
MIT-CTP-3735
ULB-TH/06-07
hep-th/0604143

# Curvature Corrections and Kac–Moody Compatibility Conditions

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**Abstract:** We study possible restrictions on the structure of curvature corrections to gravitational theories in the context of their corresponding Kac–Moody algebras, following the initial work on $E_{10}$ in Class. Quant. Grav. **22** (2005) 2849. We first emphasize that the leading quantum corrections of M-theory can be naturally interpreted in terms of *(non-gravity) fundamental weights* of $E_{10}$. We then heuristically explore the extent to which this remark can be generalized to all over-extended algebras by determining which curvature corrections are compatible with their weight structure, and by comparing these curvature terms with known results on the quantum corrections for the corresponding gravitational theories.

# 1 Introduction

The study of the BKL limits [1] of coupled gravity-matter systems near a space-like singularity has revealed an interesting 'correspondence' between the emerging cosmological billiard and over-extended Kac–Moody algebras (KMAs). The cosmological billiard describes the dynamics of a few effective degrees of freedom (spatial Kasner exponents and dilatons) moving in a space bounded by walls that become sharper as one goes towards the singularity. The unexpected connection to the theory of KMAs is that the position of these walls is related to root vectors in the root lattice of certain special KMAs called the over-extended algebras [2, 3] (for a review and references see [4]). Important aspects of the classical two-derivative action are reflected in algebraic properties of the corresponding KMAs. For example, regularity or chaos of the motion as one approaches the singularity is tied to hyperbolicity of the algebra [5]: for hyperbolic KMAs there is chaos.

In recent work devoted to M-theory [6], it was shown that the higher-derivative quantum corrections to the action admit an interpretation in terms of the Kac–Moody structure. More precisely, higher-order corrections in the curvature (Riemann or Weyl) tensor and the 4-form field strength were associated in [6] to certain negative *imaginary* roots on the $E_{10}$ root lattice. There are two reasons for the potential importance of this result, namely $(i)$ the possibility that the 'geodesic' $E_{10}/K(E_{10})$ $\sigma$-model may contain hidden information about (perturbative) higher order corrections of M-Theory to arbitrary orders, and $(ii)$ the fact that this result may allow one to understand the physical significance of imaginary roots (recall that, also on the mathematical side, this is where the main obstacles towards a better understanding of indefinite KMAs lie).

Here, we elaborate on these results and conjecturally generalize them to other over-extensions of finite-dimensional simple Lie algebras.[1] For simplicity, we only consider the pure curvature corrections, which provide the dominant terms; the generalization of our results to other types of fields ($p$-forms) is straightforward, at least in principle.[2]

We first observe that the imaginary roots that describe the quantum corrections in the $E_{10}$ case are not just any arbitrary roots but, rather, are the *dominant non-gravity weights*. More precisely, as shown in section 3 below,

[1] The particular case of over-extended $G_2$ was already analysed in [7].

[2] Corrections involving field strengths give rise to billiard walls which are 'hidden' behind those from curvature corrections [6].

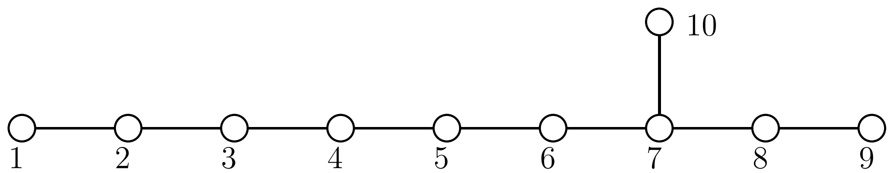


Figure 1: *Dynkin diagram of $E_{10}$ with numbering of nodes.*

the root of $E_{10}$ associated with the leading terms in the first quantum corrections $R^4$ quartic in the curvature turns out to be the fundamental weight $\Lambda_{10}$ conjugate to the 'non-gravity' root $\alpha_{10}$, i.e., to the ('exceptional') root which does not lie on the $A_9 \equiv \mathfrak{sl}(10)$ 'gravity line' (see figure 1). This property ensures that $\Lambda_{10}$ is invariant under permutations of the spatial directions (Weyl group of $\mathfrak{sl}(10)$). The leading terms in the expected subsequent quantum corrections $R^7$, $R^{10}$ *etc.* are associated with positive integer multiples of $\Lambda_{10}$ and are thus also invariant under the Weyl group of $\mathfrak{sl}(10)$.

This suggests that it might not be the root lattice which is relevant for the correction terms, but rather the *weight lattice*. We recall that the weight lattice is the lattice spanned by the fundamental weights $\Lambda_i$ obeying

$$\frac{2(\Lambda_i|\alpha_j)}{(\alpha_i|\alpha_i)} = \delta_{ij}, \tag{1}$$

where the $\alpha_i$ are the simple roots spanning the root lattice. The weight lattice always contains the root lattice as a sublattice but is generically finer. We also recall that the set of 'dominant weights' is defined by taking integral linear combinations of the fundamental weights with *non-negative* coefficients. Correspondingly, we define 'dominant non-gravity weights' by taking non-negative integral linear combinations of those fundamental weights not conjugate to roots on the 'gravity line'. Here, we shall generally define the 'gravity line' by the simple roots associated with the so-called 'symmetry walls', i.e., the Hamiltonian contributions which are related to the off-diagonal components of the metric [4].

The purpose of this paper is to explore to what extent this 'botanical observation' inspired by the M-theory/$E_{10}$ analysis (concerning the role of *dominant non-gravity weights* of the corresponding over-extended KMA), can be extended to higher-order curvature corrections in other (super)gravity models. We shall therefore systematically determine, for all over-extended

Kac–Moody algebras which higher order curvature corrections are associated with dominant non-gravity weights, and compare the results of this heuristic 'algebraic selection rule' to the currently known quantum corrections for the corresponding gravitational theories.

For $E_{10}$, the root and the weight lattices happen to coincide, and therefore the distinction between weights and roots did not play any role in the analysis of [6]. In fact, $E_{10}$ is the *only* over-extended KMA for which the root lattice (usually designated by $Q(E_{10})$) has this property: $Q(E_{10}) = \mathrm{II}_{9,1}$ is the unique even self-dual Lorentzian lattice in ten dimensions, and such lattices are known to exist only in $2 + 8n$ dimensions [8].

For other algebras, however, we will find that there are correction terms associated with genuine weights *outside* the root lattice. This fact appears to imply that these corrections cannot be described within the geodesic $\sigma$-model, and that one will have to augment the $\sigma$-model Lagrangian by additional terms related to these weights if the correspondence is to be extended to higher order corrections. *Put differently, it is only for the maximally extended $E_{10}$ model that one can argue, as was done in [6], that not only the effective low energy theory, but the entire tower of higher order corrections might be understood on the basis of a single geodesic $\sigma$-model Lagrangian.*

Even in the $E_{10}$ case, the observation that the imaginary roots that arise are actually fundamental weights sheds a new, potentially interesting, light on the structure of the subleading quantum corrections since fundamental weights are naturally linked with representations. We indicate that the pattern for these subleading terms uncovered in [6] is indeed similar to that of a lowest-weight representation (albeit a non-integrable one).

The assumption explored here that quantum corrections are controlled by dominant non-gravity weights gives us restrictions on the type of curvature corrections which are consistent with the algebraic structure. We have analysed these restrictions for all over-extended algebras $\mathfrak{g}^{++}$ in split real form. The resulting restrictions are quite satisfactory for M-theory – as originally found in [6] – but there are mismatches which remain to be understood for the other string-related cases. This is particular striking for types IIA and IIB, where some corrections known to appear do not define points on the weight lattice. Although this can be blamed on the singular field theory limits involved in passing from M-theory to the ten-dimensional models, we lack a deeper group-theoretical reason as to why one finds perfect matching in some cases and not in others.

Our paper is organized as follows. We first recall how to compute the walls for a given Lagrangian and powers of the curvature tensor in section 2. Section 3 re-examines the case of eleven-dimensional gravity and $E_{10}$ using the dominant weight perspective. In section 4 we then repeat the analysis for $DE_{10}$ and $BE_{10}$ for which the introduction of the weight lattice becomes evident. The analogous results for the remaining over-extended algebras are presented in section 5, illustrating the particular properties of the ten-dimensional hyperbolic KMAs in the class of all over-extended algebras. In section 6, we analyse $E_{10}$ under decompositions appropriate for interpretations in terms of the ten-dimensional IIA and IIB theories. Concluding remarks can be found in section 7.

# 2 Weights and curvature monomials

We follow the systematics of [4, 3] for constructing walls corresponding to any term in an effective Lagrangian of the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \sum_{m\geq 1} (\mathrm{L}^s)^m \mathcal{L}^{(m)}, \tag{2}$$

where $\mathcal{L}^{(0)}$ is the lowest order Lagrangian (quadratic in derivatives), and L is a dimensionful expansion parameter of dimension [Length] for the higher derivative correction terms $\mathcal{L}^{(m\geq 1)}$. The integer $s$ depends on the theory in question; for instance, in the case of string theories one has $s = 2$ and $\mathrm{L}^2 = \alpha'$. We will take $\mathcal{L}^{(0)}$ to be the Lagrangian of the maximally oxidized theory for a simple split Lie algebra $\mathfrak{g}$. These Lagrangians were studied and given in [9]; in particular, the maximal space-time dimension $D = (d + 1)$ was determined for all $\mathfrak{g}$. The wall forms are already completely determined by $\mathcal{L}^{(0)}$ [4, 3]. They are parametrised by a dilaton field $\varphi$ (when it exists in dimension $D$), and by the logarithmic *scale factors* $\beta^a$ (for $a = 1, \dots, d$), which appear via the following split of the spatial vielbein

$$e_m{}^a = \exp(-\beta^a)\theta_m{}^a \tag{3}$$

where the spatial frame $\theta_m{}^a$ 'freezes' near the singularity [4]. These wall forms are explicitly written as

$$w(\beta, \varphi) = \sum_{a=1}^{d} p_a \beta^a + p_\varphi \varphi. \tag{4}$$

The inner product between two such wall forms $w$ and $w'$ is determined from the Hamiltonian constraint following from the Einstein-Hilbert action in the standard way, and given by [3]

$$(w|w') = \sum_{a=1}^{d} p_a p'_a - \frac{1}{d-1}\left(\sum_{a=1}^{d} p_a\right)\left(\sum_{a=1}^{d} p'_a\right) + 2p_\varphi p'_\varphi. \tag{5}$$

For KMAs with non-symmetric generalized Cartan matrix this will yield the symmetrized form with some values different from 2 on the diagonal since not all roots have the same length. For later use, we also define the logarithm of the spatial volume factor $\det e = \sqrt{g}$, namely

$$\sigma = \sum_{a=1}^{d} \beta^a. \tag{6}$$

Near a space-like singularity, the Lagrangian (2) can be replaced by an effective Lagrangian describing the dynamics of the logarithmic scale factors $\beta^a$ and the dilaton $\varphi$ in an effective potential $\mathcal{V}$ that behaves asymptotically like a sum of (exponentially sharp) billiard walls

$$\mathcal{V}(\beta,\varphi) \sim \sum_{w} c_w e^{-2w(\beta,\varphi)} \tag{7}$$

for a number of walls $w$. In all cases, the *dominant walls* (those which have the largest contribution in the limit) are located at positions describing the fundamental Weyl chamber of the Kac–Moody-theoretic over-extension $\mathfrak{g}^{++}$ of the symmetry algebra $\mathfrak{g}$ [4].

For simplicity, we will be concerned with curvature correction terms of the type [4] (with $s = 2$ and $\mathrm{L}^2 = \alpha'$)

$$\mathcal{L}^{(N-1)} \sim \sqrt{-G}\, R^N e^{K\varphi}. \tag{8}$$

[3]Our conventions here differ from [3] by a factor of 2 for the dilaton terms since we normalise the dilaton kinetic term with a factor of 1/2 in the Einstein frame. When there are several dilaton fields $\vec{\varphi} \equiv (\varphi_1, \varphi_2, \cdots)$, the dilaton contributions in the wall forms and the wall scalar product read $\vec{p}_\varphi \cdot \vec{\varphi}$ and $2\vec{p}_\varphi \cdot \vec{p}\,'_\varphi$, respectively.

[4]Here, $R^N$ denotes some $N$th- order polynomial in the curvature tensor. Note that, while the inclusion of Ricci and scalar curvature terms do not matter for the leading terms, they do for the subleading ones [6].

As was shown in [6] the dominant contribution of such a term to the Hamiltonian density scales as

$$e^{2(N-1)\sigma+K\varphi} \tag{9}$$

in the asymptotic limit. Equating this with a Hamiltonian wall potential $e^{-2w(\beta,\varphi)}$ yields the corresponding wall form

$$w_{N,K}(\beta,\varphi) = -(N-1)\sigma - \frac{1}{2}K\varphi. \tag{10}$$

(or $w_{N,K}(\beta,\vec{\varphi}) = -(N-1)\sigma - \frac{1}{2}\vec{K}\cdot\vec{\varphi}$ in the multi-dilaton case). It is important to keep in mind that (10) is the form of the wall *in the Einstein frame.* In order to make contact with known results from string perturbation theory it will be necessary to convert such a wall into the string frame. The notion of Einstein frame exists in any dimension, and means that the Einstein Hilbert term appears without dilatonic prefactors. By contrast, the notion of string frame most readily applies in $D = 10$ dimensions, and means that the tensor-scalar part of the action reads $e^{-2\Phi}\sqrt{-\tilde{G}}(\tilde{R} + 4(\tilde{\partial}\Phi)^2)$, where $\Phi$ is the standard string dilaton field (with $g_s = e^{\Phi}$). This parametrisation is natural if there is an underlying world-sheet theory in which $\Phi$ couples to the world-sheet curvature scalar. The effective theory will then admit an expansion in genera of the world-sheet and higher order corrections at $g$-loop order in string theory come with a factor $e^{(2g-2)\Phi}$.

In *ten dimensions* the relation between the two frames, and the two dilatons, as obtained by identifying[5]

$$\sqrt{-G}\big[R - \frac{1}{2}(\partial\varphi)^2\big] = e^{-2\Phi}\sqrt{-\tilde{G}}\big[\tilde{R} + 4(\tilde{\partial}\Phi)^2\big] \tag{11}$$

reads

$$\tilde{G}_{MN} = e^{\frac{1}{2}\Phi}G_{MN} \qquad ; \quad \Phi = \varphi\,, \tag{12}$$

where $G_{MN}$ and $\tilde{G}_{MN}$ denote the metrics in the Einstein frame and the string frame, respectively. The corresponding relation for the higher order terms is therefore

$$\sqrt{-G}R^N = e^{\frac{1}{2}(N-5)\varphi}\sqrt{-\tilde{G}}\tilde{R}^N + \dots \tag{13}$$

From this we immediately read off the formula relating the coefficient $K$ multiplying the dilaton for the corresponding wall forms between the two

[5] Where the tilde on $\partial$ indicates that the partial derivatives are to be contracted with the metric $\tilde{G}^{MN}$.

frames in ten dimensions, which reads

$$K \to K_{\rm string} \equiv K + \frac{1}{2}(N-5). \tag{14}$$

In our survey of KMAs we will also make use of the generalization of this formula to $D \neq 10$ dimensions. Here, we define the string frame in any dimension $D$ by requiring that the tensor-scalar sector of the action density reads as the right-hand side of equation (11) above. This yields

$$\tilde{G}_{MN} = e^{\frac{4}{D-2}\Phi} G_{MN} \qquad ; \quad \Phi = \sqrt{\frac{D-2}{8}}\varphi\,, \tag{15}$$

instead of the $D = 10$ result (12) above. The corresponding generalization of the transformation law for the dilaton coupling coefficient reads

$$K_{\rm string} \equiv \sqrt{\frac{8}{D-2}}K + \frac{2}{D-2}(2N-D). \tag{16}$$

# 3 $E_{10}$ revisited

## 3.1 Non-gravity weight $\Lambda_{10}$ and $R^4$ correction

We briefly recall the result of [6] for $D = 11$ supergravity and $E_{10} \equiv E_8^{++}$, for which there is no dilaton present. The dominant walls (=simple roots) are given explicitly by

$$\begin{aligned}
\alpha_1 &= (0,0,0,0,0,0,0,0,-1,1) &\quad\Rightarrow\quad& \alpha_1(\beta) = -\beta^9 + \beta^{10},\\
&\vdots && \\
\alpha_9 &= (-1,1,0,0,0,0,0,0,0,0) &\quad\Rightarrow\quad& \alpha_9(\beta) = -\beta^1 + \beta^2,\\
\alpha_{10} &= (1,1,1,0,0,0,0,0,0,0) &\quad\Rightarrow\quad& \alpha_{10}(\beta) = \beta^1 + \beta^2 + \beta^3.
\end{aligned}$$

Here the components of each root are the 'covariant' coordinates $p_a$ in the $\beta^a$ basis of Eq.(4) (with $d = 10$, and without dilaton contribution). We use the numbering of nodes and simple roots indicated in fig. 1. In [6], it was demanded that any (leading) correction be associated with a root. Since there is no dilaton, this amounts to requiring $-(N-1)\sigma$ to be a root. This occurs only when $N-1$ is an integer multiple of 3. The smallest value, $N-1 = 3$, yields the imaginary, negative root $-3\sigma$, which has length squared $-10$.

Now, the crucial observation made here is that $-3\sigma$ corresponds to the non-gravity fundamental weight $\Lambda_{10}$

$$\Lambda_{10} = -(3,3,3,3,3,3,3,3,3,3) \quad \Rightarrow \quad \Lambda_{10}(\beta) = -3\sigma \tag{17}$$

conjugate to $\alpha_{10}$ by (1). Therefore, asking that the higher curvature corrections be associated with dominant non-gravity weights, i.e. of the type $a\Lambda_{10}$ for some non-negative integer $a$, reproduces the result of [6] for the $E_{10}$ prediction of curvature corrections to $D = 11$ supergravity. Indeed, equating $a\Lambda_{10} = -3a\sigma$ to $-(N-1)\sigma$ from (10) yields $N = 3a+1$, i.e. allowed curvature corrections $R^{1+3a}$, that is $R^4, R^7, R^{10}, \ldots$.

## 3.2 Subleading terms and representations

The pattern found in [6] for the subleading terms in the $R^4$ supersymmetry multiplet is also reminiscent of lowest weight (non-integrable) representations since the weights associated with these subleading terms (which are, in the $E_{10}$ case, all on the root lattice) were found to be obtained by *adding positive roots* to the dominant weight $\Lambda_{10}$. This weight pattern does correspond to (the beginning of) a representation of $E_{10}$, with $\Lambda_{10}$ as lowest weight. It is easily checked that the wall forms corresponding to the supersymmetry multiplets of the higher-order curvature corrections $R^{1+3a}$ will similarly resemble the weight patterns of representations with lowest weight $a\Lambda_{10}$.

However, these representations are not the usually considered 'integrable lowest-weight' representations [10]. Indeed, integrable lowest-weight representations must have the *negative* of dominant weights as lowest weight. This is illustrated in figure 2 .

With the convention that simple roots are in the *future* of the basic spacelike hyperplane separating positive from negative roots, the fundamental weights are located in the *past* light cone. A typical integrable lowest-weight diagram would then lie within the future light cone and would extend *upwards* from the negative of a dominant weight [6] (see upper part of figure 2). By contrast, the formal weight pattern corresponding to the curvature correction $R^{1+3a}$ extends upwards from $a\Lambda_{10}$, which lies in the middle of the past light cone. Therefore, if these weight patterns do correspond to lowest-weight representations, these must be of the non-integrable type (which has

[6]Similarly, a highest weight representation would extend downwards from a dominant weight in the past lightcone.

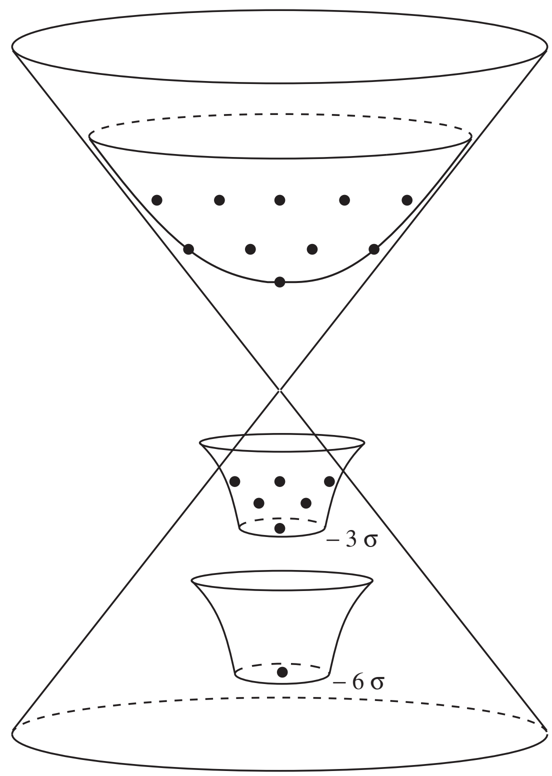


Figure 2: *Comparison between the weight pattern of a typical integrable lowest-weight representation (within the future light cone), and the weight patterns corresponding to the supersymmetry multiplets of the curvature corrections in $M-$theory. The weight pattern corresponding to $R^{1+3a}$ extends upward from the lowest weight $a\Lambda_{10} = -3a\sigma$ which lies within the past light cone.*

not been thoroughly studied because of its greater mathematical complexity). [We have not checked whether multiplicities match for the weights that are actually present.] It is interesting to point out that among the subleading terms analysed in [6], some correspond to weights $\Lambda_{10} + \alpha$ involving positive imaginary roots $\alpha$ up to the $A_9$ level 8.

It is also interesting to remark that there is another similarity between the pattern of weights entering curvature corrections, and the weight diagrams of integrable lowest-weight representations. The weight diagrams $\{\Lambda\}$ of integrable lowest-weight representations can be shown [10] to be contained in the convex hull of the quadric $\Lambda^2 = \Lambda^2_{lowest}$ passing by the lowest weight $\Lambda_{lowest}$. Rather similarly, it was 'botanically' observed in [6] that all the wall forms in the $R^4$ supermultiplet considered there are contained within a quadric defined by the equation $(\Lambda - \Lambda_{lowest})^2 = 2$. Another observation is that the enveloping hyperboloid becomes broader as one moves deeper into the past light cone. For $\Lambda_{lowest} = 2\Lambda_{10}$ the condition is $(\Lambda - \Lambda_{lowest})^2 \leq 8$. These facts are illustrated in in figure 2.

# 4 Correction terms for other rank 10 KMAs

We shall now investigate how the weight structure fits with the other over-extensions which have, in general, more than one non-gravity root.

## 4.1 $DE_{10}$

We start with $DE_{10} \equiv D_8^{++}$. This rank 10 hyperbolic KMA is associated with the bosonic part of pure type I supergravity [2, 11]. The corresponding string theory (type I') is obtained from type I string models by dropping the vector multiplet and keeping only the gravity multiplet (through a positive charge orientifold plane). Note that the bosonic sector is also identical with the low energy effective action of the closed bosonic string in ten space-time dimensions. Because $DE_{10}$ is hyperbolic, its fundamental weights are within or on the *past* lightcone. Thus, the dominant weights that are also roots are necessarily negative, imaginary roots.

There is one dilaton and the wall forms in this case are given by

$$\begin{aligned} \alpha_1 &= (0,0,0,0,0,0,0,-1,1;0) \\ &\vdots \\ \alpha_8 &= (-1,1,0,0,0,0,0,0,0;0) \\ \alpha_9 &= (1,1,0,0,0,0,0,0,0;-\tfrac{1}{2}) \\ \alpha_{10} &= (1,1,1,1,1,1,0,0,0;+\tfrac{1}{2}) \end{aligned} \tag{18}$$

The components listed here are the 'covariant' coordinates $(p_a, p_\varphi)$ in the $(\beta^a, \varphi)$ basis of Eq. (4). We use the numbering of nodes indicated in fig. 3. The first eight nodes give rise to the symmetry walls (hence they form the

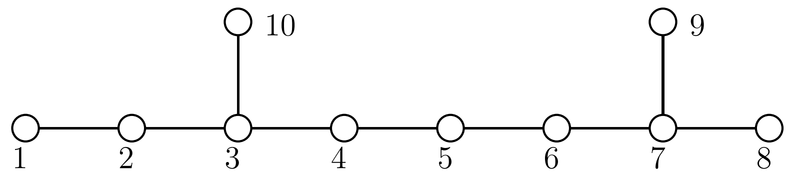

Figure 3: *Dynkin diagram of $DE_{10}$ with numbering of nodes.*

gravity line) and the nodes 9 and 10 correspond to the NSNS 2-form and its dual 6-form, respectively.

The non-gravity fundamental weights are $\Lambda_9$ and $\Lambda_{10}$; explicitly

$$\Lambda_9(\beta,\varphi) = -\sigma - \frac{3}{4}\varphi \quad , \qquad \Lambda_{10}(\beta,\varphi) = -\sigma + \frac{1}{4}\varphi. \tag{19}$$

Being a sublattice of $Q(E_{10})$ ($DE_{10}$ is a subalgebra of $E_{10}$ [12]), the root lattice $Q(DE_{10})$ is not self-dual[7]. In fact, neither $\Lambda_9$ nor $\Lambda_{10}$ is on the root lattice of $DE_{10}$. This is easy to verify for $\Lambda_{10}$, which has norm squared equal to $-1$, and can also be checked for $\Lambda_9$ (which is lightlike). Note that the non-gravity fundamental weight $\Lambda_{10}^{E_{10}}$ of $E_{10}$ is a positive linear combination of the non-gravity fundamental weights of $DE_{10}$,

$$\Lambda_{10}^{E_{10}} = \Lambda_9 + 2\Lambda_{10}. \tag{20}$$

As explained above, we shall explore here the generalized conjecture that curvature corrections correspond to positive integer combinations of non-gravity fundamental weights. Let us then consider, within the context of $DE_{10}$, the dominant weight $a\Lambda_9 + b\Lambda_{10}$, with $a, b$ some non-negative integers. Equating this form with a putative curvature form of the type of (10) gives the Einstein frame result

$$N = 1 + a + b, \qquad K = \frac{3}{2}a - \frac{1}{2}b. \tag{21}$$

In terms of string frame variables (see (14)) we obtain

$$N = 1 + a + b, \qquad K_{\text{string}} = -2 + 2a. \tag{22}$$

[7]The well-known link between the 11-dimensional metric and 3-form of M-theory on the one hand and the type-IIA 10-dimensional metric, dilaton and $p$-forms on the other hand, dictates the embedding used here. By comparing Eq.(18) with Eq.(31) below, one sees that the simple roots of $DE_{10}$ are given in terms of the simple roots of $E_{10}$ through $\alpha_i^{DE_{10}} = \alpha_i^{E_{10}}$ $(i = 1, \cdots, 8)$, $\alpha_9^{DE_{10}} = \alpha_{10}^{E_{10}}$ and $\alpha_{10}^{DE_{10}} = 2\alpha_9^{E_{10}} + 2\alpha_{10}^{E_{10}} + 3\alpha_8 + 4\alpha_7 + 3\alpha_6 + 2\alpha_5 + \alpha_4$.

This result has the following properties

- The dilaton coefficients in the string frame which are compatible with $DE_{10}$ are those which appear in the genus expansion of closed string perturbation theory, with the coefficient $a$ of $\Lambda_9$ 'counting' the number of string loops. We do not have an understanding of why the ten-dimensional hyperbolic algebra encodes this 'stringy' property outside of supergravity. We will see this rather tantalizing property also occurs for the Kac–Moody correspondants of some of the other ten-dimensional theories to be studied below.

- For fixed power $N$ of the curvature correction $R^N$ only a finite number of values for $a, b$ are allowed (since we assumed that $a\Lambda_9 + b\Lambda_{10}$ is dominant, i.e. $a, b \geq 0$). In view of the string loop interpretation above, this means that only contributions to $R^N$ from string diagrams with at most $N-1$ loops are consistent with the structure of the $DE_{10}$ weight lattice.

- The first KM-allowed curvature corrections are at order $R^2$ and arise when the pair $(a, b)$ takes the values $(0, 1)$ or $(1, 0)$. Neither of these weights is a root. The accompanying power of the dilaton in the string frame is $e^{-2\Phi}$ in the first case and $e^{0\Phi}$ in the second case suggesting an interpretation as string tree level and string one loop contribution, respectively.

- Another curious observation is that the known 'string tree level' $R^2$ correction term does not receive a contribution from $\Lambda_9$ suggesting that in fact there are correction terms related to it by $SO(9,9)$ rotations. In other words there is an (expected) sign of T-duality invariance for the corrections. Results in this direction have been obtained in [13].

Because the quantum corrections to type I' string theory have not been explicitly computed, there is not much to be added here. Our above comments are thus predictions on which terms in type I' are forbidden in the sense of not being compatible with the weight structure.

## 4.2 $BE_{10}$

Adding one Maxwell vector multiplet to pure type I supergravity yields the hyperbolic algebra $BE_{10} \equiv B_8^{++}$ [2]. Accordingly, we have the embeddings

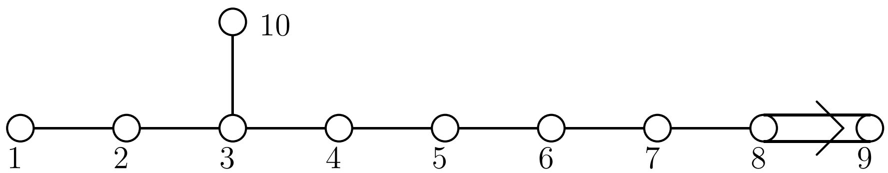

Figure 4: *Dynkin diagram of $BE_{10}$ with numbering of nodes.*

$DE_{10} \subset BE_{10}$ and $Q(DE_{10}) \subset Q(BE_{10})$, which is explicitly displayed by observing that $\alpha_9^{DE_{10}} = 2\alpha_9^{BE_{10}} + \alpha_8$. The dominant walls are taken to be

$$\begin{aligned} \alpha_1 &= (0,0,0,0,0,0,0,-1,1;0) \\ &\vdots \\ \alpha_8 &= (-1,1,0,0,0,0,0,0,0;0) \\ \alpha_9 &= (1,0,0,0,0,0,0,0,0;-\tfrac{1}{4}) \\ \alpha_{10} &= (1,1,1,1,1,1,0,0,0;+\tfrac{1}{2}) \end{aligned} \tag{23}$$

We use the numbering of nodes indicated in fig. 4, with the gravity line consisting of nodes 1 through to 8. Node 9 gives rise to the wall of the Maxwell vector field and node 10 to the 6-form dual to the NSNS 2-form.

The non-gravity fundamental weights again are $\Lambda_9$ and $\Lambda_{10}$ which are computed to be

$$\Lambda_9(\beta,\varphi) = -\sigma - \frac{3}{4}\varphi \quad , \qquad \Lambda_{10}(\beta,\varphi) = -\sigma + \frac{1}{4}\varphi. \tag{24}$$

They coincide with the fundamental non-gravity weights of $DE_{10}$. Note that $\Lambda_{10}$ is on the root lattice of $BE_{10}$ — in fact it is a root, but $\Lambda_9$ is not. Because $\Lambda_9$ and $\Lambda_{10}$ are the same as for $DE_{10}$, we again get

$$N = 1 + a + b \quad , \qquad K = \frac{3}{2}a - \frac{1}{2}b \tag{25}$$

in the Einstein frame. Therefore the $BE_{10}$ compatible string frame corrections satisfy

$$N = 1 + a + b \quad , \qquad K_{\text{string}} = -2 + 2a. \tag{26}$$

As in the pure supergravity $DE_{10}$ case, the dilaton coefficients agree with (closed) string perturbation theory. Eq. (26) also predicts that for $R^N$ only contributions from string diagrams with up to $(N-1)$ loops are compatible with the $BE_{10}$ weight lattice structure.

The dominant weight $a\Lambda_9 + b\Lambda_{10}$ is a root of $BE_{10}$ only for even (non negative) $a$'s. If one were to restrict the quantum corrections to be roots, one would miss the corrections involving an odd number of loops.

## 4.3 Heterotic and Type I cases

### 4.3.1 General Considerations

If one adds $k$ abelian vector multiplets to pure type I supergravity in ten dimensions, the relevant Kac–Moody algebra is $\mathfrak{so}(8, 8+k)^{++}$ [14], which is a non-split real form. The analysis proceeds in this case in a way very similar to that of $BE_{10}$, because it is that subalgebra that controls the (real) roots and weights.

To understand this point, consider first the familiar toroidal dimensional reduction to three spacetime dimensions. After dualization of all non-metric fields to scalars, the theory is described by the three-dimensional Einstein-Hilbert action coupled to the non-linear sigma model action for the coset space $SO(8, 8+k)/SO(8) \times SO(8+k)$. This action is most easily written down using the Iwasawa decomposition appropriate to non-split real forms [15]. This decomposition, in turn, follows from the Tits-Satake decomposition of the algebra $\mathfrak{so}(8, 8+k)$ in terms of real root spaces, which we briefly recall. The split algebra $B_8 \equiv \mathfrak{so}(8, 9)$ is a maximal split subalgebra of $\mathfrak{so}(8, 8+k)$. One can decompose over the reals $\mathfrak{so}(8, 8+k)$ in terms of representations of $\mathfrak{so}(8, 9)$. The Cartan generators of $B_8$ have indeed real eigenvalues, while the other (compact) Cartan generators of $\mathfrak{so}(8, 8+k)$ have imaginary eigenvalues and cannot be diagonalized over the reals. Because the rank of $B_8$ (known also as the real rank of $\mathfrak{so}(8, 8+k)$) is eight, the weights are 8-dimensional vectors. They turn out to coincide with the roots of $B_8$, i.e., with the weights of the adjoint representation of $B_8$, but they come with a non trivial multiplicity. Specifically, the short $B_8$-roots appear $k$ times, while the long roots are non-degenerate. And furthermore, the zero eigenvalue also appears in the spectrum. Their associated eigenvectors are the elements of $\mathfrak{so}(k)$. Corresponding to this decomposition, the coset Lagrangian for $SO(8, 8+k)/SO(8) \times SO(8+k)$ takes a form very similar to that of $SO(8, 9)/SO(8) \times SO(9)$, namely: $(i)$ there are 8 dilatons with standard kinetic term, because 8 is the real rank of $\mathfrak{so}(8, 8+k)$ i.e., the rank of its maximal split subalgebra $\mathfrak{so}(8,9)$; $(ii)$ for each positive root of $B_8$, *counting* $\mathfrak{so}(8,8+k)$ *multiplicities*, there is an axion with a kinetic term multiplied by the exponential of the corresponding root.

Coming back to the non reduced model, it is natural to conjecture that it is dual to the geodesic motion on the infinite dimensional coset space $SO(8,8+k)^{++}/K(SO(8,8+k)^{++})$. Again, the algebra $\mathfrak{so}(8,8+k)^{++}$ is a representation of its maximal split subalgebra $\mathfrak{so}(8,9)^{++} \equiv B_8^{++} \equiv BE_{10}$. The 'real roots' of $\mathfrak{so}(8,8+k)^{++}$ are the weights of that representation. They are equal to the roots of $BE_{10}$, but come with some non trivial multiplicity (over and above the Kac–Moody multiplicity of the $BE_{10}$ imaginary roots). For instance, the real root $\alpha_9$ has multiplicity $k$. It is the maximally split subalgebra $BE_{10}$ that one sees in the $\sigma$ model Lagrangian. In particular, the billiard region is the same as in the case of one Maxwell multiplet, the only difference being that the electric wall (associated with the simple root $\alpha_9$) appears $k$ times.

As the representations of $\mathfrak{so}(8,8+k)^{++}$ are characterized by real weights that are weights of $B_8^{++} \equiv BE_{10}$, the quantum corrections should be associated with weights of $BE_{10}$. Note that the non-trivial multiplicities related to the fact that the algebra is a non-split real form concerns only the terms involving the gauge fields, *which are subleading.*

$\mathfrak{so}(8,8+k)^{++}$-supergravity theories, i.e., supergravity theories with 16 supercharges, one gravity multiplet and $k$ vector multiplets, have been analyzed in the string context in [16]. It was found that string backgrounds consistent with this matter content could exist, but only in particular spacetime dimensions, which depend on $k$. For instance, the $k = 1$ theory has global anomalies except in 3 spacetime dimensions and below. For $k = 2$, the maximal spacetime dimension is 5. This suggests that it would be of interest to repeat the $BE_{10}$ weight analysis in lower dimensions. However, since the effective actions for those theories have not been much studied, we have not performed here that analysis.

If one wants to generalize the abelian group $U(1)^k$ to some non-abelian gauge group (as required for the heterotic string), one encounters the difficulty that there is no 'nice' and obvious choice of KMA that would naturally accommodate the Yang Mills gauge groups. Because the rank of the gauge groups relevant to string theory in 10 dimensions is 16, one might nevertheless argue that it is the algebra $\mathfrak{so}(8,8+16)^{++} = \mathfrak{so}(8,24)^{++}$ and thus the maximal split algebra $BE_{10}$ that controls the weight pattern. Evidence for this comes from the billiard analysis [2] and was also given in [17]. The above study of quantum corrections for $BE_{10}$ would therefore again apply.[8]

Because quantum corrections to the heterotic string and type I string models have been studied, one can check whether they are compatible with the $BE_{10}$ weight structure. The analysis proceeds differently in the two cases.

### 4.3.2 Heterotic String

The first known curvature corrections to the heterotic string are at order $R^2$ and by studying eq. (22) we see that there are two solutions compatible with $BE_{10}$ when the pair $(a, b)$ takes the values $(0, 1)$ or $(1, 0)$. The accompanying power of the dilaton in string frame is $e^{-2\Phi}$ in the first case and $e^{0\Phi}$ in the second case corresponding to string tree level and string one loop contribution. In fact the first term (tree level) has been computed in [18] and agrees with our result. The second term (one-loop), however, has been argued to be absent [19]. This shows that the algebraic constraints heuristically investigated here may play a role somewhat analogous to selection rules: they can be used to predict which terms should be absent, but not which terms should be actually present (indeed, further hidden symmetries might cancel a term allowed by general selection rules). The same situation holds for $R^3$ corrections, which are allowed at tree level, one-loop and two-loops by our algebraic constraints, but which have been argued to be absent on account of supersymmetry [19] (see also [20, 21]). Finally, $R^4$ corrections up to three loops are permitted by the $BE_{10}$ algebraic constraints. However, although $R^4$ corrections are known to be present in the heterotic effective action, their actual loop dependence is less clear [19].

### 4.3.3 Type I

To analyze the type I superstring, one must recall that the transition from the Einstein frame (where we have derived the algebraically compatible counterterms) to the string frame is different than in the heterotic case: the type I dilaton is minus the heterotic dilaton, and the spacetime metric changes accordingly. Converting the algebraically compatible counterterms to the type I string frame, one finds that the two $R^2$ corrections found above come

[8]There is a fourth hyperbolic rank 10 Kac–Moody algebra, namely $CE_{10}$. This algebra is dual to $BE_{10}$ and is a twisted overextension [14], although $CE_{10} \neq C_8^{++}$. Because it has not been associated to a field theoretical model, we shall not investigate it here.

with the (type I) dilaton powers $\exp(-\Phi)$ and $\exp(-3\Phi)$, respectively. The first term corresponds to the tree heterotic correction, while the second corresponds to the heterotic one-loop term. Only the first term is compatible with the type I effective action and furthermore, it is in perfect agreement with [19]. Combining the information that the second term must be absent in type I with heterotic-type I duality, one can argue that there is no one-loop $R^2$ term in the heterotic case. The same argument is too weak to eliminate all $R^3$ corrections, which are forbidden by supersymmetry.

# 5 Other algebras

## 5.1 Results

For the other over-extended algebras we give the result only in tabulated form, including also the results obtained for the rank 10 hyperbolic KMAs in the preceding section for completeness. We take the standard gravity lines (see for example [3]) and consider positive integral linear combinations of the non-gravity fundamental weights and equate them to the general leading correction wall form (10). We exclude $C_n^{++}$ from the list since the corresponding $(3+1)$-dimensional theory has $(n-3)$ dilaton fields which would clutter the notation. It can be checked however that the algebra allows for curvature correction $R^N$ for all values of $N$.

| Algebra | $(d+1)$ | $N$ in $\sqrt{-G}\,R^N e^{K\varphi}$ | $K$ in $\sqrt{-G}\,R^N e^{K\varphi}$ |
|---|---|---|---|
| $A_n^{++}$ | $n+3$ | $1+a$ | – |
| $B_n^{++}$ | $n+2$ | $1+a+b$ | $\frac{n-2}{\sqrt{2n}}a - \frac{2}{\sqrt{2n}}b$ |
| $D_n^{++}$ | $n+2$ | $1+a+b$ | $\frac{n-2}{\sqrt{2n}}a - \frac{2}{\sqrt{2n}}b$ |
| $G_2^{++}$ | 5 | $1+a$ | – |
| $F_4^{++}$ | 6 | $1+2a+b$ | $-\frac{1}{\sqrt{2}}b$ |
| $E_6^{++}$ | 8 | $1+2a+b$ | $-b$ |
| $E_7^{++}$ | 9 | $1+a+2b$ | $\frac{3}{\sqrt{7}}a - \frac{1}{\sqrt{7}}b$ |
| | 10 | $1+2a$ | – |
| $E_8^{++}$ | 11 | $1+3a$ | – |

As for the just quoted $C_n^{++}$ case, one sees that all the algebras in the table (except $E_8^{++}$ and $E_7^{++}$ in ten dimensions) are compatible with curvature corrections $R^N$ for all positive integers $N$. This is not very restrictive but the fact that only integers $N$ (as opposed to non integer values) are forced by the algebraic requirement was not a priori guaranteed. Furthermore, many of the corrections known to be present in the effective Lagrangians would not be allowed had we insisted on having only roots. For instance, for pure gravity in spacetime dimension $D = d + 1$, the corrections corresponding to roots are of the restricted form $R^{k(D-2)+1}$.

The $E_{10}$ case corresponds to M-theory and has been analysed above. The entry with $D = 10$ for $E_7^{++}$ in the table corresponds to a theory which is a non-supersymmetric truncation of IIB supergravity, keeping only gravity and the four-form potential. Therefore one has the same problem with writing a manifestly covariant Lagrangian for this theory in ten dimensions as with IIB supergravity and the theory is usually considered in $D = 9$, where all powers of the curvature are allowed. Here we see however that corrections in ten dimensions would be compatible with the algebra structure only for odd powers of the curvature.

In the case where there is a dilaton, non trivial restrictions on the allowed powers of the dilaton for a given $N$ do appear, as indicated in the right column of the table. However, it is difficult to test the validity of these predictions as the quantum corrections appearing in the effective Lagrangians of the corresponding theories have not been investigated.

## 5.2 String frame

The dilaton entries $K$ in the table are in the Einstein frame. The notion of string frame really makes sense only for the string-related theories, but one can nevertheless explore the consequences of converting the formulas into a would-be string frame by using Eq. (16). The result for $B_n^{++}$ and $D_n^{++}$ is

$$K_{\text{string}} = -2 + 2\,a \qquad (B_n^{++} \text{ and } D_n^{++}), \tag{27}$$

Thus, $K_{\text{string}}$ is an even integer that depends only on $a$ suggesting an interpretation in terms of a genus expansion in string perturbation theory for all $n$.

For the three exceptional algebras which give rise to a dilaton one finds after conversion to the string frame

$$K_{\text{string}} = -2 + 2\,a \qquad (F_4^{++}), \tag{28}$$

$$K_{\text{string}} = -2 + \frac{4}{3}a + \left(\frac{2}{3} - \frac{2}{\sqrt{3}}\right) b \qquad (E_6^{++}), \tag{29}$$

$$K_{\text{string}} = -2 + \left(4 + 6\sqrt{2}\right) \frac{a}{7} + \left(8 - 2\sqrt{2}\right) \frac{b}{7} \qquad (E_7^{++}). \tag{30}$$

$K_{\text{string}}$ for $F_4^{++}$ is identical to $K_{\text{string}}$ for $B_n^{++}$ and $D_n^{++}$. For the other two cases, we do not have a transparent interpretation.

# 6 Reduction of counterterms, IIA and IIB

It is conjectured that $E_{10} \equiv E_8^{++}$ is also the relevant symmetry for type IIA and type IIB supergravity in ten dimensions. Evidence for this conjecture was given in [2, 12, 22], following earlier work on the embedding of these theories into $E_{11}$ [23, 24, 25, 26]. The structure of the low derivative curvature correction terms for these theories is known from string scattering computations and is summarized for example in [27]. Somewhat unexpectedly, in both cases there is strong evidence that the first correction appears at order $R^4$ and receives contributions only from string tree level and string one-loop diagrams.[9] We will now examine these results in the light of our algebraic compatibility conditions.

## 6.1 IIA

The IIA supergravity theory can be obtained by standard dimensional reduction of the $D = 11$ theory. Reducing to Einstein frame we find that the

[9]To be sure, there are corrections coming from $D(-1)$ instantons for IIB which we do not discuss here. The absence of higher order loop corrections was partially confirmed in an explicit computation [28].

dominant walls (simple roots) of $E_{10}$ are now expressed via

$$\begin{aligned}
\alpha_1 &= (0,0,0,0,0,0,0,-1,1;0) \\
&\vdots \\
\alpha_8 &= (-1,1,0,0,0,0,0,0,0;0) \\
\alpha_9 &= (1,0,0,0,0,0,0,0,0;+\tfrac{3}{4}) \\
\alpha_{10} &= (1,1,0,0,0,0,0,0,0;-\tfrac{1}{2})
\end{aligned} \tag{31}$$

Here the first eight 'symmetry roots' define the new (non-maximal) gravity line of the $E_{10}$ Dynkin diagram (see fig. 1), corresponding to the symmetry walls of gravity in $D=10$ space-time dimensions. Unlike for $D=11$ supergravity, the simple root $\alpha_9$ is no longer associated with a symmetry wall, but now corresponds to the Kaluza Klein vector; the simple root $\alpha_{10}$ is associated with the NSNS two-form.

The relevant fundamental weights not belonging to the gravity line are now

$$\begin{aligned}
\Lambda_9 &= -(2,2,2,2,2,2,2,2,2;+\tfrac{1}{2}) \quad \Rightarrow \quad \Lambda_9(\beta,\varphi) = -2\sigma + \tfrac{1}{2}\varphi \\
\Lambda_{10} &= -(3,3,3,3,3,3,3,3,3;-\tfrac{1}{4}) \quad \Rightarrow \quad \Lambda_{10}(\beta,\varphi) = -3\sigma - \tfrac{1}{4}\varphi
\end{aligned} \tag{32}$$

Equating the dominant weight $a\Lambda_9 + b\Lambda_{10}$ with (10) corresponds to the higher order term

$$\sqrt{-G}\, R^N e^{K\varphi} \quad \text{with} \quad N = 1 + 2a + 3b\,, \;\; K = -a + \frac{b}{2}. \tag{33}$$

In string frame the dilaton exponent gives

$$K_{\text{string}} = -2 + 2b \tag{34}$$

according to (14), so that the coefficient $b$ counts the number of string loops. We see that $\Lambda_{10}$ – which corresponded to $R^4$ in $D=11$ supergravity – in the IIA basis corresponds to $(a,b)=(0,1)$, i.e. $R^4$ at one loop in string perturbation theory. Our reasoning allows for tree level terms ($b=0$) for all odd powers of the curvature. The absence of $R^3$ corrections, however, has been established both by supersymmetry arguments and explicit calculation. If one considers the known tree level term for $R^4$ one finds a wall form which is the following sum of simple roots

$$-\left(3\alpha_1+6\alpha_2+7\alpha_3+12\alpha_4+15\alpha_5+18\alpha_6+21\alpha_7+\frac{27}{2}\alpha_8+6\alpha_9+\frac{21}{2}\alpha_{10}\right). \tag{35}$$

Because of the appearance of fractional coefficients, this is not element of the weight lattice (=root lattice). Therefore, $E_{10}$ predicts correctly only the maximum loop order at which corrections can occur. One way to interpret this apparent discrepancy between the known string computations and the present KMA analysis is that the Kac–Moody model of [29] is thought to describe the *decompactified* version of M-theory, which in particular involves taking the limit in which the string coupling (or equivalently $\Phi$) tends to infinity. In this strong coupling limit only the highest genus terms of a string loop expansion survive. Indeed, only the one loop terms is known to lift to eleven dimensions.

## 6.2 IIB

One gets stranger results for type IIB string theory. This is perhaps not surprising in view of the singular field theory limit involved in getting to type IIB from the 11-dimensional model. For IIB in $D=10$ the $E_{10}$ simple roots are now represented as

$$\begin{aligned}
\alpha_1 &= (0,0,0,0,0,0,0,-1,1;0)\\
&\vdots\\
\alpha_7 &= (0,-1,1,0,0,0,0,0,0;0)\\
\alpha_{10} &= (-1,1,0,0,0,0,0,0,0;0)\\
\alpha_8 &= (1,1,0,0,0,0,0,0,0;-\tfrac{1}{2})\\
\alpha_9 &= (0,0,0,0,0,0,0,0,0;+1)
\end{aligned} \tag{36}$$

As before, the first eight are symmetry roots, but the associated 'gravity line' is now given by $\alpha_1,\ldots,\alpha_7,\alpha_{10}$, and thus differs from the IIA gravity line (compare also fig. 1). The root $\alpha_8$ corresponds to the wall generated by the NSNS 2-form, while $\alpha_9$ corresponds to the dilaton wall. Remarkably, and unlike for the IIA theory, the dilaton root $\alpha_9$ has no components involving the spatial neunbein.

Now the relevant fundamental weights are

$$\begin{aligned} \Lambda_8 &= -(4,4,4,4,4,4,4,4,4;0) \quad \Rightarrow \quad \Lambda_8(\beta,\varphi) = -4\sigma \\ \Lambda_9 &= -(2,2,2,2,2,2,2,2,2;-\tfrac{1}{2}) \quad \Rightarrow \quad \Lambda_9(\beta,\varphi) = -2\sigma + \tfrac{1}{2}\varphi \end{aligned} \tag{37}$$

Demanding that the dominant weight $a\Lambda_8 + b\Lambda_9$ be identical to the wall (10) yields corrections in the Einstein frame with

$$N = 1 + 4a + 2b, \qquad K = -b. \tag{38}$$

Conversion to the string frame gives

$$N = 1 + 4a + 2b, \qquad K_{\text{string}} = -2 + 2a, \tag{39}$$

so that the coefficient $a$ counts the number of string loops.

The first correction terms compatible with this weight pattern are $R^3$ (tree) and $R^5$ (tree and 1-loop). The pattern does not match with the known $\sqrt{-G}\,R^4$ in ten dimensions, independently of the dilatonic factor, which would require a wall form $\propto (3,3,3,3,3,3,3,3,3;*)$. One can also check directly that the latter combination does not lie on the $E_{10}$ root lattice, no matter how the dilaton factor is chosen. Incidentally, the tree level term $\sqrt{-G}\,R^4 e^{-3\varphi/2}$ in this case is again located at the same root vector (35) as in the IIA case. The one loop term is also off the weight lattice for IIB.

The correction term weight $\Lambda_{10}$ which gave sensible results for $D = 11$ and IIA (see (17) and (32)) in this basis becomes

$$\Lambda_{10} = -(4,3,3,3,3,3,3,3,3;0) \quad \Rightarrow \quad \Lambda_{10}(\beta,\varphi) = -3\sigma - \beta^1 \tag{40}$$

and is non-isotropic for IIB in $D = 10$. This might be related to the difficulties with writing a covariant Lagrangian for the IIB supergravity theory in ten space-time dimensions and we are therefore tempted to consider the situation after compactification of the theory on $S^1$ (coordinatized by $x^1$) to nine space-time dimensions (which is required to make the T-duality equivalence of IIA and IIB manifest), as is necessary in all string calculations of higher order effects in IIB theory. The 11-component of the spatial vielbein becomes another 'dilaton', and we would need to match $\sqrt{-G}\,R^4$ only with $\propto (3,3,3,3,3,3,3,3;*;*)$. This is certainly possible and will be the reduction of the one-loop term of the IIA theory but only at the price of including a non-covariant term involving this extra dilatonic factor, so far not seen in string perturbation theory.

# 7 Conclusions

In this paper, we have first shed new light on the findings of [6] concerning the quantum corrections of M-theory. We have emphasized that the imaginary roots uncovered in [6] are dominant non-gravity weights, suggesting a representation theoretic interpretation for the subleading quantum corrections.

We then 'botanically' explored the possible consequences for various maximally oxidized gravitational theories of a general conjecture linking dominant non-gravity weights to quantum curvature corrections. This conjecture has been found to have quite a few successes, but also a certain number of failures. Among the successes, let us mention that the consideration of weights (as opposed to roots) makes a difference for non self-dual lattices, and has been found to be necessary to reproduce some of the known quantum corrections. The conjecture derives further credit from the fact that the relevant KMAs appear to 'know' about string perturbation theory, via eqns. (22), (26) and (39). On the other hand, for the type I models, we found that the algebraic restrictions imposed by the conjectured symmetry are not restrictive enough: they allow terms that have not been observed. In the case of the types IIA and IIB, they also forbid terms that are known to occur — independently of whether one considers roots or weights. From this point of view, it is for the $E_{10}$-based M-theory that the matching between the algebraic constraints and the known results works most successfully (and indeed spectacularly so).

For the non string-related theories like pure gravity, the matching does appear to make sense since the known quantum corrections are reproduced (when one includes weights), but it is more difficult to test the Kac–Moody predictions because little is in general known on the effective lagrangians. It is expected that a deeper understanding of the occurrence of Kac–Moody weights could be obtained through a further analysis of the allowed $\sigma$-model counterterms, which are controlled by the invariants of the 'maximal compact subgroup' $K(G^{++})$ of $G^{++}$.

*Note added:* While this article was completed, the preprint [30] was posted. This preprint (whose reference [21] refers to part of the present work) shows that reduction to three spacetime dimensions of curvature corrections involves weights of the duality algebra $G$ which is manifest in three dimensions — and of which the Kac–Moody algebra $G^{++}$ is the overextension.

**Acknowledgements:** We thank V. Kac, K. Peeters and S. Theisen for informative discussions. We are grateful to Marie-Claude Vergne for preparing figure 2. AH and MH would like to thank the KITP of UCSB for kind hospitality during the early stages of this project. TD acknowledges the support of the European Research and Training Network 'Forces Universe' (contract number MRTN-CT-2004-005104). The work of MH is partially supported by IISN - Belgium (convention 4.4505.86), by the 'Interuniversity Attraction Poles Programme – Belgian Science Policy' and by the European Commission programme MRTN-CT-2004-005104, in which he is associated to V.U. Brussel. AK and HN are grateful for the hospitality of ULB and IHES during early stages of this work and acknowledge the support of the European Research and Training Network 'Superstrings' (contract number MRTN-CT-2004-512194). Research also supported in part by the CTP and the LNS of MIT and the U.S. Department of Energy under cooperative agreement #DE-FC02-94ER40818.

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