E_{10} AND A "SMALL TENSION EXPANSION" OF M THEORY

T. DAMOUR, M. HENNEAUX and H. NICOLAI

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Juillet 2002

IHES/P/02/48

E_{10} and a "small tension expansion" of M Theory

T. Damour

Institut des Hautes Etudes Scientifiques, 35 route de Chartres, F-91440 Bures-sur-Yvette, France

M. Henneaux

Physique Théorique et Mathématique, Université Libre de Bruxelles, C.P.231, B-1050 Bruxelles, Belgium

H. Nicolai

Max-Planck-Institut für Gravitationsphysik, Mühlenberg 1, D–14476 Golm, Germany

(July 31, 2002)

A formal "small tension" expansion of $D=11$ supergravity near a spacelike singularity is shown to be equivalent, at least up to 30th order in height, to a null geodesic motion in the infinite dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of the hyperbolic Kac-Moody group $E_{10}(\mathbb{R})$. For the proof we make use of a novel decomposition of E_{10} into irreducible representations of its $\mathrm{SL}(10,\mathbb{R})$ subgroup. We explicitly show how to identify the first four rungs of the E_{10} coset fields with the values of geometric quantities constructed from $D=11$ supergravity fields and their spatial gradients taken at some comoving spatial point.

04.65.+e 04.50.+h 11.10.Kk 11.15.-q

The consideration of limits where some (possibly dimensionful) parameter is taken to be small is often a way of revealing the hidden symmetry structure of physical theories. In [1], it was argued that the small tension limit $T_s \to 0$ of string theory gives rise to an infinite number of relations between string scattering amplitudes, indicating the presence of an enormous symmetry. In this Letter, we shall consider the bosonic sector of M Theory, and more specifically its low energy limit, $D=11$ supergravity [2], in a limit which can likewise be (intuitively) thought of as a small tension limit $T_b \to 0$, where $T_b := c^4(32\pi G_N)^{-1}$ is the bulk tension governing the propagation of small excitations (e.g. gravitational waves) in the ten-dimensional spatial geometry. Indeed, taking $T_b \to 0$ in the linearized Einstein-Hilbert action $S = \frac{1}{2}\int dT\, d^{10}x\big(\rho_b(\partial_T h_{ij})^2 - T_b(\partial_x h_{ij})^2\big)$ is equivalent to taking the limit of vanishing velocity of propagation $c = \sqrt{T_b/\rho_b}$; alternatively, it may be viewed as a strong coupling limit ($G_N \to \infty$) [3]. Physically, this limit is realized near a spacelike singularity, where the different spatial points become causally disconnected as the horizon scale $\ell_H \sim cT$ becomes smaller than their spacelike separation (T being the proper time), provided the time derivatives of the fields dominate their spatial gradients. As shown recently [4,5], this is indeed the case for the massless bosonic sector of $D=11$ supergravity. Furthermore, as $T_b \to 0$, the metric exhibits the chaotic oscillations originally discovered by Belinskii, Khalatnikov and Lifshitz (BKL) for the generic cosmological solution to Einstein's equations in four dimensions [6]. The oscillatory evolution of the metric at each spatial point can be asymptotically described as a relativistic billiard taking place in the fundamental Weyl chamber of some indefinite Kac-Moody (KM) algebra [4,5]. Chaos occurs when this KM algebra is hyperbolic, in particular for E_{10} [7].

In this Letter we extend these tantalizing results much beyond the leading order by relating a BKL-type expansion to an algebraic expansion in the height of the positive roots of the Lie algebra of E_{10}. We show how to map, up to height 30, geometrical objects of M theory onto coordinates in the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of the canonical real form of E_{10}. Under this correspondence, the time evolution of the geometric M Theory data at each spatial point is mapped, up to height 30, onto some (constrained) null geodesic motion of $E_{10}/K(E_{10})$. Our results underline the potential importance of E_{10}, whose appearance in the reduction of $D=11$ supergravity to one dimension had been conjectured already long ago by B. Julia [8], as a candidate symmetry underlying M theory (see also [9], and [10] where E_{11} was proposed as a fundamental symmetry of M Theory).

Introducing a zero-shift slicing ($N^i=0$) of the eleven-dimensional spacetime, and a *time-independent* spatial zehnbein $\theta^a(x) \equiv E^a{}_i(x)dx^i$, the metric and four form $\mathcal{F} = d\mathcal{A}$ become

$$ds^2 = -N^2(dx^0)^2 + G_{ab}\theta^a\theta^b \tag{1}$$

$$\mathcal{F} = \frac{1}{3!}\mathcal{F}_{0abc}\, dx^0 \wedge\theta^a\wedge\theta^b\wedge\theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\,\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d$$

We choose the time coordinate x^0 so that the lapse $N=\sqrt{G}$, with $G := \det G_{ab}$ (note that x^0 is not the proper time $T = \int N dx^0$; rather, $x^0 \to \infty$ as $T \to 0$). In this frame the complete evolution equations of $D=11$ supergravity read

$$\begin{aligned}\partial_0\big(G^{ac}\partial_0 G_{cb}\big) &= \tfrac{1}{6}G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \tfrac{1}{72}G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b \\ &\quad -2GR^a{}_b(\Gamma, C) \\ \partial_0\big(G\mathcal{F}^{0abc}\big) &= \tfrac{1}{144}\varepsilon^{abca_1a_2a_3b_1b_2b_3b_4}\mathcal{F}_{0a_1a_2a_3}\mathcal{F}_{b_1b_2b_3b_4}\end{aligned}$$

$$+\tfrac{3}{2}G\mathcal{F}^{de[ab}C^{c]}{}_{de} - GC^{e}{}_{de}\mathcal{F}^{dabc} - \partial_d\big(G\mathcal{F}^{dabc}\big)$$
$$\partial_0\mathcal{F}_{abcd} = 6\mathcal{F}_{0e[ab}C^{e}{}_{cd]} + 4\partial_{[a}\mathcal{F}_{0bcd]} \qquad (2)$$

where $a, b \in \{1, \ldots, 10\}$ and $\alpha, \beta \in \{0, 1, \ldots, 10\}$, and $R_{ab}(\Gamma, C)$ denotes the spatial Ricci tensor; the (frame) connection components are given by $2G_{ad}\Gamma^{d}{}_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$ with $C^{a}{}_{bc} \equiv G^{ad}C_{dbc}$ being the structure coefficients of the zehnbein $d\theta^a = \frac{1}{2}C^{a}{}_{bc}\theta^b \wedge \theta^c$. The frame derivative is $\partial_a \equiv E^{i}{}_{a}(x)\partial_i$ (with $E^{a}{}_{i}E^{i}{}_{b} = \delta^a_b$). To determine the solution at any *given* spatial point x requires knowledge of an infinite tower of spatial gradients: one should thus augment (2) by evolution equations for $\partial_a G_{bc}, \partial_a \mathcal{F}_{0bcd}, \partial_a \mathcal{F}_{bcde}$, etc., which in turn would involve higher and higher spatial gradients.

The geodesic Lagrangian on $E_{10}/K(E_{10})$ is defined by generalizing the standard Lagrangian on a finite dimensional coset space G/K, where K is a maximal compact subgroup of the Lie group G. All the elements entering the construction of $\mathcal{L}$ have natural generalizations to the case where G is the group obtained by exponentiation of a hyperbolic KM algebra. We refer readers to [11] for basic definitions and results of the theory of KM algebras, and here only recall that a KM algebra $\mathfrak{g} \equiv \mathfrak{g}(A)$ is generally defined by means of a Cartan matrix A and a set of generators $\{e_i, f_i, h_i\}$ and relations (Chevalley-Serre presentation), where $i, j = 1, \ldots, r \equiv \operatorname{rank}\mathfrak{g}(A)$. The elements $\{h_i\}$ span the Cartan subalgebra (CSA) $\mathfrak{h}$, while the e_i and f_i generate an infinite tower of raising and lowering operators, respectively. The "maximal compact" subalgebra $\mathfrak{k}$ is defined as the subalgebra of $\mathfrak{g}(A)$ left invariant under the Chevalley involution $\omega(h_i) = -h_i$, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$. In other words, $\mathfrak{k}$ is spanned by the "antisymmetric" elements $E_{\alpha,s} - E^T_{\alpha,s}$, where $E^T_{\alpha,s} \equiv -\omega(E_{\alpha,s})$ is the "transpose" of some multiple commutator $E_{\alpha,s}$ of the e_i's associated with the root α (i.e. $[h, E_{\alpha,s}] = \alpha(h)E_{\alpha,s}$ for $h \in \mathfrak{h}$). Here $s = 1, \ldots \operatorname{mult}(\alpha)$ labels the different elements of $\mathfrak{g}(A)$ having the same root α.

The σ-model is formulated in terms of a one-parameter dependent group element $\mathcal{V} = \mathcal{V}(t) \in E_{10}$ and its Lie algebra valued derivative

$$v(t) := \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}(t) \in \mathfrak{e}_{10} \equiv \operatorname{Lie} E_{10} \qquad (3)$$

In physical terms, $\mathcal{V}$ can be thought of as a vast extension of the vielbein of general relativity (an "∞-bein"), and E_{10} and $K(E_{10})$ as infinite dimensional generalizations of the $GL(d, \mathbb{R})$ and local Lorentz symmetries of general relativity. The action is $\int dt\mathcal{L}$ with

$$\mathcal{L} := n(t)^{-1}\langle v_{\rm sym}(t)|v_{\rm sym}(t)\rangle \qquad (4)$$

with a "lapse" function $n(t)$ (not to be confused with N), whose variation gives rise to the Hamiltonian constraint ensuring that the trajectory is a null geodesic. The "symmetric" projection $v_{\rm sym} := \frac{1}{2}(v + v^T)$ eliminates the component of v corresponding to a displacement "along $\mathfrak{k}$", thereby defining an evolution on the coset space $E_{10}/K(E_{10})$. $\langle .|. \rangle$ is the standard invariant bilinear form on the KM algebra [11]. We note the existence of transcendental KM invariants [12] that might be added to (4) to represent non-perturbative effects.

Because no closed form construction exists for the raising operators $E_{\alpha,s}$, nor their invariant scalar products $\langle E_{\alpha,s}|E_{\beta,t}\rangle = N^{\alpha}_{s,t}\delta^0_{\alpha+\beta}$, we have devised a recursive approach based on the decomposition of E_{10} into irreducible representations of its $\mathrm{SL}(10, \mathbb{R})$ subgroup. Let $\alpha_1, \ldots, \alpha_9$ be the nine simple roots of $A_9 \equiv sl(10)$ corresponding to the horizontal line in the E_{10} Dynkin diagram, and α_0 the "exceptional" root connected to α_3. Its dual CSA element h_0 enlarges A_9 to the Lie algebra of $\mathrm{GL}(10)$. Any positive root of E_{10} can be written as

$$\alpha = \ell\alpha_0 + \sum_{j=1}^{9} m^j\alpha_j \quad (\ell, m^j \geq 0) \qquad (5)$$

We call $\ell \equiv \ell(\alpha)$ the "level" of the root α. This definition differs from the usual one, where the (affine) level is identified with m^9 and thus counts the number of appearances of the over-extended root α_9 in α [13,14]. Hence, our decomposition corresponds to a slicing (or "grading") of the forward lightcone in the root lattice by spacelike hyperplanes, with only finitely many roots in each slice, as opposed to the lightlike slicing for the E_9 representations (involving not only infinitely many roots but also infinitely many affine representations for $m^9 \geq 2$ [13,14]).

The adjoint action of the A_9 subalgebra leaves the level $\ell(\alpha)$ invariant. The set of generators corresponding to a given level ℓ can therefore be decomposed into a (finite) number of irreducible representations of A_9. The multiplicity of α as a root of E_{10} is thus equal to the sum of its multiplicities as a weight occurring in the $\mathrm{SL}(10, \mathbb{R})$ representations. Each irreducible representation of A_9 can be characterized by its highest weight Λ, or equivalently by its Dynkin labels $(p_1, \ldots, p_9)$ where $p_k(\Lambda) := (\alpha_k, \Lambda) \geq 0$ is the number of columns with k boxes in the associated Young tableau. For instance, the Dynkin labels (001000000) correspond to a Young tableau consisting of one column with three boxes, *i.e.* the antisymmetric tensor with three indices. The Dynkin labels are related to the 9-tuple of integers $(m^1, \ldots, m^9)$ appearing in (5) (for the highest weight $\Lambda \equiv -\alpha$) by

$$S^{i3}\ell - \sum_{j=1}^{9} S^{ij}p_j = m^i \geq 0 \qquad (6)$$

where S^{ij} is the inverse Cartan matrix of A_9. This relation strongly constrains the representations that can appear at level ℓ, because the entries of S^{ij} are all positive, and the 9-tuples $(p_1, \ldots, p_9)$ and $(m_1, \ldots, m_9)$ must both consist of non-negative integers. In addition to satisfying the Diophantine equations (6), the highest weights must be roots of E_{10}, which implies the inequality

$$\Lambda^2 = \alpha^2 = \sum_{i,j=1}^{9} p_i S^{ij} p_j - \tfrac{1}{10}\ell^2 \leq 2 \qquad (7)$$

All representations occurring at level $\ell+1$ are contained in the product of the level-ℓ representations with the $\ell=1$ representation. Imposing the Diophantine inequalities (6), (7) allows one to discard many representations appearing in this product. The problem of finding a completely explicit and manageable representation of E_{10} in terms of an infinite tower of A_9 representations is thereby reduced to the problem of determining the outer multiplicities of the surviving A_9 representations, namely the number of times each representation appears at a given level ℓ. The Dynkin labels (all appearing with outer multiplicity one) for the first six levels of E_{10} are

$$\begin{aligned}
\ell=1 \quad &: \quad (001000000) \\
\ell=2 \quad &: \quad (000001000) \\
\ell=3 \quad &: \quad (100000010) \\
\ell=4 \quad &: \quad (001000001)\ ,\ (200000000) \\
\ell=5 \quad &: \quad (000001001)\ ,\ (100100000) \\
\ell=6 \quad &: \quad (100000011)\ ,\ (010001000)\ , \\
&\quad\quad (100000100)\ ,\ (000000010)
\end{aligned} \qquad (8)$$

The level $\ell \leq 4$ representations can be easily determined by comparison with the decomposition of E_8 under its A_7 subalgebra (see [15,16]) and use of the Jacobi identity, which eliminates the representations (000000001) at level three and (010000000) at level four. By use of a computer and the E_{10} root multiplicities listed in [14,17], the calculation can be carried much further [18].

From (8) we can now directly read off the $GL(10)$ tensors making up the low level elements of E_{10}. At level zero, we have the $GL(10)$ generators $K^a{}_b$ obeying $[K^a{}_b, K^c{}_d] = K^a{}_d\delta^c_b - K^c{}_b\delta^a_d$. The $\mathfrak{e}_{10}$ elements at levels $\ell = 1,2,3$ are the $GL(10)$ tensors $E^{a_1a_2a_3}$, $E^{a_1\dots a_6}$ and $E^{a_0|a_1\dots a_8}$ with the symmetries implied by the Dynkin labels (for the first three levels these represention occur for all E_n, see [19,10]). The σ-model associates to these generators a corresponding tower of "fields" (depending only on the "time" t): a zehnbein $h^a{}_b(t)$ at level zero, a three form $A_{abc}(t)$ at level one, a six-form $A_{a_1\dots a_6}(t)$ at level two, a Young-tensor $A_{a_0|a_1\dots a_8}(t)$ at level 3, etc. Writing the generic E_{10} group element in Borel (triangular) gauge as $\mathcal{V}(t) = \exp X_h(t)\cdot\exp X_A(t)$ with $X_h(t) = h^a{}_bK^b{}_a$ and $X_A(t) = \frac{1}{3!}A_{abc}E^{abc} + \frac{1}{6!}A_{a_1\dots a_6}E^{a_1\dots a_6} + \frac{1}{9!}A_{a_0|a_1\dots a_8}E^{a_0|a_1\dots a_8} + \dots$, and using the E_{10} commutation relations in $GL(10)$ form together with the bilinear form for E_{10}, we find up to third order in level

$$\begin{aligned}
n\mathcal{L} &= \tfrac{1}{4}(g^{ac}g^{bd} - g^{ab}g^{cd})\dot g_{ab}\dot g_{cd} + \tfrac{1}{2}\tfrac{1}{3!}DA_{a_1a_2a_3}DA^{a_1a_2a_3} \\
&+\tfrac{1}{2}\tfrac{1}{6!}DA_{a_1\dots a_6}DA^{a_1\dots a_6} + \tfrac{1}{2}\tfrac{1}{9!}DA_{a_0|a_1\dots a_8}DA^{a_0|a_1\dots a_8}
\end{aligned} \qquad (9)$$

where $g^{ab} = e^a{}_ce^b{}_c$ with $e^a{}_b \equiv (\exp h)^a{}_b$, and all "contravariant indices" have been raised by g^{ab}. The "covariant" time derivatives are defined by (with $\partial A \equiv \dot A$)

$$\begin{aligned}
DA_{a_1a_2a_3} &:= \partial A_{a_1a_2a_3} \\
DA_{a_1\dots a_6} &:= \partial A_{a_1\dots a_6} + 10A_{[a_1a_2a_3}\partial A_{a_4a_5a_6]} \\
DA_{a_1|a_2\dots a_9} &:= \partial A_{a_1|a_2\dots a_9} + 42A_{\langle a_1a_2a_3}\partial A_{a_4\dots a_9\rangle} \\
&\quad -42\partial A_{\langle a_1a_2a_3}A_{a_4\dots a_9\rangle} + 280A_{\langle a_1a_2a_3}A_{a_4a_5a_6}\partial A_{a_7a_8a_9\rangle}
\end{aligned} \qquad (10)$$

Here antisymmetrization $[\dots]$, and projection on the $\ell=3$ representation $\langle\dots\rangle$, are normalized with strength one (e.g. $[[\dots]] = [\dots]$). Modulo field redefinitions, all numerical coefficients in (9) and (10) are uniquely fixed by the structure of E_{10}. Our expressions are reminiscent of similar algebraic constructions in [15] and [10]. However, this is the first time that an algorithmic scheme based on a Lagrangian in terms of the invariant bilinear form on the hyperbolic KM algebra has been proposed and worked out to low orders. Likewise, the general formulas (6) and (7), and the higher level representations in (8) have not been exhibited before.

The Lagrangian (4) is invariant under a nonlinear realization of E_{10} such that $\mathcal{V}(t) \to k_g(t)\mathcal{V}(t)g$ with $g \in E_{10}$; the compensating "rotation" $k_g(t)$ being, in general, required to restore the "triangular gauge". When g belongs to the nilpotent subgroup generated by the E^{abc}, etc., this symmetry reduces to the rather obvious "shift" symmetries of (9) and no compensating rotation is needed. The latter are, however, required for the transformations generated by $F_{abc} = (E^{abc})^T$, *etc.* The associated infinite number of conserved (Noether) charges are formally given by $J = \mathcal{M}^{-1}\partial\mathcal{M}$, where $\mathcal{M} \equiv \mathcal{V}^T\mathcal{V}$. This can be formally solved in closed form as

$$\mathcal{M}(t) = \mathcal{M}(0)\cdot\exp(tJ) \qquad (11)$$

The compatibility between (11) (indicative of the integrability of (9)) and the chaotic behavior of $g_{ab}(t)$ near a spacelike singularity will be discussed elsewhere.

The main result that we report in this letter is the following: there exists a map between geometrical quantities constructed at a given spatial point x from the supergravity fields $G_{\mu\nu}(x^0,x)$ and $\mathcal{A}_{\mu\nu\rho}(x^0,x)$ and the one-parameter-dependent quantities $g_{ab}(t), A_{abc}(t),\dots$ entering the coset Lagrangian (9), under which the supergravity equations of motion (2) become equivalent, up to 30th order in height, to the Euler-Lagrange equations of (9). In the gauge (1) this map is defined by $t = x^0 \equiv \int dT/\sqrt{G}$ and

$$\begin{aligned}
g_{ab}(t) &= G_{ab}(t,x) \\
DA_{a_1a_2a_3}(t) &= \mathcal{F}_{0a_1a_2a_3}(t,x) \\
DA^{a_1\dots a_6}(t) &= -\tfrac{1}{4!}\varepsilon^{a_1\dots a_6b_1b_2b_3b_4}\mathcal{F}_{b_1b_2b_3b_4}(t,x) \\
DA^{b|a_1\dots a_8}(t) &= \tfrac{3}{2}\varepsilon^{a_1\dots a_8b_1b_2}\big(C^b{}_{b_1b_2}(x) + \tfrac{2}{9}\delta^b_{[b_1}C^c{}_{b_2]c}(x)\big)
\end{aligned} \qquad (12)$$

The expansion in height $\mathrm{ht}(\alpha) \equiv \ell+\sum m^j$, which controls the iterative validity of this equivalence, is as follows: the Hamiltonian constraint of the coset model (9) contains an infinite series of exponential coefficients $\exp\big(-2\alpha(\beta)\big)$, where α runs over all positive roots of E_{10}, and where $\beta^a \equiv -h^a{}_a$ parametrize the CSA of E_{10}. Previous work has shown that, near a spacelike singularity ($t \to \infty$), the dynamics of the supergravity fields and of truncated versions of the E_{10} coset fields is asymptotically dominated by the (hyperbolic) Toda model defined by keeping only the exponentials involving the simple roots of E_{10}. Higher roots introduce smaller and smaller corrections as t increases. The "small tension expansion" of the equations of motion is then technically defined as a formal BKL-like expansion that corresponds to such an expansion in decreasing exponentials of the Hamiltonian constraint. On the supergravity side, this expansion amounts to an expansion in gradients of the fields in appropriate frames. Level one corresponds to the simplest one-dimensional reduction of (2), obtained by assuming that both $G_{\mu\nu}$ and $\mathcal{A}_{\lambda\mu\nu}$ depend only on time [4]; levels 2 and 3 correspond to configurations of $G_{\mu\nu}$ and $\mathcal{A}_{\lambda\mu\nu}$ with a more general, but still very restricted x-dependence, so that *e.g.* the frame derivatives of the electromagnetic field in (2) drop out [20]. When neglecting terms corresponding to $\mathrm{ht}(\alpha) \geq 30$, the map (12) provides a *perfect match* between the supergravity evolution equations (2) and the E_{10} coset ones, as well as between the associated Hamiltonian constraints. (In fact, the matching extends to *all real roots* of level ≤ 3.)

It is natural to view our map as embedded in a hierarchical sequence of maps involving more and more spatial gradients of the basic supergravity fields. Our BKL-like expansion would then be a way of revealing step by step a hidden hyperbolic symmetry, implying the existence of a huge non-local symmetry of Einstein's theory and its generalizations. Although the validity of this conjecture remains to be established, we can at least show that there is "enough room" in E_{10} for all the spatial gradients. Namely, the search for affine roots (with $m^9 = 0$) in (6) and (7) reveals three infinite sets of admissible A_9 Dynkin labels $(00100000n)$, $(00000100n)$ and $(10000001n)$ with highest weights obeying $\Lambda^2 = 2$, at levels $\ell = 3n+1, 3n+2$ and $3n+3$, respectively. These correspond to three infinite towers of $\mathfrak{e}_{10}$ elements

$$E_{a_1\dots a_n}{}^{b_1b_2b_3}\ ,\ E_{a_1\dots a_n}{}^{b_1\dots b_6}\ ,\ E_{a_1\dots a_n}{}^{b_0|b_1\dots b_8} \qquad (13)$$

which are symmetric in the lower indices and all appear with outer multiplicity one (together with three transposed towers). Restricting the indices to $a_i = 1$ and $b_i \in \{2, ..., 10\}$ and using the decomposition $\mathbf{248} \to \mathbf{80} + \mathbf{84} + \overline{\mathbf{84}}$ of E_8 under its SL(9) subgroup one easily recovers the affine subalgebra $E_9 \subset E_{10}$. The appearance of higher order dual potentials (*à la* Geroch) in the E_9-based linear system for $D = 2$ supergravity [21] indeed suggests that we associate the E_{10} Lie algebra elements (13) to the higher order spatial gradients $\partial^{a_1}\cdots\partial^{a_n}A_{b_1b_2b_3}, \partial^{a_1}\cdots\partial^{a_n}A_{b_1\dots b_6}$ and $\partial^{a_1}\cdots\partial^{a_n}A_{b_0|b_1\dots b_8}$ or to some of their non-local equivalents. Of course, the elements (13) generate only a tiny subspace of $\mathfrak{e}_{10}$, suggesting the existence of further M theoretic degrees of freedom and corrections beyond $D=11$ supergravity. Finally, we note that our approach based on a height expansion can be extended to other physically relevant KM algebras, such as BE_{10} [5,22] and AE_n [7].

We thank A. Feingold, T. Fischbacher and V. Kac for informative discussions.

[1] D. Gross, Phys. Rev. Lett. **60**, 1229 (1988)

[2] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76**, 409 (1978).

[3] C. J. Isham, Proc. Roy. Soc. Lond. A **351**, 209 (1976); C. Teitelboim, in: General Relativity and Gravitation, vol 1, A. Held ed., Plenum Press, 1980.

[4] T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000); Phys. Lett. **B488**, 108 (2000)

[5] T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001)

[6] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Adv. Phys. **19**, 525 (1970); Adv. Phys. **31**, 639 (1982).

[7] T. Damour, M. Henneaux, B. Julia and H. Nicolai, Phys. Lett. **B509**, 323 (2001)

[8] B. Julia, in: Lectures in Applied Mathematics, Vol. 21, AMS-SIAM, p.335, 1985; preprint LPTENS 80/16.

[9] S. Mizoguchi, Nucl. Phys. **B528**, 238 (1998)

[10] P.C. West, Class. Quant. Grav. **18** (2001) 4443

[11] V.G. Kac, Infinite Dimensional Lie Algebras, 3rd edn., Cambridge University Press, 1990

[12] D. Peterson and V.G. Kac, Proc. Nat. Acad.Sci. USA **80**, 1778 (1983)

[13] A.J. Feingold and I.B. Frenkel, Math. Ann. **263**, 87 (1983)

[14] V.G. Kac, R.V. Moody and M. Wakimoto, in: Differential Geometric Methods in Theoretical Physics, NATO Advanced Research Workshop, eds. K. Bleuler and M. Werner, (Kluwer, 1988).

[15] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, Nucl. Phys. **B523**, 73 (1998); Nucl. Phys. **B535**, 242 (1998)

[16] K. Koepsell, H. Nicolai, and H. Samtleben, Class. Quant. Grav. **17**, 3689 (2000).

[17] O. Bärwald, R.W. Gebert and H. Nicolai, Nucl. Phys. **B510**, 721 (1998)

[18] T. Fischbacher, work in progress.

[19] N.A. Obers, B. Pioline and E. Rabinovici, Nucl. Phys. **B525**, 163 (1998)

[20] J. Demaret, J.L. Hanquin, M. Henneaux and P. Spindel, Nucl. Phys. **B252**, 538 (1985)

[21] P. Breitenlohner and D. Maison, Ann. Inst. Poincaré **46** 215 (1987); H. Nicolai, Phys. Lett. **194**, 402 (1987).

[22] E. Cremmer, B. Julia, H. Lu and C. N. Pope, arXiv:hep-th/9909099.