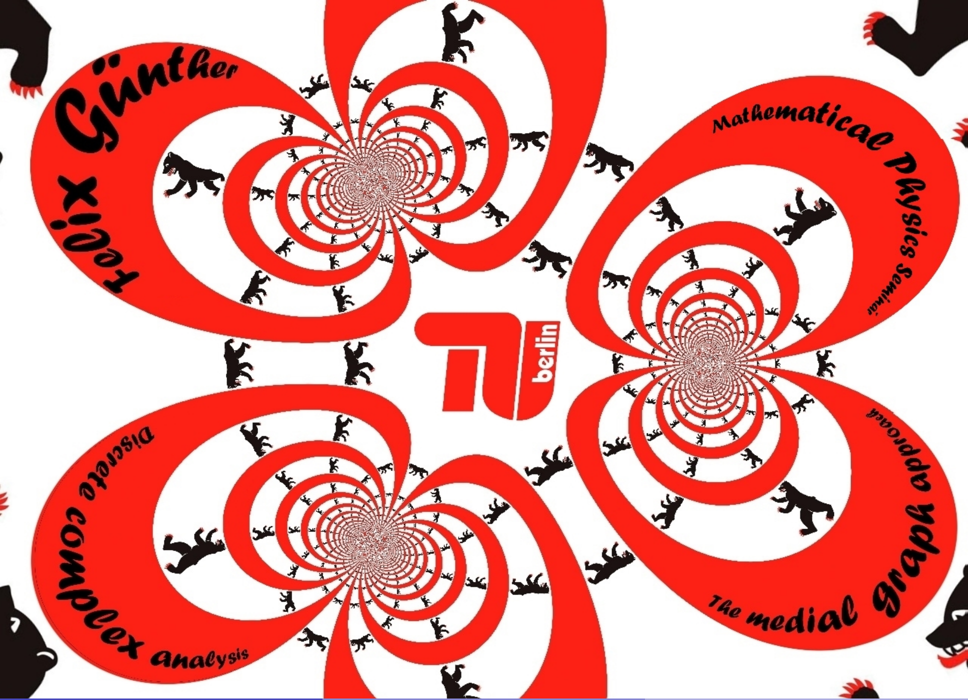
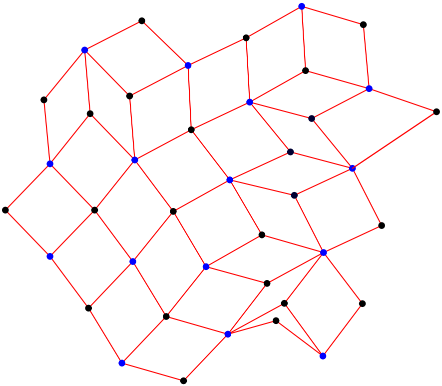
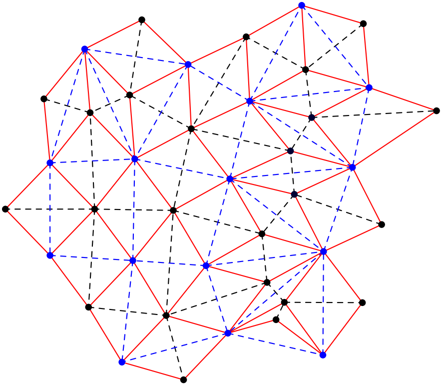

Felix Günther
Mathematical Physics Seminár
The medial graph approach
Discrete complex analysis
berlin

BIPARTITE QUAD-GRAPHS



Λ bipartite quad-graph
(strongly regular, locally finite)

Bipartite quad-graphs



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Γ graph of black diagonals

Γ^* graph of blue diagonals

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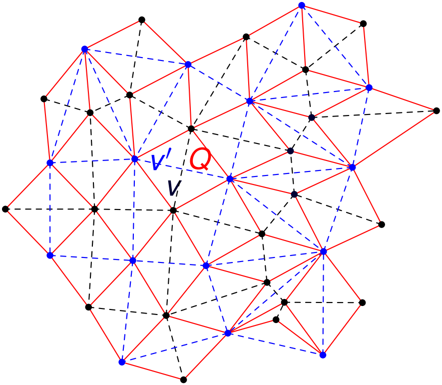


Λ bipartite quad-graph
(strongly regular, locally finite)

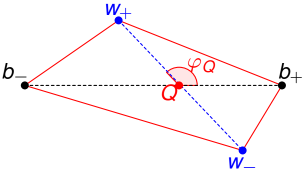
Γ graph of black diagonals

Γ^* graph of blue diagonals

Dual graph $\diamond := \Lambda^*$

Notation: $v, v' \sim Q$

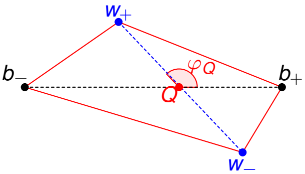
Discrete holomorphicity



$f : V(\Lambda) \to \mathbb{C}$ *discrete holomorphic* at Q iff

$$\text{(dCR)} \quad \frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}$$

Discrete holomorphicity



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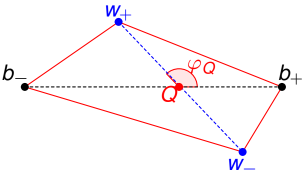

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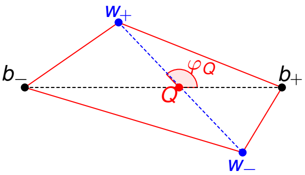
Properties:

Discrete holomorphicity



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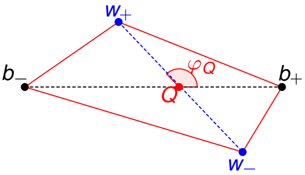
Properties:
f is discrete holomorphic iff $\bar{\partial}_\Lambda f = 0$.

Discrete holomorphicity



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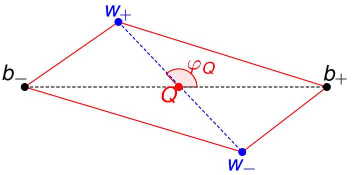
Properties:
If $f(v) = v$, then $\bar{\partial}_\Lambda f(Q) = 0$ and $\partial_\Lambda f(Q) = 1$.

Discrete holomorphicity



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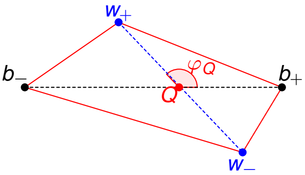
Properties:
If Q is a parallelogram and $f(v) = v^2$, then $\bar{\partial}_\Lambda f(Q) = 0$, $\partial_\Lambda f(Q) = 2Q$.

Discrete holomorphicity



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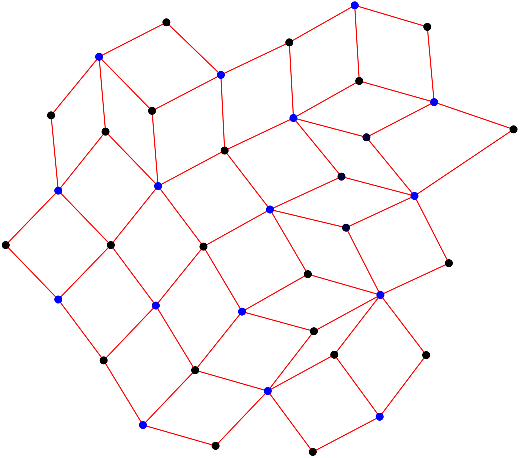
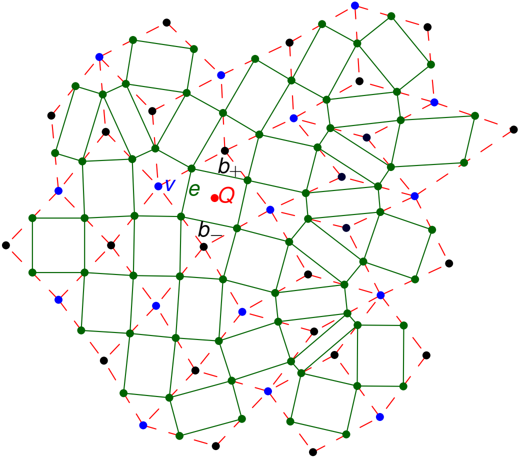
Properties:
If $\bar\partial_\Lambda f = 0$ and f is purely real/imaginary, then f is *essentially constant*.

Medial graph



Λ bipartite quad-graph

Medial graph



Λ bipartite quad-graph

X medial graph of Λ

edge e of X corresponds to pair $[Q, v] \in V(\diamond) \times V(\Lambda)$

$F(X) \cong V(\Lambda) \dot\cup V(\diamond)$

$$e = \pm \frac{b_+ - b_-}{2}$$

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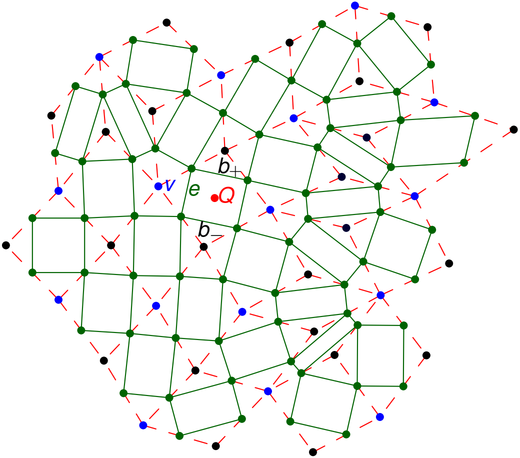

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$$e = \pm \frac{b_+ - b_-}{2}$$

Notation: For $\diamond_0 \subseteq \diamond$ connected with vertex set $V(\Lambda_0) \subseteq V(\Lambda)$, let $X_0 \subseteq X$ subgraph inside faces of $\diamond_0$.

Discrete differential forms

- functions $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$ can be extended to functions $g : F(X) \to \mathbb{C}$ by 0 on yet undefined faces

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- *discrete one-forms* $\omega : \vec{E}(X)$
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- products $g\omega$ and $g\Omega$ for $e = [Q, v]$ and face F of X:

$$\int_e g\omega = (g(Q) + g(v)) \int_e \omega,$$

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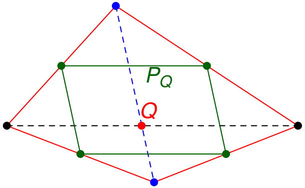
- ω of *type* $\diamond$, if $\omega = pdz + qd\bar{z}$ for $p, q : V(\diamond) \to \mathbb{C}$

Discrete derivatives



$$\partial_\Lambda f(Q) = \frac{-1}{4i\text{area}(F_Q)} \oint_{P_Q} f d\bar{z},$$

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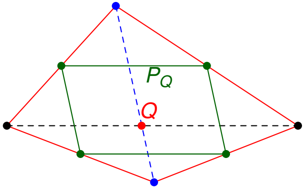
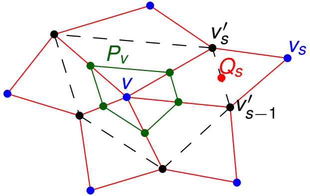
DISCRETE DERIVATIVES





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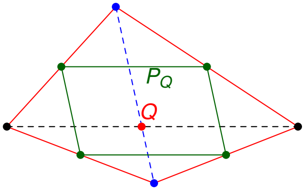

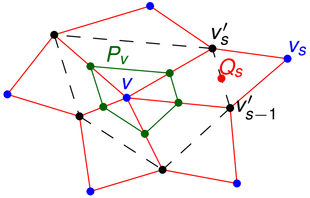

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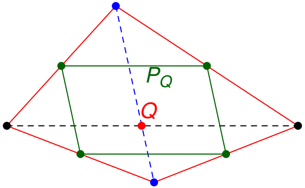
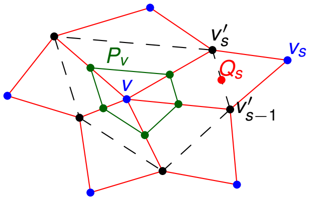
Properties:
If $h(Q) = Q = (v'_{s-1} + v'_s)/2$, $\bar{\partial}_\diamond h(v) = 0$ and $\partial_\diamond h(v) = 1$.

DISCRETE DERIVATIVES





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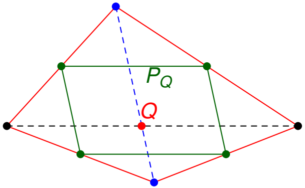
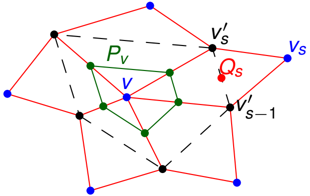
Properties:
For $f : V(\Lambda) \to \mathbb{C}$, $\partial_\diamond \bar{\partial}_\Lambda f \equiv \bar{\partial}_\diamond \partial_\Lambda f$. So $\partial_\Lambda f$ discrete holomorphic if f is.

Discrete derivatives





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Properties:
$\diamond_0$ simply-connected, $h : V(\diamond_0) \to \mathbb{C}$ discrete holomorphic. Then, there exists *discrete primitive* $f : V(\Lambda) \to \mathbb{C}$ with $\bar{\partial}_\Lambda f = 0$, $\partial_\Lambda f = h$.

Discrete scalar product

Let $g_1, g_2 : F(X) \to \mathbb{C}$. Their *discrete scalar product* is

$$\langle g_1, g_2 \rangle := -\frac{1}{i} \iint\limits_{F(X)} g_1 \overline{g_2} dz \wedge d\bar{z},$$

whenever the right hand side converges absolutely.

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where $\diamond_0 \subseteq \diamond$ is finite.

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whenever the right hand side converges absolutely.
If $f : V(\Lambda) \to \mathbb{C}$ or $h : V(\diamond) \to \mathbb{C}$ is compactly supported,

$$\langle \partial_\Lambda f, h \rangle + \langle f, \bar{\partial}_\diamond h \rangle = 0 = \langle \bar{\partial}_\Lambda f, h \rangle + \langle f, \partial_\diamond h \rangle.$$

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Proof:

$$-2i\langle \partial_\Lambda f, h \rangle - 2i\langle f, \bar{\partial}_\diamond h \rangle = \sum_{Q \in V(\diamond)} \overline{h(Q)} \oint_{P_Q} f d\bar{z} + \sum_{v \in V(\Lambda)} f(v) \oint_{P_v} \bar{h} d\bar{z}$$

$$= \oint_{P_{\mathrm{large}}} f \bar{h} d\bar{z} = 0.$$

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Proof:

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Discrete exterior derivative

Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. Define df and dh by:

$$df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z} \quad \text{and} \quad dh := \partial_\diamond h dz + \bar{\partial}_\diamond h d\bar{z}.$$

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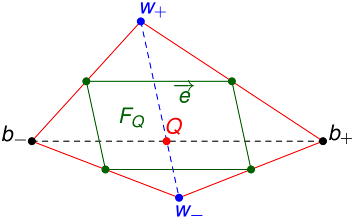

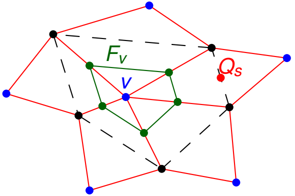

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$$d\omega|_{F_v} := 2\left(\partial_\diamond q - \bar{\partial}_\diamond p\right) dz \wedge d\bar{z} \text{ and } d\omega|_{F_Q} := 2\left(\partial_\Lambda q - \bar{\partial}_\Lambda p\right) dz \wedge d\bar{z}.$$

DISCRETE EXTERIOR DERIVATIVE

Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. Define df and dh by:

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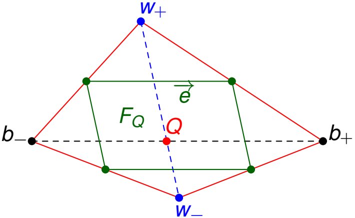

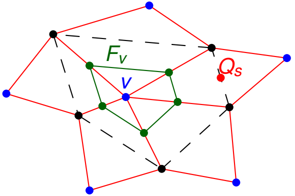

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Stokes' theorem: $\int_e df = \frac{f(w_+)+f(b_+)}{2} - \frac{f(w_+)+f(b_-)}{2}$ and $\iint_F d\omega = \int_{\partial F} \omega$.

Discrete holomorphic product

Let $f, g : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. Then,

$$ddf = 2\left(\partial_\diamond \bar{\partial}_\Lambda f - \bar{\partial}_\diamond \partial_\Lambda f\right) dz \wedge d\bar{z} = 0.$$

Thus, f discrete holomorphic iff $df = pdz$ for some $p : V(\diamond) \to \mathbb{C}$.

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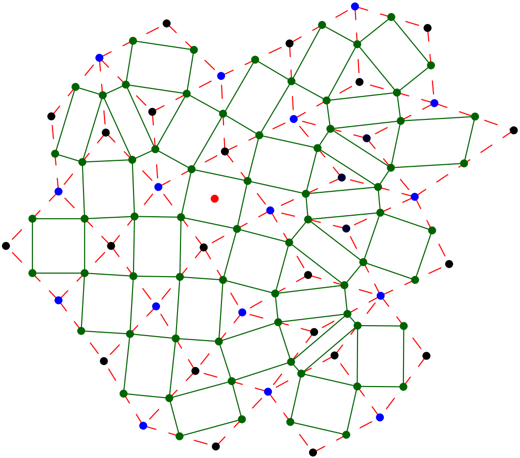

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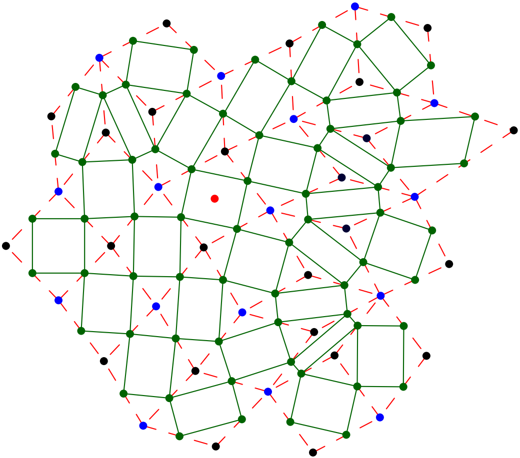

$fdg + gdf = (f\partial_\Lambda g + g\partial_\Lambda f)dz$ is closed.

Discrete holomorphic product

Let $f, g : \color{blue}{V(\Lambda)} \to \mathbb{C}$ and $h : \color{red}{V(\diamond)} \to \mathbb{C}$ discrete holomorphic. Then,

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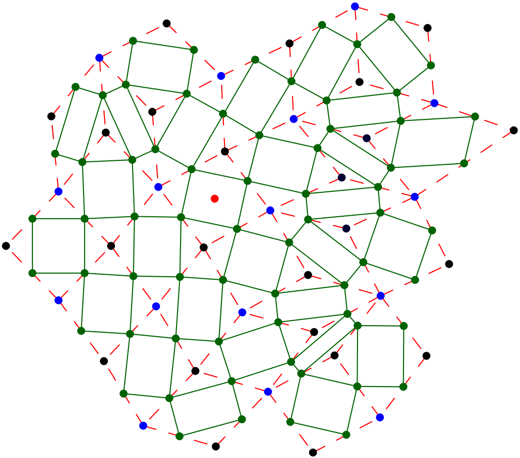

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DISCRETE WEDGE PRODUCT

Let $\omega = pdz + qd\bar{z}$ and $\omega' = p'dz + q'd\bar{z}$ discrete one-forms of type $\diamond$, $p, p', q, q' : V(\diamond) \to \mathbb{C}$. *Discrete wedge product* $\omega \wedge \omega'$ defined by

$$\begin{aligned}(\omega \wedge \omega')|_{F_Q} &= 2\left(p(Q)q'(Q) - q(Q)p'(Q)\right) dz \wedge d\bar{z} && \text{for } Q \in V(\diamond)\\ (\omega \wedge \omega')|_{F_v} &= 0 && \text{for } v \in V(\Lambda).\end{aligned}$$

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Theorem (Exterior derivative is derivation for $\wedge$)

Let $f : V(\Lambda) \to \mathbb{C}$ and ω discrete one-form of type $\diamond$. Then,

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$$d(f\omega)|_{F_v} = 2\left(f(v)\left(\partial_\diamond q\right)(v) - f(v)\left(\bar{\partial}_\diamond p\right)(v)\right) dz \wedge d\bar{z} = fd\omega|_{F_v},$$
$$d(f\omega)|_{F_Q} = 2\left(q(Q)\left(\partial_\Lambda f\right)(Q) - p(Q)\left(\bar{\partial}_\Lambda f\right)(Q)\right) dz \wedge d\bar{z} = (df \wedge \omega))|_{F_Q}.$$

Finally, $(df \wedge \omega)|_{F_v} = 0$ and $fd\omega|_{F_Q} = 0$, so $d(f\omega) = df \wedge \omega + fd\omega$.

Discrete Hodge star

Let $g : F(X) \to \mathbb{C}$, let $\omega = pdz + qd\bar{z}, \omega'$ discrete one-forms of type $\diamond$, Ω discrete two-form. The *discrete Hodge star* is defined by

$$\star g := -\frac{1}{i} g dz \wedge d\bar{z};$$
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$$0 = \iint_{F(X)} d(f \star \overline{\omega}) = \iint_{F(X)} df \wedge \star\overline{\omega} + \iint_{F(X)} fd \star \overline{\omega} = \langle df, \omega \rangle + \langle f, \star d \star \omega \rangle.$$

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- $f : V(\Lambda) \to \mathbb{C}$ discrete holomorphic. Then $\mathrm{Im}(f)$ uniquely determined by $\mathrm{Re}(f)$ up to two additive real constants.

Discrete Green's identities

Let $\diamondsuit_0 \subset \diamondsuit$ finite, and let $f, g : V(\Lambda_0) \to \mathbb{C}$.

1. $\langle f, \triangle g \rangle_{\diamondsuit_0} + \langle df, dg \rangle_{\diamondsuit_0} = \int_{\partial X_0} f \overline{\star dg}$.

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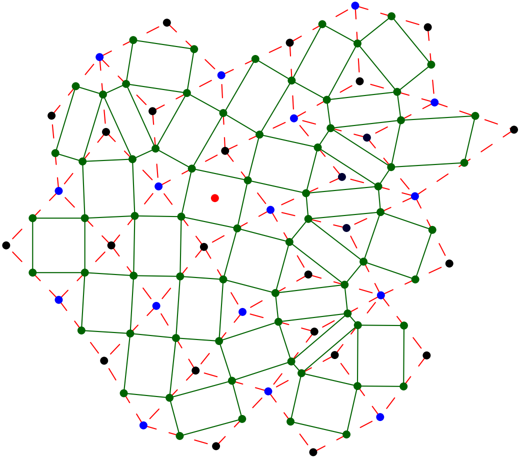
PF. $f \overline{\star(\star d \star dg)} + df \wedge \overline{\star dg} = d\left(f \overline{\star dg}\right)$.



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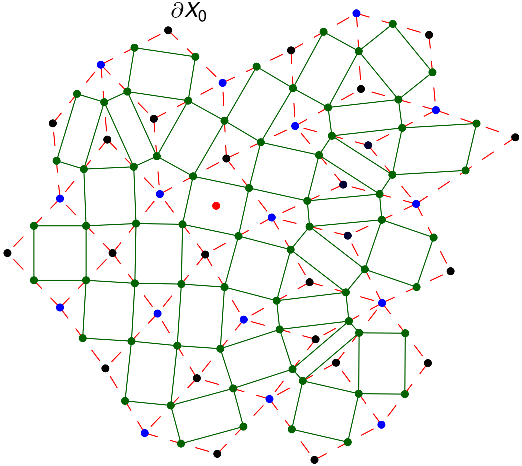

DISCRETE CAUCHY'S INTEGRAL FORMULAE

$Q_0 \in V(\diamond)$, $v_0 \in V(\Lambda)$. $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ and $K_{v_0} : V(\diamond) \to \mathbb{C}$ are called *discrete Cauchy kernels* iff for all $Q \in V(\diamond)$, $v \in V(\Lambda)$

$$\bar{\partial}_\Lambda K_{Q_0}(Q) = \delta_{QQ_0} \frac{\pi}{2\text{area}(F_Q)} \quad \text{and} \quad \bar{\partial}_\diamond K_{v_0}(v) = \delta_{vv_0} \frac{\pi}{2\text{area}(F_v)}.$$

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$$f(v_0) = \frac{1}{2\pi i} \oint_{C_{v_0}} f K_{v_0} dz, \quad \text{since:}$$

$$\frac{1}{2\pi i} \oint_{P_v} f K_{v_0} dz = \frac{2}{\pi} \text{area}(F_v) f(v) \bar{\partial}_\diamond K_{v_0}(v) = \delta_{vv_0} f(v),$$

$$\frac{1}{2\pi i} \oint_{P_Q} f K_{v_0} dz = \frac{2}{\pi} \text{area}(F_Q) \bar{\partial}_\Lambda f(Q) K_{v_0}(Q) = 0.$$

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Discrete Cauchy's integral formula for ∂f

Let $f : V(\Lambda) \to \mathbb{C}$ discrete holomorphic, $Q_0 \in V(\diamond)$.
Then, for any discrete contour C_{Q_0} in X surrounding Q_0 once in counterclockwise order, that does not contain any edge inside Q_0:

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Discrete Dirichlet energy

Let $\diamondsuit_0 \subset \diamondsuit$ be finite. For $f : V(\Lambda_0) \to \mathbb{C}$, *discrete Dirichlet energy* is

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Then, $E_{\diamondsuit_0}$ is strictly convex non-negative quadratic functional in terms of the interior values $f(v)$. Furthermore,

$$-\frac{\partial E_{\diamondsuit_0}(f)}{\partial f(v)} = 4\text{area}(F_v)\triangle f(v).$$

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Let $f_0 : V(\partial\Lambda_0) \to \mathbb{R}$. Consider vector space of $f : V(\Lambda_0) \to \mathbb{C}$ that agree with f_0 on the boundary.

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$$-\frac{\partial E_{\diamond_0}(f)}{\partial f(v)} = 4\text{area}(F_v)\triangle f(v).$$

$\Rightarrow$ Discrete Dirichlet boundary value problem uniquely solvable.

Local existence of discrete Cauchy's kernels

Let $\diamond_0 \subset \diamond$ be a disk, BV number of (boundary) vertices of $\partial\Lambda_0$.
Space of real discrete harmonic functions on $V(\Lambda_0)$ has dimension BV.

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LOCAL EXISTENCE OF DISCRETE CAUCHY'S KERNELS

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$\ker(\bar{\partial}_\diamond) \subseteq \operatorname{im}(\partial_\Lambda)$ due to existence of discrete primitives.Thus,

$$\partial_\Lambda : \mathbb{C}^{V(\Lambda_0)} \to \mathbb{C}^{V(\diamond_0)} \quad \text{is surjective.}$$

Similar arguments apply for $\bar{\partial}_\Lambda$.

Asymptotics for general quad-graphs Λ

Theorem

Assume that there exist constants $\alpha_0, E_1, E_0 > 0$, such that $\alpha \geq \alpha_0$ and $E_1 \geq e \geq E_0$ for all angles α and side lengths e.
If $f : V(\Lambda) \to \mathbb{C}$ is discrete harmonic and $f(v) = o(v^{-1/2})$, f is essentially constant.

ASYMPTOTICS FOR GENERAL QUAD-GRAPHS Λ

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PROOF.

Idea: Compare the discrete Dirichlet energy of *f* with the function that agrees with *f* on the boundary of a large disk and is zero in its interior. □

Asymptotics for parallelogram-graphs Λ

Theorem

Assume $\alpha \geq \alpha_0 > 0$ and $e/e' \geq q_0 > 0$ for all angles α and two side lengths e, e' of a parallelogram. Let $v_0 \in V(\Lambda)$, $Q_0 \in V(\diamond)$ fixed.

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Theorem

Assume $\alpha \geq \alpha_0 > 0$ *and* $e/e' \geq q_0 > 0$ *for all angles* α *and two side lengths* e, e' *of a parallelogram. Let* $v_0 \in V(\Lambda)$, $Q_0 \in V(\diamond)$ *fixed. Then, there exist discrete Green's functions* $G(\cdot; v_0)$ *and discrete Cauchy's kernels* $K(\cdot; v_0)$ *and* $K(\cdot; Q_0)$ *such that*

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$$G(v; v_0) = \frac{1}{4\pi} \log\left(|v - v_0| / |J(v, v_0)|\right) + O\left(|v - v_0|^{-2}\right),$$

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$$(\partial_\Lambda K(\cdot; Q_0))(Q) = \frac{1}{(Q - Q_0)^2} + \frac{\tau(Q, Q_0)}{J(Q, Q_0)^2} + O\left(|Q - Q_0|^{-3}\right).$$

Discrete Riemann surfaces

Discrete Riemann surfaces

Theorem

Let Σ, Σ' compact discrete Riemann surfaces of genera g, g' and $f : V(\Lambda) \to V(\Lambda')$ N-sheeted almost discrete holomorphic covering.

$$g = N(g' - 1) + 1 + b/2,$$

where $b = \sum_{v \in V(\Lambda)} b_f(v) + \sum_{Q \in V(\diamond)} b_f(Q)$ total branching number.

DISCRETE RIEMANN SURFACES

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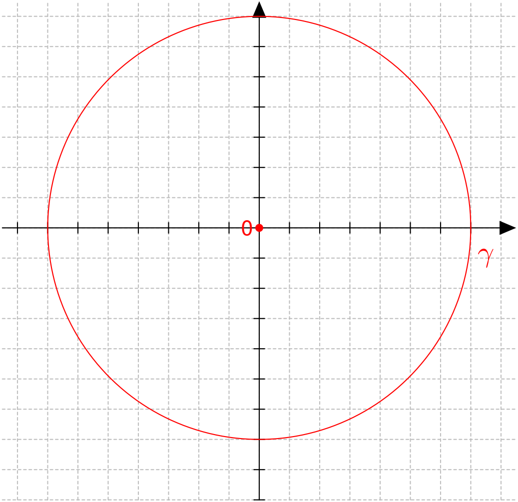
Theorem

Σ *compact discrete Riemann surface of genus g, D admissible divisor. Then,* $l(-D) = \deg D - 2g + 2 + i(D)$.

Cauchy formula for non-mathematicians



$$f(0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z} dz$$

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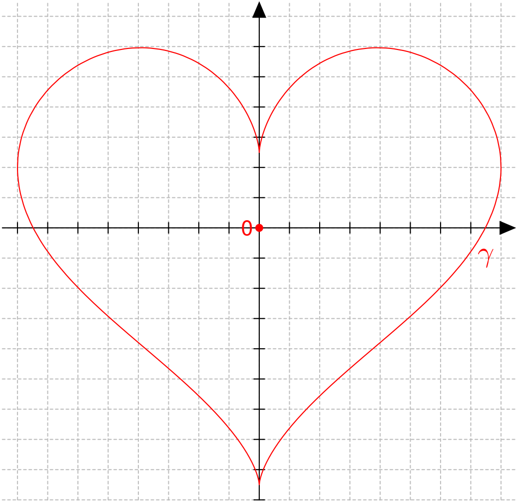

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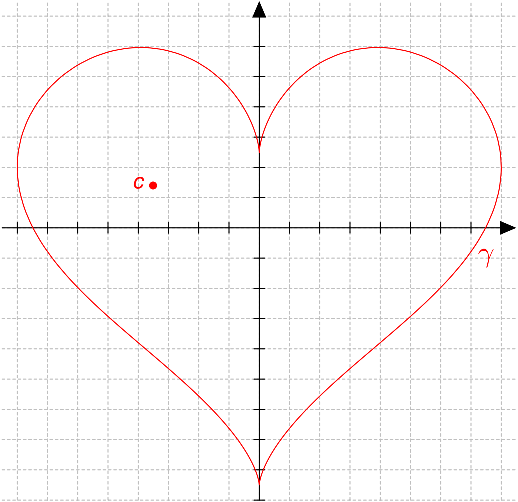

$$f(c) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - c} dz$$

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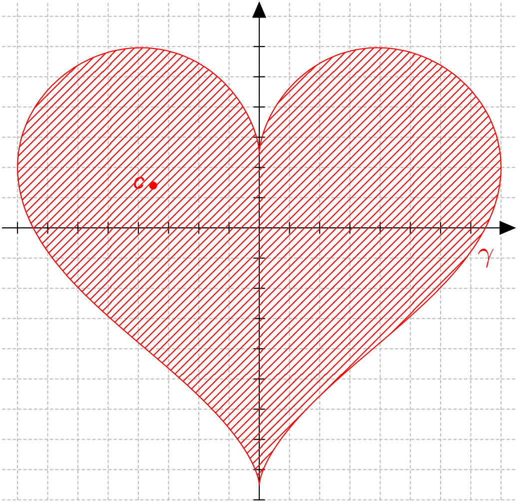

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