Young Women in Algebraic Geometry

Critical values for automorphic L-functions

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Introduction

Special values of L-functions play an important role in the Langlands program. Numerous conjectures predict that special values of L-functions reflect arithmetic properties of geometric objects. Most of these conjectures are still open and difficult to attack.

At the same time, concrete results on the special values of L-functions appear more and more in automorphic settings. For example, in [Har97], M. Harris constructed complex invariants called arithmetic automorphic periods and showed that the special values of automorphic L-function for $GL_n * GL_1$ could be interpreted in terms of these invariants.

We generalize his results in two ways. Firstly, the arithmetic automorphic periods have been defined over general CM fields. Secondly, we show that special values of arithmetic automorphic periods for $GL_n * GL_{n'}$ can be interpreted in terms of these arithmetic automorphic periods in many situations. In fact, we have found a concise formula for such critical values. This is our first main automorphic result. One possible application is to construct p-adic L-functions.

One crucial step to prove it is to show that the arithmetic automorphic periods can be factorized as products of local periods over infinite places. This was actually a conjecture of Shimura (c.f. [Shi83], [Shi88]). One possible way to show this is to define local periods geometrically and prove that special values of L-functions can be interpreted in terms of local periods. This was done by M. Harris for Hilbert modular forms in [Har93]. But it is extremely difficult to generalize his arguments to GL_n. Instead, we show that there are relations between arithmetic automorphic periods. These relations lead to a factorization which is our second main automorphic result.

We remark that the factorization is not unique. We show that there is a natural way to factorize such that the local periods are functorial for automorphic induction and base change. This is our third main automorphic result. We believe that local periods are also functorial for endoscopic transfer. We will try to prove this in the near future.

On the other hand, Deligne's conjecture related critical values for motives over $\mathbb{Q}$ and Deligne's period (c.f. [Del79]). When the motive is the restriction to $\mathbb{Q}$ of the tensor product of two motives over a CM field, we may calculate Deligne's period in terms of motivic periods defined in [Har13]. If the two motives are associated to automorphic representations of GL_n and $GL_{n'}$ respectively, we may define motivic periods which are analogues of the arithmetic automorphic periods. We get a formula of Deligne's period in terms of these motivic periods. Our main motivic result says that our formula for automorphic L-functions are at least formally compatible with Deligne's conjecture .

Notation and main results I

Let K be a quadratic imaginary field and $F \supset K$ be a CM field of degree d over K. We fix an embedding $K \hookrightarrow \mathbb{C}$. Let $\Sigma_{F;K}$ be the set of embeddings $\sigma : F \hookrightarrow \mathbb{C}$ such that $\sigma\mid_K$ is the fixed embedding.

Let E be a number field. Let $\{a(\sigma)\}_{\sigma \in Aut(\mathbb{C}/K)}$, $\{b(\sigma)\}_{\sigma\in Aut(\mathbb{C}/K)}$ be two families of complex numbers. Roughly speaking, we say $a \sim_{E;K} b$ if $a = b$ up to multiplication by elements in $E^\times$ and equivariant under G_K-action.

Let Π be a very regular conjugate self-dual cuspidal cohomological representation of $GL_n(\mathbb{A}_F)$. In particular, we know that Π_f is defined over a number field $E(\Pi)$. For any $I : \Sigma_{F;K} \to \{0, 1, \cdots, n\}$, we may define the arithmetic automorphic periods $P^{(I)}(\Pi)$ as the Petersson inner product of a rational vector in a certain cohomology space associated to a unitary group of infinity sign I. It is a non zero complex number well defined up to multiplication by elements in $E(\Pi)^\times$.

Definition(split index): Let n and n' be two positives integers.
Let Π and Π' be two regular conjugate self-dual representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ respectively. Let σ be an element of $\Sigma_{F;K}$. We denote the infinity type of Π and Π' at σ by $(z^{a_i(\sigma)}\overline{z}^{-a_i(\sigma)})_{1\leq i\leq n}$, $a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma)$ and $(z^{b_j(\sigma)}\overline{z}^{-b_j(\sigma)})_{1\leq j\leq n'}$, $b_1(\sigma) > b_2(\sigma) > \cdots > b_{n'}(\sigma)$ respectively. We assume that $a_i(\sigma) + b_j(\sigma) \neq 0$ for all $1 \leq i \leq n$ all $1 \leq j \leq n'$ and all σ.
We split the sequence $(a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma))$ with the numbers

$$-b_{n'}(\sigma) > -b_{n'-1}(\sigma) > \cdots > -b_1(\sigma).$$

This sequence is split into $n' + 1$ parts. We denote the length of each part by $sp(0, \Pi' : \Pi, \sigma), sp(1, \Pi'; \Pi, \sigma), \cdots, sp(n', \Pi'; \Pi, \sigma)$, and call them the **split indices**.

Definition (good position): We assume that $n > n'$. Let Π and Π' be as before. We say the pair(Π, Π') is **in good position** if for any $\sigma \in \Sigma_{F;K}$, the n' numbers

$$-b_{n'}(\sigma) > -b_{n'-1}(\sigma) > \cdots > -b_1(\sigma).$$

lie in different gaps between $(a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma))$.
It is equivalent to saying that $sp(i, \Pi'; \Pi, \sigma) \neq 0$ for all $0 \leq i \leq n'$ and $\sigma \in \Sigma_{F;K}$. In particular, if $n' = n - 1$, we know (Π, Π') is in good position if and only if $sp(i, \Pi'; \Pi, \sigma) = 1$ for all i and σ.

Automorphic results

Theorem 0.1 *If $m \in \mathbb{Z} + \frac{n+n'}{2}$ is critical for $\Pi \times \Pi'$ then*

$$L(m, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{nn'md} \prod_{\sigma\in\Sigma_{F;K}} (\prod_{j=0}^{n} P^{(j)}(\Pi, \sigma)^{sp(j,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} P^{(k)}(\Pi', \sigma)^{sp(k,\Pi';\Pi,\sigma)})$$

in the following cases:

1. *$n' = 1$ and m is bigger than the central value.*
2. *$n > n'$, $m \geq 1/2$ and the pair (Π, Π') is in good position.*
3. *$m = 1$, the pair (Π, Π') is regular enough.*

Theorem 0.2 *If Π is regular enough, then there exists some complex numbers $P^{(s)}(\Pi, \sigma)$ unique up to multiplication by elements in $(E(\Pi))^\times$ such that the following two conditions are satisfied:*

1. *$P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod_{\sigma\in\Sigma_{F;K}} P^{(I(\sigma))}(\Pi, \sigma)$ for all $I = (I(\sigma))_{\sigma\in\Sigma_{F;K}} \in \{0, 1, \cdots, n\}^{\Sigma_{F;K}}$*
2. *and $P^{(0)}(\Pi, \sigma) \sim_{E(\Pi);K} p(\xi_\Pi, \overline{\sigma})$*

where ξ_Π is the central character of Π, $\xi_\Pi := \xi_\Pi^{-1,c}$ and $p(\xi_\Pi, \overline{\sigma})$ is the CM period.

Theorem 0.3 *(a) Let $\mathcal{F}/F$ be a cyclic extension of CM fields of degree l and $\Pi_\mathcal{F}$ be a cuspidal representation of $GL_n(\mathbb{A}_\mathcal{F})$. We write $AI(\Pi_\mathcal{F})$ for the automorphic induction of $\Pi_\mathcal{F}$. We assume that the arithmetic automorphic periods are defined for both $AI(\Pi_\mathcal{F})$ and $\Pi_\mathcal{F}$.*
Let $I_F \in \{0, 1, \cdots, nl\}^{\Sigma_{F;K}}$. We may define $I_\mathcal{F} \in \{0, 1, \cdots, n\}^{\Sigma_{\mathcal{F};K}}$ which depends only on I_F and the infinity type of $\Pi_\mathcal{F}$. Or locally let $0 \leq s \leq nl$ be an integer and $s(\sigma)$ is an integer which depends only on s and the infinity type of $\Pi_\mathcal{F}$ at $\sigma \in \Sigma_{\mathcal{F};K}$. We have:

$$P^{(I_F)}(AI(\Pi_\mathcal{F})) \sim_{E(\Pi_\mathcal{F});K} P^{(I_\mathcal{F})}(\Pi_\mathcal{F})$$
$$\text{or locally } P^{(s)}(AI(\Pi_\mathcal{F}, \tau) \sim_{E(\Pi_\mathcal{F});K} \prod_{\sigma\mid\tau} P^{(s(\sigma))}(\Pi_\mathcal{F}, \sigma).$$

(b) Let π_F be a cuspidal representation of $GL_n(\mathbb{A}_F)$. We write $BC(\pi_F)$ for its strong base change to $\mathcal{F}$. We assume that the arithmetic automorphic periods are defined for both π_F and $BC(\pi_F)$.
Let $I_F \in \{0, 1, \cdots, n\}^{\Sigma_{F;K}}$. We write $I_\mathcal{F}$ the composition of I_F and the restriction of complex embeddings of $\mathcal{F}$ to F.
We then have:

$$P^{(I_\mathcal{F})}(BC(\pi_F)) \sim_{E(\pi_F);K} p^{I_F}(\pi_F)^l$$
$$\text{or locally } \quad P^{(s)}(BC(\pi_F), \sigma)^l \sim_{E(\pi_F);K} P^{(s)}(\pi_F, \sigma\mid_F)^l.$$

Consequently, we know

$$P^{(s)}(BC(\pi_F), \sigma) \sim_{E(\pi_F)} \lambda^{(s)}(\pi_F, \sigma)P^{(s)}(\pi_F, \sigma\mid_F).$$

where $\lambda^{(s)}(\pi_F, \sigma)$ is an algebraic number whose l-th power is in $E(\pi_F)^\times$.

Motivic result

We now introduce the motivic results. Let M, M' be motives over F with coefficients in E and E' of rank n and n' respectively. We assume that $M \otimes M'$ has no $(\omega/2, \omega/2)$-class. We may define motivic periods $Q^{(t)}(M, \sigma)$ for $0 \leq t \leq n$ and $\sigma \in \Sigma_{F;K}$. We can calculate Deligne's period of $Res_{F/\mathbb{Q}}(M \otimes M')$ in terms of these periods. If M and M' are motives associated to Π and Π', Deligne's conjecture (c.f. [Del79]) is equivalent to the following conjecture:

Conjecture 0.1 *If $m \in \mathbb{Z} + \frac{n+n'}{2}$ is critical for $\Pi \times \Pi'$ then*

$$L(m, \Pi \times \Pi') = L(m + \tfrac{n+n'-2}{2}, M \otimes M')$$
$$\sim_{E(\Pi)E(\Pi');K} (2\pi i)^{mnn'd} \prod_{\sigma\in\Sigma_{F;K}} (\prod_{j=0}^{n} Q^{(j)}(M, \sigma)^{sp(j,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} Q^{(k)}(M', \sigma)^{sp(k,\Pi';\Pi,\sigma)})$$

We see that it is compatible with Theorem 0.1. The main point to calculate Deligne's period is to fix proper basis. Deligne's period is defined by rational basis. The basis that we have fixed are not rational. But they are rational up to unipotent transformation matrices. We can still use such basis to calculate determinant.

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