FACTORIZATION OF ARITHMETIC AUTOMORPHIC PERIODS

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ABSTRACT. In this paper, we prove that the arithmetic automorphic periods for GL_n over a CM field factorize through the infinite places. This generalizes a conjecture of Shimura in 1983, and is predicted by the Langlands correspondence between automorphic representations and motives.

CONTENTS

INTRODUCTION

The aim of this paper is to prove the factorization of arithmetic automorphic periods defined as Petersson inner products of arithmetic automorphic forms on unitary Shimura varieties. This generalizes a conjecture of Shimura (c.f. Conjecture 5.12 of [Shi83]).

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We first introduce the conjecture of Shimura to illustrate our main result. Let E be a totally real field of degree d. Let J_E be the set of real embeddings of E. Let f be an arithmetic Hilbert cusp form inside a cuspidal automorphic representation π of $GL_2(\mathbb{A}_E)$. We define the period $P(\pi)$ as the Petterson inner product of f. One can show that up to multiplication by an algebraic number, the period $P(\pi)$ does not depend on the choice of f.

For each $\sigma \in J_E$, Shimura conjectured the existence of a complex number $P(\pi,\sigma)$, associated to a quaternion algebra which is split at σ and ramified at other infinite places, such that:

$$P(\pi) \sim \prod_{\sigma\in J_E} P(\pi,\sigma) \tag{0.1}$$

where the relation $\sim$ means equality up to multiplication by an algebraic number.

Furthermore, if D is a quaternion algebra and π admets a Jacquet-Langlands transfer π_D to D, we may define $P(\pi_D)$ as Petersson inner product of an algebraic form in π_D, and Shimura conjectured that:

$$P(\pi_D) \sim \prod_{\sigma\in J_E,\ \text{split for } D} P(\pi,\sigma). \tag{0.2}$$

This conjecture was proved under some local hypotheses in an important paper of M. Harris (c.f. [Har93a]) and was improved by H. Yoshida (c.f. [Yos95]). The paper of M. Harris is very long and involves many techniques which seems extremely difficult to generalize. In this paper, we prove a generalization of Shimura's conjecture (c.f. Conjecture 2.1) by a new and simpler method.

We consider representations of $GL_n(\mathbb{A}_F)$ where F is a CM field. We write J_F for the set of complex embeddings of F. We fix Σ a CM type of F, i.e., $\Sigma\subset J_F$ presents J_F modulo the action of complex conjugation.

Let F^+ be the maximal totally real subfield of F. Instead of the quaternions algebras, we consider unitary groups of rank n with respect to F/F^+. They are all inner forms of $GL_n(\mathbb{A}_F^+)$.

We use I to denote the signature of a unitary group. It can be considered as a map from Σ to $\{0,1,\cdots,n\}$. For each I, let U_I be a unitary group of signature I. We note that $U_{I,F}\cong GL_{n,F}$ as algebraic group over F. In particular, we have $U_I(\mathbb{A}_F)\cong GL_n(\mathbb{A}_F)$. We assume that Π, considered as a representation of $U_I(\mathbb{A}_F)$, descends by base change to $U_I(\mathbb{A}_{F^+})$. We refer to [Art03], [HL04], [Lab11] or [KMSW14] for details of base change.

We can then define a period $P^{(I)}(\Pi)$ as Petersson inner product of an algebraic automorphic form in the bottom degree of cohomology of the similitude unitary Shimura variety attached to U_I. The construction is given in section 2.

Conjecture 0.1. *There exists some non zero complex numbers $P^{(s)}(\Pi,\sigma)$ for all $0\leqslant s\leqslant n$ and $\sigma\in\Sigma$ such that*

$$P^{(I)}(\Pi) \sim_{E(\Pi)} \prod_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi,\sigma) \tag{0.3}$$

for any $I=(I(\sigma))_{\sigma\in\Sigma}\in\{0,1,\cdots,n\}^{\Sigma}$.

Our main theorem is the following (c.f. Theorem 3.3):

Theorem 0.1. *The above conjecture is true provided that Π is 2-regular with a global non vanishing condition which is automatically satisfied if Π is 6-regular.*

We remark that this conjecture is not as simple as it may look like even if we do not put any restriction on the complex numbers $P^{(s)}(\Pi,\sigma)$. In fact, the number of different $P^{(I)}(\Pi)$ is d^{n+1} and the number of different $P^{(I(\sigma))}(\Pi,\sigma)$ is only $d(n+1)$.

On the other hand, it is true that the choice of $P^{(I(\sigma))}(\Pi,\sigma)$ is not unique. We have specified a canonical choice in section 3.5. Similarly to Shimura's formulation, the canonical choice of $P^{(I(\sigma))}(\Pi,\sigma)$ is related to the unitary group of signature $(1,n-1)$ at σ and $(0,n)$ at other places (c.f. section 4.4 of [HL16]). The author proved that the periods $P^{(I)}(\Pi)$ as well as the local specified periods $P^{(I)}(\Pi,\sigma)$ are functorial in the sense of Langlands functoriality in [Lin15a] and [Lin15b], .

We also get a partial result with a weaker regular condition (c.f. Theorem 3.2):

Theorem 0.2. *If $n\geqslant 4$ and Π satisfies a global non vanishing condition, in particular, if Π is 3-regular, then there exists some non zero complex numbers $P^{(s)}(\Pi,\sigma)$ for all $1\leqslant s\leqslant n-1$, $\sigma\in\Sigma$ such that $P^{(I)}(\Pi)\sim_{E(\Pi)}\prod\limits_{\sigma\in\Sigma}P^{(I(\sigma))}(\Pi,\sigma)$ for all $I=(I(\sigma))_{\sigma\in\Sigma}\in\{1,2,\cdots,n-1\}^{\Sigma}$.*

Before introducing the proof, let us look at equation (0.3) that we want to prove. The left hand side involves d^{n+1} periods and the right hand side involves only $d(n+1)$ periods. This is only possible if there are many relations between the periods $P^{(I)}(\Pi)$ in the left hand side.

In fact, the first step of the our proof is to reduce Conjecture 0.1 to relations of the periods. The general argument is given in section 3.1.

The next step is then to prove these relations. The proof involves several techniques like CM periods, Whittaker periods and special values of L-functions. We use three results on special values of L-functions. The first one is due to Blasius on relations between special values of L-functions for Hecke characters and CM periods (c.f. section 1 of [Har93a]). The second one relates special values of L-functions for GL_n*GL_1 and the arithmetic automorphic periods $P^{(I)}(\Pi)$ (c.f. [GL16]). The last one is about relations between special values of L-functions for GL_n*GL_{n-1} and the Whittaker periods which is proved in [GH15] over quadratic imaginary field, and in [Gro17] for general CM fields.

The advantage of the last result is that the GL_{n-1}-representation do not need to be cuspidal. This allows us to construct auxiliary representations of GL_{n-1} more freely and leads to relations between Whittaker periods and arithmetic automorphic periods $P^{(I)}(\Pi)$ (see Theorem 3.1) which generalizes Theorem 6.7 of [GH15]. This relation already implies the partial result mentioned above.

To prove the whole conjecture, a more ingenious construction needs to be made. We construct carefully a non-cuspidal representation of $GL_{n+2}(\mathbb{A}_F)$ related to Π, and an auxiliary cuspidal representation of $GL_{n+3}(\mathbb{A}_F)$. The $GL_{n+3}(\mathbb{A}_F)$ representation is induced from Hecke characters. Hence special values of its L-function can be written in terms of CM periods by Blasius's result. The details can be found in section 3.4.

The manipulation of different special values with different auxiliary representations can give many interesting results of period relations or special values of L-functions. We refer the reader to [Lin15a], [GH15] or [Lin15b] for more examples. More recently, the author and H. Grobner proved some results on special values which implies one case of the Ichino-Ikeda conjecture up to multiplication by an algebraic number in a very general setting (c.f. [GL17]).

We remark at last that Conjecture 0.1 is predicted by motivic calculation (c.f. section 2.3 of [HL16]). More generally, the motivic calculation and the Langlands correspondence predict the existence of more automorphic periods and some finer relations between them. This will be discussed in details in a forthcoming paper of the author with H. Grobner and M. Harris.

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1. Preliminaries

1.1. **Basic Notation.** Let F be a CM field and F^+ be its maximal totally real subfield. We denote by J_F the set of embeddings of F in $\mathbb{C}$. The complex conjugation c acts on the set J_F. We say Σ a subset of J_F is a CM type if J_F is the disjoint union of Σ and Σ^c. For $\iota \in J_E$, we also write $\bar{\iota}$ for the complex conjugation of ι.

As usual, we let S be a finite set of places of F, containing all infinite places and all ramified places of any representation which will appear in the text.

Let χ be a Hecke character of F. We write χ_ι by $z^{a_\iota}\bar{z}^{a_{\bar{\iota}}}$ for $\iota \in J_F$. We say that χ is *algebraic* if $a_\iota, a_{\bar{\iota}} \in \mathbb{Z}$ for all $\iota \in J_F$. We say that χ is *critical* if it is algebraic and moreover $a_\iota \neq a_{\bar{\iota}}$ for all $\iota \in J_F$. It is equivalent to that the motive associated to χ has critical points in the sense of Deligne (cf. [Del79]). We remark that 0 and 1 are always critical points in this case.

Moreover, we write $\check{\chi}$ for the Hecke character $\chi^{c,-1}$. Apparently if χ is algebraic or critical then so is $\check{\chi}$.

For Π an algebraic automorphic representation of $GL_n(\mathbb{A}_F)$, we know that for each $\iota \in J_F$, there exists $a_{\iota,1}, \cdots, a_{\iota,n}, a_{\bar{\iota},1}, \cdots, a_{\bar{\iota},n} \in \mathbb{Z} + \frac{n-1}{2}$ such that

$$\Pi_\iota \cong \mathrm{Ind}_{B(\mathbb{C})}^{G(\mathbb{C})}[z_1^{a_{\iota,1}}\bar{z}_1^{a_{\bar{\iota},1}} \otimes \ldots \otimes z_n^{a_{\iota,n}}\bar{z}_n^{a_{\bar{\iota},n}}].$$

Here Ind refers to the normalised parabolic induction. We define the *infinity type* of Π at ι by $\{z^{a_{\iota,i}}\bar{z}^{a_{\bar{\iota},i}}\}_{1\leqslant i\leqslant n}$ (c.f. section 3.3 of [Clo90]).

Let N be any positive integer. We say that Π is *N-regular* if $|a_{\iota,i} - a_{\iota,j}| \geqslant N$ for any $\iota \in J_F$ and $1 \leqslant i < j \leqslant n$. We say Π is *regular* if it is 1 regular.

Throughout the text, we fix Σ any CM type of F. We also fix ψ an algebraic Hecke character of F with infinity type $z^1\bar{z}^0$ at each place in Σ such that $\psi\psi^c = ||\cdot||_{\mathbb{A}_K}$ (see Lemma 4.1.4 of [CHT08] for its existence). It is easy to see that the restriction of $||\cdot||_{\mathbb{A}_K}^{\frac{1}{2}}\psi$ to $\mathbb{A}_{\mathbb{Q}}^\times$ is the quadratic character associated to the extension $K/\mathbb{Q}$ by the class field theory. Consequently our construction is compatible with that in [GH15] or [GL17].

Let E be a number field. We consider it as a subfield of $\mathbb{C}$. Let x, y be two complex numbers. We say $x \sim_E y$ if $y \neq 0$ and $x/y \in E$. This relation is symmetric but not transitive unless we know both numbers involved are non-zero.

The previous relation can be defined in an equivariant way for $Aut(\mathbb{C})$-families. More precisely, let $x = \{x(\sigma)\}_{\sigma\in Aut(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma\in Aut(\mathbb{C})}$ be families of complex numbers. We say $x \sim_E y$ and equivariant under the action of $Aut(\mathbb{C}/F)$ if either $y(\sigma) = 0$ for all σ, either $y(\sigma) \neq 0$ with the following properties:

(1) $x(\sigma) \sim_{\sigma(E)} y(\sigma)$ for all $\sigma \in Aut(\mathbb{C})$;

(2) $\tau\left(\dfrac{x(\sigma)}{y(\sigma)}\right) = \dfrac{x(\tau\sigma)}{y(\tau\sigma)}$ for all $\tau \in Aut(\mathbb{C}/F)$ and all $\sigma \in Aut(\mathbb{C})$.

Lemma 1.17 of [GL17] says that if E contains F^{Gal} and $x(\sigma)$ and $y(\sigma)$ depends only on $\sigma \mid E$, then the second point above will imply the first point.

We remark that all the L-values and periods in this paper will be considered as $Aut(\mathbb{C})$-families.

1.2. Rational structures on certain automorphic representations.

Let Π be an automorphic representation of $GL_n(\mathbb{A}_F)$.

We denote by V the representation space for Π_f. For $\sigma \in Aut(\mathbb{C})$, we define another $GL_n(\mathbb{A}_{F,f})$-representation Π_f^σ to be $V \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Let $\mathbb{Q}(\Pi)$ be the subfield of $\mathbb{C}$ fixed by $\{\sigma \in Aut(\mathbb{C}) \mid \Pi_f^\sigma \cong \Pi_f\}$. We call it the **rationality field** of Π.

For E a number field, G a group and V a G-representation over $\mathbb{C}$, we say V has an **E-rational structure** if there exists an E-vector space V_E endowed with an action of G such that $V = V_E \otimes_E \mathbb{C}$ as representation of G. We call V_E an E-rational structure of V.

We denote by $\mathcal{A}lg(n)$ the set of algebraic automorphic representations of $GL_n(\mathbb{A}_F)$ which are isobaric sums of cuspidal representations as in section 1 of [Clo90].

Theorem 1.1. *(Théorème 3.13 in [Clo90])*

Let Π be a regular representation in $\mathcal{A}lg(n)$. We have that:

1. *$\mathbb{Q}(\Pi)$ is a number field.*
2. *Π_f has a $\mathbb{Q}(\Pi)$-rational structure unique up to homotheties.*
3. *For all $\sigma \in Aut(\mathbb{C})$, Π_f^σ is the finite part of a regular representation in $\mathcal{A}lg(n)$. It is unique up to isomorphism by the strong multiplicity one theorem. We denote it by Π^σ.*

Remark 1.1. *Let $n = n_1 + n_2 + \cdots + n_k$ be a partitian of positive integers and Π_i be regular representations in $\mathcal{A}lg(n_i)$ for $1 \leqslant i \leqslant k$ respectively.*

The above theorem implies that, for all $1 \leqslant i \leqslant k$, the rational field $\mathbb{Q}(\Pi_i)$ is a number field.

Let $\Pi = (\Pi_1 || \cdot ||_{\mathbb{A}_K}^{\frac{1-n_1}{2}} \boxplus \Pi_2 || \cdot ||_{\mathbb{A}_K}^{\frac{1-n_2}{2}} \boxplus \cdots \boxplus \Pi_k || \cdot ||_{\mathbb{A}_K}^{\frac{1-n_k}{2}}) || \cdot ||_{\mathbb{A}_K}^{\frac{n-1}{2}}$ be the normalized isobaric sum of Π_i. It is still algebraic.

We can see from definition that $\mathbb{Q}(\Pi)$ is the compositum of $\mathbb{Q}(\Pi_i)$ with $1 \leqslant i \leqslant k$. Moreover, if Π is regular, we know from the above theorem that Π has a $\mathbb{Q}(\Pi)$-rational structure.

1.3. Rational structures on the Whittaker model.

Let Π be a regular representation in $\mathcal{A}lg(n)$ and then its rationality field $\mathbb{Q}(\Pi)$ is a number field.

We fix a nontrivial additive character ϕ of $\mathbb{A}_F$. Since Π is an isobaric sum of cuspidal representations, it is generic. Let $W(\Pi_f)$ be the Whittaker model associated to Π_f (with respect to ϕ_f). It consists of certain functions on $GL_n(\mathbb{A}_{F,f})$ and is isomorphic to Π_f as $GL_n(\mathbb{A}_{F,f})$-modules.

Similarly, we denote the Whittaker model of Π (with respect to) ϕ by $W(\Pi)$.

Definition 1.1. ***Cyclotomic character***

There exists a unique homomorphism $\xi : Aut(\mathbb{C}) \to \widehat{\mathbb{Z}}^\times$ such that for any $\sigma \in Aut(\mathbb{C})$ and any root of unity ζ, $\sigma(\zeta) = \zeta^{\xi(\sigma)}$, called the cyclotomic character.

For $\sigma \in Aut(\mathbb{C})$, we define $t_\sigma \in (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times = \widehat{\mathcal{O}_F}^\times$ to be the image of $\xi(\sigma)$ by the embedding $(\widehat{\mathbb{Z}})^\times \hookrightarrow (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times$. We define $t_{\sigma,n}$ to be the diagonal matrix $diag(t_\sigma^{-n+1}, t_\sigma^{-n+2}, \cdots, t_\sigma^{-1}, 1) \in GL_n(\mathbb{A}_{F,f})$ as in section 3.2 of [RS08].

For $w \in W(\Pi_f)$, we define a function w^σ on $GL_n(\mathbb{A}_{F,f})$ by sending $g \in GL_n(\mathbb{A}_{F,f})$ to $\sigma(w(t_{\sigma,n}g))$. For classical cusp forms, this action is just the $Aut(\mathbb{C})$-action on Fourier coefficients.

Proposition 1.1. *(Lemma 3.2 of [RS08] or Proposition 2.7 of [GH15])*

The map $w \mapsto w^\sigma$ gives a σ-linear $GL_n(\mathbb{A}_{F,f})$-equivariant isomorphism from $W(\Pi_f)$ to $W(\Pi_f^\sigma)$.

For any extension E of $\mathbb{Q}(\Pi_f)$, we can define an E-rational structure on $W(\Pi_f)$ by taking the $Aut(\mathbb{C}/E)$-invariants.

Moreover, the E-rational structure is unique up to homotheties.

Proof. The first part is well-known (see the references in [RS08]). mahnkopf05

For the second part, the original proof in [RS08] works for cuspidal representations. The key point is to find a nonzero global invariant vector. It is equivalent to finding a nonzero local invariant vector for every finite place. Then Theorem 5.1(ii) of [JPSS81] is involved as in [GH15].

The last part follows from the one-dimensional property of the invariant vector which is the second part of Theorem 5.1(ii) of [JPSS81].

□

1.4. **Rational structures on cohomology spaces and comparison of rational structures.** Let Π be a regular representation in $\mathcal{A}lg(n)$. The Lie algebra cohomology of Π has a rational structure. It is described in section 3.3 of [RS08]. We give a brief summary here.

Let Z be the center of GL_n. Let $\mathfrak{g}_\infty$ be the Lie algebra of $GL_n(\mathbb{R} \otimes_{\mathbb{Q}} F)$. Let S_{real} be the set of real places of F, $S_{complex}$ be the set of complex places of F and $S_\infty = S_{real} \cup S_{complex}$ be the set of infinite places of F.

For $v \in S_{real}$, we define $K_v := Z(\mathbb{R})O_n(\mathbb{R}) \subset GL_n(F_v)$. For $v \in S_{complex}$, we define $K_v := Z(\mathbb{C})U_n(\mathbb{C}) \subset GL_n(F_v)$. We denote by K_∞ the product of K_v with $v \in S_\infty$, and by K_∞^0 the topological connected component of K_∞.

We fix T the maximal torus of GL_n consisting of diagonal matrices and B the Borel subgroup of G consisting of upper triangular matrices. For μ a dominant weight of $T(\mathbb{R} \otimes_{\mathbb{Q}} F)$ with respect to $B(\mathbb{R} \otimes_{\mathbb{Q}} F)$, we can define W_μ an irreducible representation of $GL_n(\mathbb{R} \otimes_{\mathbb{Q}} F)$ with highest weight μ.

From the proof of Théorème 3.13 [Clo90], we know that there exists a dominant algebraic weight μ, such that $H^*(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu) \neq 0$.

Let b be the smallest degree such that $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu) \neq 0$. We have an explicit formula for b in [RS08]. More precisely, we set r_1 and r_2 the numbers of real and complex embeddings of F respectively. We have $b = r_1 \left[\frac{n^2}{4}\right] + r_2 \frac{n(n-1)}{2}$.

We can decompose this cohomology group via the action of K_∞/K_∞^0. There exists a character ϵ of K_∞/K_∞^0 described explicitly in [RS08] such that:

(1) The isotypic component $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon)$ is one dimensional.
(2) For fixed w_∞, a generator of $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon)$, we have a $GL_n(\mathbb{A}_{F,f})$-equivariant isomorphisms:

$$\begin{aligned} W(\Pi_f) &\xrightarrow{\sim} W(\Pi_f) \otimes H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon) \\ &\xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K_\infty^0; W(\Pi) \otimes W_\mu)(\epsilon) \\ &\xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi \otimes W_\mu)(\epsilon) \end{aligned} \tag{1.1}$$

where the first map sends w_f to $w_f \otimes w_\infty$ and the last map is given by the isomorphism $W(\Pi) \xrightarrow{\sim} \Pi$.

(3) The cohomology space $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi \otimes W_\mu)(\epsilon)$ is related to the cuspidal cohomology if Π is cuspidal and to the Eisenstein cohomology if Π is not cuspidal. In both cases, it is endowed with a $\mathbb{Q}(\Pi)$-rational structure (see [RS08] for cuspidal case and [GH15] for non cuspidal case).

We denote by $\Theta_{\Pi_f,\epsilon,w_\infty}$ the $GL_n(\mathbb{A}_{F,f})$-isomorphism given in (1.1)

$$W(\Pi_f) \xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi \otimes W_\mu)(\epsilon).$$

Both sides have a $\mathbb{Q}(\Pi)$-rational structure. In particular, the preimage of the rational structure on the right hand side gives a rational structure on $W(\Pi_f)$. But the rational structure on $W(\Pi_f)$ is unique up to homotheties. Therefore, there exists a complex number $p(\Pi_f, \epsilon, w_\infty)$ such that the new map $\Theta^0_{\Pi_f,\epsilon,w_\infty} = p(\Pi_f, \epsilon, w_\infty)^{-1}\Theta_{\Pi_f,\epsilon,w_\infty}$ preserves the rational structure on both sides. It is easy to see that this number $p(\Pi_f, \epsilon, w_\infty)$ is unique up to multiplication by elements in $\mathbb{Q}(\Pi)^\times$.

Finally, we observe that the $Aut(\mathbb{C})$-action preserves rational structures on both the Whittaker models and cohomology spaces. We can adjust the numbers $p(\Pi_f^\sigma, \epsilon^\sigma, w_\infty^\sigma)$ for all $\sigma \in Aut(\mathbb{C})$ by elements in $\mathbb{Q}(\Pi)^\times$ such that the following diagram commutes:

$$\begin{tikzcd}
W(\Pi_f) \arrow[r, "p(\Pi_f,\epsilon,w_\infty)^{-1}\Theta_{\Pi_f,\epsilon,w_\infty}"] \arrow[d, "\sigma"] & H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi \otimes W_\mu)(\epsilon) \arrow[d, "\sigma"] \\
W(\Pi_f^\sigma) \arrow[r, "p(\Pi_f^\sigma,\epsilon^\sigma,w_\infty^\sigma)^{-1}\Theta_{\Pi_f^\sigma,\epsilon^\sigma,w_\infty^\sigma}"] & H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi^\sigma \otimes W_\mu^\sigma)(\epsilon^\sigma)
\end{tikzcd}$$

The proof is the same as the cuspidal case in [RS08].

In the following, we fix ϵ, w_∞ and we define the **Whittaker period** $p(\Pi) := p(\Pi_f, \epsilon, w_\infty)$. For any $\sigma \in Aut(\mathbb{C})$, we define $p(\Pi^\sigma) := p(\Pi_f^\sigma, \epsilon^\sigma, w_\infty^\sigma)$. It is easy to see that $p(\Pi^\sigma) = p(\Pi)$ for $\sigma \in Aut(\mathbb{C}/\mathbb{Q}(\Pi))$.

Moreover, the elements $(p(\Pi^\sigma))_{\sigma\in Aut(\mathbb{C})}$ are well defined up to $\mathbb{Q}(\Pi)^\times$ in the following sense: if $(p'(\Pi^\sigma))_{\sigma\in Aut(\mathbb{C})}$ is another family of complex numbers such that $p'(\Pi^\sigma)^{-1}\Theta_{\Pi_f^\sigma,\epsilon^\sigma,w_\infty^\sigma}$ preserves the rational structure and the above diagram commutes, then there exists $t \in \mathbb{Q}(\Pi)^\times$ such that $p'(\Pi^\sigma) = \sigma(t)p(\Pi^\sigma)$ for any $\sigma \in Aut(\mathbb{C})$. This also follows from the one dimensional property of the invariant vector. The argument is the same as the last part of the proof of Definition/Proposition 3.3 in [RS08].

The Whittaker periods are closely related to special values of L-functions. We refer to [Rag10], [HR15], [GH15] or [GHL16] for more details. Here we state a theorem which generalizes the main theorem of [GH15]. The proof can be found in [Gro17].

Let Π be a regular cuspidal cohomological representation of $GL_n(\mathbb{A}_F)$. Let $\Pi^\#$ be a regular automorphic cohomological representation of $GL_{n-1}(\mathbb{A}_F)$ which is the Langlands sum of cuspidal representations. Write the infinity type of Π (resp. Π') at $\sigma \in \Sigma$ by $\{z^{a_i(\sigma)}\bar{z}^{-a_i(\sigma)}\}_{1\leqslant i\leqslant n}$ (resp $\{z^{b_j(\sigma)}\bar{z}^{-b_j(\sigma)}\}_{1\leqslant j\leqslant n-1}$) with $a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma)$ (resp. $b_1(\sigma) > b_2(\sigma) > \cdots > b_{n-1}(\sigma)$). We say that the pair $(\Pi, \Pi^\#)$ is in *good position* if for all $\sigma \in \Sigma$ we have

$$a_1(\sigma) > -b_{n-1}(\sigma) > a_2(\sigma) > \cdots > -b_1(\sigma) > a_n(\sigma).$$

Theorem 1.2. *If $(\Pi, \Pi^{\#})$ is in good position, then there exists a non-zero complex number $p(m, \Pi_\infty, \Pi_\infty^{\#})$ which depends on m, Π_∞ and $\Pi_\infty^{\#}$ well defined up to $(E(\Pi)E(\Pi^{\#}))^\times$ such that for $m \in \mathbb{Z}$ with $m + \frac{1}{2}$ critical for $\Pi \times \Pi^{\#}$, we have*

$$L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} p(m, \Pi_\infty, \Pi_\infty^{\#})p(\Pi)p(\Pi^{\#}) \tag{1.2}$$

and is equivariant under the action of $Aut(\mathbb{C}/\mathbb{Q})$.

2. Arithmetic automorphic periods

2.1. **CM periods.** Let (T, h) be a Shimura datum where T is a torus defined over $\mathbb{Q}$ and $h : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{R}}$ a homomorphism satisfying the axioms defining a Shimura variety. Such pair is called a **special** Shimura datum. Let $Sh(T, h)$ be the associated Shimura variety and $E(T, h)$ be its reflex field.

Let (γ, V_γ) be a one-dimensional algebraic representation of T (the representation γ is denoted by χ in [HK91]). We denote by $E(\gamma)$ a definition field for γ. We may assume that $E(\gamma)$ contains $E(T, h)$. Suppose that γ is motivic (see *loc.cit* for the notion). We know that γ gives an automorphic line bundle $[V_\gamma]$ over $Sh(T, h)$ defined over $E(\gamma)$. Therefore, the complex vector space $H^0(Sh(T, h), [V_\gamma])$ has an $E(\gamma)$-rational structure, denoted by $M_{DR}(\gamma)$ and called the De Rham rational structure.

On the other hand, the canonical local system $V_\gamma^\nabla \subset [V_\gamma]$ gives another $E(\gamma)$-rational structure $M_B(\gamma)$ on $H^0(Sh(T, h), [V_\gamma])$, called the Betti rational structure.

We now consider χ an algebraic Hecke character of $T(\mathbb{A}_\mathbb{Q})$ with infinity type γ^{-1} (our character χ corresponds to the character ω^{-1} in *loc.cit*). Let $E(\chi)$ be the number field generated by the values of χ on $T(\mathbb{A}_{\mathbb{Q},f})$ over $E(\gamma)$. We know χ generates a one-dimensional complex subspace of $H^0(Sh(T, h), [V_\gamma])$ which inherits two $E(\chi)$-rational structures, one from $M_{DR}(\gamma)$, the other from $M_B(\gamma)$. Put $p(\chi, (T, h))$ the ratio of these two rational structures which is well defined modulo $E(\chi)^\times$.

Remark 2.1. *If we identify $H^0(Sh(T, h), [V_\gamma])$ with the set*

$$\{f \in \mathbb{C}^\infty(T(\mathbb{Q})\backslash T(\mathbb{A}_\mathbb{Q}), \mathbb{C} \mid f(tt_\infty)) = \gamma^{-1}(t_\infty)f(t), t_\infty \in T(\mathbb{R}), t \in T(\mathbb{A}_\mathbb{Q})\}$$

, then χ itself is in the rational structure inherits from $M_B(\gamma)$. See discussion from A.4 to A.5 in [HK91].

Suppose that we have two tori T and T' both endowed with a Shimura datum (T, h) and (T', h'). Let $u : (T', h') \to (T, h)$ be a map between the Shimura data. Let χ be an algebraic Hecke character of $T(\mathbb{A}_\mathbb{Q})$. We put $\chi' := \chi \circ u$ an algebraic Hecke character of $T'(\mathbb{A}_\mathbb{Q})$. Since both the Betti structure and the De Rham structure commute with the pullback map on cohomology, we have the following proposition:

Proposition 2.1. *Let χ, (T, h) and χ', (T', h') be as above. We have:*

$$p(\chi, (T, h)) \sim_{E(\chi)} p(\chi', (T', h'))$$

and is equivariant under the action of $Aut(\mathbb{C}/E(T)E(T'))$.

Remark 2.2. *In Proposition 1.4 of* [Har93b]*, the relation is up to $E(\chi); E(T, h)$ where $E(T, h)$ is a number field associated to (T, h). Here we consider the action of $G_\mathbb{Q}$ and can thus obtain a relation up to $E(\chi)$ (see the paragraph after Proposition 1.8.1 of loc.cit).*

For F a CM field and Ψ a subset of Σ_F such that $\Psi \cap \iota\Psi = \varnothing$, we can define a Shimura datum (T_F, h_Ψ) where $T_F := Res_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ is a torus and $h_\Psi : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to T_{F,\mathbb{R}}$ is a homomorphism such that over $\sigma \in \Sigma_F$, the Hodge structure induced on F by h_Ψ is of type $(-1,0)$ if $\sigma \in \Psi$, of type $(0,-1)$ if $\sigma \in \iota\Psi$, and of type $(0,0)$ otherwise.

Let χ be a motivic critical character of a CM field F. By definition, $p_F(\chi,\Psi) = p(\chi,(T_F,h_\Psi))$ and we call it a **CM period**. Sometimes we write $p(\chi,\Psi)$ instead of $p_F(\chi,\Psi)$ if there is no ambiguity concerning the base field F.

Example 2.1. *We have* $p(||\cdot||_{\mathbb{A}_K},1) \sim_{\mathbb{Q}} (2\pi i)^{-1}$. *See* (1.10.9) *on page* 100 *of* [Har97].

Proposition 2.2. *Let* $\tau \in J_F$ *and* Ψ *be a subset of* J_F *such that* $\Psi \cap \Psi^c = \varnothing$. *We take* $\Psi = \Psi_1 \sqcup \Psi_2$ *a partition of* Ψ.

We then have:

$$p(\chi_1\chi_2),\Psi) \sim_{E(\chi_1)E(\chi_2)} p(\chi_1,\Psi)p(\chi_2^\sigma,\Psi). \tag{2.1}$$

$$p(\chi,\Psi_1 \sqcup \Psi_2) \sim_{E(\chi)} p(\chi,\Psi_1)p(\chi,\Psi_2). \tag{2.2}$$

$$p(\chi,\Psi) \sim_{E(\chi)} p(\bar{\chi},\bar{\Psi}). \tag{2.3}$$

All the relations are equivariant under the action of $Aut(\mathbb{C}/F^{Gal})$.

Proof. All the equations in Proposition 2.2 come from Proposition 2.1 by certain maps between Shimura data as follows:

(1) The diagonal map $(T_F,h_\Psi) \to (T_F \times T_F, h_\Psi \times h_\Psi)$ pulls (χ_1,χ_2) back to $\chi_1\chi_2$.
(2) The multiplication map $T_F \times T_F \to T_F$ sends h_{Ψ_1}, h_{Ψ_2} to $h_{\Psi_1 \sqcup \Psi_2}$.
(3) The Galois action $\theta : H_F \to H_F$ sends h_Ψ to h_{Ψ^θ}.
(4) The norm map $(T_F, h_{\{\tau\}}) \to (T_{F_0}, h_{\{\tau|_{F_0}\}})$ pulls η back to $\eta \circ N_{\mathbb{A}_F/\mathbb{A}_{F,0}}$.

□

The special values of an L-function for a Hecke character over a CM field can be interpreted in terms of CM periods. The following theorem is proved by Blasius. We state it as in Proposition 1.8.1 in [Har93b] where ω should be replaced by $\check{\omega} := \omega^{-1,c}$ (for this erratum, see the notation and conventions part on page 82 in the introduction of [Har97]),

Theorem 2.1. *Let* F *be a CM field and* F^+ *be its maximal totally real subfield. Put* d *the degree of* F^+ *over* $\mathbb{Q}$.

Let χ *be a motivic critical algebraic Hecke character of* F *and* Φ_χ *be the unique CM type of* F *which is compatible with* χ.

For m *a critical value of* χ *in the sense of Deligne (c.f.* [Del79]*), we have*

$$L(\chi,m) \sim_{E(\chi)} (2\pi i)^{md} p(\check{\chi},\Phi_\chi)$$

equivariant under action of $Aut(\mathbb{C}/F^{Gal})$.

Remark 2.3. *Let* $\{\sigma_1,\sigma_2,\cdots,\sigma_n\}$ *be any CM type of* F. *Let* $(\sigma_i^{a_i}\overline{\sigma}_i^{-w-a_i})_{1\leqslant i\leqslant n}$ *denote the infinity type of* χ *with* $w = w(\chi)$. *We may assume* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. *We define* $a_0 := +\infty$ *and* $a_{n+1} := -\infty$ *and define* $k := max\{0 \leqslant i \leqslant n \mid a_i > -\dfrac{w}{2}\}$. *An integer* m *is critical for* χ *if and only if*

$$max(-a_k+1, w+1+a_{k+1}) \leqslant m \leqslant min(w+a_k, -a_{k+1}). \tag{2.4}$$

2.2. **Construction of cohomology spaces.** Let I be the signature of a unitary group U_I of dimension n with respect to the extension F/F^+. Let V_I be the associated Hermitian space. We can consider I as a map from Σ to $\{0, 1, \cdots, n\}$. We write $s_\sigma := I(\sigma)$ and $r_\sigma := n - I(\sigma)$ for all $\sigma \in \Sigma$.

Denote $\mathbb{S} := Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. We define the rational similitude unitary group defined by

$$GU_I(R) := \{g \in GL(V_I \otimes_{\mathbb{Q}} R) | (gv, gw) = \nu(g)(v, w), \nu(g) \in R^*\} \tag{2.5}$$

where R is any $\mathbb{Q}$-algebra.

We know that $GU_I(\mathbb{R})$ is isomorphic to a subgroup of $\prod\limits_{\sigma\in\Sigma} GU(r_\sigma, s_\sigma)$ defined by the same *similitude*. We can define a homomorphism $h_I : \mathbb{S}(\mathbb{R}) \to GU_I(\mathbb{R})$ by sending $z \in \mathbb{C}$ to $\left(\begin{pmatrix} zI_{r_\sigma} & 0 \\ 0 & \bar{z}I_{s_\sigma} \end{pmatrix}\right)_{\sigma\in\Sigma}$.

Let X_I be the $GU_I(\mathbb{R})$-conjugation class of h_I. We know (GU_I, X_I) is a Shimura datum with reflex field E_I and dimension $2\sum\limits_{\sigma\in\Sigma} r_\sigma s_\sigma$. The Shimura variety associated to (GU_I, X_I) is denoted by Sh_I.

Let $K_{I,\infty}$ be the centralizer of h_I in $GU_I(\mathbb{R})$. Via the inclusion $GU_I(\mathbb{R}) \hookrightarrow \prod\limits_{\sigma\in\Sigma} GU(r_\sigma, s_\sigma) \subset \mathbb{R}^{+,\times} \prod\limits_{\sigma\in\Sigma} U(n, \mathbb{C})$, we may identify $K_{I,\infty}$ with

$$\{(\mu, \begin{pmatrix} u_{r_\sigma} & 0 \\ 0 & v_{s_\sigma} \end{pmatrix}_{\sigma\in\Sigma}) \mid u_{r_\sigma} \in U(r_\sigma, \mathbb{C}), v_{s_\sigma} \in U(s_\sigma, \mathbb{C}), \mu \in \mathbb{R}^{+,\times}\}$$

where $U(r, \mathbb{C})$ is the standard unitary group of degree r over $\mathbb{C}$. Let H_I be the subgroup of $K_{I,\infty}$ consisting of the diagonal matrices in $K_{I,\infty}$. Then it is a maximal torus of $GU_I(\mathbb{R})$. Denote its Lie algebra by $\mathfrak{h}_I$.

We observe that $H_I(\mathbb{R}) \cong \mathbb{R}^{+,\times} \times \prod\limits_{\sigma\in\Sigma} U(1, \mathbb{C})^n$. Its algebraic characters are of the form

$$(w, (z_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n}) \mapsto w^{\lambda_0} \prod_{\sigma\in\Sigma}\prod_{i=1}^{n} z_i(\sigma)^{\lambda_i(\sigma)}$$

where $(\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n})$ is a $(nd+1)$-tuple of integers with $\lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \lambda_i(\sigma)$ (mod 2).

Recall that $GU_I(\mathbb{C}) \cong \mathbb{C}^\times \prod_{\sigma\in\Sigma} GL_n(\mathbb{C})$. We fix B_I the Borel subgroup of $GU_{I,\mathbb{C}}$ consisting of upper triangular matrices. The highest weights of finite-dimensional irreducible representations of $K_{I,\infty}$ are tuples $\Lambda = (\Lambda_0, (\Lambda_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n})$ such that $\Lambda_1(\sigma) \geqslant \Lambda_2(\sigma) \geqslant \cdots \geqslant \Lambda_{r_\sigma}(\sigma)$, $\Lambda_{r_\sigma+1}(\sigma) \geqslant \cdots \geqslant \Lambda_n(\sigma)$ for all σ and $\Lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \Lambda_i(\sigma)$ (mod 2).

We denote the set of such tuples by $\Lambda(K_{I,\infty})$. Similarly, we write $\Lambda(GU_I)$ for the set of the highest weights of finite-dimensional irreducible representations of GU_I. It consists of tuples $\lambda = (\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n})$ such that $\lambda_1(\sigma) \geqslant \lambda_2(\sigma) \geqslant \cdots \lambda_n(\sigma)$ for all σ and $\lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \lambda_i(\sigma)$ (mod 2).

We take $\lambda \in \Lambda(GU_I)$ and $\Lambda \in \Lambda(K_{I,\infty})$.

Let V_λ and V_Λ be the corresponding representations. We define a local system over Sh_I:

$$W_\lambda^\triangledown := \varprojlim_{K} GU_I(\mathbb{Q})\backslash V_\lambda \times X \times GU_I(\mathbb{A}_{\mathbb{Q},f})/K$$

and an automorphic vector bundle over Sh_I

$$E_\Lambda := \varprojlim_{K} GU_I(\mathbb{Q})\backslash V_\Lambda \times GU_I(\mathbb{R}) \times GU_I(\mathbb{A}_{\mathbb{Q},f})/KK_{I,\infty}$$

where K runs over open compact subgroup of $GU_I(\mathbb{A}_{\mathbb{Q},f})$.

The automorphic vector bundles E_Λ are defined over the reflex field E.

The local systems $W_\lambda^\triangledown$ are defined over K. The Hodge structure of the cohomology space $H^q(Sh_I, W_\lambda^\triangledown)$ is not pure in general. But the image of $H_c^q(Sh_I, W_\lambda^\triangledown)$ in $H^q(Sh_I, W_\lambda^\triangledown)$ is pure of weight $q-c$. We denote this image by $\bar{H}^q(Sh_I, W_\lambda^\triangledown)$.

Note that all cohomology spaces have coefficients in $\mathbb{C}$ unless we specify its rational structure over a number field.

2.3. **The Hodge structures.** The results in section 2.2 of [Har94] give a description of the Hodge components of $\bar{H}^q(Sh_I, W_\lambda^\triangledown)$.

Denote by R^+ the set of positive roots of $H_{I,\mathbb{C}}$ in $GU_I(\mathbb{C})$ and by R_c^+ the set of positive compact roots. Define $\alpha_{j,k} = (0,\cdots,0,1,0,\cdots,0,-1,0,\cdots,0)$ for any $1 \leqslant j < k \leqslant n$. We know $R^+ = \{(\alpha_{j_\sigma,k_\sigma})_{\sigma\in\Sigma} \mid 1 \leqslant j_\sigma < k_\sigma \leqslant n\}$ and $R_c^+ = \{(\alpha_{j_\sigma,k_\sigma})_{\sigma\in\Sigma} \mid j_\sigma < k_\sigma \leqslant r_\sigma \text{ or } r_\sigma + 1 \leqslant j_\sigma < k_\sigma\}$.

Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \left(\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-1}{2}\right)\right)_\sigma$.

Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be Lie algebras of $GU_I(\mathbb{R})$, $K_{I,\infty}$ and $H(\mathbb{R})$. Write W for the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and W_c for the Weyl group $W(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We can identify W with $\prod_{\sigma\in\Sigma}\mathfrak{S}_n$ and W_c with $\prod_{\sigma\in\Sigma}\mathfrak{S}_{r_\sigma}\times\mathfrak{S}_{s_\sigma}$ where $\mathfrak{S}$ refers to the standard permutation group. For $w \in W$, we write the length of w by $l(w)$.

Let $W^1 := \{w \in W | w(R^+) \supset R_c^+\}$ be a subset of W. By the above identification, $(w_\sigma)_\sigma \in W^1$ if and only if $w_\sigma(1) < w_\sigma(2) < \cdots < w_\sigma(r_\sigma)$ and $w_\sigma(r_\sigma+1) < \cdots < w_\sigma(n)$ One can show that W^1 is a set of coset representatives of shortest length for $W_c\backslash W$.

Moreover, for λ a highest weight of a representation of GU_I, one can show easily that $w * \lambda := w(\lambda+\rho) - \rho$ is the highest weight of a representation of $K_{I,\infty}$. More precisely, if $\lambda = (\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n})$, then $w*\lambda = (\lambda_0, ((w*\lambda)_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$ with $(w*\lambda)_i(\sigma) = \lambda_{w_\sigma(i)}(\sigma) + \frac{n+1}{2} - w_\sigma(i) - (\frac{n+1}{2} - i) = \lambda_{w_\sigma(i)}(\sigma) - w_\sigma(i) + i$.

Remark 2.4. *The results of* [Har94] *tell us that there exists*

$$\bar{H}^q(Sh_I, W_\lambda^\triangledown) \cong \bigoplus_{w\in W^1} \bar{H}^{q;w}(Sh_I, W_\lambda^\triangledown) \tag{2.6}$$

a decomposition as subspaces of pure Hodge type $(p(w,\lambda), q-c-p(w,\lambda))$. We now determine the Hodge number $p(w,\lambda)$.

We know that $w*\lambda$ is the highest weight of a representation of $K_{I,\infty}$. We denote this representation by $(\rho_{w*\lambda}, W_{w*\lambda})$. We know that $\rho_{w*\lambda}\circ h_I|_{\mathbb{S}(\mathbb{R})} : \mathbb{S}(\mathbb{R}) \to K_{I,\infty} \to GL(W_{w*\lambda})$ is of the form $z \mapsto z^{-p(w,\lambda)}\bar{z}^{-r(w,\lambda)}I_{W_{w*\lambda}}$ with $p(w,\lambda)$, $r(w,\lambda) \in \mathbb{Z}$. The first index $p(w,\lambda)$ is the Hodge type mentioned above.

Recall that the map

$$h_I|_{\mathbb{S}(\mathbb{R})}:\mathbb{S}(\mathbb{R}) \rightarrow K_{I,\infty}\subset \mathbb{R}^{+,\times}\times U(n,\mathbb{C})^{\Sigma} \tag{2.7}$$
$$z \mapsto \left(|z|, \begin{pmatrix} \frac{z}{|z|}I_{r_\sigma} & 0 \\ 0 & \frac{\bar{z}}{|z|}I_{s_\sigma}\end{pmatrix}_{\sigma\in\Sigma}\right)$$

and the map

$$\rho_{w*\lambda}: K_{I,\infty} \rightarrow GL(W_{w*\lambda})$$
$$(w, diag(z_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n}) \mapsto w^{\lambda_0}\prod_{\sigma\in\Sigma}\prod_{i=1}^{n} z_i(\sigma)^{(w*\lambda)_i(\sigma)}$$

where $diag(z_1,z_2,\cdots,z_n)$ means the diagonal matrix of coefficients $z_1,z_2,\cdots,z_n$.

Therefore we have:

$$\begin{aligned} & z^{-p(w,\lambda)}\bar{z}^{-r(w,\lambda)} \\ = \ & |z|^{\lambda_0}\prod_{\sigma\in\Sigma}\Big(\prod_{1\leqslant i\leqslant r_\sigma}\big(\frac{z}{|z|}\big)^{(w*\lambda)_i(\sigma)}\prod_{r_\sigma+1\leqslant i\leqslant n}\big(\frac{\bar{z}}{|z|}\big)^{(w*\lambda)_i(\sigma)} \\ = \ & (z^{\frac{1}{2}}\bar{z}^{\frac{1}{2}})^{\lambda_0-\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}(w*\lambda)_i(\sigma)} z^{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant r_\sigma}(w*\lambda)_i(\sigma)}\bar{z}^{\sum\limits_{\sigma\in\Sigma}\sum\limits_{r_\sigma+1\leqslant i\leqslant n}(w*\lambda)_i(\sigma)}\end{aligned}$$

Since $(w*\lambda)_i(\sigma)=\lambda_{w_\sigma(i)}(\sigma)-w_\sigma(i)+i$ and then $\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}(w*\lambda)_i(\sigma)=\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)$, we obtain that:

$$\begin{aligned} p(w,\lambda) &= \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0}{2}-\sum_{\sigma\in\Sigma}\sum_{1\leqslant i\leqslant r_\sigma}(w*\lambda)_i(\sigma) \\ &= \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0}{2}-\sum_{\sigma\in\Sigma}\sum_{1\leqslant i\leqslant r_\sigma}(\lambda_{w_\sigma(i)}(\sigma)-w_\sigma(i)+i) \end{aligned}\tag{2.8}$$

The method of toroidal compactification gives us more information on $\bar{H}^{q;w}(Sh_I, W_\lambda^\triangledown)$. We take $j: Sh_I\hookrightarrow \widetilde{Sh}_I$ to be a smooth toroidal compactification. Proposition 2.2.2 of [Har94] tells us that the following results do not depend on the choice of the toroidal compactification.

The automorphic vector bundle E_Λ can be extended to $\widetilde{Sh}_I$ in two ways: the canonical extension E_Λ^{can} and the sub canonical extension E_Λ^{sub} as explained in [Har94]. Define:

$$\bar{H}^q(Sh_I,E_\Lambda)=Im(H^q(\widetilde{Sh}_I,E_\Lambda^{sub})\rightarrow H^q(\widetilde{Sh}_I,E_\Lambda^{can})).$$

Proposition 2.3. *There is a canonical isomorphism*

$$\bar{H}^{q;w}(Sh_I,W_\lambda^\triangledown)\cong \bar{H}^{q-l(w)}(Sh_I,E_{w*\lambda})$$

Let $D=2\sum\limits_{\sigma\in\Sigma}r_\sigma s_\sigma$ be the dimension of the Shimura variety. We are interested in the cohomology space of degree $D/2$. Proposition 2.2.7 of [Har97] also works here:

Proposition 2.4. *The space $\bar{H}^{D/2}(Sh_I,W_\lambda^\triangledown)$ is naturally endowed with a K-rational structure, called the de Rham rational structure and noted by $\bar{H}^{D/2}_{DR}(Sh_I,W_\lambda^\triangledown)$. This* *rational structure is endowed with a K-Hodge filtration $F^{\cdot}\bar{H}^{D/2}_{DR}(Sh_I, W^{\triangledown}_{\lambda})$ pure of weight $D/2-c$ such that*

$$F^p\bar{H}^{D/2}_{DR}(Sh_I, W^{\triangledown}_{\lambda})/F^{p+1}\bar{H}^{D/2}_{DR}(Sh_I, W^{\triangledown}_{\lambda})\otimes_K \mathbb{C} \cong \bigoplus_{w\in W^1, p(w,\lambda)=p} \bar{H}^{D/2;w}(Sh_I, W^{\triangledown}_{\lambda}).$$

Moreover, the composition of the above isomorphism and the canonical isomorphism

$$\bar{H}^{D/2;w}(Sh_I, W^{\triangledown}_{\lambda}) \cong \bar{H}^{D/2-l(w)}(Sh_I, E_{w*\lambda})$$

is rational over K.

Holomorphic part: Let $w_0 \in W^1$ defined by

$$w_0(\sigma)(1,2,\cdots,r_\sigma; r_{\sigma+1},\cdots,n)_{\sigma\in\Sigma} = (s_{\sigma+1},\cdots,n;1,2,\cdots,s_\sigma)$$

for all $\sigma\in\Sigma$. It is the only longest element in W^1. Its length is $D/2$.

We have a K-rational isomorphism

$$\bar{H}^{D/2;w_0}(Sh_I, W^{\triangledown}_{\lambda}) \cong \bar{H}^0(Sh_I, E_{w_0*\lambda}). \tag{2.9}$$

We can calculate the Hodge type of $\bar{H}^{D/2;w_0}(Sh_I, W^{\triangledown}_{\lambda})$ as in Remark 2.4.

By definition we have

$$w_0*\lambda = (\lambda_0, (\lambda_{s_\sigma+1}(\sigma)-s_\sigma,\cdots,\lambda_n(\sigma)-s_\sigma;\lambda_1(\sigma)+r_\sigma,\cdots,\lambda_{s_\sigma}(\sigma)+r_\sigma)_{\sigma\in\Sigma}). \tag{2.10}$$

By the discussion in Remark 2.4, the Hodge number

$$p(w_0,\lambda) = \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0+D}{2} - \sum_{\sigma\in\Sigma}(\lambda_{s_\sigma+1}(\sigma)+\cdots+\lambda_n(\sigma)).$$

From equation (2.8), it is easy to deduce that $p(w_0,\lambda)$ is the only largest number among $\{p(w,\lambda)\mid w\in W^1\}$. Therefore

$$F^{p(w_0,\lambda)}(Sh_I, W^{\triangledown}_{\lambda})\otimes_K\mathbb{C} \cong \bar{H}^0(Sh_I, E_{w_0*\lambda}). \tag{2.11}$$

Moreover, as mentioned in the above proposition, we know that the above isomorphism is K-rational.

We call $\bar{H}^{D/2;w_0}(Sh_I, W^{\triangledown}_{\lambda}) \cong \bar{H}^0(Sh_I, E_{w_0*\lambda})$ the **holomorphic part** of the Hodge decomposition of $\bar{H}^{D/2}(Sh_I, W^{\triangledown}_{\lambda})$. It is isomorphic to the space of holomorphic cusp forms of type $(w_0*\lambda)^{\vee}$.

Anti-holomorphic part: The only shortest element in W^1 is the identity with the smallest Hodge number

$$p(id,\lambda) = \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0}{2} - \sum_{\sigma\in\Sigma}(\lambda_1(\sigma)+\cdots+\lambda_{r_\sigma}(\sigma)).$$

We call $\bar{H}^{D/2;id}(Sh_I, W^{\triangledown}_{\lambda}) \cong \bar{H}^{D/2}(Sh_I, E_{\lambda})$ the **anti-holomorphic part** of the Hodge decomposition of $\bar{H}^{D/2}(Sh_I, W^{\triangledown}_{\lambda})$.

2.4. **Complex conjugation.** We specify some notation first.

Let $\lambda = (\lambda_0, (\lambda_1(\sigma) \geqslant \lambda_2(\sigma) \geqslant \cdots \geqslant \lambda_n(\sigma))_{\sigma\in\Sigma}) \in \Lambda(GU_I)$ as before. We define $\lambda^c := (\lambda_0, (-\lambda_n(\sigma) \geqslant -\lambda_{n-1}(\sigma) \geqslant \cdots \geqslant -\lambda_1(\sigma))_{\sigma\in\Sigma})$ and $\lambda^\vee := (-\lambda_0, (-\lambda_n(\sigma) \geqslant -\lambda_{n-1}(\sigma) \geqslant \cdots \geqslant -\lambda_1(\sigma))_{\sigma\in\Sigma})$. They are elements in $\Lambda(GU_I)$. Moreover, the representation V_{λ^c} is the complex conjugation of V_λ and the representation $V_{\lambda^\vee}$ is the dual of V_λ as GU_I-representation.

Similarly, for $\Lambda = (\Lambda_0, (\Lambda_1(\sigma) \geqslant \cdots \geqslant \Lambda_{r_\sigma}(\sigma), \Lambda_{r_\sigma+1}(\sigma) \geqslant \cdots \geqslant \Lambda_n(\sigma))_{\sigma\in\Sigma}) \in \Lambda(K_{I,\infty})$, we define $\Lambda^* := (-\Lambda_0, (-\Lambda_{r_\sigma}(\sigma) \geqslant \cdots \geqslant -\Lambda_1(\sigma), -\Lambda_n \geqslant \cdots \geqslant -\Lambda_{r_\sigma+1})_{\sigma\in\Sigma})$.

We know V_{Λ^*} is the dual of V_Λ as K_I-representation. We sometimes write the latter as $\breve{V_\Lambda}$.

We define I^c by $I^c(\sigma) = n - I(\sigma)$ for all $\sigma \in \Sigma$. We know $V_{I^c} = -V_I$ and $GU_{I^c} \cong GU_I$. The complex conjugation gives an anti-holomorphic isomorphism $X_I \xrightarrow{\sim} X_{I^c}$. This induces a K-antilinear isomorphism

$$\bar{H}^{D/2}(Sh_I, W_\lambda^\nabla) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_{I^c}, W_{\lambda^c}^\nabla). \tag{2.12}$$

In particular, it sends holomorphic (resp. anti-holomorphic) elements with respect to (I, λ) to those respect to (I^c, λ^c). If we we denote by w_0^c the longest element related to I^c then we have K-antilinear rational isomorphisms

$$c_{DR}: \quad \bar{H}^0(Sh_I, E_{w_0*\lambda}) \xrightarrow{\sim} \bar{H}^0(Sh_{I^c}, E_{w_0^c*\lambda^c}) \tag{2.13}$$

$$\bar{H}^{D/2}(Sh_I, E_\lambda) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_{I^c}, E_{\lambda^c}) \quad . \tag{2.14}$$

The Shimura datum (GU_I, h) induces a Hodge structure of wights concentrated in $\{(-1,1), (0,0), (1,-1)\}$ which corresponds to the Harish-Chandra decomposition induced by h on the Lie algebra: $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

Let $\mathfrak{P} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$. Let $\mathcal{A}$ (resp. $\mathcal{A}_0$, $\mathcal{A}_{(2)}$) be the space of automorphic forms (resp. cusp forms, square-integrable forms) on $GU_I(\mathbb{Q})\backslash GU_I(\mathbb{A}_{\mathbb{Q}})$.

We have inclusions for all q:

$$H^q(\mathfrak{g}, K_{I,\infty}; \mathcal{A}_0 \otimes V_\lambda) \subset \bar{H}^q(Sh_I, V_\lambda^\nabla) \subset H^q(\mathfrak{g}, K_{I,\infty}; \mathcal{A}_{(2)} \otimes V_\lambda)$$
$$H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_0 \otimes V_\Lambda) \subset \bar{H}^q(Sh_I, E_\Lambda) \subset H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_{(2)} \otimes V_\Lambda).$$

The complex conjugation on the automorphic forms induces a K-antilinear isomorphism:

$$c_B: \bar{H}^0(Sh_I, E_{w_0*\lambda}) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee}) \tag{2.15}$$

More precisely, we summarize the construction in [Har97] as follows.
Automorphic vector bundles:

We recall some facts on automorphic vector bundles first. We refer to page 113 of [Har97] and [Har85] for notation and further details.

Let (G, X) be a Shimura datum such that its special points are all CM points. Let $\check{X}$ be the compact dual symmetric space of X. There is a surjective functor from the category of G-homogeneous vector bundles on $\check{X}$ to the category of automorphic vector bundles on $Sh(G, X)$. This functor is compatible with inclusions of Shimura data as explained in the second part of Theorem 4.8 of [Har85]. It is also rational over the reflex field $E(G, X)$.

Let $\mathcal{E}$ be an automorphic vector bundle on $Sh(G,X)$ corresponding to $\mathcal{E}_0$, a G-homogeneous vector bundle on $\check{X}$. Let (T,x) be a special pair of (G,X), i.e. (T,x) is a sub-Shimura datum of (G,X) with T a maximal torus defined over $\mathbb{Q}$. Since the functor mentioned above is compatible with inclusions of Shimura datum, we know that the restriction of $\mathcal{E}$ to $Sh(T,x)$ corresponds to the restriction of $\mathcal{E}_0$ to $\check{x} \in \check{X}$ by the previous functor. Moreover, by the construction, the fiber of $\mathcal{E} \mid_{Sh(T,x)}$ at any point of $Sh(T,x)$ is identified with the fiber of $\mathcal{E}_0$ at $\check{x}$. The $E(\mathcal{E}) \cdot E(T,x)$-rational structure on the fiber of $\mathcal{E}_0$ at $\check{x}$ then defines a rational structure of $\mathcal{E} \mid_{Sh(T,x)}$ and called the **canonical trivialization** of $\mathcal{E}$ associated to (T,x).

Complex conjugation on automorphic vector bundles:

Let $\mathcal{E}$ be as in page 112 of [Har97] and $\overline{\mathcal{E}}$ be its complex conjugation. The key step of the construction is to identify $\overline{\mathcal{E}}$ with the dual of $\mathcal{E}$ in a rational way.

More precisely, we recall Proposition 2.5.8 of the *loc.cit* that there exists a non-degenerate $G(\mathbb{A}_{\mathbb{Q},f})$-equivariant paring of real-analytic vector bundle $\mathcal{E} \otimes \overline{\mathcal{E}} \to \mathcal{E}_\nu$ such that its pullback to any CM point is rational with respect to the canonical trivializations.

We now explain the notion $\mathcal{E}_\nu$. Let $h \in X$ and K_h be the stabilizer of h in $G(\mathbb{R})$. We know $\mathcal{E}$ is associated to an irreducible complex representation of K_h, denoted by τ in the *loc.cit*. The complex conjugation of τ can be extended as an algrebraic representation of K_h, denoted by τ'. We know τ' is isomorphic to the dual of τ and then there exists ν, a one-dimensional representation K_h, such that a K_h-equivariant rational paring $V_\tau \otimes V_{\tau'} \to V_\nu$ exists. We denote by $\mathcal{E}_\nu$ the automorphic vector bundle associated to V_ν.

In our case, we have $(G,X) = (GU_I, X_I)$, $h = h_I$ and $K_h = K_{I,\infty}$. Let $\tau = \Lambda = w_0 * \lambda$ and $\mathcal{E} = E_\Lambda$. As explained in the last second paragraph before Corollary 2.5.9 in the *loc.cit*, we may identify the holomorphic sections of V_Λ with holomorphic sections of the dual of $\overline{V_\Lambda}$. The complex conjugation then sends the latter to the anti-holomorphic sections of $\widecheck{V_\Lambda} = V_{\Lambda^*}$. The latter can be identified with harmonic (0,d)-forms with values in $\mathbb{K} \otimes E_\Lambda$ where $\mathbb{K} = \Omega^{D/2}_{Sh_I}$ is the canonical line bundle of Sh_I.

By 2.2.9 of [Har97] we have $\mathbb{K} = E_{(0,(-s_\sigma,\cdots,-s_\sigma,r_\sigma,\cdots,r_\sigma)_{\sigma\in\Sigma})}$ where the number of $-s_\sigma$ in the last term is r_σ. Therefore, complex conjugation gives an isomorphism:

$$c_B : \bar{H}^0(Sh_I, E_\Lambda) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_I, E_{\Lambda^*+0,(-s_\sigma,\cdots,-s_\sigma,r_\sigma,\cdots,r_\sigma)_{\sigma\in\Sigma}}). \tag{2.16}$$

Recall equation (2.10) that

$$\Lambda = w_0 * \lambda = (\lambda_0, (\lambda_{s_\sigma+1}(\sigma) - s_\sigma, \cdots, \lambda_n(\sigma) - s_\sigma; \lambda_1(\sigma) + r_\sigma, \cdots, \lambda_{s_\sigma}(\sigma) + r_\sigma)_{\sigma\in\Sigma}).$$

We have

$$\Lambda^* = (-\lambda_0, (-\lambda_n(\sigma) + s_\sigma, \cdots, -\lambda_{s_\sigma+1}(\sigma) + s_\sigma; -\lambda_{s_\sigma}(\sigma) - r_\sigma, \cdots, -\lambda_1(\sigma) + r_\sigma)_{\sigma\in\Sigma}). \tag{2.17}$$

Therefore, $\Lambda^* + (0, (-s_\sigma, \cdots, -s_\sigma, r_\sigma, \cdots, r_\sigma)_{\sigma\in\Sigma}) = \lambda^\vee$. We finally get equation (2.15).

Similarly, if we start from the anti-holomorphic part, we will get a K-antilinear isomorphism which is still denoted by c_B:

$$c_B : \bar{H}^{D/2}(Sh_I, E_\lambda) \xrightarrow{\sim} \bar{H}^0(Sh_I, E_{w_0*\lambda^\vee}) \tag{2.18}$$

which sends anti-holomorphic elements with respect to λ to holomorphic elements for $\lambda^\vee$.

2.5. **The rational paring.** Let $\Lambda \in \Lambda(K_{I,\infty})$. We write $V = V_\Lambda$ in this section for simplicity. As in section 2.6.11 of [Har97], we denote by $\mathbb{C}_\Lambda$ the corresponding highest weight space. We know $\Lambda^* := \Lambda^\# - (2\Lambda_0, (0))$ is the tuple associated to $\check{V}$, the dual of this K_I representation. We denote by $\mathbb{C}_{-\Lambda}$ the lowest weight of $\check{V}$.

The restriction from V to $\mathbb{C}_\Lambda$ gives an isomorphism

$$Hom_{K_{I,\infty}}(V, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \xrightarrow{\sim} Hom_H(\mathbb{C}_\Lambda, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V) \tag{2.19}$$

where $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V$ is the V-isotypic subspace of $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))$.

Similarly, we have

$$Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \xrightarrow{\sim} Hom_H(\mathbb{C}_{-\Lambda}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_{\check{V}}) \tag{2.20}$$

Proposition 2.6.12 of [Har97] says that up to a rational factor the perfect paring

$$Hom_H(\mathbb{C}_\Lambda, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V) \times Hom_H(\mathbb{C}_{-\Lambda}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_{\check{V}}) \tag{2.21}$$

given by integration over the diagonal equals to restriction of the canonical paring (c.f. (2.6.11.4) of [Har97])

$$\begin{aligned}
& Hom_{K_{I,\infty}}(V, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \times Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\
\to\ & Hom_{K_{I,\infty}}(V \otimes \check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\
\to\ & Hom_{K_{I,\infty}}(\mathbb{C}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\
\to\ & \mathbb{C}.
\end{aligned} \tag{2.22}$$

We identify $\Gamma^\infty(Sh_I, E_\Lambda)$ with $Hom_{GU_I K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$ and regard the latter as subspace of $Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$.

The above construction gives a K-rational perfect paring between holomorphic sections of E_Λ and anti-holomorphic sections of E_{Λ^*}.

If $\Lambda = w_0 * \lambda$, as we have seen in Section 2.4 that the anti-holomorphic sections of E_{Λ^*} can be identified with harmonic $(0, d)$-forms with values in $E_{\lambda^\vee}$.

We therefore obtain a K-rational perfect paring

$$\Phi = \Phi^{I,\lambda} : \bar{H}^0(Sh_I, E_{w_0*\lambda}) \times \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee}) \to \mathbb{C}. \tag{2.23}$$

In other words, there is a rational paring between the holomorphic elements for (I, λ) and anti-holomorphic elements for $(I, \lambda^\vee)$.

It is easy to see that the isomorphism $Sh_I \xrightarrow{\sim} Sh_{I^c}$ commutes with the above paring and hence:

Lemma 2.1. *For any $f \in \bar{H}^0(Sh_I, E_{w_0*\lambda})$ and $g \in \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})$, we have $\Phi^{I,\lambda}(f, g) = \Phi^{I^c,\lambda^c}(c_{DR}f, c_{DR}g)$.*

The next lemma follows from Corollary 2.5.9 and Lemma 2.8.8 of [Har97].

Lemma 2.2. *Let $0 \neq f \in \bar{H}^0(Sh_I, E_{w_0*\lambda})$. We have $\Phi(f, c_B f) \neq 0$.*

More precisely, if we consider f as an element in

$$Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$$

then by (2.20) and the fixed trivialization of $\mathbb{C}_{-w_0*\lambda}$, we may consider f as an element in $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$. We have:

$$\Phi(f, c_B f) = \pm i^{\lambda_0} \int_{GU_I(\mathbb{Q})Z_{GU_I}(\mathbb{A}_\mathbb{Q})\backslash GU_I(\mathbb{A}_\mathbb{Q})} f(g)\overline{f}(g)||\nu(g)||^c dg. \tag{2.24}$$

Recall that $\nu(\cdot)$ is the similitude defined in (2.5).

Similarly, if we start from anti-holomorphic elements, we get a paring:

$$\Phi^{-} = \Phi^{I,\lambda,-} : \bar{H}^{D/2}(Sh_I, E_\lambda) \times \bar{H}^0(Sh_I, E_{w_0 * \lambda^\vee}) \to \mathbb{C}. \tag{2.25}$$

We use the script $^-$ to indicate that is anti-holomorphic. It is still c_{DR} stable. For $0 \neq f^- \in \bar{H}^{D/2}(Sh_I, E_\lambda)$, we also know that $\Phi^-(f^-, c_B f^-) \neq 0$.

2.6. **Arithmetic automorphic periods.** Let π be an irreducible cuspidal representation of $GU_I(\mathbb{A}_\mathbb{Q})$ defined over a number field $E(\pi)$. We may assume that $E(\pi)$ contains the quadratic imaginary field K.

We assume that π is cohomological with type λ, i.e. $H^*(\mathfrak{g}, K_{I,\infty}; \pi \otimes W_\lambda) \neq 0$.

For M a $GU_I(\mathbb{A}_{\mathbb{Q},f})$-module, define the K-rational π_f-isotypic components of M by

$$M^\pi := Hom_{GU_I(\mathbb{A}_{F,f})}(Res_{E(\pi)/K}(\pi_f), M) = \bigoplus_{\tau \in \Sigma_{E(\pi)}} Hom(\pi_f^\tau, M).$$

Therefore, if M has a K-rational structure then M^π also has a K-rational structure.

As in section 2.4, we have inclusions:

$$H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_0^\pi \otimes V_\Lambda) \subset \bar{H}^q(Sh_I, E_\Lambda)^\pi \subset H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_{(2)}^\pi \otimes V_\Lambda).$$

Under these inclusions, c_B sends $\bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi$ to $\bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})^{\pi^\vee}$.

These inclusions are compatible with those K-rational structures and then induce K-rational parings

$$\Phi^\pi : \bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi \times \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})^{\pi^\vee} \to \mathbb{C} \tag{2.26}$$

$$\text{and } \Phi^{-,\pi} : \bar{H}^{D/2}(Sh_I, E_\lambda)^\pi \times \bar{H}^0(Sh_I, E_{w_0 * \lambda^\vee})^{\pi^\vee} \to \mathbb{C}. \tag{2.27}$$

Definition 2.1. *Let β be a non zero K-rational element of $\bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi$. We define the* ***holomorphic arithmetic automorphic period associated to*** *β by $P^{(I)}(\beta, \pi) := (\Phi^\pi(\beta^\tau, c_B \beta^\tau))_{\tau \in \Sigma_{E(\pi)}}$. It is an element in $(E(\pi) \otimes_K \mathbb{C})^\times$.*

Let γ be a non zero K-rational element of $\bar{H}^{D/2}(Sh_I, E_\lambda)^\pi$. We define the ***anti-holomorphic arithmetic automorphic period associated to*** *γ by $P^{(I),-}(\gamma, \pi) := (\Phi^{-,\pi}(\gamma^\tau, c_B \gamma^\tau))_{\tau \in \Sigma_{E(\pi)}}$. It is an element in $(E(\pi) \otimes_K \mathbb{C})^\times$.*

Definition-Lemma 2.1. *Let us assume now π is tempered and π_∞ is discrete series representation. In this case, $\bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi$ is a rank one $E(\pi) \otimes_K \mathbb{C}$-module (c.f. [KMSW14]).*

We define the ***holomorphic arithmetic automorphic period*** *of π by $P^{(I)}(\pi) := P^{(I)}(\beta, \pi)$ by taking β any non zero rational element in $\bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi$. It is an element in $(E(\pi) \otimes_K \mathbb{C})^\times$ well defined up to $E(\pi)^\times$-multiplication.*

We define $P^{(I),-}(\pi)$ the ***anti-holomorphic arithmetic automorphic period*** *of π similarly.*

Lemma 2.3. *We assume that π is tempered and π_∞ is discrete series representation. Let β be a non zero rational element in $\bar{H}^0(Sh_I, E_{w_0 * \lambda})^\pi$ and $\beta^\vee$ be a non zero rational element in $\bar{H}^0(Sh_I, E_\lambda^\vee)^{\pi^\vee}$.*

We have $c_B(\beta) \sim_{E(\pi)} P^{(I)}(\pi) \beta^\vee$.

Proof. It is enough to notice that $\Phi^{\pi}(\beta,\beta^{\vee})\in E(\pi)^{\times}$.

$\square$

Lemma 2.4. *If π is tempered and π_{∞} is discrete series representation then we have:*

(1) $P^{(I^c)}(\pi^c)\sim_{E(\pi)}P^{(I)}(\pi)$.
(2) $P^{(I)}(\pi^{\vee})*P^{(I),-}(\pi)\sim_{E(\pi)}1$.

Proof. The first part comes from Lemma 2.1 and the fact that c_{DR} preserves rational structures.

For the second part, recall that the following two parings are actually the same:

$$\Phi^{\pi^{\vee}}:\bar{H}^0(Sh_I,E_{w_0*\lambda^{\vee}})^{\pi^{\vee}}\times\bar{H}^{D/2}(Sh_I,E_{\lambda})^{\pi}\to\mathbb{C} \tag{2.28}$$

$$\text{and }\Phi^{-,\pi}:\bar{H}^{D/2}(Sh_I,E_{\lambda})^{\pi}\times\bar{H}^0(Sh_I,E_{w_0*\lambda^{\vee}})^{\pi^{\vee}}\to\mathbb{C}. \tag{2.29}$$

We take β a rational element in $\bar{H}^0(Sh_I,E_{w_0*\lambda^{\vee}})^{\pi^{\vee}}$ and γ a rational element in $\bar{H}^{D/2}(Sh_I,E_{\lambda})^{\pi}$. We may assume that $\Phi^{\pi^{\vee}}(\beta^{\tau},\gamma^{\tau})=\Phi^{-,\pi}(\gamma^{\tau},\beta^{\tau})=1$ for all $\tau\in\Sigma_{E(\pi)}$.

By definition $p^{(I)}(\pi^{\vee})=(\Phi^{\pi^{\vee}}(\beta^{\tau},c_B\beta^{\tau}))_{\tau\in\Sigma_{E(\pi)}}$. Since $\bar{H}^{D/2}(Sh_I,E_{\lambda})^{\pi}$ is a rank one $E(\pi)\otimes\mathbb{C}$-module, there exists $C\in(E(\pi)\otimes_K\mathbb{C})^{\times}$ such that $(c_B\beta^{\tau})_{\tau\in\Sigma_{E(\pi)}}=C(\gamma^{\tau})_{\tau\in\Sigma_{E(\pi)}}$. Therefore $p^I(\pi^{\vee})=C(\Phi^{\pi^{\vee}}(\beta^{\tau},\gamma^{\tau}))_{\tau\in\Sigma_{E(\pi)}}=C$.

On the other hand, since $c_B^2=Id$, we have $(c_B\gamma^{\tau})_{\tau\in\Sigma_{E(\pi)}}=C^{-1}(\beta^{\tau})_{\tau\in\Sigma_{E(\pi)}}$. We can deduce that $p^{(I),-}(\pi)=C^{-1}$ as expected.

$\square$

Definition 2.2. *We say I is **compact** if $U_I(\mathbb{C})$ is. In other words, I is compact if and only if $I(\sigma)=0$ or n for all $\sigma\in\Sigma$.*

Corollary 2.1. *If I is compact then $P^{(I)}(\pi)\sim_{E(\pi)}P^{(I),-}(\pi)$. We have $P^{(I)}(\pi^{\vee})*P^{(I)}(\pi)\sim_{E(\pi)}1$.*

Proof. If I is compact, then $w_0=Id$. The anti-holomorphic part and holomorphic part are the same. We then have $P^{(I)}(\pi)\sim_{E(\pi)}P^{(I),-}(\pi)$. The last assertion comes from Lemma 2.4.

$\square$

The following theorem is Theorem 4.3.3 of [GL16] which generalizes the main theorem of [Gue16] and [Har97]:

Theorem 2.2. *Let Π be a regular, conjugate self-dual, cohomological, cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ which descends to $U_I(\mathbb{A}_{F^+})$ for any I. We denote the infinity type of Π at $\sigma\in\Sigma$ by $(z^{a_i(\sigma)}\overline{z}^{-a_i(\sigma)})_{1\leqslant i\leqslant n}$.*

Let η be an algebraic Hecke character of F with infinity type $z^{a(\sigma)}\overline{z}^{b(\sigma)}$ at $\sigma\in\Sigma$. We know that $a(\sigma)+b(\sigma)$ is a constant independent of σ, denoted by $-\omega(\eta)$.

We suppose that $a(\sigma)-b(\sigma)+2a_i(\sigma)\neq 0$ for all $1\leqslant i\leqslant n$ and $\sigma\in\Sigma$. We define $I:=I(\Pi,\eta)$ to be the map on Σ which sends $\sigma\in\Sigma$ to $I(\sigma):=\#\{i:a(\sigma)-b(\sigma)+2a_i(\sigma)<0\}$.

Let $m\in\mathbb{Z}+\dfrac{n-1}{2}$. If $m\geqslant\dfrac{1+\omega(\eta)}{2}$ is critical for $\Pi\otimes\eta$, we have:

$$L(m,\Pi\otimes\eta)\sim_{E(\Pi)E(\eta)}(2\pi i)^{mnd}P^{(I(\Pi,\eta))}(\Pi)\prod_{\sigma\in\Sigma}p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}. \tag{2.30}$$

and is equivariant under the action of F^{Gal}. Here $E(\Pi)$ is the compositum of all $E(\pi)$ when I varies among all the signatures.

The aim of this paper is to prove the following conjecture which generalizes a conjecture of Shimura ([Shi83]):

Conjecture 2.1. *There exists some non zero complex numbers $P^{(s)}(\Pi,\sigma)$ for all $0 \leqslant s \leqslant n$ and $\sigma \in \Sigma$ such that $P^{(I)}(\Pi) \sim_{E(\Pi)} \prod\limits_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi,\sigma)$ for all $I = (I(\sigma))_{\sigma\in\Sigma} \in \{0,1,\cdots,n\}^{\Sigma}$.*

3. Factorization of arithmetic automorphic periods and a conjecture

3.1. **Basic lemmas.** Let X, Y be two sets and Z be a multiplicative abelian group. We will apply the result of this section to $Z = \mathbb{C}^{\times}/E^{\times}$ where E is a proper number field.

Lemma 3.1. *Let f be a map from $X \times Y$ to Z. The following two statements are equivalent:*

(1) *There exists two maps $g : X \to Z$ and $h : Y \to Z$ such that $f(x,y) = g(x)h(y)$ for all $(x,y) \in X \times Y$.*
(2) *For all $x, x' \in X$ and $y, y' \in Y$, we have $f(x,y)f(x',y') = f(x,y')f(x',y)$.*

Moreover, if the above equivalent statements are satisfied, the maps g and h are unique up to scalars.

Proof. The direction that 1 implies 2 is trivial. Let us prove the inverse. We fix any $y_0 \in Y$ and define $g(x) := f(x,y_0)$ for all $x \in X$. We then fix any $x_0 \in X$ and define $h(y) := \dfrac{f(x_0,y)}{g(x_0)} = \dfrac{f(x_0,y)}{f(x_0,y_0)}$.

For any $x \in X$ and $y \in Y$, Statement 2 tells us that $f(x,y)f(x_0,y_0) = f(x,y_0)f(x_0,y)$. Therefore $f(x,y) = f(x,y_0) \times \dfrac{f(x_0,y)}{f(x_0,y_0)} = g(x)h(y)$ as expected.

□

Let n be a positive integer and $X_1,\cdots,X_n$ be some sets. Let f be a map from $X_1 \times X_2 \times \cdots \times X_n$ to Z.

The following corollary can be deduced from the above Lemma by induction on n.

Corollary 3.1. *The following two statements are equivalent:*

(1) *There exists some maps $f_k : X_k \to Z$ for $1 \leqslant k \leqslant n$ such that $f(x_1,x_2,\cdots,x_n) = \prod\limits_{1\leqslant k\leqslant n} f_k(x_k)$ for all $x_k \in X_k$, $1 \leqslant k \leqslant n$.*
(2) *Given any $x_j, x'_j \in X_j$ for each $1 \leqslant j \leqslant n$, we have*
$$\begin{aligned} &f(x_1,x_2,\cdots,x_n) \times f(x'_1,x'_2,\cdots,x'_n) \\ = &f(x_1,\cdots,x_{k-1},x'_k,x_{k+1},x_n) \times f(x'_1,\cdots x'_{k-1},x_k,x'_{k+1},\cdots,x'_n) \end{aligned}$$
for any $1 \leqslant k \leqslant n$.

Moreover, if the above equivalent statements are satisfied then for any $\lambda_1,\cdots,\lambda_n \in Z$ such that $\lambda_1\cdots\lambda_n = 1$, we have another factorization $f(x_1,\cdots,x_n) = \prod\limits_{1\leqslant i\leqslant n}(\lambda_i f_i)(x_i)$. Each factorization of f is of the above form.

We fix $a_i \in X_i$ for each i and $c_1,\cdots,c_n \in Z$ such that $f(a_1,\cdots,a_n) = c_1\cdots c_n$. If the above equivalent statements are satisfied then there exists a unique factorization such that $f_i(a_i) = c_i$.

Remark 3.1. *If $\#X_k \geqslant 3$ for all k, it is enough to verify the condition in statement 2 of the above corollary in the case $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$.*

In fact, when $\#X_k \geqslant 3$ for all k, for any $1 \leqslant j \leqslant n$ and any $y_j, y'_j \in X_j$, we may take $x_j \in X_j$ such that $x_j \neq y_j$, $x_j \neq y'_j$.

We fix any $1 \leqslant k \leqslant n$. If statement 2 is verified when $x_j \neq x'_j$ for all j then for any $y_k \neq y'_k$, we have

$$\begin{aligned}
& f(y_1, y_2, \cdots, y_n) f(y'_1, y'_2, \cdots, y'_n) f(x_1, x_2, \cdots, x_n) \\
= \; & f(y_1, y_2, \cdots, y_n) f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) \times \\
& \quad f(x_1, \cdots, x_{k-1}, y'_k, x_{k+1}, \cdots x_n) \\
= \; & f(y_1, y_2, \cdots, y_n) f(x_1, \cdots, x_{k-1}, y'_k, x_{k+1}, \cdots x_n) \times \\
& \quad f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) \\
= \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) f(x_1, \cdots, x_{k-1}, y_k, x_{k+1}, \cdots, x_n) \times \\
& \quad f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) \\
= \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) f(y'_1, \cdots y'_{k-1}, y_k, y'_{k+1}, \cdots, y'_n) f(x_1, x_2, \cdots, x_n).
\end{aligned}$$

We have assumed $y_k \neq y'_k$ to guarantee that each time we apply the formula in Statement 2, the coefficients satisfy $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$.

Therefore

$$\begin{aligned}
& f(y_1, y_2, \cdots, y_n) \times f(y'_1, y'_2, \cdots, y'_n) \\
= \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) \times f(y'_1, \cdots y'_{k-1}, y_k, y'_{k+1}, \cdots, y'_n)
\end{aligned}$$

if $y_k \neq y'_k$. If $y_k = y'_k$, this formula is trivially true.

We conclude that we can weaken the condition in Statement 2 of the above Corollary to $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$ when $\#X_k \geqslant 3$ for all k. We will verify this weaker condition in the application to the factorization of arithmetic automorphic periods.

3.2. **Relation of the Whittaker period and arithmetic periods.** Let Π be a regular cuspidal representation of $GL_n(\mathbb{A}_F)$ as in Theorem 2.2 with infinity type $(z^{a_i(\sigma)}\overline{z}^{-a_i(\sigma)})_{1\leqslant i\leqslant n}$ at $\sigma \in \Sigma$. We may assume that $a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma)$ for all $\sigma \in \Sigma$.

Recall that we say Π is N**-regular** if $a_i(\sigma) - a_{i+1}(\sigma) \geqslant N$ for all $1 \leqslant i \leqslant n-1$ and $\sigma \in \Sigma$.

Theorem 3.1. *For $1 \leqslant i \leqslant n-1$, let I_u be a map from Σ to $\{1, \cdots, n-1\}$. There exists a non-zero complex number $Z(\Pi_\infty)$ depending only on the infinity type of Π, such that if for any $\sigma \in \Sigma$, each number inside $\{1, 2, \cdots, n-1\}$ appears exactly once in $\{I_u(\sigma)\}_{1\leqslant i\leqslant n-1}$, then we have:*

$$p(\Pi) \sim_{E(\Pi)E(\Pi^\#)} Z(\Pi_\infty) \prod_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi) \tag{3.1}$$

provided Π is 3-regular or certain central L-values are non-zero.

Proof. Let us assume at first that n is even.

For each σ and u, let $k_u(\sigma)$ be an integer such that $I_u(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma)\}$.

Since n is even, $a_i(\sigma) \in \mathbb{Z} + \dfrac{1}{2}$ for all $1 \leqslant i \leqslant n$ and all $\sigma \in \Sigma$. The condition on I_u implies that for all $\sigma \in \Sigma$, the numbers $\{k_u(\sigma) \mid 1 \leqslant u \leqslant n-1\}$ lie in the $n-1$ gaps between $-a_n(\sigma) > -a_{n-1}(\sigma) > \cdots > -a_1(\sigma)$.

For $1 \leqslant u \leqslant n-1$, let χ_u be an algebraic conjugate self-dual Hecke character of F with infinity type $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$ at $\sigma \in \Sigma$.

We define $\Pi^{\#}$ to be the Langlands sum of χ_u, $1 \leqslant u \leqslant n-1$. It is an algebraic regular automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. Then the pair $(\Pi, \Pi^{\#})$ is in good position. By Proposition 1.2 we have

$$L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} p(\Pi)p(\Pi^{\#})p(m, \Pi_{\infty}, \Pi^{\#}_{\infty}) \tag{3.2}$$

where $p(m, \Pi_{\infty}, \Pi^{\#}_{\infty})$ is a complex number which depends on m, Π_{∞} and $\Pi^{\#}_{\infty}$.

Since $\Pi^{\#}$ is the Langlands sum of χ_u, $1 \leqslant u \leqslant n-1$, we have

$$L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) = \prod_{1\leqslant u\leqslant n-1} L(\frac{1}{2}+m, \Pi \times \chi_u).$$

We then apply Theorem 2.2 to the right hand side and get:

$$\begin{aligned} L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) &= \prod_{1\leqslant u\leqslant n-1} L(\frac{1}{2}+m, \Pi \times \chi_u) \\ \sim_{E(\Pi)E(\Pi^{\#})} & \prod_{1\leqslant u\leqslant n-1} [(2\pi i)^{d(m+\frac{1}{2})n} P^{(I(\Pi,\chi_u))}(\Pi) \prod_{\sigma\in\Sigma} p(\widecheck{\chi_u}, \sigma)^{I_u(\sigma)} p(\widecheck{\chi_u}, \overline{\sigma})^{n-I_u(\sigma)}] \end{aligned}$$

Recall that $I(\Pi, \chi_u)(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma)\} = I_u(\sigma)$ for any $\sigma \in \Sigma$ and $1 \leqslant u \leqslant n-1$.

Note that χ_u is conjugate self-dual, we have $p(\widecheck{\chi_u}, \overline{\sigma}) \sim_{E(\Pi^{\#})} p(\widecheck{\chi_u^c}, \sigma) \sim_{E(\Pi^{\#})} p(\widecheck{\chi_u^{-1}}, \sigma) \sim_{E(\Pi^{\#})} p(\widecheck{\chi_u}, \sigma)^{-1}$. We deduce that:

$$L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} (2\pi i)^{d(m+\frac{1}{2})n(n-1)} \prod_{1\leqslant u\leqslant n-1} [P^{(I_u)}(\Pi) \prod_{\sigma\in\Sigma} p(\widecheck{\chi_u}, \sigma)^{2I_u(\sigma)-n}] \tag{3.3}$$

By Thoerem Whittaker period theorem CM, there exists a constant $\Omega(\Pi^{\#}_{\infty}) \in \mathbb{C}^{\times}$ well defined up to $E(\Pi^{\#})^{\times}$ such that

$$p(\Pi^{\#}) \sim_{E(\Pi^{\#})} \Omega(\Pi^{\#}_{\infty}) \prod_{1\leqslant u<v\leqslant n-1} L(1, \chi_u\chi_v^{-1}). \tag{3.4}$$

By Blasius's result, we have:

$$L(1, \chi_u\chi_v^{-1}) \sim_{E(\Pi^{\#})} (2\pi i)^d \prod_{\sigma\in\Sigma} p(\widecheck{\chi_u\chi_v^{-1}}, \sigma')$$

where the embedding σ' is defined as follows: if $k_u(\sigma) < k_v(\sigma)$ then $\sigma' = \sigma$ and $p(\widecheck{\chi_u\chi_v^{-1}}, \sigma') \sim_{E(\chi_u)} p(\widecheck{\chi_u}, \sigma)p(\widecheck{\chi_v}, \sigma)^{-1}$; otherwise $\sigma' = \overline{\sigma}$ and $p(\widecheck{\chi_u\chi_v^{-1}}, \sigma') \sim_{E(\chi_u)} p(\widecheck{\chi_u}, \sigma)^{-1}p(\widecheck{\chi_v}, \sigma)$.

Therefore, the Whittaker period $p(\Pi^{\#})$

$$\sim_{E(\Pi^{\#})} (2\pi i)^{\frac{d(n-1)(n-2)}{2}} \Omega(\Pi^{\#}_{\infty}) \prod_{1\leqslant u\leqslant n-1} \prod_{\sigma\in\Sigma} p(\widecheck{\chi_u}, \sigma)^{\#\{v|k_v(\sigma)>k_u(\sigma)\}-\#\{v|k_v(\sigma)<k_u(\sigma)\}} \tag{3.5}$$

We know $\#\{v \mid k_v(\sigma) < k_u(\sigma)\} = n-2-\#\{v \mid k_v(\sigma) > k_u(\sigma)\}$.

Moreover, by definition of $k_u(\sigma)$ we have $\#\{v \mid k_v(\sigma) > k_u(\sigma)\} = \#\{i \mid -a_i(\sigma) > k_u(\sigma)\} - 1 = I_u(\sigma) - 1$. Therefore,

$$\#\{v \mid k_v(\sigma) > k_u(\sigma)\} - \#\{v \mid k_v(\sigma) < k_u(\sigma)\} = 2I_u(\sigma) - n \tag{3.6}$$

We compare equations (3.2), (3.3), (3.5) and (3.6). If Π is 3-regular we may take $m = 1$ and then $L(\frac{1}{2} + m, \Pi \times \Pi^{\#})$ is automatically non-zero, otherwise we take $m = 0$ and we assume that $L(\frac{1}{2}, \Pi \times \Pi^{\#}) \neq 0$. We obtain that:

$$\begin{aligned} & (2\pi i)^{d(m+\frac{1}{2})n(n-1)} \prod_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi) \\ \sim_{E(\Pi)E(\Pi^{\#})} \quad & (2\pi i)^{\frac{d(n-1)(n-2)}{2}} p(\Pi)\Omega(\Pi_\infty^{\#})p(m, \Pi_\infty, \Pi_\infty^{\#}). \end{aligned}$$

Hence we have $p(\Pi) \sim_{E(\Pi)E(\Pi^{\#})}$

$$(2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi_\infty^{\#})^{-1}p(m, \Pi_\infty, \Pi_\infty^{\#})^{-1} \prod_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi).$$

If we take

$$Z(m, \Pi_\infty, \Pi'_\infty) := (2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi_\infty^{\#})^{-1}p(m, \Pi_\infty, \Pi_\infty^{\#})^{-1}$$

then $p(\Pi) \sim_{E(\Pi)E(\Pi^{\#})} Z(m, \Pi_\infty, \Pi'_\infty) \prod\limits_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi)$.

In particular, we have that $Z(m, \Pi_\infty, \Pi'_\infty)$ depends only on Π_∞.

We may define:

$$Z(\Pi_\infty) := Z(m, \Pi_\infty, \Pi'_\infty) = (2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi_\infty^{\#})^{-1}p(m, \Pi_\infty, \Pi_\infty^{\#})^{-1}. \tag{3.7}$$

It is a non-zero complex number well defined up to elements in $E(\Pi)^\times$.

We deduce that:

$$p(\Pi) \sim_{E(\Pi)E(\Pi^{\#})} Z(\Pi_\infty) \prod_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi). \tag{3.8}$$

Now assume that n is odd. We keep the notation in the above section We have $a_i(\sigma) \in \mathbb{Z}$ for all $1 \leqslant i \leqslant n$ and all $\sigma \in \Sigma$. In this case, we take integers $k_u(\sigma)$ such that $I_u(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma) + \frac{1}{2}\}$.

We still let χ_u be an algebraic conjugate self-dual Hecke character of F with infinity type $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$ at $\sigma \in \Sigma$.

Recall that ψ is an algebraic Hecke character of F with infinity type z^1 at each $\sigma \in \Sigma$ such that $\psi\psi^c = ||\cdot||_{\mathbb{A}_F}$. We take $\Pi^{\#}$ to be the Langlands sum of $\chi_u\psi||\cdot||_{\mathbb{A}_F}^{-\frac{1}{2}}$, $1 \leqslant u \leqslant n-1$. It is an algebraic regular automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. The conditions of Theorem 1.2 hold.

We repeat the above process for Π and $\Pi^{\#}$ and get

$$\begin{aligned}
&L(\frac{1}{2}+m,\Pi\times\Pi^{\#})\\
\sim_{E(\Pi)E(\Pi^{\#})}\ &(2\pi i)^{dmn(n-1)}\prod_{1\leqslant u\leqslant n-1}[P^{(I(\Pi,\chi_u\psi))}\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{2I_u(\sigma)-n}]\times\\
&\prod_{\sigma\in\Sigma}(p(\check{\psi},\sigma)^{\sum_{1\leqslant u\leqslant n-1}I_u(\sigma)}p(\check{\psi}^c,\sigma)^{\sum_{1\leqslant u\leqslant n-1}(n-I_u(\sigma))})
\end{aligned}$$

where $I_u := I(\Pi,\chi_u\psi)$ with $I_u(\sigma)=\#\{i\mid -a_i(\sigma)>k_u(\sigma)+\frac{1}{2}\}$.

We see $\prod\limits_{1\leqslant u\leqslant n-1} I_u(\sigma)=\dfrac{n(n-1)}{2}$ and $\sum_{1\leqslant u\leqslant n-1}(n-I_u(\sigma))=\dfrac{n(n-1)}{2}$.

We then have

$$\begin{aligned}
&\prod_{\sigma\in\Sigma}(p(\check{\psi},\sigma)^{\sum_{1\leqslant u\leqslant n-1}I_u(\sigma)}p(\check{\psi}^c,\sigma)^{\sum_{1\leqslant u\leqslant n-1}(n-I_u(\sigma))})\\
\sim_{E(\psi)}\ &\prod_{\sigma\in\Sigma}p(\widetilde{\psi\psi^c},\sigma)^{\frac{n(n-1)}{2}}\sim_{E(\psi)}\prod_{\sigma\in\Sigma}p(||\cdot||_{\mathbb{A}_F}^{-1},\sigma)^{\frac{n(n-1)}{2}}\sim_{E(\psi)}(2\pi i)^{\frac{dn(n-1)}{2}}.
\end{aligned}$$

We verify that the equation (3.5) and (3.6) remain unchanged. We can see that equation (3.1) still holds here.

□

3.3. **Factorization of arithmetic automorphic periods: restricted case.**

We consider the function $\prod\limits_{\sigma\in\Sigma}\{0,1,\cdots,n\}\to\mathbb{C}^\times/E(\Pi)^\times$ which sends $(I(\sigma))_{\sigma\in\Sigma}$ to $P^{(I)}(\Pi)$.

In this section, we will prove the above conjecture restricted to $\{1,2,\cdots,n-1\}^\Sigma$. More precisely, we will prove that

Theorem 3.2. *If $n\geqslant 4$ and Π satisfies a global non vanishing condition, in particular, if Π is 3-regular, then there exists some non zero complex numbers $P^{(s)}(\Pi,\sigma)$ for all $1\leqslant s\leqslant n-1$, $\sigma\in\Sigma$ such that $P^{(I)}(\Pi)\sim_{E(\Pi)}\prod\limits_{\sigma\in\Sigma}P^{(I(\sigma))}(\Pi,\sigma)$ for all $I=(I(\sigma))_{\sigma\in\Sigma}\in\{1,2,\cdots,n-1\}^\Sigma$.*

Proof. For all $\sigma\in\Sigma$, let $I_1(\sigma)\neq I_2(\sigma)$ be two numbers in $\{1,2,\cdots,n-1\}$. We consider I_1,I_2 as two elements in $\{1,2,\cdots,n-1\}^\Sigma$.

Let σ_0 be any element in Σ. We define $I_1',I_2'\in\{1,2,\cdots,n-1\}^\Sigma$ by $I_1'(\sigma):=I_1(\sigma)$, $I_2'(\sigma):=I_2(\sigma)$ if $\sigma\neq\sigma_0$ and $I_1'(\sigma_0):=I_2(\sigma_0)$, $I_2'(\sigma_0):=I_1(\sigma_0)$.

By Remark 3.1, it is enough to prove that

$$P^{(I_1)}(\Pi)P^{(I_2)}(\Pi)\sim_{E(\Pi)}P^{(I_1')}(\Pi)P^{(I_2')}(\Pi).$$

Since $I_1(\sigma)\neq I_2(\sigma)$ for all $\sigma\in\Sigma$, we can always find $I_3,\cdots,I_{n-1}\in\{1,2,\cdots,n-1\}^\Sigma$ such that for all $\sigma\in\Sigma$, the $(n-1)$ numbers $I_u(\sigma)$, $1\leqslant u\leqslant n-1$ run over $1,2,\cdots,n-1$. In other words, conditions in Theorem 3.1 are verified.

By Theorem 3.1, we have

$$p(\Pi)\sim_{E(\Pi)}Z(\Pi_\infty)P^{(I_1)}(\Pi)P^{(I_2)}(\Pi)\prod_{3\leqslant u\leqslant n-1}P^{(I_u)}(\Pi).$$

On the other hand, it is easy to see that I_1', I_2', $I_3, \cdots, I_{n-1}$ also satisfy conditions in Theorem 3.1. Therefore

$$p(\Pi) \sim_{E(\Pi)} Z(\Pi_\infty) P^{(I_1')}(\Pi) P^{(I_2')}(\Pi) \prod_{3 \leqslant u \leqslant n-1} P^{(I_u)}(\Pi).$$

We conclude at last $P^{(I_1)}(\Pi)P^{(I_2)}(\Pi) \sim_{E(\Pi)} P^{(I_1')}(\Pi)P^{(I_2')}(\Pi)$ and then the above theorem follows.

□

Corollary 3.2. *If* Π *satisfied the conditions in the above theorem then we have:*

$$p(\Pi) \sim_{E(\Pi)} Z(\Pi_\infty) \prod_{\sigma \in \Sigma} \prod_{1 \leqslant i \leqslant n-1} P^{(i)}(\Pi, \sigma) \tag{3.9}$$

3.4. **Factorization of arithmetic automorphic periods: complete case.** In this section, we will prove Conjecture 2.1 when Π is regular enough. More precisely, we have

Theorem 3.3. *Conjecture 2.1 is true provided that* Π *is* 2*-regular and satisfies a global non vanishing condition which is automatically satisfied if* Π *is* 6*-regular.*

Proof. If $n = 1$, Conjecture 2.1 is known as multiplicity of CM periods (see Proposition 2.2). We may assume that $n \geqslant 2$. The set $\{0, 1, \cdots, n\}$ has at least 3 elements and then Remark 3.1 can apply.

For all $\sigma \in \Sigma$, let $I_1(\sigma) \neq I_2(\sigma)$ be two numbers in $\{0, 1, \cdots, n\}$. We have $I_1, I_2 \in \{0, 1, 2, \cdots, n\}^\Sigma$.

Let σ_0 be any element in Σ. We define $I_1', I_2' \in \{0, 1, 2, \cdots, n\}^\Sigma$ as in the proof of Theorem 3.2.

It remains to show that

$$P^{(I_1)}(\Pi)P^{(I_2)}(\Pi) \sim_{E(\Pi)} P^{(I_1')}(\Pi)P^{(I_2')}(\Pi). \tag{3.10}$$

Let us assume that n is odd at first. Since Π is 2-regular, we can find χ_u a conjugate self-dual algebraic Hecke character of F such that $I(\Pi, \chi_u) = I_u$ for $u = 1, 2$. We denote the infinity type of χ_u at $\sigma \in \Sigma$ by $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$, $u = 1, 2$. We remark that $k_1(\sigma) \neq k_2(\sigma)$ for all σ since $I_1(\sigma) \neq I_2(\sigma)$.

Let $\Pi^\#$ be the Langlands sum of Π, χ_1^c and χ_2^c. We write the infinity type of $\Pi^\#$ at $\sigma \in \Sigma$ by $(z^{b_i(\sigma)}\overline{z}^{-b_i(\sigma)})_{1 \leqslant i \leqslant n+2}$ with $b_1(\sigma) > b_2(\sigma) > \cdots > b_{n+2}(\sigma)$. The set $\{b_i(\sigma), 1 \leqslant i \leqslant n+2\} = \{a_i(\sigma), 1 \leqslant i \leqslant n\} \cup \{-k_1(\sigma), -k_2(\sigma)\}$.

Let $\Pi^\diamondsuit$ be a cuspidalconjugate self-dual cohomological representation of $GL_{n+3}(\mathbb{A}_F)$ with infinity type $(z^{c_i(\sigma)}\overline{z}^{-c_i(\sigma)})_{1 \leqslant i \leqslant n+3}$ such that $-c_{n+3}(\sigma) > b_1(\sigma) > -c_{n+2}(\sigma) > b_2(\sigma) > \cdots > -c_2(\sigma) > b_{n+2}(\sigma) > -c_1(\sigma)$ for all $\sigma \in \Sigma$. We may assume that $\Pi^\diamondsuit$ has definable arithmetic automorphic periods.

Proposition 1.2 is true for $(\Pi^\diamondsuit, \Pi^\#)$. Namely,

$$L(\frac{1}{2} + m, \Pi^\diamondsuit \times \Pi^\#) \sim_{E(\Pi^\diamondsuit)E(\Pi^\#)} p(\Pi^\diamondsuit)p(\Pi^\#)p(m, \Pi^\diamondsuit_\infty, \Pi^\#_\infty). \tag{3.11}$$

We know

$$L(\frac{1}{2} + m, \Pi^\diamondsuit \times \Pi^\#) = L(\frac{1}{2} + m, \Pi^\diamondsuit \times \Pi)L(\frac{1}{2} + m, \Pi^\diamondsuit \times \chi_1^c)L(\frac{1}{2} + m, \Pi^\diamondsuit \times \chi_2^c) \tag{3.12}$$

For $u = 1$ or 2, by Theorem 2.2 and the fact that χ_u is conjugate self-dual, we have

$$L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\chi_u)$$
$$\sim_{E(\Pi^{\diamondsuit})E(\Pi^{\#})} (2\pi i)^{(\frac{1}{2}+m)d(n+3)}P^{I(\Pi^{\diamondsuit},\chi_u^c)}(\Pi)\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{-2I(\Pi^{\diamondsuit},\chi_u^c)(\sigma)+(n+3)}.$$

Thoerem Whittaker period theorem CM implies that

$$p(\Pi^{\#})\sim_{E(\Pi^{\#})}\Omega(\Pi^{\#}_{\infty})p(\Pi)L(1,\Pi\otimes\chi_1)L(1,\Pi\otimes\chi_2)L(1,\chi_1\chi_2^c) \tag{3.13}$$

where $\Omega(\Pi^{\#}_{\infty})$ is a non zero complex numbers depend on $\Pi^{\#}_{\infty}$.

By Theorem 2.2 again, for $u = 1, 2$, we have

$$L(1,\Pi\times\chi_u)\sim_{E(\Pi^{\#})}(2\pi i)^{dn}P^{I(\Pi,\chi_u)}\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{2I(\Pi,\chi_u)(\sigma)-n}. \tag{3.14}$$

Moreover, $L(1,\chi_1\chi_2^c)\sim_{E(\Pi^{\#})}(2\pi i)^d\prod_{\sigma\in\Sigma}p(\widetilde{\chi_1},\sigma)^{t(\sigma)}p(\widetilde{\chi_2},\sigma)^{-t(\sigma)}$ where $t(\sigma)=1$ if $k_1(\sigma)<k_2(\sigma)$, $t(\sigma)=-1$ if $k_1(\sigma)>k_2(\sigma)$.

Lemma 3.2. *For all $\sigma\in\Sigma$,*

$$-2I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)+(n+3)=2I(\Pi,\chi_1)(\sigma)-n+t(\sigma),$$
$$-2I(\Pi^{\diamondsuit},\chi_2^c)(\sigma)+(n+3)=2I(\Pi,\chi_1)(\sigma)-n-t(\sigma).$$

Proof of the lemma: By definition we have

$$I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)=\#\{1\leqslant i\leqslant n+3\mid -c_i(\sigma)>-k_1(\sigma)\}.$$

Recall that $-c_{n+3}(\sigma)>b_1(\sigma)>-c_{n+2}(\sigma)>b_2(\sigma)>\cdots>-c_2(\sigma)>b_{n+2}(\sigma)>-c_1(\sigma)$ and $\{b_i(\sigma),1\leqslant i\leqslant n+2\}=\{a_i(\sigma),1\leqslant i\leqslant n\}\cup\{-k_1(\sigma),-k_2(\sigma)\}$.

Therefore

$$\begin{aligned}I(\Pi^{\diamondsuit},\chi_1^c)(\sigma) &= \#\{1\leqslant i\leqslant n+2\mid b_i(\sigma)>-k_1(\sigma)\}+1\\ &= \#\{1\leqslant i\leqslant n\mid a_i(\sigma)>-k_1(\sigma)\}+\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}+1.\end{aligned}$$

By definition we have

$$I(\Pi,\chi_1)(\sigma)=\#\{1\leqslant i\leqslant n\mid -a_i(\sigma)>k_1(\sigma)\}=n-\#\{1\leqslant i\leqslant n\mid a_i(\sigma)>-k_1(\sigma)\}.$$

Therefore, $I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)=n-I(\Pi,\chi_1)(\sigma)+\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}+1$. Hence we have $-2I(\Pi^{\diamondsuit},\chi_1^c)(\sigma))+(n+3)=2I(\Pi,\chi_1)(\sigma)-n+1-2\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}$.

It is easy to verify that $1-2\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}=t(\sigma)$. The first statement then follows and the second is similar to the first one.

□

We deduce that if $L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi^{\#})\neq 0$, then

$$L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi)(2\pi i)^{(1+2m)d(n+3)}P^{I(\Pi^{\diamondsuit},\chi_1^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},\chi_2^c)}(\Pi^{\diamondsuit})$$
$$\sim_{E(\Pi^{\diamondsuit})E(\Pi^{\#})}(2\pi i)^{d(2n+1)}p(\Pi^{\diamondsuit})\Omega(\Pi^{\#}_{\infty})p(m,\Pi^{\#}_{\infty},\Pi^{\#}_{\infty})P^{I(\Pi,\chi_1)}(\Pi)P^{I(\Pi,\chi_2)}(\Pi).$$

Now let χ_1', χ_2' be two conjugate self-dual algebraic Hecke characters of F such that $\chi'_{1,\sigma}=\chi_{1,\sigma}$ and $\chi'_{2,\sigma}=\chi_{2,\sigma}$ for $\sigma\neq\sigma_0$, $\chi'_{1,\sigma_0}=\chi_{2,\sigma_0}$ and $\chi'_{2,\sigma_0}=\chi_{1,\sigma_0}$.

We take $\Pi^{\#\#}$ as Langlands sum of Π, ${\chi_1'}^c$ and ${\chi_2'}^c$. Since the infinity type of $\Pi^{\#\#}$ is the same with $\Pi^{\#}$, we can repeat the above process and we see that equation (3.15) is true for $(\Pi^{\diamondsuit}, \Pi^{\#\#})$. Observe that most terms remain unchanged.

Comparing equation (3.15) for $(\Pi^{\diamondsuit}, \Pi^{\#})$ and that for $(\Pi^{\diamondsuit}, \Pi^{\#\#})$, we get

$$\frac{P^{I(\Pi^{\diamondsuit},{\chi'_1}^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},{\chi'_2}^c)}(\Pi^{\diamondsuit})}{P^{I(\Pi^{\diamondsuit},\chi_1^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},\chi_2^c)}(\Pi^{\diamondsuit})} \sim_{E(\Pi^{\diamondsuit})E(\Pi)} \frac{P^{I(\Pi,\chi'_1)}(\Pi)P^{I(\Pi,\chi'_2)}(\Pi)}{P^{I(\Pi,\chi_1)}(\Pi)P^{I(\Pi,\chi_2)}(\Pi)}. \tag{3.15}$$

By construction, $I(\Pi, \chi_u) = I_u$ and $I(\Pi, \chi'_u) = I'_u$ for $u = 1, 2$. Hence to prove (3.10), it is enough to show the left hand side of the above equation is a number in $E(\Pi^{\diamondsuit})^\times$.

There are at least two ways to see this. We observe that $I(\Pi^{\diamondsuit}, {\chi'_1}^c)(\sigma) = I(\Pi^{\diamondsuit}, {\chi_1}^c)(\sigma)$, $I(\Pi^{\diamondsuit}, {\chi'_2}^c)(\sigma) = I(\Pi^{\diamondsuit}, {\chi_2}^c)(\sigma)$ for $\sigma \neq \sigma_0$ and $I(\Pi^{\diamondsuit}, {\chi'_1}^c)(\sigma_0) = I(\Pi^{\diamondsuit}, {\chi_2}^c)(\sigma_0)$, $I(\Pi^{\diamondsuit}, {\chi'_2}^c)(\sigma_0) = I(\Pi^{\diamondsuit}, {\chi_1}^c)(\sigma_0)$. Moreover, these numbers are all in $\{1, 2, \cdots, (n+3)-1\}$. Theorem 3.2 gives a factorization of the holomorphic arithmetic automorphic periods through each place. In particular, it implies that the left hand side of (3.15) is in $E(\Pi^{\diamondsuit})^\times$ as expected.

One can also show this by taking $\Pi^{\diamondsuit}$ an automorphic induction of a Hecke character. We can then calculate $L(\frac{1}{2}+m, \Pi^{\diamondsuit} \times \chi_u^c)$ in terms of CM periods. Since the factorization of CM periods is clear, we will also get the expected result.

When n is even, we consider $\Pi^{\#}$ the Langlands sum of Π, $(\chi_1\psi||\cdot||^{-1/2})^c$ and $(\chi_2\psi||\cdot||^{-1/2})^c$ where χ_1, χ_2 are two suitable algebraic Hecke characters of F. We follow the above steps and will get the factorization in this case. We leave the details to the reader. $\square$

3.5. **Specify the factorization.** Let us assume that Conjecture 2.1 is true. We want to specify one factorization.

We denote by I_0 the map which sends each $\sigma \in \Sigma$ to 0. By the last part of Corollary 3.1, it is enough to choose $c(\Pi, \sigma) \in (\mathbb{C}/E(\Pi))^\times$ which is G_K-equivariant such that $P^{(I_0)}(\Pi) \sim_{E(\Pi)} \prod\limits_{\sigma\in\Sigma} c(\Pi, \sigma)$. Then there exists a unique factorization of $P^{(\cdot)}(\Pi)$ such that $P^{(0)}(\Pi, \sigma) = c(\Pi, \sigma)$. We may then define the **local arithmetic automorphic periods** $P^{(s)}(\Pi, \sigma)$ as an element in $\mathbb{C}^\times/(E(\pi))^\times$.

In this section, we shall prove $P^{(I_0)}(\Pi) \sim_{E(\Pi)} p(\breve{\xi_\Pi}, \overline{\Sigma}) \sim_{E(\Pi)} \prod\limits_{\sigma\in\Sigma} p(\breve{\xi_\Pi}, \overline{\sigma})$. Therefore, we may take $c(\Pi, \sigma) = p(\breve{\xi_\Pi}, \overline{\sigma})$.

More generally, we will see that:

Lemma 3.3. *If I is compact then $P^{(I)}(\Pi) \sim_{E(\Pi)} \prod\limits_{I(\sigma)=0} p(\breve{\xi_\Pi}, \overline{\sigma}) \times \prod\limits_{I(\sigma)=n} p(\breve{\xi_\Pi}, \sigma)$.*

This lemma leads to the following theorem:

Theorem 3.4. *If Conjecture 2.1 is true, in particular, if conditions in Theorem 3.3 are satisfied, then there exists some complex numbers $P^{(s)}(\Pi, \sigma)$ unique up to multiplication by elements in $(E(\Pi))^\times$ such that the following two conditions are satisfied:*

(1) $P^{(I)}(\Pi) \sim_{E(\Pi)} \prod\limits_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi, \sigma)$ *for all* $I = (I(\sigma))_{\sigma\in\Sigma} \in \{0, 1, \cdots, n\}^\Sigma$,

(2) *and* $P^{(0)}(\Pi, \sigma) \sim_{E(\Pi)} p(\breve{\xi_\Pi}, \overline{\sigma})$

where ξ_Π is the central character of Π.

Moreover, we know $P^{(n)}(\Pi, \sigma) \sim_{E(\Pi)} p(\breve{\xi_\Pi}, \sigma)$ or equivalently $P^{(0)}(\Pi, \sigma) \times P^{(n)}(\Pi, \sigma) \sim_{E(\Pi)}$ 1.

Proof of Lemma 3.3: Recall that $D/2 = \sum_{\sigma\in\Sigma} I_\sigma(n-I_\sigma) = 0$ since I is compact.

Let T be the center of GU_I. We have

$$T(\mathbb{R}) \cong \{(z_\sigma) \in (\mathbb{C}^\times)^\Sigma \mid |z_\sigma| \text{ does not depend on } \sigma\}.$$

We define a homomorphism $h_T : \mathbb{S}(\mathbb{R}) \to T(\mathbb{R})$ by sending $z \in \mathbb{C}$ to $((z)_{I(\sigma)=0}, (\overline{z})_{I(\sigma)=n})$.

Since I is compact, we see that h_I is the composition of h_T and the embedding $T \hookrightarrow GU_I$. We get an inclusion of Shimura varieties: $Sh_T := Sh(T, h_T) \hookrightarrow Sh_I = Sh(GU_I, h_I)$.

Let ξ be a Hecke character of K such that $\Pi^\vee \otimes \xi$ descends to π, a representation of $GU_I(\mathbb{A}_\mathbb{Q})$, as before. We write $\lambda \in \Lambda(GU_I)$ the cohomology type of π. We define $\lambda^T := (\lambda_0, (\sum_{1\leqslant i\leqslant n} \lambda_i(\sigma))_{\sigma\in\Sigma})$. Since π is irreducible, it acts as scalars when restrict to T. This gives π^T, a one dimensional representation of $T(\mathbb{A}_\mathbb{Q})$ which is cohomology of type λ^T. We denote by V_{λ^T} the character of $T(\mathbb{R})$ with highest weight λ^T.

The automorphic vector bundle E_λ pulls back to the automorphic vector bundle $[V_{\lambda^T}]$ (see [HK91] for notation) on Sh_T.

Let β be an element in $\bar{H}^0(Sh_I, E_\lambda)^\pi$. We fix a non zero $E(\pi)$-rational element in π and then we can lift β to ϕ, an automorphic form on $GU_I(\mathbb{A}_\mathbb{Q})$.

There is an isomorphism $H^0(Sh_T, [V_{\lambda^T}]) \xrightarrow{\sim} \{f \in \mathbb{C}^\infty(T(\mathbb{Q})\backslash T(\mathbb{A}_\mathbb{Q}), \mathbb{C} \mid f(tt_\infty)) = \pi^T(t_\infty)f(t), t_\infty \in T(\mathbb{R}), t \in T(\mathbb{A}_\mathbb{Q})\}$ (c.f. [HK91]). We send β to the element in $H^0(Sh_T, [V_{\lambda^T}])^{\pi^T}$ associated to $\phi|_{T(\mathbb{A}_\mathbb{Q})}$.

We then obtain rational morphisms

$$\bar{H}^0(Sh_I, E_\lambda)^\pi \xrightarrow{\sim} H^0(Sh_T, [V_{\lambda^T}])^{\pi^T} \tag{3.16}$$

$$\text{and similarly} \quad \bar{H}^0(Sh_I, E_{\lambda^\vee})^{\pi^\vee} \xrightarrow{\sim} H^0(Sh_T, [V_{\lambda^{T,\vee}}])^{\pi^{T,\vee}}. \tag{3.17}$$

These morphisms are moreover isomorphisms. In fact, since both sides are one dimensional, it is enough to show the above morphisms are injective. Indeed, if ϕ, a lifting of an element in $\bar{H}^0(Sh_I, E_\lambda)^\pi$, vanishes at the center, in particular, it vanishes at the identity. Hence it vanishes at $GU_I(\mathbb{A}_{\mathbb{Q},f})$ since it is an automorphic form. We observe that $GU_I(\mathbb{A}_{\mathbb{Q},f})$ is dense in $GU_I(\mathbb{Q})\backslash GU_I(\mathbb{A}_\mathbb{Q})$. We know $\phi = 0$ as expected.

We are going to calculate the arithmetic automorphic period. Let β be rational. We take a rational element $\beta^\vee \in \bar{H}^0(Sh_I, E_{\lambda^\vee})^{\pi^\vee}$ and lift it to an automorphic form $\phi^\vee$. We have $c_B(\phi) \sim_{E(\pi)} P^{(I)}(\pi)\phi^\vee$ by Lemma 2.3.

For the torus, by Remark 2.1, we know

$$\phi^\vee|_{T(\mathbb{A}_\mathbb{Q})} \sim_{E(\pi)} p(Sh(T, h_T), \pi^T)^{-1}(\phi|_{T(\mathbb{A}_\mathbb{Q})})^{-1}.$$

Recall that $c_B(\phi) = \pm i^{\lambda_0}\overline{\phi}||\nu(\cdot)||^{\lambda_0}$. Therefore $(c_B(\phi))|_{T(\mathbb{A}_\mathbb{Q})} = \pm i^{\lambda_0}(\phi|_{T(\mathbb{A}_\mathbb{Q})})^{-1}$. We then get

$$i^{\lambda_0}P^{(I)}(\pi) \sim_{E(\pi)} p(Sh(T, h_T), \pi^T). \tag{3.18}$$

We now set $T^\# := Res_{K/\mathbb{Q}}T_K$. We have $T^\# \cong Res_{K/\mathbb{Q}}\mathbb{G}_m \times Res_{F/\mathbb{Q}}\mathbb{G}_m$. In particular, $T^\#(\mathbb{R}) \cong \mathbb{C}^\times \times (\mathbb{R} \otimes_\mathbb{Q} F)^\times \cong \mathbb{C}^\times \times (\mathbb{C}^\times)^\Sigma$.

We define $h_{T^\#} : \mathbb{S}(\mathbb{R}) \to T^\#(\mathbb{R})$ to be the composition of h_T and the natural embedding $T(\mathbb{R}) \to T^\#(\mathbb{R})$. We know $h_{T^\#}$ sends $z \in \mathbb{C}^\times$ to $(z\overline{z}, (z)_{I(\sigma)=0}, (\overline{z})_{r(\sigma)=0})$. The embedding $(T, h_T) \to (T^\#, h_{T^\#})$ is a map between Shimura datum.

We observe that $\pi^{T,\#} := ||\cdot||^{-\lambda_0} \times \xi_{\Pi}^{-1}$ is a Hecke character on $T^{\#}$. Its restriction to T is just π^T. By Proposition 2.1, we have $p(Sh(T,h_T),\pi^T) \sim_{E(\pi)} p(Sh(T^{\#},h_{T^{\#}}),\pi^{T^{\#}})$.

By the definition of CM period and Proposition 2.2, we have

$$p(Sh(T^{\#},h_{T^{\#}}),\pi^{T^{\#}}) \sim_{E(\pi)} (2\pi i)^{\lambda_0} \prod_{I(\sigma)=0} p(\xi_{\Pi}^{-1},\sigma) \prod_{I(\sigma)=n} p(\xi_{\Pi}^{-1},\overline{\sigma}). \tag{3.19}$$

Since ξ_{Π} is conjugate self-dual, we have $p(\xi_{\Pi}^{-1},\overline{\sigma}) \sim_{E(\Pi)} p(\xi_{\Pi},\sigma)$.
By equation (3.18), we get:

$$i^{\lambda_0} P^{(I)}(\pi) \sim_{E(\pi)} (2\pi i)^{\lambda_0} \prod_{I(\sigma)=0} p(\xi_{\Pi}^{-1},\sigma) \prod_{I(\sigma)=n} p(\xi_{\Pi},\sigma). \tag{3.20}$$

Recall that by definition $P^{(I)}(\Pi) \sim_{E(\Pi)} (2\pi)^{-\lambda_0} P^{(I)}(\pi)$, we get finally

$$\begin{aligned} P^{(I)}(\Pi) &\sim_{E(\Pi)} \prod_{I(\sigma)=0} p(\xi_{\Pi}^{-1},\sigma) \times \prod_{I(\sigma)=n} p(\xi_{\Pi},\sigma) \\ &\sim_{E(\Pi)} \prod_{I(\sigma)=0} p(\widecheck{\xi_{\Pi}},\overline{\sigma}) \times \prod_{I(\sigma)=n} p(\widecheck{\xi_{\Pi}},\sigma). \end{aligned}$$

The last formula comes from the fact that ξ_{Π} is conjugate self-dual.

$\square$

Remark 3.2. *If $n=1$ and $\Pi=\eta$ is a Hecke character, we obtain that: $P^{(0)}(\eta,\sigma) \sim_{E(\eta)} p(\check{\eta},\overline{\sigma})$ and similarly $P^{(1)}(\eta,\sigma) \sim_{E(\eta)} p(\check{\eta},\sigma)$.*

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