PERIOD RELATIONS FOR AUTOMORPHIC INDUCTION AND APPLICATIONS, I
RELATIONS DE PÉRIODES POUR INDUCTION AUTOMORPHE ET APPLICATIONS, I

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Abstract. Let K be a quadratic imaginary field. Let Π (resp. Π') be a regular algebraic cuspidal representation of $GL_n(\mathbb{A}_K)$ (resp. $GL_{n-1}(\mathbb{A}_K)$) which is moreover cohomological and conjugate self-dual. When Π is a cyclic automorphic induction of a Hecke character χ over a CM field, we show relations between automorphic periods of Π defined by Harris and those of χ. Consequently, we refine a formula given by Grobner and Harris for critical values of the Rankin-Selberg L-function $L(s, \Pi \times \Pi')$. This completes the proof of an automorphic version of Deligne's conjecture in certain case.

Résumé. Soit K un corps quadratique imaginaire. Soit Π (resp. Π') une représentation cuspidale régulière algébrique de $GL_n(\mathbb{A}_K)$ (resp. $GL_{n-1}(\mathbb{A}_K)$) qui est, de plus, cohomologique et auto-duale. Si Π est une induction automorphe cyclique d'un caractère de Hecke χ sur un corps CM, on montre les relations entre les périodes automorphes de Π définies par Harris et celles de χ. Par conséquent, on affine une formule de Grobner et Harris pour les valeurs critiques de $L(s, \Pi \times \Pi')$, L étant la fonction de Rankin-Selberg. Cela complète la démonstration d'une version automorphe de la conjecture de Deligne dans certains cas.

Keywords: automorphic periods, automorphic induction, L-functions, Deligne's conjecture
Mots clés: périods automorphes, induction automorphe, fonctions L, conjecture de Deligne

Introduction

In [Har97], M. Harris has defined complex invariants, called automorphic periods, for certain automorphic representations over quadratic imaginary field. We believe that these periods are functorial. In this note, we treat the case when the representation is a cyclic automorphic induction of a Hecke character over a CM field. More precisely, let K be a quadratic imaginary field and $F \supset K$ be a CM field which is cyclic over K. Let χ be certain Hecke character of F and $\Pi(\chi)$ be the automorphic induction of χ with respect to F/K. We show relations between automorphic periods of $\Pi(\chi)$ and CM periods of χ. Our main result is Theorem 3.2 below.

These relations allow us to simplify a formula obtained by Grobner and Harris on the critical values for the Rankin-Selberg L-function of $\Pi \times \Pi'$ where Π and Π' are certain automorphic representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$ (c.f. [GH]). We first refine the formula in the case that Π and Π' are both induced from characters and then to more general cases. We see finally that our result is compatible with Deligne's conjecture.

1. Notation and conventions

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Let $K \subset \overline{\mathbb{Q}}$ be a quadratic imaginary field and n be an integer at least 2. Let ε_K be the Artin character of $\mathbb{A}_{\mathbb{Q}}$ associated to the extension $K/\mathbb{Q}$. We fix ψ an algebraic Hecke character of K with infinity type $z^1\overline{z}^0$ such that $\psi\psi^c = ||\cdot||_{\mathbb{A}_K}$. The existence follows from Lemma 4.1.4 in [CHT08].

Let F^+ (resp. F'^+) be a totally real field of degree n (resp. $n-1$) over $\mathbb{Q}$. We set $F = KF^+$ (resp. $F' = KF'^+$) a CM field. We put $L = F \otimes_K F'$. It is easy to see that L is a CM field of degree $n(n-1)$ over K.

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Let $\iota \in G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ be the complex conjugation. We may consider it as an element of $Gal(F/F^{+})$ or $Gal(F'/F'^{+})$. For any number field E, let Σ_E be the set of complex embeddings of E. For $z \in \mathbb{C}$, we write $\bar{z}$ for its complex conjugation. For $\sigma \in \Sigma_F$, we define $\bar{\sigma} := \iota \circ \sigma$ the complex conjugation of σ.

Let Φ be a subset of Σ_F. We say that Φ is a **CM type** of F if $\Phi \cup \iota\Phi = \Sigma_F$ and $\Phi \cap \iota\Phi = \varnothing$. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ be the elements of Σ_F which are the identity on K. We know $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ is a CM type of F.

Let χ be a Hecke character of F with infinity type $\chi_\infty(z) = \prod\limits_{i=1}^{n} \sigma_i(z)^{a_i}\bar{\sigma}_i(z)^{b_i}$. We suppose that χ is **algebraic** which implies that $a_i,\ b_i \in \mathbb{Z}$ and $a_i + b_i = -w(\chi)$, an integer independent of i, and **critical**, i.e. $a_i \neq b_i$ for all i. We can then define Φ_χ, a unique CM type associated to χ, as follows: for each i, $\sigma_i \in \Phi_\chi$ if $a_i < b_i$, otherwise $\bar{\sigma}_i \in \Phi_\chi$. In this case, we say that χ is **compatible** with Φ_χ.

For such χ, one can define $E(\chi_\infty) \subset \mathbb{C}$, a number field, as in (1.1.2) of [Har93]. It is the field of definition of $\sum(a_i\sigma_i + b_i\overline{\sigma_i}) \in \mathbb{Z}^{\Sigma_F}$. We denote by $E(\chi)$ the field generated by the values of χ on $\mathbb{A}_{F,f}$ over $E(\chi_\infty)$. It is still a number field. We assume that $E(\chi)$ contains F for simplicity of notation.

For any $\Psi \subset \Sigma_F$ such that $\Psi \cap \iota\Psi = \varnothing$, one can associate a non zero complex number $p_F(\chi, \Psi)$ which is well defined modulo $E(\chi)^\times$ (c.f. the appendix of [HK91]). We call it a **CM period**. Sometimes we write $p(\chi, \Psi)$ instead of $p_F(\chi, \Psi)$ if there is no ambiguity concerning the base field F.

The special values of an L-function for a Hecke character over a CM field can be interpreted in terms of CM periods. The following theorem is proved by Blasius. We state it as in Proposition 1.8.1 in [Har93] where ω should be replaced by $\breve{\omega} := \omega^{-1,c}$ (for this erratum, see the notation and conventions part on page 82 in [Har97]).

Theorem 1.1. *Let χ be as before. We denote D_{F^+} the absolute discriminant of F^+. If an integer m is critical for χ in the sense of Deligne, we have*

$$(L(\chi^\sigma, m))_{\sigma \in \Sigma_{E(\chi)}} \sim_{E(\chi)} D_{F^+}^{1/2}(2\pi i)^{mn}(p(\check{\chi}^\sigma, \Phi_{\chi^\sigma}))_{\sigma \in \Sigma_{E(\chi)}}.$$

We now introduce the notation $\sim_{E(\chi)}$ in previous theorem. Let E be a finite extension of K. We identify $\mathbb{C}^{\Sigma_E}$ with $E \otimes \mathbb{C}$ by the inverse of the map which sends $t \otimes z$ to $(\sigma(t)z)_{\sigma \in \Sigma_E}$ for all $t \in E$ and $z \in \mathbb{C}$. This is a morphism of algebras where the multiplication on the former is the usual multiplication through each coordinates. Similarly, let $\Sigma_{E;K}$ be the subset of Σ_E containing embeddings of E into $\mathbb{C}$ which are the identity on K. We may identify $\mathbb{C}^{\Sigma_{E;K}}$ with $E \otimes_K \mathbb{C}$.

Definition 1.1. *Let A, B be two elements in $E \otimes \mathbb{C}$ (resp. $E \otimes_K \mathbb{C}$). We say that $A \sim_E B$ (resp. $A \sim_{E;K} B$) if one of the following conditions is satisfied: (i) $A = 0$, (ii) $B = 0$ or (iii) $A, B \in (E \otimes \mathbb{C})^\times$ (resp. $(E \otimes_K \mathbb{C})^\times$) with $AB^{-1} \in E^\times \subset (E \otimes \mathbb{C})^\times$ (resp. $(E \otimes_K \mathbb{C})^\times$).*

Note that this relation is symmetric but not transitive unless we know that everything is non zero.

Let $(a(\sigma))_{\sigma \in G_K}$ be some complex numbers such that $a(\sigma) = a(\sigma')$ if $\sigma|_E = \sigma'|_E$ for any $\sigma, \sigma' \in G_K$.For example, for $E = E(\chi)$ and s a complex number, the values $(L(s, \chi^\sigma))_{\sigma \in G_K}$ satisfy the above condition. We can define $a(\sigma)$ for $\sigma \in \Sigma_{E;K}$ by taking $\tilde{\sigma}$, any lift of σ in G_K, and defining $a(\sigma)$ to be $a(\tilde{\sigma})$. We consider $(a(\sigma))_{\sigma \in \Sigma_{E;K}}$ as an elements in $\mathbb{C}^{\Sigma_{E;K}}$.

Definition-Lemma 1.1. *Let $b(\sigma)_{\sigma \in G_K}$ be some complex numbers with the same property as $a(\sigma)_{\sigma \in G_K}$. We assume $b(\sigma) \neq 0$ for all $\sigma \in G_K$. We fix $\sigma_0 \in \Sigma_{E;K}$. We then have $(a(\sigma))_{\sigma \in \Sigma_{E;K}} \sim_{E;K} (b(\sigma))_{\sigma \in \Sigma_{E;K}}$ if and only if $\dfrac{a(\sigma_0)}{b(\sigma_0)} \in \overline{\mathbb{Q}}$ and $\tau\left(\dfrac{a(\sigma_0)}{b(\sigma_0)}\right) = \dfrac{a(\tau\sigma_0)}{b(\tau\sigma_0)}$ for all $\tau \in G_K$.*

*In this case, we say $a \sim_E b$ **equivariant under action of** G_K. In particulier, $\dfrac{a(\sigma)}{b(\sigma)} \in E$ for all $\sigma \in G_K$.*

At last, we introduce certain notation concerning Hecke characters of K.

Definition 1.2. *For η an algebraic Hecke character of K with infinity type $z^{a(\eta)}\bar{z}^{b(\eta)}$, we define:*

- $\check{\eta} = \eta^{-1,c}$ *a Hecke character of K.*
- $\tilde{\eta}(z) = \eta(z)/\eta(\bar{z})$ *a Hecke character of K.*
- η_0 *the Hecke character of $\mathbb{Q}$ such that* $\eta\eta^c = (\eta_0 \circ N_{\mathbb{A}_K/\mathbb{A}_\mathbb{Q}})||\cdot||^{a(\eta)+b(\eta)}$.
- $\eta^{(2)} = \eta^2/\eta_0 \circ N_{\mathbb{A}_K/\mathbb{A}_\mathbb{Q}}$.

2. Unitary similitude group and base change

In this section, we recall a result on base change of representations for similitude unitary groups. Let G be a connected quasi-split reductive group over $\mathbb{Q}$ and $G' = Res_{K/\mathbb{Q}}G_K$. Roughly speaking, the base change is a correspondence from certain automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ to certain automorphic representations of $G'(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A}_K)$. We refer to Section 26 of [Art03] for more details.

Over a local field, this correspondence can be defined concretely for unramified representations (c.f. [Min11]) and is in fact a map from the set of unramified representations of G to that of G'. This allows us to give a precise definition for global base change. For π an admissible irreducible representation of $G(\mathbb{A}_\mathbb{Q})$, we say Π, a representation of $G(\mathbb{A}_K)$, is a **(weak) base change** of π if for all v, a finite place of $\mathbb{Q}$ at which π is unramified and G is quasi-split, and for all w, a place of K over v, Π_w is the base change of π_v. In this case, we say Π **descends to** π by base change.

For example, if v is a place of $\mathbb{Q}$ split in K. Let w be a place of K above v. We know $\mathbb{Q}_v \cong K_w$ and hence $G(\mathbb{Q}_v) = G(K_w)$. The local base change map is the identity.

Let $r, s \in \mathbb{N}$ such that $r + s = n$. Fix q_1, q_2 two places of $\mathbb{Q}$ which are split in K and inert in F^+. Let V be a n-dimensional vector space over K. The calculation of local invariants of unitary groups in chapter 2 of [Clo91] shows that there exists a hermitian form on V with respect to $K/\mathbb{Q}$ such that the associated unitary group over $\mathbb{Q}$ is quasi-split at all finite places except q_1 and q_2, ramified at one or two places between q_1 and q_2 and has infinity sign (r, s). We denote $U(r, s)$ this unitary group and write $GU(r, s)$ for the associated similitude group.

One can show that $GU(r, s)_K \cong U(r, s)_K \times \mathbb{G}_{m,K}$. In particular, $GU(\mathbb{A}_K) \cong GL_n(\mathbb{A}_K) \times \mathbb{A}_K^\times$. For Π a cuspidal representation of $GL_n(\mathbb{A}_K)$ and ξ a Hecke character of K, $\Pi \otimes \xi$ defines a cuspidal representation of $GU(\mathbb{A}_K)$. Conversely, by the tensor product theorem, every irreducible automorphic representation of $GU(\mathbb{A}_K)$ can be written in the form $\Pi \otimes \xi$. Moreover, Π and ξ are unique up to isomorphisms.

Let us consider now the base change for $G = GU(r, s)$. Theorem 2.1.2 and Theorem 3.1.2 of [HL04] tells us when $\Pi \otimes \xi$ descends to a representation of $G(\mathbb{A}_\mathbb{Q})$. In this note, we start with a representation of $GL_n(\mathbb{A}_K)$. The following lemma will be useful (c .f. Lemma VI.2.10 of [HT01]):

Lemma 2.1. *Let Π be a conjugate self-dual cuspidal representation of $GL_n(\mathbb{A}_K)$. We assume Π is cohomological and supercuspidal at places over q_1 and q_2. There always exists ξ, a Hecke character of K, such that $\Pi \otimes \xi$ descends to a representation of $G(\mathbb{A}_\mathbb{Q})$.*

3. Automorphic period

In this note, a **motive** M simply means a pure motive for absolute Hodge cycles in the sense of Deligne. We refer the reader to [Del79] for detailed definitions. We recall that an integer m is **critical** for M if neither $L_\infty(M, s)$ nor $L_\infty(\check{M}, 1 - s)$ has a pole at $s = m$ where $\check{M}$ is the dual of M. In this case, we say m is **critical** for M.

The **Hodge type** of M is defined by the set $T = T(M)$ consisting of pairs (p, q) such that $M^{p,q} \neq 0$. We assume that M is pure, namely there exists an integer w such that $p + q = w$ for all $(p, q) \in T(M)$. In [Del79], the author has determined the critical points in terms of the Hodge type of M.

Let $n \geqslant 1$ be an integer, K be a quadratic imaginary field and $\Pi = \Pi_f \otimes \Pi_\infty$ be a regular cohomological cuspidal representation of $GL_n(\mathbb{A}_K)$. We denote V the representation space for Π_f. For $\sigma \in Aut(\mathbb{C})$, we define another $GL_n(\mathbb{A}_{K,f})$-representation Π_f^σ to be $V \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Let $\mathbb{Q}(\Pi)$ be the subfield of $\mathbb{C}$ fixed by $\{\sigma \in Aut(\mathbb{C}) \mid$

$\Pi_f^\sigma \cong \Pi_f\}$. We call it the **rationality field** of Π. This is in fact a number field and Π_f has a rational structure on $\mathbb{Q}(\Pi)$. In other words, there exists V, a $GL_n(\mathbb{A}_{\mathbb{Q},f})$-module over $\mathbb{Q}(\Pi)$, such that $\Pi_f = V \otimes_{\mathbb{Q}(\Pi)} \mathbb{C}$ as $GL_n(\mathbb{A}_{\mathbb{Q},f})$-module.

Moreover, for all $\sigma \in Aut(\mathbb{C})$, Π_f^σ is the finite part of a cuspidal representation of $GL_n(\mathbb{A}_K)$ which is unique by the strong multiplicity one theorem, denoted by Π^σ.We know that Π^σ is determined by $\sigma|_{\mathbb{Q}(\Pi)} : \mathbb{Q}(\Pi) \hookrightarrow \mathbb{C}$. Therefore, we may define Π^σ for any $\sigma \in \Sigma_{Q(\Pi)}$ by lifting σ to an element in $Aut(\mathbb{C})$. In particular, we may define Π^σ for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ or $\sigma \in \Sigma_E$ where E is an extension of $\mathbb{Q}(\Pi)$.

When Π is cohomological and conjugate self-dual, M. Harris has proved that there exists a motive associated to Π of rank n over K with coefficients in a number field. By restriction of scalars from K to $\mathbb{Q}$, we obtain (c.f. [Har97]) that:

Theorem 3.1. *There exists E a finite extension of $\mathbb{Q}(\Pi)$ and M a regular pure motive of rank $2n$ over $\mathbb{Q}$ with coefficients in E such that $L(s, M, \sigma) = L(s + \frac{1-n}{2}, \Pi^\sigma)$ for all $\sigma : E \hookrightarrow \mathbb{C}$.*

Harris has also defined automorphic periods $P^{(s)}(\Pi)$ for certain integers $0 \leqslant s \leqslant n$ which is a complex number defined up to multiplication by an element in $E^\times$. If Π is supercuspidal at each places over q_1 and q_2, the automorphic period can be defined for every $0 \leqslant s \leqslant n$. More precisely, $P^{(s)}$ is defined when there exists ξ, a Hecke character of K, such that $\Pi \otimes \xi$ descends to a representation of $GU_{n-s,s}(\mathbb{A}_{\mathbb{Q}})$. With the supercuspidal condition, we know that this is true by Lemma 2.1. We assume this condition on Π throughout this note. Harris proved that special values of the automorphic L-function can be interpreted in terms of automorphic periods:

Theorem 3.2. *Let Π be as before with its infinity type $(z^{a_i}\overline{z}^{-a_i})_{1\leqslant i\leqslant n}$. Let η be an algebraic Hecke character of K with infinity type $\eta_\infty(z) = z^a \bar{z}^b$ such that for all $1 \leqslant i \leqslant n$, $b - a \neq 2a_i$.*

Write $\eta^c = \widetilde{\beta}\alpha$. Here α, β are Hecke characters of K with $\alpha_\infty(z) = z^\kappa$ and $\beta_\infty(z) = z^{-k}$, $\kappa, k \in \mathbb{Z}$. Define $s = s(\eta^c, \Pi^\vee) = \#\{i \mid a - b + 2a_i < 0\}$.

For $m \in \mathbb{Z}$ critical for $M(\Pi) \otimes M(\eta)$ and satisfies $m \geqslant \frac{n-\kappa}{2} = \frac{n-a-b}{2}$, we have

$$L(m, M(\Pi) \otimes M(\eta)) \sim_{E(\Pi)E(\beta)E(\alpha)} (2\pi i)^{(m-\frac{n-1}{2})n} \mathcal{G}(\varepsilon_K)^{[\frac{n}{2}]} P^{(s)}(\Pi)[(2\pi i)^\kappa \mathcal{G}(\alpha_0)]^s[(2\pi i)^k p(\check{\beta}^{(2)}\check{\alpha}, 1)]^{n-2s}$$

equivariant under action of G_K. Here $\mathcal{G}$ refers to the Gauss sum.

Proposition 3.1. *Let Π be as in Theorem 3.2. For any fixed integer $0 \leqslant s \leqslant n$, there exists an algebraic Hecke character η and an integer m as in Theorem 3.2 such that $s(\eta^c, \Pi^\vee) = s$ and $L(m, M(\Pi) \otimes M(\eta)) \neq 0$.*

In [GH], the authors gave an interpretation of special values of L-function for $GL_n \times GL_{n-1}$ over K. Let Π and Π' be two cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$ which satisfy the conditions in Theorem 3.2 and some regular conditions (c.f. *loc. cit*). We have

Theorem 3.3. *Let m be a non negative integer. If $m + n - 1$ is critical for $M(\Pi) \otimes M(\Pi')$, then*

$$L(m + \frac{1}{2}, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi')} p(m, \Pi_\infty, \Pi'_\infty) Z(\Pi_\infty) Z(\Pi'_\infty) \prod_{j=1}^{n-1} P^{(j)}(\Pi) \prod_{k=1}^{n-2} P^{(k)}(\Pi')$$

equivariant under action of G_K.

Here $p(m, \Pi_\infty, \Pi'_\infty)$ is a complex number depending only on m, Π_∞ and Π'_∞ (c.f. Proposition 6.4 of loc. cit); $Z(\Pi_\infty)$ (resp. $Z(\Pi'_\infty)$) is a complex number depending only on Π_∞ (resp. Π'_∞) (c.f. Theorem 6.7 of loc. cit).

4. Period relations for automorphic induction of Hecke characters

In this section, we consider the representation induced from Hecke characters. Let χ be a regular algebraic conjugate self-dual Hecke character of F. Here conjugate self-dual means $\chi^{-1} = \chi^c$.

We also make the hypothesis that:

Hypothesis 4.1. *For any v a place of K over q_1 or q_2, $\chi_v \neq \chi_v^\tau$ for all $\tau \in Gal(F_v/K_v)$.*

Under this hypothesis, $\Pi(\chi)$, the automorphic induction of χ from $GL_1(\mathbb{A}_F)$ to $GL_n(\mathbb{A}_K)$, is supercuspidal at all places over q_1 or q_2 (c.f. Proposition 2.4 of [Har98]).

Definition-Lemma 4.1. *Let χ be as above. We define $\Pi_\chi := \Pi(\chi)$ if the degree of F over K is odd; $\Pi_\chi := \Pi(\chi) \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$ otherwise where ψ is a Hecke character of K defined in Section 1.*

We have that Π_χ is a regular algebraic cuspidal which satisfies all the conditions in Theorem 3.2.

Up to finite extension, we may assume $E(\Pi_\chi) = E(\chi)$. We define $\Phi_{s,\chi}$, a CM type of F as follows: for each i such that a_i is one of the s smallest numbers in $\{a_i, 1 \leqslant i \leqslant n\}$, we have $\sigma_i \in \Phi_{s,\chi}$; otherwise $\bar{\sigma}_i \in \Phi_{s,\chi}$.

Theorem 4.1. *Let n be an integer. Let $F = F^+K$ with F^+ a totally real field of degree n over $\mathbb{Q}$ and K a quadratic imaginary field. Assume that F is cyclic over K. Let χ be a regular conjugate self-dual algebraic Hecke character of F satisfying Hypothesis 4.1. We have that the automorphic period of $\Pi = \Pi_\chi$ satisfies:*

$$P^{(s)}(\Pi) \sim_{E(\chi)} D_{F^+}^{1/2}\mathcal{G}(\varepsilon_K)^{-[\frac{n}{2}]}p(\check{\chi}, \Phi_{s,\chi}) \text{ if } n \text{ is odd}$$

$$P^{(s)}(\Pi) \sim_{E(\chi)E(\psi)} D_{F^+}^{1/2}(2\pi i)^{-\frac{n}{2}}\mathcal{G}(\varepsilon_K)^{-[\frac{n}{2}]}p(\check{\chi}, \Phi_{s,\chi})p(\psi)^s p(\psi^c)^{n-s} \text{ if } n \text{ is even}$$

equivariant under action of G_K.

This is the main result of this note. The idea is simple. We fix $0 \leqslant s \leqslant n$ an integer. We take η and m as in Proposition 3.1. When n is odd, we have $L(m, \Pi_\chi \otimes \eta) = L(m, \chi \otimes \eta \circ N_{\mathbb{A}_F/\mathbb{A}_K})$ by automorphic induction and with both sides non zero. We may simplify the left hand side of this equation by Theorem 3.2 and the right hand side by Blasius's result. The CM periods of η appeared in both sides unsurprisingly coincide and we then deduce the above result.

5. Application: simplification of Archimedean local factors

We can now refine the Archimedean local factors in 3.3 first in the case where Π and Π' come from a Hecke character and then for general Π and Π'.

We take χ and χ' two algebraic regular conjugate self-dual Hecke characters of F and F' who satisfy Hypothesis 4.1 and some regular conditions. We may apply Theorem 3.3 to $\Pi_\chi \times \Pi'_\chi$. Our main result Theorem 4.1 allows us to replace the automorphic periods by CM periods and we get:

$$p(m, \Pi_\infty, \Pi'_\infty)Z(\Pi_\infty)Z(\Pi'_\infty) \sim_{KE(\chi_\infty)E(\chi'_\infty)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}$$

provided that $L(m + \frac{1}{2}, \Pi \times \Pi')$ does not vanish. This is always true when $m > 0$ since in this case, m is in the absolutely convergent range.

Note that the above result concerns only the infinity type. The following lemma allows us to generalize it.

Lemma 5.1. *If Π is an algebraic cuspidal representation of $GL_n(K)$ then there exists χ an algebraic Hecke character of F which satisfies Hypothesis 4.1 such that $\Pi_\infty \cong \Pi_{\chi,\infty}$. Furthermore, if Π is conjugate self-dual, we may have in addition that χ is conjugate self-dual.*

Note that an extra condition on the non vanishing of L-function will be needed when $m = 0$:

Hypothesis 5.1. *For Π and Π' conjugate self-dual algebraic cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$, there exists Hecke characters χ and χ' of F and F' such that χ and χ' are as in the previous lemma and $L(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) \neq 0$.*

Theorem 5.1. *Let Π and Π' be cuspidal representations of $GL_n(\mathbb{A}_K)$ which are very regular, cohomological, conjugate self-dual, supercuspidal over at least two places of $\mathbb{Q}$ that split in K.*

Let $m \geqslant 0$ be an integer such that $m+n-1$ is critical for $M(\Pi)\otimes M(\Pi')$. If $m=0$, we assume moreover Hypothesis 5.1.

We then have the following equation equivariant under action of G_K:

$$p(m, \Pi_\infty, \Pi'_\infty) Z(\Pi_\infty) Z(\Pi'_\infty) \sim_{KE(\Pi_\infty)E(\Pi'_\infty)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}.$$

Consequently, we have, equivariant under action of G_K,

$$L(m+\frac{1}{2}, \Pi\times\Pi') \sim_{E(\Pi)E(\Pi')} (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{j=1}^{n-1} P^{(j)}(\Pi) \prod_{k=1}^{n-2} P^{(k)}(\Pi').$$

Remark 5.1. *The above result is compatible with the Deligne conjecture and M. Harris's calculation on the Deligne period.*

Recall that the Deligne conjecture predicts

$$L(n-1+m, M(\Pi)\otimes M(\Pi')) \sim c^+(M(\Pi)\otimes M(\Pi')(n-1+m))$$

where $c^+(\cdot)$ is Deligne's period defined in [Del79].

The equation (4.12) of [GH] gives

$$c^+(M(\Pi)\otimes M(\Pi')(n-1+m)) \sim (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{j=1}^{n-1} P_{\leqslant j}(\Pi) \prod_{k=1}^{n-2} P_{\leqslant k}(\Pi')$$

(see chapter 4 of [GH] for the notion). From the discussion after Theorem 4.27 in [GH] we see that $P^{(s)} \sim P_{\leqslant s}$ in our case.

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