RESEARCH PROJECT: GEOMETRY AND TOPOLOGY OF SPHERICAL VARIETIES, GROMOV-WITTEN THEORY, AND BIVARIANT THEORIES

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Contents

1. Introduction

Let G be a connected complex reductive group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. A G-variety is called *spherical* if it is normal and contains a dense B-orbit. Examples include flag varieties, symmetric spaces, and group embeddings (Section 2). For an up-to-date discussion of spherical varieties, and a comprehensive bibliography, see [Ti] and [Pe]. If X is a spherical G-variety, then it has only a finite number of B-orbits and G-orbits. In particular, T acts on X with only finitely many fixed points. These features make spherical varieties particularly well suited for applying GKM theory, a set of tools developed by Goresky-Kottwitz-MacPherson in the topological setting [GKM], and later generalized to equivariant Chow groups [Br2], equivariant algebraic K-theory [VV], and equivariant operational theories [G3], [G4]. Through this method, substantial information about the geometry and topology of a spherical variety can be obtained by restricting one's attention to the induced T-action.

The aim of this project is threefold:

• To further study the equivariant intersection cohomology, Chow groups and K-theory of possibly singular spherical varieties; in particular, those that are T-skeletal (for us a T-skeletal variety is a complete T-variety with only finitely many fixed points and invariant curves). From this viewpoint, projective group embeddings stand out as a very natural class of examples. The main tool for this part is GKM theory.

• To explore the several connections between the cohomology of spherical varieties, Gromov-Witten theory and Mori theory. Many of these relations arise from the quantum to classical principle, as well as explicit descriptions of the cones of effective curves and divisors, and associated Cox rings. This elaborates on ongoing joint work with Nicolas Perrin (Heinrich-Heine-Universität Düsseldorf).

• To relate equivariant operational K-theory to equivariant operational Chow groups via the notion of equivariant multiplicities at isolated fixed points. This would provide an explicit form of the equivariant (operational) Riemann-Roch, and a more geometric description of these bivariant theories, specially in the case of spherical varieties. This builds on ongoing joint work with Dave Anderson (Ohio State University) and Sam Payne (Yale University).

The project presented here seeks to contribute to a better understanding of the geometry and topology of spherical varieties. Previous results by the author in this direction include:

• A description of the equivariant cohomology and equivariant Chow groups of rationally smooth group embeddings [G2], [G5], extending previous work by Danilov [D], De Concini-Procesi [DP-1], [DP-2], Brion [Br3], [Br2], and Strickland [St].

• A description of the equivariant operational Chow groups of *all* complete spherical varieties [G3], generalizing Brion's in the smooth case [Br2]. Also, some criteria for Poincaré duality in (operational) Chow groups are provided in [G5]. These are inspired by similiar results in equivariant cohomology [Br5].

• A localization theorem for equivariant operational K-theory; moreover, the analogue of the Chang-Skjelbred property holds for all complete T-schemes in this setting [G4]. This yields a description of the equivariant operational K-theory of *all* projective group embeddings, extending previous work by Uma [U], in the smooth case, and Anderson-Payne [AP], in the case of toric varieties.

2. Summary of past work

Notation: The ground field $\Bbbk$ is algebraically closed, of characteristic zero. By a scheme we mean a separated scheme of finite type. A variety is a reduced scheme. When discussing cohomology, we take the ground field to be the complex numbers. We denote by T an algebraic torus. For a T-scheme X, we denote by X^T the fixed point subscheme and by $i_T : X^T \to X$ the natural inclusion. If H is a closed subgroup of T, we similarly denote by $i_H : X^H \to X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \to X^H$ for the natural (T-equivariant) inclusion. We denote by Δ the character group of T, and by S the symmetric algebra over $\mathbb{Q}$ of the abelian group Δ. The quotient field of S is denoted by $\mathcal{Q}$. The representation ring of T is denoted $R(T)$.

We denote by G a connected reductive group with Borel subgroup B and maximal torus $T \subset B$. We write W for the Weyl group of (G, T). An irreducible algebraic variety is called an *embedding* of G, or a *group embedding*, if it is a normal $G \times G$-variety containing an open orbit isomorphic to G, where $G \times G$ acts on G by left and right multiplication. When G is a torus, we get back the notion of toric varieties. Group embeddings are spherical $G \times G$-varieties (by the Bruhat decomposition). Affine embeddings of G are nothing but reductive monoids having G as group of units. An affine algebraic monoid is called *reductive* if it is irreducible, normal, and its unit group is a reductive group. See [R6], [Ti]. Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m \to T$ such that $\lim_{t \to 0} \epsilon(t) = 0$. Moreover, the quotient space $\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/\epsilon(\mathbb{G}_m)$ is a projective embedding of the quotient group $G/\epsilon(\mathbb{G}_m)$. Notably, projective embeddings of connected reductive groups are exactly the projectivizations of reductive monoids [R1]. An embedding of G is called simple if it has only one closed $G \times G$-orbit.

2.1. Equivariant cohomology of rationally smooth projective group embeddings.

Cohomology is considered with rational coefficients. Let X be a T-skeletal variety. The main purpose of GKM theory is to identify the image of the pullback $i_T^* : H_T^*(X) \to H_T^*(X^T)$, assuming X is *equivariantly formal* (i.e. X has no cohomology in odd degrees). GKM theory has been mostly applied to smooth projective T-skeletal varieties, because of the Bialynicki-Birula decomposition [B]. Furthermore, the GKM data issued from the fixed points and invariant curves has been explicitly obtained for some interesting subclasses: flag varieties [C], toric varieties [Br2], and *regular embeddings* of reductive groups [Br3]. In the case of singular varieties, GKM theory has been applied to Schubert varieties [C] and to *rationally smooth (projective) group embeddings*, due to the author's work [G1, G2] (a complex variety of pure dimension n is rationally smooth if the local cohomology at any point is the same as the local cohomology of $\mathbb{C}^n$). Rationally smooth group embeddings satisfy Poincaré duality and form a class larger than that of smooth embeddings.

The contents of [G1, G2] are summarized as follows. Let $X = \mathbb{P}_\epsilon(M)$ be projective group embedding. Then X is $T \times T$-skeletal. Using the theory of reductive monoids, we identify explicitly the GKM data of X (i.e. $T \times T$-fixed points, $T \times T$-stable curves and the corresponding characters of $T \times T$) in terms of the combinatorial data of M. Now suppose that X is rationally smooth. Then its GKM data quantifies the sought-after image of $i^*_{T\times T} : H^*_{T\times T}(X) \to H^*_{T\times T}(X^{T\times T})$. Moreover, X comes equipped with both a canonical decomposition into rationally smooth cells and a filtration by equivariantly formal closed subvarieties (this builds on previous work of Renner [R2, R4]). The major result of [G2] describes $H^*_{T\times T}(X)$ and $H^*_{G\times G}(X)$ explicitly in terms of roots, idempotents, and underlying monoid data. We also characterize those rationally smooth projective embeddings X whose equivariant cohomology is obtained via restriction to the associated toric variety $\mathbb{P}_\epsilon(\overline{T})$, where $\overline{T}$ is the Zariski closure of T in M. Such characterization is given in terms of the closed $G \times G$-orbits of X. Our results increase the effectiveness of GKM theory as a tool in embedding theory.

2.2. **On a notion of rational smoothness for Chow groups. Equivariant Poincaré duality.** Chow groups are taken with $\mathbb{Q}$-coefficients. Let X be a T-scheme. Denote by $A^T_*(X)$ the T-equivariant Chow group of X ([EG]). When X is smooth, the group $A^T_*(X)$ admits a natural product by intersection of cycles. In this case, we denote by $A^*_T(X)$ the corresponding ring. The T-equivariant Chow ring of a point identifies to S. Pullback via projection to a point gives a natural map $S \to A^T_*(X)$. In this way, $A^T_*(X)$ becomes a S-module. A fixed point $x \in X$ is called *non-degenerate* if all weights of T in the tangent space T_xX are non-zero. A fixed point $x \in X$ is called *attractive* if there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ and a Zariski neighborhood U of x, such that $\lim_{t\to 0} \lambda(t) \cdot y = x$ for all points y in U. Clearly, attractive fixed points are non-degenerate. Let X be a complete equidimensional T-scheme with finite fixed point set. By the localization theorem [Br2], we can uniquely assign to each $x \in X^T$ a rational function $e_x(X) \in \mathcal{Q}$ (the *T-equivariant multiplicity of X at x*) such that $[X] = \sum_{x\in X^T} e_x(X)[x]$ in $A^T_*(X) \otimes_S \mathcal{Q}$. Then $e_x(X)$ is either zero or a homogeneous rational function of degree $-\dim X$. If $x \in X$ is an attractive fixed point, then $e_x(X) \neq 0$. See [Br2].

Let X be an *affine* T-variety with an attractive fixed point x. It follows from [Br4] that $X = \{y \in X \mid \lim_{t\to 0} \lambda(t)\, y = x_0\}$, for a suitable $\lambda : \mathbb{G}_m \to T$. We say that (X, x) is an attractive cell in this situation. If (X, x) is an attractive cell, then the geometric quotient $\mathbb{P}(X) := [X \setminus \{x\}]/\mathbb{G}_m$ exists; we call it the *link* at x. It is a projective variety [Br4]. In [G1] we studied the links of rationally smooth attractive cells. If (X, x) is rationally smooth, then $\mathbb{P}(X)$ is a rational cohomology complex projective space. Many important results on the equivariant cohomology of T-varieties admitting a paving by rationally smooth cells are given in [G1], for instance, such varieties have no cohomology in odd degrees and their equivariant cohomology is a free S-module. The purpose of [G5] is to provide analogues of these notions, and a version of Poincaré duality, in the context of equivariant Chow groups. This program was started in [G3]. Next is a survey of the main results of [G5].

Definition 2.1. Let (X, x) be an attractive cell of dimension n. We say that X is an *algebraic rational cell* if and only if $A_*(\mathbb{P}(X)) \simeq A_*(\mathbb{P}^{n-1})$.

To illustrate how this "local" notion is well-suited for the study of more general schemes, we recall some notions from [B]. Let X be a complete T-scheme. Assume that X^T is finite, and let $X^T = \{x_1, \ldots, x_m\}$. Pick a generic one-parameter subgroup $\gamma : \mathbb{G}_m \to T$, that is, $X^\gamma = X^T$. For each $i = 1, \ldots, m$, define $W_i(\gamma) := \{x \in X \mid \lim_{t\to 0} \gamma(t)x = x_i\}$. Clearly $X = \bigsqcup W_i(\gamma)$, and each $W_i(\gamma)$ is a locally closed T-invariant subscheme of X. The decomposition $\{W_i(\gamma)\}$ is called the Bialynicki-Birula decomposition, or BB-decomposition, of X (associated to γ), and the $W_i(\gamma)$'s are called *cells* of the decomposition. A BB-decomposition $\{W_i(\gamma)\}$ is called *filtrable*, if there exists a finite increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_m$ of T-invariant closed subschemes of X such that:

a) $X_0 = \emptyset$, $X_m = X$,

b) $X_j \setminus X_{j-1}$ is a cell of the decomposition $\{W_i(\gamma)\}$, for each $j = 1, \ldots, m$.

Definition 2.2. Let X be a complete T-variety. We say that X is $\mathbb{Q}$-*filtrable* if the following two conditions hold: (1) the fixed point set X^T is finite, (2) there exists a generic $\gamma : \mathbb{G}_m \to T$ such that the associated BB-decomposition of X is filtrable and the corresponding cells are algebraic rational cells.

The next result shows that algebraic rational cells are a good substitute for the notion of affine space in the study of equivariant Chow groups of singular varieties.

Theorem 2.3 ([G5])**.** *Let X be a $\mathbb{Q}$-filtrable, complete, T-variety. Then the T-equivariant Chow group $A_*^T(X)$ is a free S-module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $W_i(\gamma)$. Furthermore, the ordinary Chow group $A_*(X)$ is also freely generated by the classes of the cell closures $\overline{W_i(\gamma)}$.*

Remarkably, rationally smooth projective group embeddings are examples of $\mathbb{Q}$-filtrable varieties [G5]. This yields purely algebraic proofs of the topological results of [G2]. More generally, if X is a projective spherical variety such that the associated BB-cells are rationally smooth, then X is $\mathbb{Q}$-filtrable, and so Theorem 2.3 applies. In general, however, the class of $\mathbb{Q}$-filtrable varieties is strictly larger than that of rationally smooth T-varieties [G5].

For a T-scheme X, let $\mathrm{op}A_T^*(X)$ be the (rational) T-equivariant operational Chow group of X (or T-equivariant Chow cohomology of X). See [FM], [EG]. Subsection 2.3 and [G3, Appendix] show that these operational groups are well-adapted to the study of singular spaces.

Theorem 2.4 ([G3])**.** *Let X be a complete G-spherical variety. Then the equivariant Kronecker duality map*

$$\mathcal{K}_T : \mathrm{op}A_T^*(X) \longrightarrow Hom_S(A_*^T(X), S) \qquad \alpha \mapsto (\beta \mapsto \int_X (\beta \cap \alpha))$$

is an isomorphism of S-modules. In particular, $\mathrm{op}A_T^*(X)$ is a torsion free finitely generated S-module.

For the analogue of Theorem 2.4 in the case of ordinary Chow cohomology, see [FMSS]. For a complete equidimensional T-scheme X, there is an equivariant Poincaré duality map:

$$\mathcal{P}_T : \mathrm{op}A_T^k(X) \to A_{n-k}^T(X), \quad z \mapsto z \cap [X],$$

where $n = \dim X$. The article [G5] also addresses the following question: Let X be a complete G-spherical variety. When is $\mathcal{P}_T$ an isomorphism? An answer is given next.

Theorem 2.5 ([G5])**.** *Let X be a complete equidimensional T-variety with finite fixed point set. If all equivariant multiplicities are non-zero (e.g. all fixed points are attractive), then the equivariant Poincaré duality map is injective. If moreover X is G-spherical, $\mathbb{Q}$-filtrable, and $\dim_{\mathbb{Q}} A_k(X) = \dim_{\mathbb{Q}} A_{n-k}(X)$ for all k, then the equivariant Poincaré duality map is also surjective.*

The previous theorem is modeled after the topological results of Brion [Br5]. His results characterize Poincaré duality in equivariant cohomology. My work is inspired by his, and it is a contribution towards characterizing Poincaré duality in intersection theory.

2.3. **Localization for equivariant operational K-theory.** Vezzosi and Vistoli [VV] established GKM theory for the equivariant algebraic K-theory of smooth T-schemes. The purpose of [G4] is to establish a version of GKM-theory for the equivariant K-theory of *singular* T-schemes. Let X be a T-scheme. Let $K^0_T(X)$ denote the Grothendieck ring of equivariant vector bundles on X. Let $K_0^T(X)$ denote the Grothendieck group of equivariant coherent sheaves on X. This is a module for the ring $K^0_T(X)$. If X is smooth, then $K^0_T(X) \simeq K_0^T(X)$, see [Th1]. The T-equivariant K-theory of a point identifies to $R(T)$, so pullback via projection to a point gives a natural map $R(T) \to K^0_T(X)$. In this way, $K^0_T(X)$ becomes an $R(T)$-algebra and $K_0^T(X)$ an $R(T)$-module.

In general, especially for singular varieties, the K-theory groups are difficult to compute. In [FM] Fulton-MacPherson devised a machinery that produces a "cohomology" theory out of a homology theory. This "cohomology" has all the formal properties one could hope for, and it is well suited for the study of singular schemes. Taking as input $K_0^T(-)$, Anderson and Payne [AP] obtain a theory that is very well suited for computations. The T-equivariant operational K-theory ring of X, denoted $\mathrm{op}K^0_T(X)$, is an associative commutative ring with unit, and the functor $\mathrm{op}K^0_T(-)$ is contravariant. Roughly speaking, elements of $\mathrm{op}K^0_T(X)$ can be thought of as certain operators defined on the structure sheaves $\mathcal{O}_{\overline{Tx}}$ of the T-orbit closures $\overline{Tx} \subseteq X$ (and their equivariant resolutions), see [AP]. There are natural $R(T)$-linear maps $K^0_T(X) \to \mathrm{op}K^0_T(X)$ and $\mathrm{op}K^0_T(X) \to K_0^T(X)$. Put together, they provide a factorization of the canonical map $K^0_T(X) \to K_0^T(X)$. If X is smooth, these maps are all isomorphisms [AP].

The article [G4] is motivated by the following situation. Let X be a (complete) T-skeletal variety. Assume, for simplicity, that each T-invariant irreducible curve has exactly two fixed points (e.g. X is equivariantly embedded in a normal T-variety). In this setting it is possible to define a ring $PE_T(X)$ of *piecewise exponential* functions. Indeed, let $K^0_T(X^T) = \oplus_{x\in X^T} R_x$, where R_x is a copy of $R(T)$. We then define $PE_T(X)$ as the subalgebra of $K^0_T(X^T)$ given by

$$PE_T(X) = \{(f_1, \dots, f_m) \in \oplus_{x\in X^T} R_x \mid f_i \equiv f_j \mod 1 - e^{-\chi_{i,j}}\}$$

where x_i and x_j are the two distinct fixed points in the closure of the one-dimensional T-orbit $C_{i,j}$, and T acts on $C_{i,j}$ via the character $\chi_{i,j}$. This character is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite). If X is smooth, then GKM theory states that $i_T^* : K^0_T(X) \to K^0_T(X^T)$ induces an isomorphism between $K^0_T(X)$ and $PE_T(X)$. Moreover, $K^0_T(X)$ is a free $R(T)$-module of rank $|X^T|$.

Thus far, it is clear that to any complete T-skeletal variety X one can associate the ring $PE_T(X)$, regardless of whether X is smooth or not. (In fact, if X is a projective G-embedding, then $PE_{T\times T}(X)$ has been explicitly identified in [G2].) Nonetheless, when X is singular, $PE_T(X)$ does not always describe $K^0_T(X)$. This phenomena yields some natural *questions:* Let X be a T-skeletal variety. What kind of information does $PE_T(X)$ encode? If not equivariant K-theory, is it still reasonable to expect that $PE_T(X)$ encodes certain geometric information that is *common* to all possible T-equivariant resolution of singularities of X? The work of Payne [P] and Anderson-Payne [AP], inspired in turn by the works of Fulton-MacPherson-Sottile-Sturmfels [FMSS] and Totaro [To], gives a positive answer to these questions *when X is a toric variety.* Namely, the GKM data (i.e. $PE_T(X)$) of a toric variety encodes all the information needed to reconstruct $\mathrm{op}K^0_T(X)$. This positive result motivates [G4]. There I show that Anderson-Payne's assertion on toric varieties holds more generally for *all* T-skeletal varieties. In fact, I obtain the following extensions of [VV]:

Localization theorem [G4]**.** *Let X be a T-scheme. If X is complete, then the pullback $i_T^* : \mathrm{op}K^0_T(X) \to \mathrm{op}K^0_T(X^T)$ is injective, and it becomes surjective over the quotient field of $R(T)$.*

Next is a criteria that dates back to the work of Chang-Skjelbred [CS] in equivariant cohomology.

CS property [G4]**.** *Let X be a T-scheme. If X is complete, then the image of i_T^* equals the intersection of the images of $i_{T,H}^* : \mathrm{op}K_T^0(X^H) \to \mathrm{op}K_T^0(X^T)$, where H runs over all subtori of codimension one in T.*

In particular, the CS property yields:

GKM theorem [G4]**.** If X is a T-skeletal variety, then $\mathrm{op}K_T^0(X) \simeq PE_T(X)$.

Together with the combinatorial results of [G2], this extends Anderson and Payne's work on toric varieties to *all* projective group embeddings [G4, Theorems 6.2, 6.4]. It is worth noting that the previous results easily adapt to equivariant operational Chow groups. This is used in [G3, Section 4] to describe the equivariant operational Chow rings of *all* complete, possibly singular, spherical varieties. In particular, if X is an equivariantly formal complete G-spherical variety, then there is a natural isomorphism between $\mathrm{op}A_T^*(X)$ and $H_T^*(X)$, as their images in $\mathrm{op}A_T^*(X^T) \simeq H_T^*(X^T)$ are canonically isomorphic. Thus the equivariant operational Chow groups of rationally smooth spherical varieties are free S-modules.

3. Research Proposal

3.1. **Relation between equivariant and ordinary operational theories.** Let X be a T-scheme. Let $\mathrm{op}A_T^*(X)$ be the T-equivariant Chow cohomology of X. Let $\mathrm{op}A^*(X)$ be the ordinary Chow cohomology of X. There is a canonical map $i^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A^*(X)$, see e.g. [EG]. But this map, unlike its counterpart in equivariant Chow groups, is not surjective in general, and its kernel is not necessarily generated in degree one, not even for toric varieties [KP]. This becomes an issue when translating results from equivariant to non-equivariant Chow cohomology. The same issue shows up in equivariant operational K-theory.

My goal is to investigate the relation between equivariant and ordinary operational theories, *via* equivariant multiplicities (Sections 2.2 and 4.1). This approach was already successfully taken in the case when X is a complete toric variety, by work of Katz and Payne [KP], or a $\mathbb{Q}$-filtrable spherical variety, by the author's work [G3, Corollary 3.9]. In this project, I consider the case of an arbitrary complete G-spherical variety X. Under this assumption, X satisfies T-equivariant Kronecker duality (Theorem 2.4). Thus elements $c \in \mathrm{op}A_T^*(X)$ can be thought of as S-linear functionals on $A_*^T(X)$; moreover, they are completely determined by their values $c(\beta) := \int_X c \cap \beta$ on B-invariant cycles $\beta \in A_*^T(X)$, cf. [Br2, Prop. 6.1]. By the localization theorem, $c(\beta)$ can be computed in terms of equivariant multiplicities. We should assess how such multiplicities come together to define $i^*(c) \in \mathrm{op}A^*(X) \simeq \mathrm{Hom}_{\mathbb{Q}}(A_*(X), \mathbb{Q})$. When X is a toric variety, the authors of [KP] obtain explicit formulas for $i^*(c)$ in terms of equivariant multiplicites. It would be interesting to obtain such formulas for more general, possibly singular, projective embeddings of G. These formulas seem to depend only on the combinatorial invariants of the underlying monoids. A related question is: to what extent one can recover the combinatorial description of $\mathrm{op}A^*(X)$ in [FMSS], from the GKM-like description of $\mathrm{op}A_T^*(X)$ in [G3, Theorem 4.8]? These questions can also be elaborated/answered in the context of equivariant operational K-theory, for which we are developing a notion of equivariant multiplicities at isolated fixed points (Section 4.1).

3.2. **Geometric interpretation of equivariant operational theories.** Let X be a complete equidimensional T-variety with isolated fixed points. Assume that all equivariant multiplicities are nonzero (e.g. all fixed points are attractive). Then the equivariant Poincaré duality map $\mathcal{P}_T : \mathrm{op}A_T^k(X) \to A_{n-k}^T(X)$, $z \mapsto z \cap [X]$, is injective (Theorem 2.5). Thus $\mathrm{op}A_T^*(X) \subseteq A_*^T(X)$. In particular, this holds for Schubert varieties, toric varieties and simple projective group embeddings (because in these cases all fixed points are known to be attractive).

My plan is to describe $\mathrm{op}A^*_T(X)$ as a subgroup of $A^T_*(X)$ in terms of T-invariant cycles on X, especially in the case of simple group embeddings. Notice that $\mathrm{op}A^*_T(X)$ carries an additional ring structure. So a related task is to assess the effect of this "abstract" product on the associated (geometric) cycles. Solutions to these problems would yield a geometric interpretation of operational Chow groups, at least in the three cases mentioned above, and those where Poincaré duality holds (Theorem 2.5).

Also, I would like to understand the action of $\mathrm{op}A^*_T(X)$ on $A^T_*(X)$ for T-skeletal spherical varieties X, in light of Brion's description of the intersection pairing between curves and divisors on spherical varieties [Br1]. This should also provide a geometric interpretation of the coefficients arising from the cap and cup product formulas of [FMSS] and [G3], and generators and relations for the algebra $\mathrm{op}A^*_T(X)$, the dual of $A^T_*(X)$.

These problems can be formulated in the setting of equivariant operational K-theory. Indeed, in the situation above, $\mathrm{op}K^0_T(X) \subseteq K^T_*(X)$, cf. Section 4.1. Another one of my goals is to find explicitly $\mathrm{op}K^0_T(X)$ as a submodule of $K^T_*(X)$, and get a geometric interpretation of the former in terms of certain equivariant coherent sheaves on X.

3.3. **Equivariant intersection cohomology of projective group embeddings.** *Intersection cohomology* is a powerful tool for the study of singular algebraic varieties. It was developed by Goresky and MacPherson in [GM1, GM2]. Later on, Brylinski [Bry] and Joshua [Jo] developed *Equivariant intersection cohomology*, a variant of intersection cohomology that incorporates the group action into its definition. Let X be a G-variety. The *equivariant intersection cohomology* $IH^*_G(X)$, for the middle perversity, is defined to be the hypercohomology $\mathbb{H}^*(EG \times_G X; IC^G(X))$, where $IC^G(X)$ is the equivariant cohomology complex of X, and EG is the classifying space of G. See [Bry] and [Jo] for details. $IH^*_G(X)$ is a graded module over $H^*(BG)$, the equivariant cohomology ring of a point. If X is rationally smooth, then both its singular cohomology $H^*(X)$ and its intersection cohomology $IH^*(X)$ agree. There is also a sheaf version of GKM theory for equivariant intersection cohomology, due to the work of Braden and MacPherson [BM].

Our goal is to write down, explicitly and combinatorially, the equivariant intersection cohomology of projective group embeddings, using the GKM theory of [BM], without any assumptions on rational smoothness, so as to allow more general singularities. This is a further step in the program started in [G2]. The outcome should reveal certain aspects of the geometry of projective group embeddings which are not captured by equivariant bivariant theories.

Let $X = \mathbb{P}_\epsilon(M)$ be a projective group embedding (refer to Section 2 for notation). As mentioned before, one can find explicitly the *GKM graph* of X, i.e. the finite graph build from the $T \times T$-fixed points and $T \times T$-invariant curves of X, regardless of whether or not X is rationally smooth [G2]. Next, consider $IH^*_{T\times T}(X)$, the $T \times T$-equivariant intersection cohomology of X. The results of [Bry] and [GKM] imply that $X = \mathbb{P}_\epsilon(M)$ is equivariantly formal for intersection cohomology, that is, $IH^*_{T\times T}(X)$ is a free $H^*_{T\times T}(pt)$-module. Using the results of [BM] one expects the isomorphism

$$IH^*_{T\times T}(X) \simeq \Gamma(X),$$

where $\Gamma(X)$ is the ring of global sections of a *GKM sheaf* Γ defined on the GKM graph of X. The problem here is to fully understand Γ in terms of the local structure of the underlying monoid M. The structure of the various stalks Γ_x, for $x \in X$, needs to be determined as well. I anticipate that a thorough analysis of the results of [G2], [Br7], [R3] and [BM] should shed some light into the innermost structure of the sheaf Γ. The work of P. Fiebig ([F], [J]) will also play a crucial role in this program, specially on its applications to representation theory. For toric varieties $\mathcal{Y}$, the ring $IH^*_T(\mathcal{Y})$ is known, by work of Barthel-Brasselet-Fieseler-Kaup [BBFK]. Our project offers a generalization of their results, and complements our previous work [G3], [G4] and [G5]. This is our first step towards describing explicitly the equivariant intersection cohomology of singular spherical varieties.

3.4. **The cone of projective group embeddings with the same Cox ring.** Let X be a normal irreducible projective variety whose divisor class group $\mathrm{Cl}(X)$ is finitely generated (e.g. X is a projective spherical G-variety). The *Cox ring* of X is the $\mathrm{Cl}(X)$-graded $\Bbbk$-algebra

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

See [EKW], [H], [Co] for details. The ring $\mathcal{R}(X)$ is a unique factorization domain [EKW]. Other terminologies are *total coordinate ring* [EKW] and *factorial hull* [R2]. $\mathbb{Q}$-factorial projective varieties with finitely generated Cox ring are characterized in [HK] using Mori theory. If X is a toric variety, then $\mathcal{R}(X)$ is a polynomial ring [Co]. More generally, if X is a spherical G-variety, then $\mathcal{R}(X)$ is a finitely generated $\Bbbk$-algebra, and it carries a natural grading by the classes of effective divisors, see e.g. [Ti]. The Cox ring of spherical varieties has been described by Brion [Br6]. Another description is given in recent work of Gagliardi [Ga].

My interest in Cox rings comes from the following situation, first considered by Renner [R2]. Let $X = \mathbb{P}(M)$ be a projective group embedding, and assume that M is semisimple (i.e. G has a one-dimensional center $Z(G)$). In [R2], Renner shows that $\mathcal{R}(X)$ is determined by a certain arrangement $\mathbb{H}$ of hyperplanes in the space of rational characters $\Delta(T)\otimes\mathbb{Q}$ of a maximal torus T of $G_0 = G/Z(G)$. If G_0 is simply connected and X its wonderful compactification, then $\mathcal{R}(X)$ is the coordinate ring of Vinberg's enveloping monoid $\mathrm{Env}(G)$. In general, associated with X is a certain *cone* $\mathcal{H} \subseteq \mathrm{Cl}(X)$. Each $\delta \in \mathcal{H}$ corresponds to a semisimple monoid M_δ with $\mathcal{R}(\mathbb{P}(M_\delta)) = \mathcal{R}(X)$, that is, $\mathbb{P}(M_\delta)$ and X differ only by $G\times G$-orbits of codimension two or more. In particular, Renner calculates $\mathcal{H}$ explicitly in the case when X is the wonderful compactification of G_0. The cone $\mathcal{H}$ seems to be what is called the *moving cone* in [HK].

My goal is to further investigate the interesting relationship between the results of [R2] and those of [HK]. In particular, I plan to calculate explicitly the cone $\mathcal{H}$ in other cases of interest, namely, those when $X = \mathbb{P}(M)$ is simple and rationally smooth. Recall that a simple embedding is uniquely characterized by its closed orbit [R1]. Let $X = \mathbb{P}(M)$ be a simple embedding with unique closed orbit $G/P_J \times G/P_J^-$; here J is a subset of $\mathfrak{S}$ (the simple reflections of W), and P_J, P_J^- are opposite parabolic subgroups associated to J. In this case, we say that X is of *type* J, and write X_J instead of X. In [R5] Renner gives a complete list of all J's that yield rationally smooth simple embeddings. If $J = \emptyset$, then $X_\emptyset$ is the wonderful compactification of G_0. If X_J is rationally smooth, then $\mathrm{Pic}(X_J)_\mathbb{Q} \simeq \mathrm{Cl}(X_J)_\mathbb{Q}$ (this follows from Poincaré duality, Theorem 2.5). Thus the number of $G\times G$-stable divisors in X_J is exactly $\mathfrak{S}\setminus J$. The aim is to describe the cone $\mathcal{H}_J$ of such X_J in terms of $\mathfrak{S}\setminus J$, *and* to make more precise the description of $\mathcal{R}(X_J)$. By [Br6] we can think of the various $\mathcal{H}_J$ as being "deformations" of $\mathcal{H}_\emptyset$. Nevertheless, I would like to obtain a more precise geometric picture. For instance, in type A_n, choosing $J = \{s_2, \ldots, s_{n-1}\}$, where s_i is a simple permutation, gives $X = \mathbb{P}^{(n+1)^2-1}$, and thus $\mathcal{R}(X) \simeq \Bbbk[M_{n+1}]$, a polynomial ring in $(n+1)^2$ variables. So how do we explain the deformation of $\Bbbk[Env(GL_{n+1})]$ to $\Bbbk[M_{n+1}]$ in terms of the cones $\mathcal{H}_J$, $\mathcal{H}_\emptyset$ and the corresponding hyperplane arrangements $\mathbb{H}_J$, $\mathbb{H}_\emptyset$? Does it amount to "wall-crossing"?

4. Current Related Work

4.1. **Equivariant operational Riemann-Roch and equivariant multiplicities in K-theory (joint work with Dave Anderson and Sam Payne).** Let X be a complete T-scheme with finitely many fixed points. In virtue of Thomason's localization theorem for

$K_0^T(-)$ [Th2, Theorem 2.1], the following identity holds in $\mathcal{Q}(\Delta)$, the quotient field of $\mathbb{Z}[\Delta]$:

$$[\mathcal{O}_X] = \sum_{x \in X^T} EK(x, X)[\mathcal{O}_x],$$

where the various $EK(x, X)$ are (possibly zero) rational functions on $(\Delta_{\mathbb{Q}})^*$. Following the nomenclature of [Br2, Section 4.2] we call $EK(x, X)$ the *K-theoretic equivariant multiplicity of X at x*. If all the $EK(x, X)$ are non-zero, then the natural map $\mathrm{op}K_T^0(X) \to K_0^T(X)$ is injective (cf. [G5, Lemma 6.6]). We anticipate that the $EK(x, X)$'s are non-zero whenever x is an attractive fixed point of X, because, in that case, $EK(x, X)$ is related to the formal character of $k[X_x]$, where X_x is the unique open affine T-stable neighborhood of x (cf. [Br2]). The notion of K-theoretic equivariant multiplicity at attractive fixed points is already present in the study of flag varieties (see e.g. [BBM]). For complete toric varieties, Schubert varieties, and simple projective group embeddings, our claim would imply that the natural map $\mathrm{op}K_T^0(X) \to K_0^T(X)$ is always injective (this deeply contrasts with the behaviour of the map $K_T^0(X) \to K_0^T(X)$, whose kernel could be rather large, cf. [AP]). In current collaboration with Dave Anderson (Ohio State University) and Sam Payne (Yale University), we are developing these ideas, and exploring the behaviour of K-theoretic equivariant multiplicities under proper pushforward. We are also assessing the effect that the (operational) Riemann-Roch morphism has on equivariant multiplicites. Further applications include (i) to understand the behaviour of the forgetful maps $K_T^0(X) \to K^0(X)$ and $\mathrm{op}K_T^0(X) \to \mathrm{op}K(X)$ via equivariant multiplicities, (ii) to describe the equivariant operational K-theory of T-varieties where the fixed point subscheme has positive dimension.

4.2. **Computation of structure constants of the equivariant cohomology and/or K-theory of rationally smooth projective group.** Let $X = \mathbb{P}_\epsilon(M)$ be a projective group embedding. Assuming X is rationally smooth, [G2] gives a precise combinatorial description of $H^*_{G\times G}(X)$ in terms of finite combinatorial invariants of M. An explicit algebra presentation of $H^*_{G\times G}(X)$, in terms of generators and relations (Schubert calculus), is still an open problem. Some partial results have been obtained by Strickland [St] and Uma [U], for the case of regular embeddings. However, such polynomial generators are not known for more general (e.g. rationally smooth) embeddings. Our current project is to compute, explicitly, generators and relations for $H^*_{G\times G}(X)$ using the abstract tools developed in [G2], [G5] and [R5]. The outcome of this task has many concrete applications in Gromov-Witten theory, specifically, in the description of the quantum cohomology of projective group embeddings (Subsection 4.4). As a first step, we deal with the case of simple group embeddings. Examples to keep in mind are the wonderful compactification X' of a semisimple group G', with unique closed orbit $G'/B \times G'/B^-$, and the projective space $\mathbb{P}^{(n+1)^2-1}$, a simple embedding of $PSL(n+1, \mathbb{C})$, with unique closed orbit $\mathbb{P}^n \times \mathbb{P}^n$.

Let $X = \mathbb{P}_\epsilon(M)$ be a simple embedding of type J (cf. Section 3.4). For convenience, denote by $\mathcal{O}$ its unique closed orbit $G/P_J \times G/P_J^-$. Now assume that X is rationally smooth (for a complete list of all J's that yield rationally smooth simple embeddings, see [R5]). From [G2] it follows that $H^*_{G\times G}(X)$ is a subring of $H^*_{G\times G}(\mathcal{O})$. It is well-known that $H^*_{G\times G}(\mathcal{O})$ is a polynomial ring. The work of Lascoux and Schützenberger [L-S] provides a finite list of polynomials $\{\Phi_{w,v}\}$, indexed over $W \times W$, that generate $H^*_{G\times G}(\mathcal{O})$. These polynomials are the *Schubert polynomials* and play a crucial role in the representation theory of reductive groups. However, $H^*_{G\times G}(X)$ is a proper subring of $H^*_{G\times G}(\mathcal{O})$ subject to certain non-trivial relations. That is, a suitable "deformation" of these polynomials is needed. For wonderful embeddings, the results of Strickland [St] and Uma [U] yield the right correction factor. Moreover, one can read off the sought-after polynomials from a pair of Bruhat graphs. The natural question is: what about other simple embeddings? Except for a few cases, this question has remained unsolved. We plan to solve it as follows:

(1) Construction of generalized Schubert polynomials $\{\Psi_{v,w}\}$: these polynomials will be indexed by a finite set $W^J \times W^J$, a quotient of $W \times W$, which depends on J. When $J = \emptyset$, then $W^J = W$. If J is not empty, then W^J is no longer a group. Using the results of [R3], one can define a descent set $D^J(v)$, for each $v \in W^J$. Our polynomials will include a correction factor coming from the various $D^J(v)$.
(2) Construction of combined divided differences operators $\{\partial_s\}$, for each $s \in W^J$. This is based on ideas of Lascoux and Newton [L]. Using such operators we construct our polynomials $\{\Psi_{v,w}\}$ by looking at the product of two finite graphs, a procedure that mimics Schubert calculus on flag varieties.
(3) Finally, we provide a neat geometric interpretation of our generalized Schubert polynomials as equivariant pushforwards of characteristic classes associated to the boundary divisors of X. This is possible due to Section 2.2: equivariant (co)homology and equivariant Chow (co)homology agree on X [G5]. This yields the desired geometric picture of the algebra $H^*_{G\times G}(X)$.

These results are being collected in my preprint "Group embeddings, divided difference operators and Schubert calculus".

4.3. **Quantum cohomology of spherical varieties (joint work with Nicolas Perrin).** Let X be a smooth complex projective variety and let $d \in H_2(X, \mathbb{Z})$. Then there exists an algebraic variety $\mathcal{M}_{g,d,n}(X)$ parametrizing the set of all stable maps from a genus g curve C, with n marked points $(p_i)_{i\in[1,n]}$, to X. This variety is endowed with many combinatorial properties, in particular, there is an evaluation map $\mathrm{ev}_i : \mathcal{M}_{g,d,n}(X) \to X$, for each marked point p_i. For cohomology classes $(\alpha_i)_{i\in[1,n]}$, one can define the Gromov-Witten invariant

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle_{g,d} = \int_{[\mathcal{M}_{g,d,n}(X)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i)$$

These invariants, for genus 0, built a quantum potential

$$\Phi_q(\gamma) = \sum_{d,n\geq 3} \frac{q^d}{n!} \langle \gamma^n \rangle_{0,d},$$

where $\gamma \in H^*(X, \mathbb{C})$. By work of Ruan and Tian (see e.g. [M]), this potential satisfies the DWVV equation, which means that the quantum product $\star$ defined by $(\alpha \star \alpha', \alpha'') = \partial_{\alpha,\alpha',\alpha''}\Phi(\gamma)$, where $(\, , \,)$ is the usual intersection pairing, is associative. The *big quantum cohomology ring* is the ring obtained from the above quantum product. It is a deformation of the usual cohomology ring. Note that, for defining quantum cohomology, we only consider rational curves (i.e. we take into account Gromov-Witten invariants of genus 0). Remarkably, Kontsevich and Manin ([KM]) showed that if $H^*(X, \mathbb{C})$ is generated in degree 2, then one can *reconstruct* from the Gromov-Witten invariants involving only $n = 3$ points all the Gromov-Witten invariants involving more points. This reduction technique plays a crucial role in applications. The quantum cohomology ring obtained by considering only 3-points Gromov-Witten invariants is called the *small quantum cohomology ring* of X, denoted $QH^*(X)$. Quantum cohomology is intensively studied because of its applications to enumerative problems in algebraic geometry. See [K], [M].

In current collaboration with Nicolas Perrin (Heinrich-Heine-Universität Düsseldorf), we plan to describe the (equivariant) quantum cohomology of certain spherical varieties (e.g., group embeddings, symmetric spaces) and provide explicit formulas for the underlying Schubert calculus. Some important results in this direction have been obtained by Chaput, Manivel and Perrin for the case of minuscule homogeneous spaces and rational homogeneous spaces [CMP08, CMP07, CMP10, CMP09]. A crucial ingredient here is the *quantum to classical principle* [CMP08], [BKT]. Loosely speaking, this principle allows to compute $QH^*(X)$, where X is a sufficiently nice space, in terms of the singular cohomology of a suitable replacement

Y_d, which, in many cases, turns out to be an *spherical variety*. Using the results from Section 2 and Subsection 4.2, we anticipate a GKM presentation of $H_T^*(Y_d)$ and $H^*(Y_d)$. Such description would be a fundamental step towards implementing a Schubert calculus in $QH^*(X)$. Indeed, we aim at providing (geometric) bases for these rings and formulas to multiply them.

4.4. **Spherical varieties, rational smoothness and GKM-theory.** Let X be a spherical G-variety. Two interesting problems are:

(1) *Classification of rationally smooth spherical varieties.* Spherical varieties are classified combinatorially, in terms of colors and fans [Ti]. The subclass of rationally smooth group embeddings has been characterized combinatorially by Renner ([R4], [R5]). By appropriately interpreting Renner's results in the larger setup of spherical varieties, we would like to provide a combinatorial characterization of rational smoothness for spherical varieties. It is also relevant to determine the cases where they admit a paving by rationally smooth cells. In such cases, equivariant (intersection) cohomology and equivariant Chow cohomology agree [G5].

(2) *Characterization of T-skeletal spherical varieties and description of the associated GKM-data.* Although X^T is finite, in general X need not be T-skeletal, i.e., there might be a codimension one subtorus $H \subset T$ such that X^H has dimension two (this is the largest dimension X^H could have [Br2, Prop. 7.1]). Is it possible to read off the property of being T-skeletal from any combinatorial invariant of X? The subclass of T-skeletal spherical varieties encompasses, but is not limited to, flag varieties [C], group embeddings [G2] and smooth spherical varieties of minimal rank [BJ-2]. Mainly, we are interested in the following concrete setup. Let X be a projective G-embedding. Then $X = \mathbb{P}_\epsilon(M)$, for certain reductive monoid M. Let $\sigma : G \to G$ be an involution, and let G act on X via $g \cdot x \mapsto (g, \sigma(g)) \cdot x$. By construction, $\mathcal{O} = G \cdot [1] = G/G^\sigma$ is a symmetric space contained in X, where $G^\sigma = \{g \in G \,|\, \sigma(g) = g\}$. Taking Y_σ to be the Zariski closure of $\mathcal{O}$ in X, we get a spherical variety. So natural questions are: how are the T-curves in Y_σ related to the combinatorial data of M?, how many are there?, do they correspond to "twistings" of the rank-two elements of the Renner monoid of M? what are the associated characters? We anticipate that a generalization of the techniques of Brion-Joshua [BJ-2], Renner [R3] and the author [G2] yield the desired answers. Recent work of Can-Howe-Renner [CHR] is also relevant here. Note that when G is semisimple of adjoint type and X its wonderful compactification, this construction dates back to the work of Littelmann-Procesi [LP].

4.5. **Equivariant topological K-theory and rational smoothness.** Let $T = (\mathbb{C}^*)^n$ be an algebraic torus and let $\mathbf{T} = (S^1)^n$ be its compact subtorus. Consider a compact $\mathbf{T}$-space X. It is customary to define $K_{\mathbf{T}}(X)$, the equivariant topological K-theory of X, as the Grothendieck ring of $\mathbf{T}$-equivariant complex vector bundles over X. Similarly, if X is a locally compact $\mathbf{T}$-space, one defines $K_{\mathbf{T}}(X) := \widetilde{K_{\mathbf{T}}}(X^+)$, the reduced $\mathbf{T}$-equivariant K-theory of the one-point compactification of X, see [AtSe]. Denote by $K^*_{\mathbf{T}}$ the equivariant K-theory of a point.

In this setting, following a suggestion of Graham [Gr], we obtain

Theorem 4.1. *Let X be a compact T-space with finitely many fixed points. Then X is equivariantly formal if and only if $K^*(X) \otimes \mathbb{Q} \simeq K^*_{\mathbf{T}}(X) \otimes_{K^*_{\mathbf{T}}} \mathbb{Q}$.*

In [G1] we studied a special class of equivariantly formal spaces, namely, *topologically $\mathbb{Q}$-filtrable varieties* (the topological counterpart of Definition 2.2). A projective T-variety X is topologically $\mathbb{Q}$-filtrable if (i) X^T is finite, (ii) the associated BB-decomposition (for some generic $\lambda : \mathbb{G}_m \to T$) is filtrable and (iii) the corresponding BB-cells W_i are rationally smooth. The BB-cells W_i of a topologically $\mathbb{Q}$-filtrable variety are called *rational cells* [G1].

Let X be a topologically $\mathbb{Q}$-filtrable variety. Motivated by Theorem 4.1 and [G1] we would like to determine $K^*_{\mathbf{T}}(X) \otimes \mathbb{Q}$. In particular, is there a GKM-description of $K^*_{\mathbf{T}}(X)$? If X is smooth, this description can be found in [At], [VV] and [U]. However, very little is known when X is singular. If T is one dimensional, then it is possible to reconstruct $K^*_{\mathbf{T}}(X)$ from the K-theory of the strata under certain conditions.

Theorem 4.2. *Notation being as above, suppose $T = \mathbb{C}^*$. Then $K^*_{\mathbf{T}}(W_i) \otimes \mathbb{Q} \simeq K^*_{\mathbf{T}} \otimes \mathbb{Q}$ if and only if $(W_i)^\alpha$ is rationally smooth for every $\alpha \in T$ of finite order, where $(W_i)^\alpha$ is the subvariety fixed by α. Moreover, if $K^*_{\mathbf{T}}(W_i) \otimes \mathbb{Q} \simeq K^*_{\mathbf{T}} \otimes \mathbb{Q}$ for every i, then $K^*_{\mathbf{T}}(X)$ is a free $K^*_{\mathbf{T}}$-module of rank $|X^T|$.*

This can be thought of as a generalization of the equivariant Bott periodicity [At]. Similar statements hold for higher dimensional tori. The main difficulty here is the presence of singularities, for even when X has no cohomology in odd degrees, X^α might have non-trivial odd cohomology. This is the crucial reason why we cannot simply apply the results of [RK] or [VV], for they deal with smooth T-varieties, and in those cases the loci of points fixed by a finite group is also smooth. On the other hand, the conditions of Theorem 4.2 are too rigid, and do not hold in generic situations. For example, letting $T = \mathbb{C}^*$ act on $W = \{z^2 = xy\}$ via $t \cdot (x, y, z) = (t^2x, t^4y, t^3z)$ gives $W^{\langle -1 \rangle}$ as the union of two coordinate axes, a reducible space which is by no means rationally smooth. In fact, one obtains $K^*_T(W) \simeq R[T] \oplus R[\mathbb{Z}_2]$. A detailed study of this fundamental example yields

Theorem 4.3. *Let (X, x_0) be a rational cell. Let $S \subset T$ be a generic one parameter subgroup of T. Assume that S acts freely on $X - \{x_0\}$. Then*

$$K^*_S(X) \otimes \mathbb{C} \simeq \mathbb{C}[q, q^{-1}].$$

As the previous theorems illustrate, a good control of the loci of points fixed by a finite group is required. We turn to the class of group embeddings for a better understanding of this phenomenon. Let M and N be two reductive monoids. Following Renner [R2], we write $M \sim_0 N$ if there is a reductive monoid L and finite dominant morphisms $L \to M$ and $L \to N$ of algebraic monoids. It is easy to check that this is indeed an equivalent relation. Moreover, by [R2], M is rationally smooth if and only if $M \sim_0 \prod_i M_{n_i}(\mathbb{C})$. Now let $X = \mathbb{P}_\epsilon(M)$ be a rationally smooth group embedding. Then the associated BB-cells, say C_r, are reductive monoids satisfying $C_r \sim_0 \prod_i M_{n_i}(\mathbb{C})$ (*the monoid cells*). In ongoing work, we are studying the stability of equivariant K-theory under finite surjective morphisms of monoids. This is the key step to concretely describe $K^*_{\mathbf{T}\times\mathbf{T}}(C_r)$ in terms of the representation rings of the non-trivial isotropy subgroups. The corresponding problem in Chow groups has been solved in [G5]. Ultimately, we anticipate a nice combinatorial description of $K^*_{\mathbf{T}\times\mathbf{T}}(X)$, after inverting certain primes associated to the non-trivial isotropy subgroups of the monoid cells, and the fibre-cardinality of the finite dominant morphisms involved. A tightly related problem is to understand the $T \times T$-equivariant K-theory of coherent sheaves on X. We anticipate many relations with equivariant operational K-theory, via Kronecker duality [AP] and equivariant multiplicities (Section 4.1). M. Brion has pointed out to the author that determining the structure of the fixed varieties $(G/P \times G/P^-)^{(h,s)}$, where (h, s) are finite-order elements in $T \times T$, and P, P^- form a pair of opposite parabolic subgroups, is of particular interest, specially on the relations to Theorems 4.2 and 4.3. This would not only shed some light into the structure of $X^{(h,s)}$, when $X = \mathbb{P}_\epsilon(M)$, but also provide an interesting setup for applications.

My results are being collected in the preprint "Equivariant K-theory, rational smoothness and projective embeddings of algebraic groups". In future work, and using the notion of regular differential forms [Br5], I would like to explore to what extent $K^*_{\mathbf{T}\times\mathbf{T}}(X)$ is isomorphic to the ring of global sections of an equivariant De Rham sheaf defined on the GKM-graph of X (compare [F]).

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