FINAL REPORT ON RESEARCH VISIT TO MPIM

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Contents

1. Introduction and Motivation

Let G be a connected reductive group defined over an algebraically closed field k of characteristic zero. Let B be a Borel subgroup of G. A normal G-variety is called *spherical* if it contains a dense B-orbit. Examples include flag varieties, symmetric spaces, and group embeddings. Recall that a normal irreducible variety X is called an *embedding* of G, or a *group embedding*, if X is a $G \times G$-variety containing an open orbit isomorphic to G. Due to the Bruhat decomposition, group embeddings are spherical $G \times G$-varieties. Substantial information about the topology of a group embedding can be obtained by restricting one's attention to the induced action of a maximal torus $T \subset B$ of G. When $G = B = T$, we get back the notion of toric varieties.

Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m^* \to T$, with image Z, such that $\lim_{t \to 0} \epsilon(t) = 0$. Moreover, the quotient space

$$\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z$$

is a normal projective variety on which $G \times G$ acts via $(g, h) \cdot [x] = [gxh^{-1}]$. Hence, $\mathbb{P}_\epsilon(M)$ is a normal projective embedding of the quotient group G/Z. These varieties were introduced by Renner in his study of algebraic monoids ([R2], [R3]). Notably, normal projective embeddings of connected reductive groups are exactly the projectivizations of algebraic monoids [T].

Goresky, Kottwitz and MacPherson [GKM] developed a theory, nowadays called GKM theory, that makes it possible to describe the equivariant cohomology of certain T-*skeletal* varieties: complete algebraic varieties upon which an algebraic torus T acts with a finite number of fixed points and weighted invariant curves. (For instance, projective group embeddings are $T \times T$-skeletal.) Let X be a T-skeletal variety and denote by X^T the fixed point set. The main purpose of GKM theory is to identify the image of the functorial map $i^* : H_T^*(X) \to H_T^*(X^T)$, assuming X is *equivariantly formal*. GKM theory has been mostly applied to smooth projective T-skeletal varieties, because of the Bialynicki-Birula decomposition [B]. Furthermore, the GKM data issued from the fixed points and invariant curves has been explicitly obtained for some interesting cases: flag varieties (Carrell [C]), toric varieties (Brion [Br1]) and *regular embeddings* of reductive groups (Brion [Br2] and Uma [U]). In the case of singular varieties, GKM theory has been applied to Schubert varieties [C] and to *rationally smooth* projective group embeddings, due to the author's work [G1, G2].

Because of its power as a computational tool, GKM theory has been implemented in other equivariant "cohomology" theories on T-schemes. For instance, Brion [Br1] established GKM theory for equivariant Chow groups, Vezzosi-Vistoli [VV] did the same for equivariant algebraic K-theory, and Krishna [Kri] provided the tool in equivariant algebraic cobordism. Nevertheless, in all of these generalizations, a crucial assumption on *smoothness* of the ambient space needs to be made.

Our aim is to establish a version of GKM-theory for the equivariant K-theory of *singular* varieties (Section 3). For convenience of the reader, we briefly recall some basic notions. Let X be a T-scheme. Let $K^0_T(X)$ denote the Grothendieck group of T-equivariant vector bundles on X. This is a ring, with the product given by the tensor product of equivariant vector bundles. Let $K_0^T(X)$ denote the Grothendieck group of T-equivariant coherent sheaves on X. This is a module for the ring $K^0_T(X)$. If we identify the representation ring $R(T)$ with $K^0_T(pt)$, then pullback by the projection $X \to pt$ gives a natural map $R(T) \to K^0_T(X)$. In this way, $K^0_T(X)$ becomes an $R(T)$-algebra and $K_0^T(X)$ an $R(T)$-module. The functor $K^0_T(-)$ is contravariant with respect to arbitrary equivariant maps. In contrast, $K_0^T(-)$ is covariant for equivariant proper morphisms and contravariant for equivariant flat maps. If X is smooth, then every T-equivariant coherent sheaf has a finite resolution by T-equivariant locally free sheaves, and thus $K^0_T(X) \simeq K_0^T(X)$ ([Th1]).

In general, the K-theory groups are difficult to compute. In the case of singular varieties, they can be quite large [AP, Introduction]. In the smooth case, however, there are three powerful theorems that allow many computations and important comparison theorems of Riemann-Roch type. The first one is the localization theorem of Borel-Atiyah-Segal type [Th2] (see also [VV] for higher equivariant K-theory).

Localization theorem of Borel-Atiyah-Segal type. *Let X be a smooth complete T-scheme. Let X^T be the subscheme of fixed points and let $i_T : X^T \to X$ be the natural inclusion. Then the pullback map $i_T^* : K^0_T(X) \to K^0_T(X^T)$ is injective, and it becomes surjective over the quotient field of $R(T)$.*

Let X be a smooth complete T-scheme. The second fundamental theorem in this context identifies the image of i_T^* inside $K^0_T(X^T) \simeq K^0(X^T) \otimes R(T)$. To state it, we introduce some notation. Let $H \subset T$ be a subtorus of codimension one. Observe that i_T factors as $i_{T,H} : X^T \to X^H$ followed by $i_H : X^H \to X^T$. Thus, the image of i_T^* is contained in the image of $i_{T,H}^*$. In symbols,

$$\mathrm{Im}[i_T^* : K^0_T(X) \to K^0_T(X^T)] \subseteq \bigcap_{H \subset T} \mathrm{Im}[i_{T,H}^* : K^0_T(X^H) \to K^0_T(X^T)],$$

where the intersection runs over all codimension-one subtori H of T. This observation leads to a complete description of the image of i_T^* (see [VV]). This criteria dates back to the work of Chang-Skjelbred [CS] in equivariant cohomology.

CS property. *Let X be a smooth complete T-scheme. Then the image of the injective map $i_T^* : K^0_T(X) \to K^0_T(X^T)$ equals the intersection of the images of $i_{T,H}^* : K^0_T(X^H) \to K^0_T(X^T)$, where H runs over all subtori of codimension one in T.*

Now let X be a (complete) T-skeletal variety. For convenience, we assume that each T-invariant irreducible curve has exactly two fixed points (e.g. X is equivariantly embedded in a normal T-variety). In this setting, it is possible to define a ring $PE_T(X)$ of *piecewise exponential* functions. Indeed, let $K^0_T(X^T) = \oplus_{x \in X^T} R_x$, where R_x is a copy of the representation ring $R(T)$. We then define $PE_T(X)$ as the subalgebra of $K^0_T(X^T)$ given by

$$PE_T(X) = \{(f_1, \ldots, f_m) \in \oplus_{x \in X^T} R_x \,|\, f_i \equiv f_j \mod 1 - e^{-\chi_{i,j}}\}$$

where x_i and x_j are the two distinct fixed points in the closure of the one-dimensional T-orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $C_{i,j}$. This character is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite). In light of the CS property, one obtains ([VV, U]):

GKM theorem. *Let X be a smooth T-skeletal variety. Then $i_T^* : K_T^0(X) \to K_T^0(X^T)$ induces an isomorphism between $K_T^0(X)$ and $PE_T(X)$. If X is also projective, then $K_T^0(X)$ is a free $R(T)$-module of rank $|X^T|$.*

Thus far, it is clear that to any complete T-skeletal variety X we can associate the ring $PE_T(X)$, regardless of whether X is smooth or not. (In fact, if $\mathbb{P}_\epsilon(M)$ is a group embedding, then $PE_{T\times T}(\mathbb{P}_\epsilon(M))$ has been explicitly written down in [G2].) Nonetheless, as it steams from the previous facts, $PE_T(X)$ does not always describe $K_T^0(X)$. This phenomena yields a natural *question*. Let X be a T-skeletal variety. What kind of information does $PE_T(X)$ describe? If not equivariant K-theory, is it still reasonable to expect that $PE_T(X)$ encodes certain topological/geometric information that is *common* to all possible T-equivariant resolution of singularities of X? The work of Payne [P] and Anderson-Payne [AP], inspired, in turn, by the works of Fulton-MacPherson-Sottile-Sturmfels [FMSS] and Totaro [T], gives a positive answer to these questions *when X is a toric variety*. Namely, the GKM data (i.e. $PE_T(X)$) of a toric variety encodes all the information needed to reconstruct "Bott-Chern operators" defined on the structure sheaves $\mathcal{O}_{\overline{Tx}}$ of the T-orbit closures $\overline{Tx} \subseteq X$ (and their equivariant resolutions). This positive result motivates us. In the pages to follow we will show that Anderson-Payne's assertion on toric varieties holds more generally for *all* T-skeletal varieties. We also obtain a version of Poincaré duality for the equivariant Chow groups of singular spherical varieties.

2. Statement of the problems and results

During my research visit to the Max-Planck-Institut für Mathematik (October-December 2013), I worked on the following problems:

(1) To address the questions posted at the end of the Introduction, I establish localization theorems of Borel-Atiyah-Segal type for Fulton-MacPherson's equivariant operational Chow rings [Fu], [EG] and Anderson-Payne's operational K-theory [AP]. Consequently, I establish GKM theory for equivariant operational K-theory (see Section 3). This complements my results in [G3].

(2) I provide a notion of rational smoothness for the Chow groups of possibly singular schemes. My results yield a version of Poincaré duality for the operational Chow rings of possibly singular group embeddings. This extends some results on simplicial toric varieties to more general varieties. Brion's notion of equivariant multiplicities [Br1] as well as Renner's combinatorial classification of group embeddings [R1], [R3] play a crucial role in my work.

In future work I plan to apply the machinery developed in items (1) and (2) to better understand the geometry, topology and combinatorics of spherical varieties. Below is a more detailed account of the progress I have made on these problems.

3. Localization theorem of Borel-Atiyah-Segal type for equivariant operational theories

Fulton-MacPherson [Fu] devised a machinery that produces a "cohomology" theory out of a homology theory. This "cohomology" has all the formal properties one could hope for, and it is well suited for the study of singular schemes. Taking as input $K_0^T(-)$, Anderson-Payne [AP] obtained a theory that is very well suited for computations. Let X be a T-scheme. The *T-equivariant operational K-theory* ring of X, denoted $opK_T^0(X)$, is defined as follows:

an element $c \in opK_T^0(X)$ is a collection of homomorphisms $c_f : K_0^T(Y) \to K_0^T(Y)$ for every T-map $f : Y \to X$. These homomorphisms must be compatible with (equivariant) proper pushforward, flat pullback and Gysin morphisms [AP]. For any X, the ring structure on $opK_T^0(X)$ is given by composition of such homomorphisms. With this product, $opK_T^0(X)$ becomes an associative commutative ring with unit. Moreover, $opK_T^0(X)$ is contravariantly functorial in X. Other salient functorial properties of $opK_T^0(-)$ are:

(a) For any X, there is a canonical homomorphism $K_T^0(X) \to opK_T^0(X)$ of $R(T)$-algebras, sending a class γ to the operator $[\gamma]$ which acts via $[\gamma]_g = g^*\gamma \cdot \xi$, for any T-map $g : Y \to X$ and $\xi \in K_0^T(Y)$. There is also a canonical map $opK_T^0(X) \to K_0^T(X)$ defined by $c \mapsto c_{\mathrm{id}_X}[\mathcal{O}_X]$, where $\mathcal{O}_X$ is the structure sheaf of X. Put together, they provide a factorization of the canonical homomorphism $K_T^0(X) \to K_0^T(X)$.

(b) When X is smooth, the homomorphisms $K_T^0(X) \to opK_T^0(X) \to K_0^T(X)$ are all isomorphisms of $R(T)$-modules.

(c) If $\pi : \tilde{X} \to X$ is an equivariant envelope (that is, any T-invariant subvariety of X is the birational image of a T-invariant subvariety of $\tilde{X}$), then the following sequence is exact

$$0 \to opK_T^0(X) \to opK_T^0(\tilde{X}) \to opK_T^0(\tilde{X} \times_X \tilde{X}).$$

In my preprint [G4] I use property (c) above, together with resolution of singularities and the fact that $\pi : \tilde{X}^H \to X^H$ is also an envelope (for any subtorus $H \subset T$), to establish:

(I) The localization theorem of Borel-Atiyah-Segal type for $opK_T^0(X)$, whenever X is a complete T-scheme.

(II) The CS property for $opK_T^0(X)$, where X is any complete T-scheme.

(III) If X is a singular T-skeletal variety, then $opK_T^0(X) \simeq PE_T(X)$. Moreover, if X is G-spherical, then, using a result of [AP], the $R(T)$-module structure on $opK_T^0(X)$ is determined by the identity

$$opK_T^*(X) \simeq Hom_{R(T)}(K_0^T(X), R(T))$$

(see [G3] for the corresponding statement in operational Chow groups). In particular, when X is G-spherical, we show via "topology" that the ring $PE_T(X)$ is a finitely generated $R(T)$-module.

Together with the combinatorial results of [G2], this yields an explicit extension of Anderson-Payne's work on toric varieties to *all* projective group embeddings. Our results are being collected in the preprint [G4], to be available on the arxiv by mid-January 2014 (a copy will also be sent to MPIM). We should remark that our arguments easily adapt to equivariant operational Chow groups (with $\mathbb{Q}$-coefficients). This is used next.

4. On a notion of rational smoothness for Chow groups. Equivariant Poincaré duality.

Let X be a T-scheme. Denote by $A_*^T(X)$ the T-equivariant Chow group of X ([EG]). Throughout this section, Chow groups are taken with $\mathbb{Q}$-coefficients. When X is smooth, the group $A_*^T(X)$ admits a natural product by intersection of cycles. In this case, we denote by $A_T^*(X)$ the corresponding ring. Set $S := A_T^*(pt)$. It can be identified with the polynomial ring on the character group of T. Pullback by the projection $X \to pt$ gives a natural map $S \to A_*^T(X)$. In this way, $A_*^T(X)$ becomes a S-module (or a S-algebra if X is smooth).

Let X be a T-scheme. Call a fixed point $x \in X$ *non-degenerate* if all weights of T in the tangent space T_xX are non-zero. Likewise, call a fixed point $x \in X$ *attractive* if there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ and a Zariski neighborhood U of x, such that $\lim_{t\to 0} \lambda(t) \cdot y = x$ for all points y in U. Clearly, attractive fixed points are non-degenerate. To study possibly singular varieties, Brion [Br1] developed a notion of *equivariant multiplicity* at non-degenerate fixed points. This notion is crucially used in our work.

Now let X be an *affine* T-variety with an attractive fixed point x. It follows from [Br3] that $X = \{y \in X \mid \lim_{t\to 0} \lambda(t)\, y = x_0\}$, for a suitable one-parameter subgroup λ. We say that (X, x) is an *attractive cell* in this situation. If (X, x) is an attractive cell, then the geometric quotient

$$\mathbb{P}(X) := [X \setminus \{x\}]/\mathbb{G}_m$$

exists and we call it the *link* at x. This is a projective variety since X is assumed to be affine. In [G1] we studied the links of *complex* rationally smooth cells. Recall that a complex algebraic variety X, of dimension n, is called *rationally smooth* if

$$H^m(X, X - \{y\}) = (0) \ \text{ if } m \neq 2n, \quad \text{and} \ \ H^{2n}(X, X - \{y\}) = \mathbb{Q}.$$

for all $x \in X$. Such varieties satisfy Poincaré duality with rational coefficients. If (X, x) is a complex rational cell, then $\mathbb{P}(X)$ is a rational cohomology complex projective space. Many important results on the equivariant cohomology of T-varieties admitting a paving by rational cells are provided in [G1], for instance, such varieties have no cohomology in odd degrees and their equivariant cohomology is a free S-module. Our goal is to provide analogues of these notions, and a version of Poincaré duality, in the context of equivariant Chow groups. This program was started in [G3].

Definition. Let (X, x) be an attractive cell of dimension n. We say that X is an *algebraic rational cell* if and only if

$$A_*(\mathbb{P}(X)) \simeq A_*(\mathbb{P}^{n-1}).$$

To see how this "local" notion is well-suited for the study of more general schemes, we need to introduce a few extra tools from [B]. Let X be a complete T-scheme and let X^T be the subscheme of fixed points. Assume that X^T consists of finitely many isolated points. Let $X^T = \{x_1, \ldots, x_m\}$. Pick a generic one-parameter subgroup $\gamma : \mathbb{G}_m \to T$, i.e. $X^\gamma = X^T$. For each $i = 1, \ldots, m$, define $W_i(\gamma) := \{x \in X \mid \lim_{t\to 0} \gamma(t)x = x_i\}$. Clearly $X = \bigsqcup W_i(\gamma)$, and each $W_i(\gamma)$ is a locally closed T-invariant subscheme of X. The decomposition $\{W_i(\gamma)\}$ is called the *BB-decomposition of* X (associated to γ), and the $W_i(\gamma)$'s are called *cells* of the decomposition. A BB-decomposition $\{W_i(\gamma)\}$ is called *filtrable*, if there exists a finite increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_m$ of T-invariant closed subschemes of X such that:

a) $X_0 = \emptyset$, $X_m = X$,

b) $X_j \setminus X_{j-1}$ is a cell of the decomposition $\{W_i(\gamma)\}$, for each $j = 1, \ldots, m$.

Definition. Let X be a variety equipped with a T-action. We say that X is $\mathbb{Q}$-*filtrable* if the following two conditions hold: (1) the fixed point set X^T is finite, and (2) there exists a generic one-parameter subgroup $\gamma : \mathbb{G}_m \to T$ for which the associated BB-decomposition of X is filtrable and the corresponding cells are (affine) algebraic rational cells.

The next result, recorded in [G3], shows that algebraic rational cells are a good substitute for the notion of affine space in the study of equivariant Chow groups of singular varieties.

Theorem. *Let X be a $\mathbb{Q}$-filtrable T-variety. Then the rational T-equivariant Chow group $A^T_*(X)_{\mathbb{Q}}$ is a free $S_{\mathbb{Q}}$-module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $W_i(\gamma)$. Furthermore, the ordinary rational Chow group $A_*(X)_{\mathbb{Q}}$ is also freely generated by the classes of the cell closures $\overline{W_i(\gamma)}$.*

Remarkably, when $k = \mathbb{C}$, examples of projective $\mathbb{Q}$-filtrable varieties are rationally smooth group embeddings. This yields purely algebraic proofs of the topological results of [G2]. We should point out that the class of $\mathbb{Q}$-filtrable varieties is strictly larger than that of rationally smooth T-varieties (see [G5] for more details).

Section 3 shows that Fulton-MacPherson's equivariant operational Chow groups are well-adapted to the study of singular spaces. A more precise statement is given below.

Theorem [G3]**.** Let X be a G-spherical variety. Let T be a maximal torus of G. Then the equivariant Kronecker duality map

$$\mathcal{K}_T : opA_T^*(X) \longrightarrow Hom_S(A_*^T(X), A_*^T(pt)) \qquad \alpha \mapsto (\beta \mapsto \int_X (\beta \cap \alpha))$$

is an isomorphism of S-modules. In particular, $opA_T^*(X)$ is a torsion free finitely generated S-module.

For a T-scheme X, there is also an equivariant Poincaré duality map:

$$\mathcal{P}_T : opA_T^k(X) \to A_{n-k}^T(X), \quad z \mapsto z \cap [X].$$

Our upcoming paper [G5] is motivated by this question: Let X be G-spherical variety. Assume that X is $\mathbb{Q}$-filtrable. When is $\mathcal{P}_T$ an isomorphism? An answer is given next.

Theorem [G5]. *Let X be a $\mathbb{Q}$-filtrable spherical variety. If all equivariant multiplicities are non-zero (e.g. all fixed points are attractive), then the equivariant Poincaré duality map is injective. If moreover the Chow homology Betti numbers satisfy $A_k(X) = A_{n-k}(X)$, then the equivariant Poincaré duality map is also surjective (over $\mathbb{Q}$).*

In [G5] we give some combinatorial characterizations of $\mathbb{Q}$-filtrability, and provide a description of the operational Chow rings of spherical varieties (generalizing [Br1]). The results of [G5] are modeled after the topological results of Brion [Br4]. His results characterize Poincaré duality in equivariant cohomology. My work is inspired by his, and it is a contribution towards characterizing Poincaré duality in intersection theory. The pioneering work of Vistoli [Vis] on Alexander schemes is another source of inspiration for us. The paper [G5] will be available on the arxiv by the end of January 2014 (and a copy will be sent to MPIM).

5. Upcoming preprints and delivered talks

(1) Upcoming preprints (written almost entirely at MPIM):
   (a) Localization in equivariant operational theories and the Chang-Skjelbred principle [G4].
   (b) On a notion of rational smoothness for intersection theory [G5].
   (c) Equivariant intersection theory on group embeddings.

(2) Delivered talks:
   (a) December 5th: Heinrich-Heine-Universität Düsseldorf.
   Organizer: Nicolas Perrin.
   Title: Equivariant intersection theory of group embeddings.
   (b) December 10th: MPIM, Germany. Seminar on Algebra, Geometry and Physics.
   Organizer: Yu. Manin.
   Title: Equivariant Chow cohomology of spherical varieties.

6. Visitors and other projects planned

(1) From October 30th to November 4th, Matteo Paganin (Sabancı Üniversitesi, Istanbul) visited me at MPIM. Our plan is to understand the equivariant operational algebraic cobordism [GK] of singular spherical varieties, using my version of GKM theory. Since equivariant algebraic cobordism is a universal cohomology theory, we anticipate that the results of this project would shed some more light on the structure of the associated formal group laws. We plan to work on explicit examples.

(2) In future collaboration with Nicolas Perrin (Mathematisches Institut, Düsseldorf), we plan to describe the (equivariant) quantum cohomology and quantum K-theory of certain spherical varieties (e.g., group embeddings, symmetric spaces) and provide explicit formulas for the underlying Schubert calculus. Some important results in this direction have been obtained by Chaput, Manivel and Perrin for the case of minuscule

homogeneous spaces and rational homogeneous spaces. A crucial ingredient here is the *quantum to classical principle.* This principle allows to compute $QH^*(X)$, where X is a sufficiently nice space, in terms of the singular cohomology of a suitable replacement Y_d, which, in many cases, turns out to be a spherical variety. By my previous work, we anticipate a GKM presentation of $H_T^*(Y_d)$ and $H^*(Y_d)$. Such description would be a fundamental step towards implementing a Schubert calculus in $QH^*(X)$. Indeed, we aim at providing (geometric) bases for these rings and formulas to multiply them.

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