Arithmetic surfaces and successive minima

Bowen lectures

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Lecture one

Minkowski's theorem and arithmetic surfaces

1. Successive minima

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Definition. – A *euclidean lattice* is a pair

$$\bar{\Lambda} = (\Lambda, h)$$

of a free $\mathbb{Z}$-module Λ of finite rank N, and a scalar product h on the real vector space $\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

1. Successive minima

If we choose an orthonormal basis of $\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ we get

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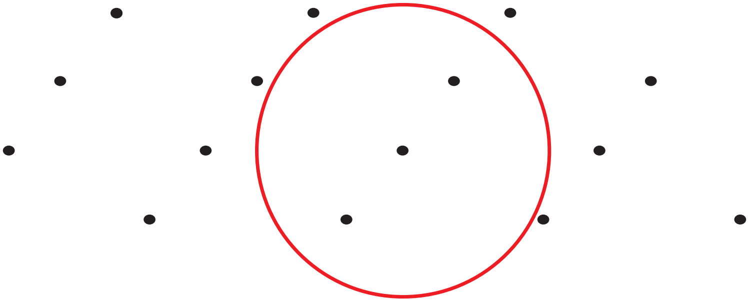

1. Successive minima

Let

$$\|x\| = \sqrt{h(x, x)}$$

be the norm defined by *h*.

One can attach to $\bar{\Lambda}$ several real invariants.
First one can look at the smallest vectors $x \in \Lambda$, $x \neq 0$:

$$\mu_1(\bar{\Lambda}) := \mathrm{Inf}\left\{\log\|x\|,\ x \in \Lambda - \{0\}\right\}.$$

1. Successive minima

More generally, if $1 \leq k \leq N$, we let

$$\mu_k(\bar{\Lambda}) := \mathrm{Inf}\,\{\mu \in \mathbb{R} \;/\; \exists\, x_1, \ldots, x_k \in \Lambda\,,$$

linearly independent in $\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, such that

$$\log \|x_i\| \leq \mu \quad \text{for all} \quad i \leq k\}\,.$$

These numbers are called the (logarithms of the) *successive minima* of $\bar{\Lambda}$.

Note that

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Minkowski gave an estimate for the sum of the successive minima.

1. Successive minima

Let us equip $\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ with its Lebesgue measure. The *arithmetic degree* of $\bar{\Lambda}$ is the real number

$$\widehat{\deg}\,(\bar{\Lambda}) = -\log \mathrm{vol}\left(\frac{\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}}{\Lambda}\right) .$$

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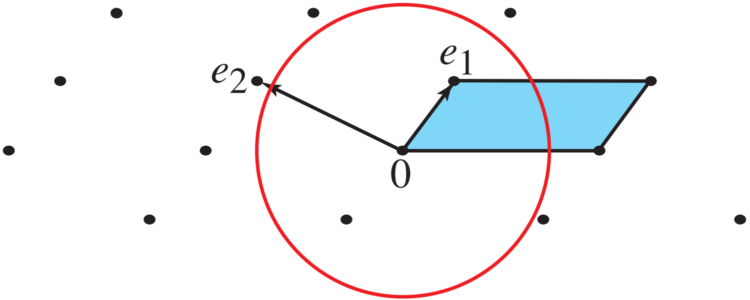

1. Successive minima

Theorem (Minkowski).

$$0 \leq \mu_1(\bar{\Lambda}) + \mu_2(\bar{\Lambda}) + \ldots + \mu_N(\bar{\Lambda}) + \widehat{\deg}\,(\bar{\Lambda}) \leq C\,,$$

where $C = N\log(2) - \log(V_N)$, and $V_N :=$ volume of the unit ball in $\mathbb{R}^N$.

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Theorem (Minkowski).

$$0 \leq \mu_1(\bar{\Lambda}) + \mu_2(\bar{\Lambda}) + \ldots + \mu_N(\bar{\Lambda}) + \widehat{\deg}\,(\bar{\Lambda}) \leq C\,,$$

Corollary. – If $\widehat{\deg}\,(\bar{\Lambda})$ is large enough there exists $x \in \Lambda - \{0\}$ such that $\|x\| \leq 1$.

2. Arithmetic surfaces

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We are ultimately interested in *diophantine equations*, i.e. integral solutions of polynomial equations with integral coefficients.

2. Arithmetic surfaces

Let $F \in \mathbb{Z}[u, v]$ be a polynomial in two variables. In order to study the solutions of the equation

$$F(x, y) = 0\,, \quad x, y \in \mathbb{Z}$$

we can first:

1) For every prime p consider the congruence

$$F(x, y) \equiv 0 \pmod{p}.$$

This leads to the study of the *scheme*

$$X/p = \mathrm{Spec}\,(\mathbb{F}_p[u, v]/(F))$$

over the finite field $\mathbb{F}_p$.

2) Consider the set $X(\mathbb{C})$ of solutions of

$$F(x,y) = 0\,, \quad x,y \in \mathbb{C}\,.$$

This set $X(\mathbb{C})$ is a complex curve that we can study by means of *complex geometry*.

2. Arithmetic surfaces

Our goal is to do 1) and 2) simultaneously.

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$$\left.\begin{array}{c}\text{Algebraic geometry of}\\ \text{schemes over } \mathbb{Z}\\ \\ \text{Hermitian complex geometry}\end{array}\right\}\ \text{Arakelov geometry.}$$

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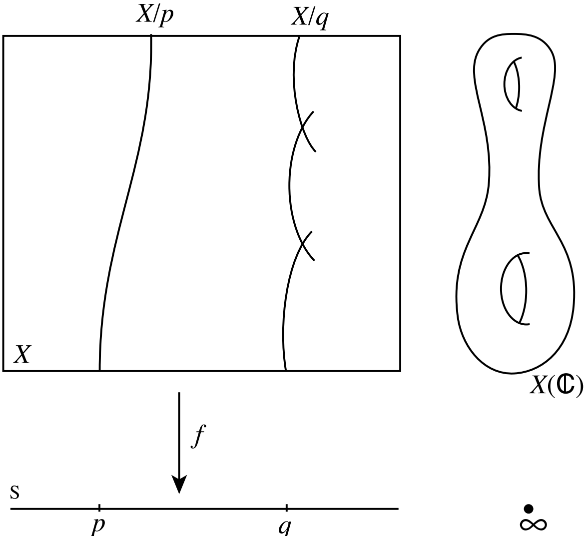
We shall assume that $X(\mathbb{C})$ is connected of genus $g \geq 2$.

2. Arithmetic surfaces



2. Arithmetic surfaces

Our main object of study will be hermitian vector bundles over X.

Definition. – An *hermitian vector bundle* over X is a pair

$$\overline{E} = (E, h)$$

where:

i) E is an algebraic vector bundle of rank N over X;

ii) h is a C^∞ hermitian metric on the restriction $E_{\mathbb{C}}$ of E to $X(\mathbb{C})$. (Furthermore h is invariant under complex conjugation) .

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When $N = 1$ we get *hermitian line bundles*.

Let $\bar{L}$ and $\bar{M}$ be two hermitian line bundles on X. Arakelov defined a real number

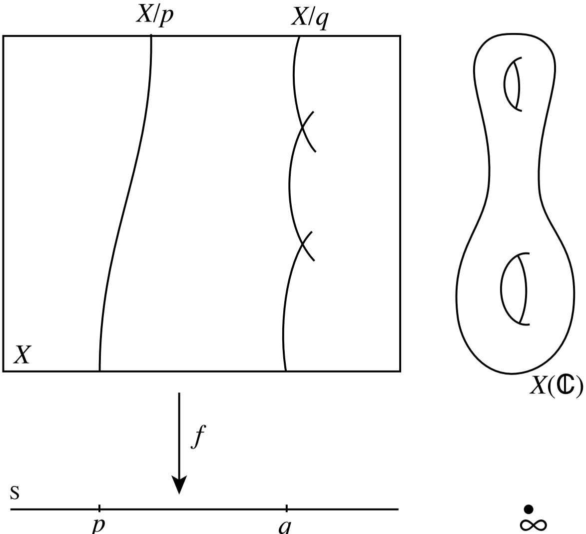
$$\boxed{\bar{L} \cdot \bar{M} \in \mathbb{R}}$$

called the *arithmetic intersection number* of $\bar{L}$ and $\bar{M}$. We shall give a formula for this number tomorrow.

Example. – Let $\omega_{X/S} =$ *relative dualizing sheaf* of X/S
$:=$ the unique algebraic line bundle on X such that, if $U = X -$ double points,

$$(\omega_{X/S})_{|U} = (\Omega^1_{X/S})_{|U} \, .$$

2. Arithmetic surfaces



2. Arithmetic surfaces

Arakelov defined a canonical metric on $\omega_{X/S}$. We let

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The real number

$$\bar{\omega}^2 = \bar{\omega} \cdot \bar{\omega}$$

is a fundamental invariant of X.

3. Some conjectures

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Theorem (Fermat,...,Wiles). – When $n \geq 3$ the equation

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has no nontrivial solution.

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Conjecture. There exists positive constants α and β such that, if

$$A + B = C$$

and $gcd(A, B, C) = 1$, the following inequality holds:

$$ABC \leq \beta\Big(\prod_{p|ABC} p\Big)^{\alpha}.$$

Theorem (Parshin; Moret-Bailly). – A good upper bound for $\bar{\omega}^2$ implies the *ABC* conjecture.

4. The Arakelov metric

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First we get as follows a volume form μ on M.

Endow the vector space $\Gamma(M, \Omega^1)$ with a scalar product by the formula

$$\langle \omega, \eta \rangle = i \int_M \omega \wedge \bar{\eta} \, .$$

If $\omega_1, \ldots, \omega_g$ is an orthonormal basis of $\Gamma(M, \Omega^1)$ for this scalar product, we let

$$\mu = \frac{i}{g} \sum_{\alpha=1}^{g} \omega_\alpha \wedge \bar{\omega}_\alpha \, .$$

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Clearly

$$\int_M \mu = 1 \, .$$

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Given a point $P \in M$ we let $g_P(x)$ be the L^1-function on M such that

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One can show that, if $P \neq Q$,

$$g_P(Q) = g_Q(P)\,.$$

4. The Arakelov metric

Let $\Delta \subset M \times M$ be the diagonal, $\mathcal{O}(\Delta)$ the associated line bundle, and $1_\Delta \in \Gamma(M \times M, \mathcal{O}(\Delta))$ its canonical section. We define a smooth metric on $\mathcal{O}(\Delta)$ by the formula

$$(-\log \|1_\Delta\|)\,(P, Q) = \frac{1}{2}\, g_P(Q)\,.$$

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If $u : M \to M \times M$ is the diagonal embedding, and $\mathcal{O}(-\Delta)$ the dual of $\mathcal{O}(\Delta)$, there is a canonical isomorphism

$$u^*\,\mathcal{O}(-\Delta) \simeq \Omega^1\,.$$

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4. The Arakelov metric

The *Arakelov metric* on Ω^1 is the pull-back by u^* of the metric on $\mathcal{O}(-\Delta)$ which is dual to the one we defined on $\mathcal{O}(\Delta)$.

Lecture two

A vanishing theorem in Arakelov geometry

1. The self intersection of $\bar{\omega}$

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Let $S = \mathrm{Spec}\,(\mathbb{Z})$ and

$$\begin{array}{c} X \\ \downarrow f \\ S \end{array}$$

a semi-stable curve over S such that $X(\mathbb{C})$ is connected of genus $g \geq 2$.

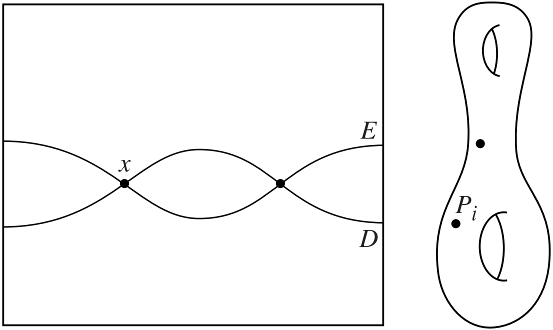
1. The self intersection of $\bar{\omega}$

Given two hermitian line bundles $\bar{L}$ and $\bar{M}$ over X, Arakelov defined

$$\bar{L} \cdot \bar{M} \in \mathbb{R} .$$

Assume that L and M have global sections s and t and that $D = \mathrm{div}\,(s)$ and $E = \mathrm{div}\,(t)$ are irreducible horizontal divisors.

1. The self intersection of $\bar{\omega}$



1. The self intersection of $\bar{\omega}$

Then

$$\bar{L} \cdot \bar{M} \;=\; \sum_{x \in D \cap E} \log \# \frac{\mathcal{O}_x}{\langle s, t \rangle}$$
$$-\; \sum_i \log \|t(P_i)\| - \int_{X(\mathbb{C})} \log \|s\| \, c_1(\bar{M}_{\mathbb{C}}) \,,$$

where $\langle s, t \rangle \subset \mathcal{O}_x$ is the submodule generated by s and t,

$$D_{|X(\mathbb{C})} = \sum_i P_i$$

and

$$c_1(\bar{M}_{\mathbb{C}}) = \text{first Chern form of } \bar{M}_{\mathbb{C}} \,.$$

1. The self intersection of $\bar{\omega}$

This number has the following properties:

a) $\bar{L} \cdot \bar{M} = \bar{M} \cdot \bar{L}$

b) $(\bar{L}_1 + \bar{L}_2) \cdot \bar{M} = \bar{L}_1 \cdot \bar{M} + \bar{L}_2 \cdot \bar{M}$

c) For any positive number $a > 0$

$$(L, a\, h_L) \cdot \bar{M} = (L, h_L) \cdot \bar{M} - \frac{1}{2} \log(a) \deg(M_{\mathbb{C}})\,.$$

1. The self intersection of $\bar{\omega}$

Let

$$\bar{\omega} = (\omega_{X/S}\,,\ \text{Arakelov metric})$$

be the relative dualizing sheaf. Then

$$\bar{\omega}^2 \geq 0 \qquad \text{(Faltings, 1984)}$$
$$\bar{\omega}^2 > 0 \qquad \text{(Ullmo, 1998).}$$

1. The self intersection of $\bar{\omega}$

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$$\begin{array}{ll} \bar{\omega}^2 \geq 0 & \text{(Faltings, 1984)} \\ \bar{\omega}^2 > 0 & \text{(Ullmo, 1998).} \end{array}$$

Corollary (Szpiro). – If C is a curve of genus ≥ 2 over a number field F and $C \hookrightarrow J$ an embedding of C in its jacobian, the set $C(\bar{F})$ is discrete in $J(\bar{F})$.

1. The self intersection of $\bar{\omega}$

Instead of $X/\mathrm{Spec}\,(\mathbb{Z})$ we could consider a semi-stable curve $X/\mathrm{Spec}\,(\mathcal{O}_F)$ where F is a number field. Let Δ_F be the discriminant of F over $\mathbb{Q}$.

1. The self intersection of $\bar{\omega}$

Conjecture (Parshin; Moret-Bailly).

$$\bar{\omega}^2 \leq \alpha \log |\Delta_F| + \beta \, [F : \mathbb{Q}] \,,$$

where the constants α and β are bounded in projective families.

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where the constants α and β are bounded in projective families.

This conjecture implies *ABC* and effective Mordell.

2. *ABC* implies no Siegel zeroes

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Let $d > 0$, χ the non-trivial character of $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$ and $L(\chi, s)$ its Dirichlet L-function.

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Let $d > 0$, χ the non-trivial character of $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$ and $L(\chi, s)$ its Dirichlet L-function.

Theorem (Stark-Granville; Elkies). – *A strong version of the ABC conjecture for number fields implies that there exists a constant* $c > 0$ *such that* $L(\chi, s)$ *has no zero in the real interval*

$$1 - \frac{c}{\log(d)} < s \leq 1\,.$$

3. Statement of the main result

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Let $S = \mathrm{Spec}\,(\mathbb{Z})$ and X/S an arithmetic surface as above. Fix an integer $n \geq 1$ and let

$$\bar{L} = \bar{\omega}^{\otimes n}\,.$$

Let L^{-1} = dual of L.

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Consider the euclidean lattice

$$\bar{H}^1 := (H^1(X, L^{-1})/\text{torsion},\ L^2\text{-metric}).$$

3. Statement of the main result

Theorem 1. – There exists a constant $C(g, n)$ such that, if $1 \leq k \leq (g-1)\,n$,

$$\mu_k(\bar{H}^1) \geq \frac{n+k}{4g(g-1)}\,\bar{\omega}^2 - C(g, n)\,.$$

Remarks. –

1) The rank N of the lattice H^1 is

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hence Theorem 1 deals with about half of the successive minima.

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2) The sum $\mu_1(\bar{H}^1) + \ldots + \mu_N(\bar{H}^1)$ is known by Minkowski's theorem and the *arithmetic Riemann-Roch* theorem.

3) Theorem 1 extends to the case $X/\mathrm{Spec}\,(\mathcal{O}_F)$.

4) By Serre duality

$$H^1(X, L^{-1})^* = H^0(X, \omega_{X/S} \otimes L)$$

so we get upper bounds for successive minima of $(H^0(X, \omega_{X/S} \otimes L), L^2$-metric).

4. The semi-stable case

Under the assumption of Theorem 1 we let

$$e \in H^1(X, L^{-1}) = \mathrm{Ext}\,(L, \mathcal{O}_X)$$

be such that $e_{\mathbb{C}} = e_{|X(\mathbb{C})} \neq 0$.

The class e defines an extension of vector bundles over X

$$0 \to \mathcal{O}_X \to E \to L \to 0\,. \qquad (1)$$

Let $E_{\mathbb{C}} = E_{|X(\mathbb{C})}$.

Definition. – $E_{\mathbb{C}}$ is *semi-stable* if, for any line bundle $M_{\mathbb{C}} \subset E_{\mathbb{C}}$, we have

$$\deg(M_{\mathbb{C}}) \leq \frac{\deg(E_{\mathbb{C}})}{2}\,.$$

4. The semi-stable case

We now assume that $E_{\mathbb{C}}$ is semi-stable.

Theorem (Miyaoka; Moriwaki). For any choice of an hermitian metric on E we have

$$\hat{c}_1(\bar{E})^2 \leq 4\,\hat{c}_2(\bar{E})\,.$$

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Theorem (Miyaoka; Moriwaki). For any choice of an hermitian metric on E we have

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Here

$$\hat{c}_1(\bar{E})^2 = \det(\bar{E})^2$$

and $\hat{c}_2(\bar{E})$ = second Chern number of $\bar{E}$ (Gillet-S.).

Goal: find the metric on E which gives the best inequality.

On $\mathcal{O}_{X(\mathbb{C})} = \mathbb{C}$ we take the trivial metric.
On L we take some metric h_L, to be specified later.

4. The semi-stable case

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Choose a C^∞-splitting of (1):

$$0 \to \mathbb{C} \to E_{\mathbb{C}} \overset{\overset{\sigma}{\curvearrowleft}}{\to} L_{\mathbb{C}} \to 0\,.$$

We get a C^∞ isomorphism

$$E_{\mathbb{C}} \overset{C^\infty}{\simeq} \mathbb{C} \oplus L_{\mathbb{C}}\,.$$

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Choose on E the metric such that

$$\bar{E}_{\mathbb{C}} \simeq \bar{\mathbb{C}} \overset{\perp}{\oplus} \bar{L}_{\mathbb{C}}\,.$$

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Under these assumptions we get

$$\det(\bar{E}) = \bar{L}$$

therefore

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On the other hand, we may think that

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BUT σ is not holomorphic hence

$$\hat{c}_2(\bar{E}) = -\frac{1}{2} \int_{X(\mathbb{C})} \tilde{c}_2 \,,$$

where $\tilde{c}_2$ is a Bott-Chern secondary class.

4. The semi-stable case

Let

$$\bar{\partial}_E : C^\infty(X(\mathbb{C}), E_\mathbb{C}) \to A^{01}(X(\mathbb{C}), E_\mathbb{C})$$

be the Cauchy-Riemann operator. Since

$$E_\mathbb{C} \overset{C^\infty}{\simeq} \mathbb{C} \oplus L_\mathbb{C}$$

we get

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_\mathbb{C} & \alpha \\ 0 & \bar{\partial}_L \end{pmatrix} ,$$

where $\alpha \in A^{01}(X(\mathbb{C}), L_\mathbb{C}^{-1})$.

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where $\alpha \in A^{01}(X(\mathbb{C}), L_{\mathbb{C}}^{-1})$.

Let

$$\alpha^* \in A^{10}(X(\mathbb{C}), L_{\mathbb{C}})$$

be the adjoint of α. We have

4. The semi-stable case

$$\tilde{c}_2 = \frac{1}{2\pi i}\, \alpha^* \alpha \quad \text{in} \quad A^{11}(X(\mathbb{C}))\,.$$

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The class of α in

$$H^1(X(\mathbb{C}), L_{\mathbb{C}}^{-1}) = A^{01}(X(\mathbb{C}), L_{\mathbb{C}}^{-1})/\mathrm{Im}(\bar{\partial})$$

is

$$[\alpha] = e_{\mathbb{C}}\,.$$

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is

$$[\alpha] = e_{\mathbb{C}}\,.$$

Furthermore, if we change the splitting σ of (1), α is replaced by $\alpha + \bar{\partial}(\beta)$. So we can choose σ such that α is the harmonic representative of $e_{\mathbb{C}}$.

4. The semi-stable case

In that case we get

$$-\frac{1}{2}\int_{X(\mathbb{C})}\tilde{c}_2 = -\frac{1}{2}\int_{X(\mathbb{C})}\frac{\alpha^*\alpha}{2\pi i} = \frac{1}{2}\,\|e\|^2\,,$$

where $\|e\|$ is the L^2-norm of $e_{\mathbb{C}}$.

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where $\|e\|$ is the L^2-norm of $e_{\mathbb{C}}$.

So we have obtained

$$\bar{L}\cdot\bar{L} \leq 2\,\|e\|^2\,.$$

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Now, let us replace the metric h_L on L by $a\,h_L$, where $a = \|e\|^2$.

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Now, let us replace the metric h_L on L by $a\,h_L$, where $a = \|e\|^2$. We get

$$\begin{aligned} \|e\|^2 &\mapsto a^{-1}\|e\|^2 = 1 \\ \bar{L}\cdot\bar{L} &\mapsto \bar{L}\cdot(L, a\,h_L) = \bar{L}^2 - \tfrac{1}{2}\log(a)\deg(L_{\mathbb{C}})\,. \end{aligned}$$

Now, let us replace the metric h_L on L by $a\,h_L$, where $a = \|e\|^2$. We get

$$\begin{array}{rcl} \|e\|^2 & \mapsto & a^{-1}\|e\|^2 = 1 \\ \bar{L}\cdot\bar{L} & \mapsto & \bar{L}\cdot(L, a\,h_L) = \bar{L}^2 - \frac{1}{2}\log(a)\deg(L_{\mathbb{C}})\,. \end{array}$$

Therefore

$$\bar{L}^2 \leq \deg(L_{\mathbb{C}})\log\|e\| + 2\,.$$

4. The semi-stable case

If $\bar{L} = \bar{\omega}^{\otimes n}$, we get

$$\deg(L_{\mathbb{C}}) = 2(g-1)\, n\,,$$

and

$$\log \|e\| \geq \frac{n\,\bar{\omega}^2}{4(g-1)} - C'(g,n)\,,$$

for any $e \in H^1$, $e_{\mathbb{C}} \neq 0$.

4. The semi-stable case

This implies

$$\log \|e\| \geq \frac{n+k}{4g(g-1)}\,\bar{\omega}^2 - C(g,n)$$

if $k \leq (g-1)\,n$.

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Hence

$$\mu_1(\bar{H}^1) \geq \frac{n+k}{4g(g-1)} - C(g,n)\,.$$

Lecture three

Secant varieties and successive minima

Let $X/S = \mathrm{Spec}\,(\mathbb{Z})$ be a semi-stable curve such that $X(\mathbb{C})$ is connected of genus $g \geq 2$. Fix an integer $n \geq 1$, let $\bar{L} = \bar{\omega}^{\otimes n}$, and let

$$\bar{H}^1 = (H^1(X, L^{-1})/\text{torsion},\ L^2\text{-metric})\,.$$

1. The first minimum

Let $X/S = \mathrm{Spec}\,(\mathbb{Z})$ be a semi-stable curve such that $X(\mathbb{C})$ is connected of genus $g \geq 2$. Fix an integer $n \geq 1$, let $\bar{L} = \bar{\omega}^{\otimes n}$, and let

$$\bar{H}^1 = (H^1(X, L^{-1})/\text{torsion},\ L^2\text{-metric})\,.$$

Theorem 1. There exists a constant $C(g, n)$ such that, if $1 \leq k \leq (g-1)\,n$,

$$\mu_k(\bar{H}^1) \geq \frac{n+k}{4g(g-1)}\,\bar{\omega}^2 - C(g, n)\,.$$

1. The first minimum

Let $e \in H^1(X, L^{-1})$ be such that $e_{\mathbb{C}} \neq 0$, and let

$$0 \to \mathcal{O}_X \to E \to L \to 0 \qquad (2)$$

be the extension of class e.

1. The first minimum

Assume that $E_{\mathbb{C}}$ is *not semi-stable*.

Let $M_{\mathbb{C}} \subset E_{\mathbb{C}}$ be a line bundle of maximal degree. In particular

$$\deg(M_{\mathbb{C}}) > \deg(E_{\mathbb{C}})/2\,.$$

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$$\deg(M_{\mathbb{C}}) > \deg(E_{\mathbb{C}})/2 .$$

The line bundle $M_{\mathbb{C}}$ is unique, therefore it is defined over $\mathbb{Q}$, and there exists a line bundle $M \subset E$ on X such that

$$M_{\mathbb{C}} = M_{|X(\mathbb{C})} .$$

Consider the composite

$$M \to E \to L\,.$$

It is not zero, otherwise $M \subset \mathcal{O}_X$ hence $\deg(M_{\mathbb{C}}) \leq 0$. Therefore there exists an effective divisor D on X such that

$$M = L(-D)\,.$$

1. The first minimum

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$$M = L(-D)\,.$$

We get another exact sequence

$$0 \to L(-D) \to E \to \mathcal{O}(D)\,\mathcal{I}_Z \to 0\,, \tag{3}$$

where $Z \subset X$ is a closed subset of dimension zero and $\mathcal{I}_Z$ its ideal of definition.

1. The first minimum

We choose as follows the metrics:

- On L we choose the metric $a\,h_L$, where h_L is the Arakelov metric on $\bar{\omega}^{\otimes n}$, and $a = \|e\|^2$.
- On $E_{\mathbb{C}}$ we choose the same metric as in the semi-stable case.
- On $\mathcal{O}(D)$ we choose the metric defined by Arakelov. Let $\bar{D} = (\mathcal{O}(D)$, Arakelov metric).
- On $L(-D) = L \otimes \mathcal{O}(D)^{-1}$ we take the induced metric.

1. The first minimum

We shall compute $\hat{c}_2(\bar{E}) \in \mathbb{R}$ in two ways.

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From the exact sequence (2) we get

$$\hat{c}_2(\bar{E}) = -\frac{1}{2}\int_{X(\mathbb{C})} \tilde{c}_2 = \frac{1}{2}$$

as was explained in the semi-stable case.

From the exact sequence (3) and our choice of metrics we get

$$\hat{c}_2(\bar{E}) = \bar{D}(\bar{L}-\bar{D}) - d\log\|e\| - \frac{1}{2}\int_{X(\mathbb{C})} \tilde{c}_2' + \log \# \Gamma(X, \mathcal{O}_Z)\,,$$

where $d = \deg(D_{\mathbb{C}})$ and $\tilde{c}_2'$ is the Bott-Chern class attached to (3).

1. The first minimum

Lemma.

$$\int_{X(\mathbb{C})} \tilde{c}_2' \leq 0 \,.$$

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We conclude that

$$\bar{D}(\bar{L} - \bar{D}) - d \log \|e\| \leq \frac{1}{2} \, . \tag{4}$$

Now we apply the *arithmetic Hodge index* theorem (Faltings, Hriljac). Let

$$F_\infty := (\mathcal{O}_X, |\cdot|\exp(1))\,.$$

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We get

$$\det\begin{pmatrix} \bar{L}^2 & \bar{L}\cdot\bar{D} & \bar{L}\cdot F_\infty \\ \bar{L}\cdot\bar{D} & \bar{D}^2 & \bar{D}\cdot F_\infty \\ \bar{L}\cdot F_\infty & \bar{D}\cdot F_\infty & F_\infty^2 \end{pmatrix} \geq 0\,.$$

If $m = \deg(L_{\mathbb{C}})$ we have

$$F_\infty^2 = 0\,, \quad \bar{L}\cdot F_\infty = m\,, \quad \bar{D}\cdot F_\infty = d\,.$$

So we get

$$d^2\,\bar{L}^2 - 2\,md\,\bar{L}\,\bar{D} + m^2\,\bar{D}^2 \leq 0\,. \tag{5}$$

1. The first minimum

From (4) and (5) we deduce

$$d^2\,\bar{L}^2 - 2\,md\,\bar{L}\,\bar{D} + m^2\,\bar{L}\,\bar{D} \leq \frac{m^2}{2} + m^2\,d\log\|e\|\,.$$

1. The first minimum

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$$e(\bar{L}) := \operatorname*{Inf}_{D' \geq 0} \frac{\bar{L} \cdot \bar{D}'}{\deg(D'_{\mathbb{C}})} .$$

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Since $d > \frac{m}{2}$ we get

$$d^2 \, \bar{L}^2 + (m^2 - 2\, md)\, d\, e\,(\bar{L}) \leq m^2\, d \log \|e\| + \frac{m^2}{2}$$

$$d(\bar{L}^2 - 2\, m\, e\,(\bar{L})) + m^2\, e\,(\bar{L}) \leq m^2 \log \|e\| + \frac{m^2}{2d} \, .$$

1. The first minimum

But

$$\bar{L}^2 \geq 2\, m\, e(\bar{L}) \qquad \text{(Zhang)},$$

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$$\bar{L}^2 + (m^2 - 2m)\, e(\bar{L}) \leq m^2 \log \|e\| + m^2 \,.$$

Since $\bar{L} = \bar{\omega}^{\otimes n}$ we have

$$m = 2n\,(g-1)$$

and

$$e(\bar{L}) = n\, e\,(\bar{\omega})\,.$$

1. The first minimum

But

$$e(\bar{\omega}) \geq \frac{\bar{\omega}^2}{4g(g-1)} \qquad \text{(Szpiro).}$$

So we conclude that

$$\log \|e\| \geq \frac{n+1}{4g(g-1)}\,\bar{\omega}^2 - C(g,n)$$

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$$\mu_1(\bar{H}^1) \geq \frac{n+1}{4g(g-1)}\,\bar{\omega}^2 - C(g,n)\,.$$

2. Secant varieties

Let $C \subset \mathbb{P}^{N-1}(\mathbb{C})$ be a smooth projective curve of genus g and $d \geq 1$ an integer. When $P_1, \ldots, P_d \in C$ are d distinct points, we let

$$\langle P_1, \ldots, P_d \rangle := \text{linear span of } P_1, \ldots, P_d \subset \mathbb{P}^{N-1}(\mathbb{C})\,.$$

The *d-th secant variety* of C is

$$\Sigma_d := \text{Zariski closure of} \bigcup_{(P_i)\in C^d} \langle P_1, \ldots, P_d \rangle .$$

Theorem (Voisin). – Assume $C \subset \mathbb{P}^{N-1}(\mathbb{C})$ is defined by a complete linear system of degree $2g - 2 + m$ with $m > 2d + 2$. Then, if $A \subset \Sigma_d$ is a linear subvariety,

$$\dim(A) \leq d - 1\,.$$

3. Higher minima

We go back to the proof of Theorem 1 in the unstable case.

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Consider the embedding

$$X(\mathbb{C}) \subset \mathbb{P}(H^0(X(\mathbb{C}), L \otimes \omega)^*) = \mathbb{P}^{N-1}(\mathbb{C}) \,.$$

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Consider the embedding

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If $e \in H^1(X, L^{-1}) = H^0(X, L \otimes \omega)^*$ and $e_{\mathbb{C}} \neq 0$, we let

$$\dot{e}_{\mathbb{C}} \in \mathbb{P}^{N-1}(\mathbb{C})$$

be its image and we consider the extension defined by e:

$$0 \to \mathcal{O}_X \to E \to L \to 0 .$$

Let $M = L(-D) \subset E$ as above,

$$d = \deg\left(D_{\mathbb{C}}\right), \quad m = \deg\left(L_{\mathbb{C}}\right).$$

By hypothesis $d < \frac{m}{2}$.

Lemma 1. The following assertions are equivalent:

a) $\dot{e}_{\mathbb{C}} \in \Sigma_{d_0}$

b) $d \leq d_0$.

3. Higher minima

Now, let us fix k, $k \leq (g-1)\,n$ and $\mu \in \mathbb{R}$. Let $e_1, \ldots, e_k \in H^1$ be k vectors, linearly independent in $H^1 \bigotimes\limits_{\mathbb{Z}} \mathbb{Q}$, such that

$$\log \|e_i\| \leq \mu\,, \quad i = 1, \ldots, k\,.$$

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$$\log \|e_i\| \leq \mu\,, \quad i = 1, \ldots, k\,.$$

Consider the linear span

$$A = \langle e_1, \ldots, e_k \rangle \subset \mathbb{P}^{N-1}(\mathbb{C})\,.$$

3. Higher minima

We have

$$\dim(A) = k - 1 \, .$$

So, by the theorem of Voisin,

$$A \not\subset \Sigma_{k-1} \, .$$

Lemma 2. – There exists $e = \sum_{i=1}^{k} n_i \, e_i \in H^1$ such that

a) $\dot{e}_{\mathbb{C}} \notin \Sigma_{k-1}$

b) $\log \|e\| \leq \mu + \mathrm{cst}.$

3. Higher minima

By Lemma 1 with $d_0 = k - 1$ we get

$$\boxed{d \geq k}.$$

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we obtain

$$\mu_k(\bar{H}^1) \geq \frac{k+n}{4g(g-1)}\,\bar{\omega}^2 - C(g,n)\,.$$

q.e.d.