Rigidity of Black Holes

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PREAMBLES I, II

PREAMBLE I

General setting

Assume $S \subset B$ two different connected, open, domains and u_1, u_2 **smooth** solutions of an equation $\mathcal{P}(u) = 0$ in B.

- **Non uniqueness:** $u_1 \equiv u_2$ in S but $u_1 \neq u_2$ in B.
- **Well posedness:** u_1, u_2 "close" in $S \Rightarrow u_1, u_2$ "close" in B.
- **Unique continuation:** $u_1 \equiv u_2$ in $S \Rightarrow u_1 \equiv u_2$ in B.
  However, u_1 may be "close" to u_2 in S, but completely different in B.

Example

$B = B(0,2), S = B(0,1/2) \subset \mathbb{R}^2$, $u_n = (x+iy)^n$ are solutions to $\Delta u = 0$, **small** in S but **large** in B.

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PREAMBLE I (Pseudo-convexity)

Theorem(Calderon-Hörmander)

Given $\mathcal{P}$ and $S = \{h < 0\}$, $dh \neq 0$, there exists a condition on h, called **pseudo-convexity** (with respect to $\mathcal{P}$) which, if satisfied at $p \in \partial S$, $\Rightarrow$**unique continuation at** p.

g-pseudo-convexity

Defining function h is pseudo-convex at p for,

$$\mathcal{P} = \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_b + B^\alpha\mathbf{D}_\alpha + C$$

If $\quad X^\alpha X^\beta \mathbf{D}_\alpha\mathbf{D}_\beta h(p) < 0, \ \forall X \in T_p(\mathcal{M}), \quad \mathbf{g}(X,X) = X(h) = 0,$

Alinhac-Baouendi example

If $h = |x|^2 - 1$ in $\mathbb{R}^{1+2}$, $S = \{h < 0\}$ and $p \in \partial S$, there exists non-vanishing smooth V, ϕ, vanishing in S and verifying $\Box\phi + V\phi = 0$ in a neighborhood of p.

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PREAMBLE I (A model problem)

Theorem[Ionescu-Kl(2008)]

Let $\mathbf{E} = \{(t,x) \in \mathbb{R}^{1+d} : |x| > |t| + 1\}$, $\phi \in C^2$ solution of

$$\begin{cases} \Box\phi = A\phi + \sum_{l=0}^{d} B^l \cdot \partial_l \phi & A, B^l \in C^0(\mathbb{R}^{1+d}). \\ \phi|_{\partial \mathbf{E}} = 0 & \end{cases}$$

Then, $\phi = 0$ on $\overline{\mathbf{E}}$.

Proof[Carleman estimates]

For any $\phi \in C_0^\infty(\mathbf{E})$, $\lambda > 0$ sufficiently large

$$\lambda \cdot \|e^{-\lambda f} \cdot \phi\|_{L^2} + \|e^{-\lambda f} \cdot D\phi\|_{L^2} \leq C\lambda^{-1/2} \cdot \|e^{-\lambda f} \cdot \Box\phi\|_{L^2},$$

with $f = \log\left((|x| - 1/2)^2 - t^2\right)$

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Problem

Given a smooth pseudo-riemannian $(\mathbf{M}, \mathbf{g})$, an open subset $O \subseteq \mathbf{M}$ and a smooth Killing vector-field Z in O. Under what assumptions does Z extend (uniquely) as a Killing vector-field in $\mathbf{M}$?

Nomizu's theorem

If $\mathbf{g}$ is **real analytic** $\mathbf{M}$ and O are connected and, $\mathbf{M}$ is simply connected $\Rightarrow$ Extension holds true.

Remark

The metric is not assumed to satisfy any **specific equation**. No assumptions are needed about the boundary of $O \subset \mathbf{M}$ and the result is **global** with only minimal assumptions on the topology

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MAIN NO HAIR CONJECTURE

Kerr spacetimes

Kerr $\mathcal{K}(a, m)$, $0 \leq a \leq m$

$$-\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

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Key properties of the Kerr spaces

- Solutions of the Einstein vacuum equations.
- **P1** Killing vector field $\mathbf{T} = \partial_t$, timelike at "infinity" ,
- **P2** Geometric properties: asymptotic flatness, smooth bifurcate sphere, global hyperbolicity,
- **P3** Non-degenerate if $0 \le a < m$,
- **P4** Killing vector-field $\mathbf{Z} = \partial_\phi$, with closed orbits,
- **P5** Real-analytic.

Definition: A vacuum space-time verifying **P1** $-$ $-$**P3** is called a *regular, non-degenerate, stationary vacuum.*

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Main Conjecture "Black holes have no hair"

Conjecture

If $(\mathbf{M}^4, \mathbf{g}, \mathbf{T})$ is *regular, non-degenerate, stationary vacuum* $\Rightarrow$ its domain of outer communication is isometric to the domain of outer communication of a Kerr spacetime $\mathcal{K}(a, m)$, $0 \leq a < m$.

- (Carter 1971): axially symmetric black holes have only 2 degrees of freedom.
- (Robinson 1975): Conjecture holds in the case of axially symmetric black holes.
- (Hawking 1973): Conjecture holds in the case of **real-analytic** spacetimes.

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LOCAL RIGIDITY

Hawking's rigidity theorem

Hawking

The event horizon of a **real analytic**, stationary, regular, vacuum spacetime is a Killing horizon, i.e. the space-time admits **another** Killing field normal to the event horizon

Main ideas

- Follows from the tangency of T to the horizon that there must exist an infinitesimal Killing field normal to the horizon.
- **(Nomizu' s Theorem)** M **real analytic,** pseudo-riemannian, simply connected, $O \subset M$, connected, open. Then any Killing v-field in O extends to a Killing field in M.

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Conjecture

Important

Analyticity should be proved not assumed !

Results without analyticity

- (Ionescu-Kl(2008)) Conjecture holds provided that a scalar identity is assumed to be satisfied on the bifurcation sphere.
- (Alexakis-Ionescu-Kl(2009)) Conjecture holds provided that the spacetime is assumed to be “close” to a Kerr spacetime.
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Local Extension

- (Alexakis–Ionescu–Kl(2009) Hawking's rigidity theorem is true, locally, in a neighborhood of a **non-degenerate bifurcate horizon**
- (Ionescu–Kl(2011) Extension of Killing vector-fields fails near points away from the bifurcate sphere of the horizon.

Main ideas

- (Friedrich–Racz–Wald) Construct the Hawking vector-field $\mathbf{K}$ in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. Have $[L, \mathbf{K}] = cL$
- Extend the vector-field $\mathbf{K}$ to a full neighborhood of S by solving a transport equation $[L, \mathbf{K}] = cL$.
- Show that the extended $\mathbf{K}$ is Killing by a **unique continuation** argument.

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Ionescu-Klainerman(2011)

Theorem 1

$(\mathbf{M}, \mathbf{g})$ Ricci flat, pseudo-riemannian manifold; (O, Z) verify:

- **A1** There exists a smooth v-field L geodesic in $\mathbf{M}$ $(\mathbf{D}_L L = 0)$,
- **A2** Z Killing v-field in O, $[L, Z] = c_0 L$.

If ∂O is **strongly pseudo-convex** $\Rightarrow$ Z extends as a Killing vector-field to a neighborhood of p.

Pseudo-convexity

$O \subset \mathbf{M}$ is strongly pseudo-convex at $p \in \partial O$ if it admits defining function f at p, s.t. for any $X \neq 0 \in T_p(\mathbf{M})$, $X(f)(p) = 0$ and $\mathbf{g}(X, X) = 0$, we have

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Theorem 2.

(O, Z) as before with ∂O smooth, **null hypersurface** in a neighborhood of $p \in \partial O$. Also $c_0 = 0$ and L null, transversal to ∂O.

$\Rightarrow$

There exists U_p and a Ricci flat, Lorentz metric, $\mathbf{g}'$ in U_p, such that $\mathbf{g}' = \mathbf{g}$ in $O \cap U_p$, but Z **does not admit an extension** as a smooth Killing vector-field for $\mathbf{g}$ in U_p.

Main Idea

Construct a null hypersurface transversal to $\mathcal{N}$ and solve a characteristic Cauchy problem.

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Main ideas of Theorem 1

- Define

$$\begin{aligned} \pi_{\alpha\beta} &:= (\mathcal{L}_{\mathbf{K}}\mathbf{g})_{\alpha\beta} \\ W_{\alpha\beta\mu\nu} &:= (\mathcal{L}_{\mathbf{K}}\mathbf{R})_{\alpha\beta\mu\nu} - (B * \mathbf{R})_{\alpha\beta\mu\nu}. \end{aligned}$$

- Prove a system of wave/transport equations of the form

$$\begin{aligned} \Box_{\mathbf{g}} W &= \mathcal{M}(W, \mathbf{D}W, \pi, \mathbf{D}\pi), \\ \mathbf{D}_L \pi &= \mathcal{M}(W, \mathbf{D}W, \pi, \mathbf{D}\pi), \\ \mathbf{D}_L(\mathbf{D}\pi) &= \mathcal{M}(W, \mathbf{D}W, \pi, \mathbf{D}\pi). \end{aligned}$$

- Use a **unique continuation** argument to conclude that W, π vanish in a neighborhood of Z.
- Role of **pseudo-convexity**

Main ideas of Theorem 1

- Define

$$\pi_{\alpha\beta} := (\mathcal{L}_{\mathbf{K}}\mathbf{g})_{\alpha\beta}$$
$$W_{\alpha\beta\mu\nu} := (\mathcal{L}_{\mathbf{K}}\mathbf{R})_{\alpha\beta\mu\nu} - (B * \mathbf{R})_{\alpha\beta\mu\nu}.$$

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GLOBAL RESULTS

Unique continuation in Kerr

Theorem (Ionescu–Kl)

Assume W, A, B, C verify

$$\begin{cases} \square_{\mathbf{g}} W = A \cdot W + B \cdot \mathbf{D} W; \\ \mathcal{L}_{\mathbf{T}} W = C \cdot W, \end{cases}$$

in a Kerr space $\mathbf{K}(a, m)$, $0 \leq a < m$.
Unique continuation holds across the level sets of h if the following $\mathbf{T}$-**conditional pseudo-convexity** property holds:

$$\mathbf{T}(h) = 0;$$
$$X^\alpha X^\beta \mathbf{D}_\alpha \mathbf{D}_\beta h < 0 \quad \text{if} \quad X^\alpha X_\alpha = X(h) = X^\alpha \mathbf{T}_\alpha = 0.$$

The function $h = r$, in the Boyer-Lindquist coordinates, verifies it.

Conjecture

Important

Analyticity should be proved not assumed !

Results without analyticity

- (Ionescu-Kl(2008)) Conjecture holds provided that a scalar identity is assumed to be satisfied on the bifurcation sphere.
- (Alexakis-Ionescu-Kl(2009)) Conjecture holds provided that the spacetime is assumed to be “close” to a Kerr spacetime.
- Both theorems have been extended by Willie Wong and Yu Pi to the case of Einstein-Maxwell equations (Kerr-Newman)

Ionescu-Kl(2008)

Strategy

Want a tensor $\mathcal{S}$, analogous to the Riemann tensor $\mathbf{R}$,

- It describes locally the Kerr spaces,
- It satisfies a suitable geometric equation of the form

$$\square_{\mathbf{g}}\mathcal{S} = A \cdot \mathcal{S} + B \cdot \mathbf{D}\mathcal{S}.$$

Mars-Simon tensor

Given a stationary space-time $(\mathbf{M}^4, \mathbf{g}, \mathbf{T})$.

$$\mathcal{S}_{\alpha\beta\mu\nu} = \mathcal{R}_{\alpha\beta\mu\nu} + 6(1-\sigma)^{-1}\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2 \cdot \mathcal{I}_{\alpha\beta\mu\nu}\big).$$

complex, self-dual Weyl field verifying

$$\mathbf{D}^\rho \mathcal{S}_{\rho\alpha\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\beta \mathcal{S}_{\beta\rho\gamma\lambda}(\mathcal{F}_\alpha{}^\rho \delta_\mu^\gamma \delta_\nu^\lambda - (2/3)\mathcal{F}^{\gamma\lambda}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}).$$

Mars-Simon tensor

- Killing 2−form $F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta, \quad \mathcal{F} = F + iF^*$
- Ernst 1-form $\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu},$
- Ernst potential $\mathbf{D}_\mu \sigma = \sigma_\mu, \qquad \sigma \to 1$ at asymptotic infinity.
- Mars–Simon tensor

$$\mathcal{S}_{\alpha\beta\mu\nu} = \mathcal{R}_{\alpha\beta\mu\nu} + 6(1-\sigma)^{-1}\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2 \cdot \mathcal{I}_{\alpha\beta\mu\nu}\big).$$

$$\mathbf{D}^\rho \mathcal{S}_{\rho\alpha\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\beta \mathcal{S}_{\beta\rho\gamma\lambda}(\mathcal{F}_\alpha{}^\rho \delta_\mu^\gamma \delta_\nu^\lambda - (2/3)\mathcal{F}^{\gamma\lambda}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}).$$

Thus it satisfies a wave equation of the form

$$\square_{\mathbf{g}} \mathcal{S}_{\alpha_1\ldots\alpha_4} = \mathcal{S}_{\beta_1\ldots\beta_4} A_{\alpha_1\ldots\alpha_4}{}^{\beta_1\ldots\beta_4} + \mathbf{D}_\mu \mathcal{S}_{\beta_1\ldots\beta_4} B_{\alpha_1\ldots\alpha_4}{}^{\mu\beta_1\ldots\beta_4}.$$

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Main results

Ionescu-Kl(2008)

The domain of outer communication $\mathbf{E}$ of a regular stationary vacuum $(\mathbf{M}, \mathbf{g}, \mathbf{T})$ is locally isometric to the domain of outer communication of a Kerr spacetime, provided that the identity

$$-4m^2\mathcal{F}^2 = (1-\sigma)^4$$

holds on the bifurcation sphere S_0.

Kerr

$$\sigma = 1 - \frac{2m}{r + ia\cos\theta}, \qquad \mathcal{F}^2 = -\frac{4m^2}{(r + ia\cos\theta)^4}.$$

Main results

Alexakis–Ionescu–Kl

The domain of outer communication $\mathbf{E}$ of a regular stationary vacuum $(\mathbf{M}, \mathbf{g}, \mathbf{T})$ is isometric to the domain of outer communication of a Kerr spacetime, provided that the smallness condition

$$|(1-\sigma)\mathcal{S}(\mathbf{T}, e_\alpha, e_\beta, e_\gamma)| \le \bar{\varepsilon}$$

holds along a Cauchy hypersurface in $\mathbf{E}$, for some sufficiently small $\bar{\varepsilon}$.

Main idea

Extend a Killing vector-field across a $\mathbf{T}$-conditional pseudoconvex hypersurface in an Einstein vacuum, using a unique continuation argument for a system of wave equations coupled with transport equations.