NATO ASI

CENTRE DE PHYSIQUE DES HOUCHES

M. T. Béal-Monod and D. Thoulouze Directors

2 June - 21 June 1982

RAYONNEMENT GRAVITATIONNEL

GRAVITATIONAL RADIATION

edited by

NATHALIE DERUELLE AND TSVI PIRAN

INSTITUT D'ÉTUDES AVANCÉES DE L'OTAN
NATO ADVANCED STUDY INSTITUTE

1983

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM · NEW YORK · OXFORD

GRAVITATIONAL RADIATION AND THE MOTION OF COMPACT BODIES

Thibaut Damour

Groupe d'Astrophysique Relativiste
Equipe de Recherche du C.N.R.S. n° 176
Observatoire de Paris-Meudon
92190 Meudon (France)

Using a post-Minkowskian approximation method supplemented by a technique of asymptotic matching, we obtain the general relativistic gravitational field outside two compact bodies (neutron stars or black holes). The equations of orbital motion of the compact bodies are deduced from the vacuum field equations by an Einstein-Infeld-Hoffmann-Kerr type approach simplified by the use of complex analytic continuation. The same process of analytic continuation allows one to push the accuracy of the calculations up to the third order: gravitational field containing cubic non-linearities and equations of motion deduced from the quartically non-linear vacuum Einstein equations. The equations of motion are explicitly written in Newtonian-like form as an expansion in powers of the inverse velocity of light up to the fifth order inclusively. The equations of motion up to c^{-4} are deduced from a generalized Lagrangian. The construction of Noetherian quantities conserved up to c^{-4} allows one to separate and investigate the c^{-5} secular kinematical effects caused by the finite velocity of propagation of gravity (Laplace-Eddington effect or "radiation damping"). These results agree with the phenomena observed in the Hulse-Taylor pulsar PSR 1913 + 16.

1. MOTIVATION

Two of the most remarkable features of Einstein's gravitational equations are:

1) their "hyperbolicity" (presence of propagation effects at a finite velocity),

2) their (infinite) non-linearity ("gravity generates gravity and influences its propagation").

It was soon realized (Einstein 1916) that the first feature implied the existence of wave-like solutions of the "linearized" vacuum field equations. Later these "linearized" waves were shown to be associated with an outgoing "energy flux" far from the system given, in the case of slow sources, by the famous "quadrupole formula" (Einstein 1918, see the lectures of K.S. Thorne and M. Walker in these proceedings):

$$\text{Energy flux} = -\frac{1}{5Gc^5}\,(d^3Q_{ik}/dt^3)^2, \tag{1}$$

where Q_{ik} is the quadrupole moment of the radiating source:

$$Q_{ik} = G\int d^3x\,\rho\,(x^i x^k - \frac{1}{3}x^2 f^{ik}), \tag{2}$$

G being Newton's constant, c the velocity of light, ρ the mass density and f^{ik} the Kronecker symbol (in these lectures we shall denote the Minkowski metric by f^{ab} with $f^{00} = -1$ and $f^{ik} = +\,(\text{Kronecker})^{ik}$).

It was first realized by Weyl (1921) and Eddington (1924) that the second feature, non-linearity, together with the very special structure of Einstein's equations implied that, not only was the gravitational field determined by the sources, but the equations of motion of the sources could also be deduced from the field equations (see section 3 below). Consequently Eddington (1924) pointed out that the existing derivations of (1) and of a correlated "energy loss" in the source could not be automatically applied to a binary star where non-linear effects are essential because they provide the binding of the system. He even pointed out that one should take into account the cubic non-linearity of Einstein's equations in any radiation damping calculation (though the situation is somewhat simpler for an "energy flux" calculation where, as shown by Landau and Lifshitz (1941), only quadratic non-linearities need to be considered and where a simple argument leads back to (1-2)). Subsequently many authors, starting with the pioneering work of Hu (1947), tried to include the effects due to the propagation of gravity in the equations of motion of gravitationally bound matter, hoping to get some kind of radiation reaction on the motion of the sources of the gravitational field. However the sought-for effect is so small that one must include many highly non-linear contributions before being sure that the result is complete. Many conflicting results were obtained but, in the early seventies, an agreement was reached between some detailed, albeit incomplete, calculations, valid only for systems where the gravitational field is weak everywhere, and a heuristic argument ("balance" between the "energy flux" (1) lost in the "wave zone" and a loss of the "near zone" "Newtonian energy" of the system) (see section 3 and the criticisms in section 15).

Although in the meantime, the search for gravitational waves had begun with the pioneering work of J. Weber (see the lectures of D. Blair, V. Braginsky and R. Drever), nevertheless the successful detection on Earth of gravitational wave signals seems still far ahead of us. In 1979, however, Taylor, Fowler and McCulloch (1979) reported the observations of a secular acceleration of the mean orbital longitude of the binary pulsar PSR 1913 + 16 : i.e., in other words, a secular diminution of the time of return to the periastron (see the lectures of D. Eardley).

While this effect had been qualitatively and quantitatively predicted on the basis of the above-mentioned heuristic argument (Esposito and Harrisson 1975, Wagoner 1975), it had not been validly demonstrated to be a consequence of Einstein's theory: on one hand because the detailed calculations were not complete enough to control all the terms of the equations of motion and were plagued by mathematical inconsistencies, and, on the other hand, because the methods of calculation did not apply to a system, like the binary pulsar, containing "compact" objects, i.e. objects with a radius $\sim G(\text{mass})/c^2$ and therefore with very strong self gravitational fields (the inconclusiveness of these calculations, as well as of other derivations, is further discussed in Sections 3 and 15). It was therefore necessary to re-examine afresh the problem of motion of two bodies in General Relativity by a method able:

1) to deal with the interaction of compact objects,
2) to include all the propagation effects of gravity without running into mathematical inconsistencies at higher orders,
3) to push the calculations up to the inclusion of all the non-linear terms which are greater than or equal to the "radiation damping" effects.

The aim of these lectures is to present a new method which fulfils all these requirements and to describe the results that have been obtained from it. Our final result justifies the conclusions obtained from the earlier heuristic arguments, and hence, in conjunction with Taylor's observations of the binary pulsar, provides a profound confirmation of the non-linear hyperbolic structure of Einstein's theory. Hence it provides also an indirect confirmation of the existence of gravitational waves.

2. OUTLINE OF THESE LECTURES

Section 3 reviews the history of the problem of motion with special emphasis on the birth and evolution of the concepts that will be useful in these lectures: asymptotic matching, radiation damping, Post-Minkowskian Approximation schemes, deduction of the equations of motion from the vacuum field equations. It contains also a brief discussion of the Post-Newtonian Approximation schemes. A short history of the other useful concepts: analytic continuation and action principles will be found in sections 7 and 13 respectively. Section 13 contains also a brief discussion of the clarifying -- though perhaps deceptively simple -- analogy between the electromagnetic and gravitational interactions ("radiation reaction", "least action principles," "conservation laws").

Section 4 presents a new strategy for analyzing and solving the problem of the motion of two compact objects. The first step of this strategy, which deals with the tidal effects on a compact object, is presented in Section 5. Section 6 uses the information from the tidal distortion of the gravitational field outside (but "near") each compact object to provide necessary boundary conditions for the gravitational field in the empty region of space-time which is outside two world tubes containing the two compact objects.

Section 7 presents a brief introduction to a mathematical tool which will be extremely useful in the following: the analytic continuation of certain integrals.

Section 8 shows how to get a particular solution for the external ("harmonically relaxed") gravitational field (outside the two world tubes of section 6) by means of a well-defined algorithm based on analytic continuation. This solution takes into account the cubic non-linearity of the "harmonic" Einstein equations.

Section 9 deduces the equations of motion from the vacuum field equations by demanding that the "harmonicity conditions" be satisfied. These "integrability conditions" applied to the quartically non-linear Einstein equations provide the third order equations of motion. These equations of motion are calculated thanks to analytic continuation, from the cubically non-linear external field only.

Section 10 shows that the particular solution of the external problem constructed in Sections 8–9, because it fulfils the necessary boundary conditions of Section 6, is the unique solution of Einstein's equations which matches two compact objects.

Section 11 describes the transformation of "retarded" equations of motion into predictive Poincaré invariant ones.

Section 12 deduces the Newtonian-like equations of motion of two compact objects, complete up to terms of order $(v/c)^5$ ($2^{1/2}$Post-Newtonian level).

Section 13 describes the deduction of the second post-Newtonian equations of motion from a "generalized" Lagrangian.

Section 14 deduces from the preceding "generalized" Lagrangian some quantities which are constant at the $(v/c)^4$ level.

Section 15 uses the preceding "conserved quantities" as a tool for studying secular kinematical effects in a binary system. The Laplace-Eddington effect is obtained in complete agreement with the observations of PSR 1913 + 16.

The Appendix contains some technical details. The references are listed by alphabetical order at the end: for instance: M. ABRAHAM (1903) Ann. d. Phys. 10, 156...

Nota Bene: pay attention to a peculiarity of notation introduced for typographical reasons: I tried to avoid as far as possible all Greek, Gothic,... letters. Therefore the space-time indices: 0,1,2,3 will be denoted by a,b,c,d,e,f,g,h, the space indices: 1,2,3 by i,j,k,...and the Gothic metric by $\underline{g}^{ab}$.

3. DIGEST OF THE HISTORY OF THE PROBLEM OF MOTION

In 1687, I. Newton showed how the orbital motion of approximately spherical extended objects could be well-approximated by the motion of point masses. This is a very important result of Newtonian physics whose extension to General Relativity is highly non-trivial, as was pointed out by M. Brillouin (1922). M. Brillouin called this schematization of an extended body by a point mass with disappearance of all internal structure: "le principe d'effacement" ("effacing principle;" perhaps a more picturesque name would be: "the Cheshire cat principle"). In Newtonian physics the proof of this "effacing principle" makes an essential use of:

1) the linearity of the gravitational field as a function of the matter distribution (which allows one to define and separate the self-field and the external field);
2) the Action and Reaction principle (which allows one to define the center of mass and to ignore the contribution of the self-field to its motion);
3) Newton's theorem on the attraction of spherical bodies.

More specifically, for a binary system constituted of non-rotating nearly spherical bodies of masses m and m', one deduces from 1) that the main correction to the point mass idealization will come from the tidal field $Gm'd^{-3}r$ (where G is Newton's constant, r is the distance away from the center of mass of the first object m, and d is the distance between the two objects). If b denotes the radius of the first object, the tidal field will deform slightly its shape: $\delta b/b = h(m'/d^3)(b^3/m)$, where h, the first Love (1909) number, is a dimensionless quantity of order unity. This deformation induces in turn a small quadrupole moment: $Q = k\, m'b^5d^{-3}$, where k, the second Love number, is a dimensionless quantity of order unity ($h = 3/5$ and $k = 4/15$ for the Earth). Finally this tidally induced quadrupole moment will create a small correction to Newton's law for point masses: $\delta F/F \sim k\,(b/d)^5$. Therefore as long as the radii of the objects are much smaller than their mutual distances, their internal structure (if they are not rotating) will be utterly negligible. We shall show in Section 5 how this result of "effacing" can be extended to Einstein's theory even, and in fact most accurately, in the case of compact objects, i.e. when the radius $b \sim Gm/c^2$. But as we shall not be able to use 1) and 2) above, we shall need a completely different approach to show that the very strong "self field" of the compact object does not contribute to its orbital motion.

From 1798 to 1825, P.S. Laplace published his "Traité de Mécanique Céleste" which contains at least two ideas which will be of importance for the following. (Incidentally, it is interesting to note that during the preparation of his famous treatise Laplace was greatly helped, especially for numerical calculations, by a collaborator: Alexis Bouvard. According to Arago, Bouvard was born "in an obscure village in a valley of the Alps, not very far from Saint-Gervais;" in fact this village was Les Houches!). The first idea introduced by Laplace is now called "asymptotic matching". Laplace was investigating the shape of a large drop of mercury resting on a plane. The basic partial differential equation that must be satisifed by the height $z = z(x,y)$ of the drop is (as first determined by Laplace):

$$(1 + (z,_x)^2 + (z,_y)^2)^{-3/2}((1 + (z,_y)^2)z,_{xx} + (1 + (z,_x)^2)z,_{yy} - 2z,_x z,_y z,_{xy}) - a.z = 0. \quad (1)$$

(a being a positive constant: $2g/A$).

(1) is a highly non-linear partial differential equation and therefore there is little hope of solving it exactly. But the idea of Laplace is that there are two regimes where (1) is amenable to a perturbative treatment:

a) when the slope is small $(z,_x)^2+(z,_y)^2 \ll 1$, (1) reduces to:

$$z,_{xx} + z,_{yy} - a.z = \text{(non-linear terms)}, \quad (2)$$

which can be solved iteratively because the leading terms are linear.

b) at the boundary of the drop the slope is not necessarily small (it even becomes infinite) but the fact that the radius of the drop is large allows one to approximate the two-dimensional problem (1) by a one-dimensional problem ($z,_Y = 0$ for a local choice of coordinates X,Y at the boundary):

$$(1 + (z,_X)^2)^{-3/2} z,_{XX} - a.z = 0, \quad (3)$$

which can be further transformed by posing: $z,_X = \tan(u(X))$ which allows one to treat the infinite slope points. Then one can look for approximations better than (3). Finally Laplace writes that the solutions of (2) and (3) must match asymptotically, that is to say, that they must approximately coincide in an open domain which is near the boundary but where the slope is small. This determines completely the unknown constants that appear in the solutions of (2) and (3). In the following sections we shall use a similar method and get similar results when dealing with Einstein's equations for two compact objects instead of (1).

The second idea introduced by Laplace is that if gravity propagates with a finite velocity c, then there should exist corrections to Newton's law for the motion of a planet which:

A) are of order v/c where v is the velocity of a planet.

B) are of a "damping" type $(-k\, v^i)$

C) will cause a shrinkage and a circularization of the orbit together with a secular acceleration of the mean orbital longitude.

Moreover, Laplace pointed out that only the last effect would be observable. The "known" secular acceleration of the moon led him to conclude that the velocity of propagation of gravity is at least seven million times the velocity of light! As we shall see explicitly in Section 12, the conclusion A) is not correct, the first order effect is compensated and the corrections due to the velocity of propagation of gravity are an expansion of the type: $(v/c)^2 + (v/c)^4 + (v/c)^5 + \ldots$ (the fact that it starts at order 2, therefore making a value of c equal to the velocity of light compatible with the astronomical observations was proved very generally by H. Poincaré in June 1905 in his attempt to describe a general Lorentz invariant gravitational interaction). However, we shall prove in Section 15 that conclusion B) is essentially correct for the $(v/c)^5$ term (the conclusion that the "residual Laplace effect" was contained in the $(v/c)^5$ term was first reached by Eddington (1924, see below) in his study of the radiation damping of a spinning rod). Finally, we shall prove that conclusion C) is not modified by the $(v/c)^2 + (v/c)^4$ terms but that it leads precisely to the observed secular orbital acceleration of the binary pulsar. Therefore from this point of view, this acceleration, that we shall call the "Laplace-Eddington effect," is a direct confirmation that gravity propagates with the speed of light and therefore an indirect confirmation of the existence of gravitational waves.

The concept of a "damping" force associated with an interaction which propagates with a finite velocity was to find its first full elaboration in electromagnetic theory. Apparently H.A. Lorentz was the first to mention, in 1892, the existence of such a "résistance." What is very interesting for the following, and premonitory of what was going to happen much later in the gravitational case is that:

1) Lorentz obtained this result by a direct calculation of the resultant force on a small extended accelerated particle due to its self field;

2) His result:

$$F^i = e^2 c^{-3} \ddot{v}^i , \quad (4)$$

was wrong by a factor 3/2!

This illustrates the difficulty of any direct calculation of a self force even in a linear theory. In fact it seems that the correct result was first obtained by a heuristic argument based on energy conservation due, not too unexpectedly, to M. Planck in 1897:

$$F^i = \frac{2}{3} e^2 c^{-3} \ddot{v}^i. \quad (5)$$

Lorentz corrected his first result (4) in 1902 and published the first complete direct calculation in 1903 (see also his book in 1909). The relativistic generalization of this celebrated "radiation reaction" force was first obtained by a heuristic argument based on energy and linear momentum conservation by M. Abraham (1903, 1904):

$$F^a = \frac{2}{3} e^2 (\ddot{u}^a - \dot{u}^2 u^a). \quad (6)$$

Then G.A. Schott (1912, 1915) obtained (6) by a direct relativistic self-field calculation.

In 1916 Einstein's theory of gravitation started its brilliant career. It was immediately clear that one would often have to resort to approximation schemes in order to draw physical conclusions from this theory. Indeed, in this same year the two main types of approximation schemes that were going to be used from then on were first clearly expounded. Einstein (1916) introduced an iterative method for solving his equations whose zeroth approximation is a flat Minkowski metric (denoted here and in the following by f_{ab}:-,+,+,+). In order to get a first approximation, Einstein transforms his equations into a diagonal hyperbolic partial differential system by using the celebrated "harmonic" coordinate condition in linearized form (this condition, sometimes named after De Donder, Lanczos, or even Lorentz, was apparently first suggested to Einstein in a private communication by W. De Sitter, see Einstein (1916)):

$$g_{ab} = f_{ab} + \underset{1}{g}_{ab} + \ldots, \tag{7}$$

$$f^{bc}(\underset{1}{g}_{ab,c} - \tfrac{1}{2}\underset{1}{g}_{bc,a}) = 0. \tag{8}$$

$$\Box\, \underset{1}{g}_{ab} = -16\pi G c^{-2}(T_{ab} - \tfrac{1}{2} f_{ab} f^{cd} T_{cd}) + \underset{2}{N}_{ab}, \tag{9}$$

where: g_{ab} is the sought-for metric of space-time, $\Box$ is the usual d'Alembertian, or wave operator: $\Delta - c^{-2}\partial_t^2$, T_{ab} is the stress-energy tensor (divided by c^2 so that it has the dimension of a mass density), the indices a,b,c,d = 0,1,2,3 and where all the non-linear and higher order terms are hidden in $\underset{2}{N}_{ab}$ and were neglected by Einstein (as well as by most other authors until fairly recently!). Equations (7-9) do not constitute an iterative algorithm because a partial differential equation has many solutions. One needs some extra conditions. Einstein augmented the algorithm by the prescription that one should solve (9) by means of the flat space retarded Green function (Lorentz's "retarded potentials"); this amounts essentially to imposing the Kirchhoff-Sommerfeld "no-incoming-radiation" boundary conditions to $\underset{1}{g}_{ab}$ (see e.g. Fock (1959) section 92). These conditions are, more generally, for $h_{ab} = g_{ab} - f_{ab} = \underset{1}{g}_{ab} + \ldots$, that rh_{ab} and $rh_{ab,c}$ be bounded and that:

$$\lim_{\substack{r\to+\infty \\ t+r/c=\text{const.}}} \left((rh_{ab})_{,r} + \frac{1}{c}(rh_{ab})_{,t} \right) = 0. \tag{10}$$

Note that this condition is imposed in the infinite past at infinite distances, such that t + r/c = const. This limiting process: $r \to +\infty$, $t \to -\infty$ with t + r/c = const. is often referred to as: going to (Minkowski) $\mathscr{I}^-$: scri minus. This condition should not be imposed, as is often mistakenly believed, at infinite distances but at constant time or even worse in the infinite future: t - r/c = const. ("wrong" outgoing wave condition instead of the "correct" no-incoming radiation condition (10)). In the following I shall refer to the complete algorithm (7,8,9,10) as a Post-Minkowskian Approximation scheme (sometimes abbreviated to PMA scheme). This terminology is, in my opinion, much better than the often used "Fast Motion Approximation" because, as is clear from the preceding discussion, the magnitude of the velocities of the sources of the gravitational field does not play any role; what is important is the constant use of the geometry of the Minkowski space-time and above all of its causality properties (use of "retarded" potentials versus the use of "instantaneous" potentials in the Post-Newtonian, sometimes called "Slow Motion," Approximation scheme to be described next). We emphasize this point here because we shall later use a Post-Minkowsian Approach together, at some point, with the simplifying assumption of "slow relative motion" (a Minkowski invariant concept) when treating the highest (cubic) non-linearities of Einstein's equations.

Simultaneously with Einstein, J. Droste (1916) and W. DeSitter (1916), motivated by the (urgent) necessity of working out the main astronomical consequences of Einstein's theory, which meant in particular estimating the first relativistic corrections in the solar system (and not just the perihelion precession of a planet around a fixed center) introduced an alternative iterative method for solving Einstein's equations which tried to keep as close as possible to Newtonian concepts. In particular they introduced from the start the assumption not only that the velocities were small: v/c <<1, but, most importantly, that the time derivatives of the gravitational field were smaller, by a factor of order v, than the space derivatives. They formalized this assumption by saying: "$g_{ij} - f_{ij}$ (i,j = 1,2,3) and $g_{00} - f_{00}$ will be of the first order, g_{0i} will be of order 3/2... The velocities $\dot{x}_i = dx_i/cdt$ are of the order 1/2..., a differentiation with respect to x_0=ct increases the order of smallness by 1/2, and such a term as , e.g., $g_{00,00}$ is of the second order..." (De Sitter 1916, p. 155-156): a very modern way of defining what is now called a Post-Newtonian Approximation scheme (abbreviation: PNA scheme). In contrast with Einstein they did not introduce a clearly defined coordinate condition right from the beginning, but they introduced coordinate conditions progressively. However the main distinction is that their assumption about the smallness of time derivatives led them to perturbative equations of the following type:

$$\Delta \underset{n\text{th order}}{g_{ab}} = \text{source terms} + \text{terms known from preceding approximations}. \tag{11}$$

Now one should beware of the fact that, contrary to what is often believed, the appearance of a Laplacian in (11) does not mean that (11) is essentially non-equivalent to (9) where a D'Alembertian appeared. In fact, the two partial differential systems (11) and (9) are, in a perturbative sense, equivalent. Nevertheless, the physical results that one will deduce from them will be different if one supplements the Post-Newtonian Approximation scheme by a prescription for solving (11) which is not equivalent to the preceding use of "retarded potentials" i.e. to the satisfaction of the Kirchhoff-Sommerfeld conditions (10). Droste and DeSitter solved (11) by using the usual inverse of the Laplacian: the familiar "instantaneous potentials." Then they checked explicitly that their (Post-Newtonian) result was equivalent, at the order they considered, to the corresponding Post-Minkowskian one. It was realized only much later that this equivalence does not persist at higher approximations and, worse, that the reiterated use of "instantaneous potentials" (even somewhat corrected for taking into account odd-time retardation) leads always to divergent integrals at some order of approximation. This means that it is an intrinsically inconsistent method of solving Einstein's equations. On the other hand, however, there are hints that "retarded potentials" can be indefinitely reiterated. The reason why "instantaneous potentials" necessarily lead to divergent integrals when used in a non-linear theory is that they correspond to expanding in powers of the retardation time r/c where r is the distance between the source point and the field point. But even the simplest retarded field: $\Phi = S(t - r/c)/r$ yields, when so expanded:

$$\frac{S(t-r/c)}{r} = \frac{S(t)}{r} - \frac{1}{c}\dot{S}(t) + \frac{1}{c^2}\frac{\ddot{S}(t)r}{2} - \frac{1}{c^3}\frac{\dddot{S}(t)r^2}{6} + \dots \qquad (11')$$

Therefore terms which grow like positive powers of r will appear and will generate similar terms in the right hand side of (11) which in turn cause (infrared) divergent integrals. A possible cure for this is to realize that (11') is valid only in the near zone ($r \ll \lambda$) and that the solutions of (11) can only be meaningful in the near zone. Then one must somehow match the general solution of (11) to a wave zone solution (see "Asymptotic Matching" below).

The next important step in the history of the problem of motion was the realization that the equations of motion of a body as a whole could be deduced from the vacuum field equations together with some knowledge of the structure of the gravitational field around the body under consideration. This was first understood by H. Weyl (1921) but his method, as well as the one used by Einstein and Grommer (1927), can only be applied to the test particle case (in the sense that the object considered has a negligible influence on the "external field," though such a "test" object is permitted to have a strong self field). In 1938, Einstein, Infeld and Hoffman, in a celebrated paper, succeeded in implementing this idea in the case of comparable masses. Taking advantage of the freedom of choice of the coordinate system, they imposed the following conditions on the "gothic metric" perturbation:

$$\underline{h}^{ab} := g^{1/2} g^{ab} - f^{ab}\,, \qquad (12)$$

$$\underline{h}^{oi}{}_{,i} + \underline{h}^{00}{}_{,0} = 0, \qquad (13a)$$

$$\underline{h}^{ik}{}_{,k} = 0. \qquad (13b)$$

Then they introduced a Post-Newtonian scheme:

$$\underline{h}^{ab}(x) = (1/c^2)\,{}^2\underline{h}^{ab}(x) + (1/c^3)\,{}^3\underline{h}^{ab}(x) + \dots, \qquad (14a)$$

$${}^n\underline{h}^{ab}{}_{,0} \sim {}^{n+1}\underline{h}^{ab}\,. \qquad (14b)$$

(14b) is formalized by writing time as: $x^0 = ct$. Basically the expansion parameter used by them, here denoted $1/c$, is the square root of the expansion parameter implicitly used by Droste and De Sitter. Introducing the nth reiterated field, i.e. the sum of the first n approximations:

$${}^{(n)}\underline{h}^{ab} = (1/c^2)\,{}^2\underline{h}^{ab} + \dots + (1/c^n)\,{}^n\underline{h}^{ab}, \qquad (15)$$

the Einstein vacuum field equations gave:

$${}^{(n+2)}\underline{h}^{00}{}_{,ss} = {}^{(n+2)}N^{00}(\dots), \qquad (16a)$$

$${}^{(n+1)}\underline{h}^{oi}{}_{,ss} = {}^{(n+1)}N^{oi}(\dots), \qquad (16b)$$

$${}^{(n+2)}\underline{h}^{ik}{}_{,ss} = {}^{(n+2)}N^{ik}({}^{(n+1)}\underline{h}^{oi},\dots), \qquad (16c)$$

where the dots in the right hand sides denote the lower approximations. Now the basic idea of their method is that if we calculated somehow the preceding approximation: ${}^{(n)}\underline{h}^{oo}$, ${}^{(n-1)}\underline{h}^{oi}$, ${}^{(n)}\underline{h}^{ik}$ then we would have now 14 equations constraining the 10 unknowns of the next approximation: the 10 (Poisson) field equations (16) and the 4 coordinate conditions (13). The system is "overdetermined". They showed that this "overdetermination" implied some further constraints on the preceding approximation. If the preceding approximation was expressed as a functional of the motion of the "particles" which are the sources of the field then these new constraints coming from the next approximation provide precisely the equations of motion of the "particles." Einstein, Infeld and Hoffmann found a very beautiful way of extracting from the overdetermined system (13) and (16), necessary constraints on the preceding approximation (which appears in the right hand sides of (16)). Using (13), they transformed (16b) and (16c) so as to exhibit some "curls" in the left hand sides: for instance (15b) became:

$$({}^{(n+1)}\underline{h}^{oi}{}_{,s} - {}^{(n+1)}\underline{h}^{os}{}_{,i})_{,s} = {}^{(n+1)}N^{oi} + {}^{(n)}\underline{h}^{oo}{}_{,oi}. \qquad (15b')$$

The left hand side of (15b') is the "curl" of the "curl" of the vector $A^i = {}^{(n+1)}\underline{h}^{oi}$. Then they used the well-known fact that the "flux" integral of a curl over any closed 2-surface is identically zero (even if the vector A becomes singular at some points within the surface) to write necessary constraints on the preceding approximation in the form of <u>2-surface integrals</u>:

$$\int ({}^{(n+1)}N^{oi} + {}^{(n)}h^{oo}{}_{,oi})\, dS_i = 0, \qquad (17a)$$

$$\int {}^{(n+2)}N^{ik} dS_k = 0. \qquad (17b)$$

The beauty of this result is that the equations of motion of a "particle" are given by some integrals over any two-surface enclosing the "particle." This means that we do not need to know precisely the internal structure of the "particle": it could be endowed with a very strong self gravitational field, it could be a black hole or a naked <u>singularity</u>; even in the most singular case the equations of motion are given by <u>finite</u> two surface integrals. This means also that we are replacing a knowledge of the precise internal structure by a knowledge of the structure of the gravitational field around the "particle" as is evident from the fact that we need to know ${}^{(n)}\underline{h}^{oo}, \ldots$ in order to compute the surface integrals (17). Einstein, Infeld and Hoffmann achieved this by prescribing a partly implicit set of rules for solving the preceding approximations: ${}^{(n)}\underline{h}^{oo}, \ldots$, so that it was not clear to what kind of objects their results could be applied. Later, in Sections 5-6, we shall use a technique of asymptotic matching to transform the knowledge of the internal structure of the source "particles" (tidally distorted compact objects) into <u>some</u> knowledge of the behavior of the gravitational field around the "particle," and we shall show that the latter partial knowledge is in fact sufficient to determine the gravitational field and the equations of motion of the objects. On the other hand, the theoretical beauty of the surface integral formulation of the equations of motion as well as its usefulness when dealing with strong field sources (no need to deal with the strong fields inside the object) are plagued by a very serious technical drawback: the surface integrals (17) are very complicated to compute explicitly. Einstein, Infeld and Hoffmann worked out "only" the first relativistic correction (v^2/c^2) to the (Newtonian) equations of motion and this entailed calculations so long that they could not publish them and that they deposited a detailed manuscript at the Institute for Advanced Study. Now the reader should remember that we need to push the calculation up to the order v^5/c^5

in order to try to explain the observed kinematical behaviour of PSR 1913 + 16! Later, in section 9, we shall see how the use of a mathematical trick (analytic continuation) allows one to tremendously simplify the calculations while still essentially respecting the spirit of the Einstein-Infeld-Hoffmann method, that is, deducing the nth approximated <u>equations of motion</u> from the <u>integrability conditions</u> of the (n+1) approximated <u>vacuum field equations</u>.

Now that we have described, in statu nascendi, the concepts that are going to be important for the following sections (except for two of them that are dealt with in sections 7 and 13) let us briefly sketch their evolution in time:

DEDUCTION of the equations of MOTION from VACUUM field equations was further investigated by Einstein and Infeld (1940, 1949) and Infeld and Schild (1949). A great clarification of the structure of the method, a correction of several flaws in the original method, as well as its extension to the Post-Minkowskian Approximation case is due to the excellent work of Kerr (1959, 1960). Infeld (1954, 1957) and Infeld and Plebanski (1960) discovered "experimentally," so to speak, that the calculations could be greatly simplified by the formal use of "good delta functions" playing the role of effective sources of the gravitational field. But they could never give any sound theoretical basis to their use of these "delta functions," even on a formal level where only consistency is required. Their only justification, and this is true for all the other works using "delta functions" until very recently, rested on the "experimental" agreement with the Einstein-Infeld-Hoffmann results at order $1/c^2$. However in the last two years, the constraints coming from the structure of Einstein's equations when one requires formal consistency of the use of "delta functions" have been investigated by Bel, Damour, Deruelle, Ibañez and Martin (1981). Moreover, one possible sound theoretical justification for their use in computing the gravitational field, together with a proof that the equations of motion deduced from them were effectively the integrability conditions of the next approximation, has been given by Damour (1980) (see also section 9 below). Other methods using only vacuum field equations to get the equations of motion and which are the descendants of the Weyl (1921) approach are quoted in the next paragraph.

ASYMPTOTIC MATCHING was introduced in General Relativity by V. Fock (1959), section 87), who matched a Post-Newtonian near zone expansion to a wave zone expansion. It was first applied to the problem of motion by F.K. Manasse and J.A. Wheeler (Manasse 1963) in a study of the tidal distortion and the radial infall of a small black hole into a large one. In 1969 Burke and Thorne proposed the use of asymptotic matching to the wave zone as a way of getting boundary conditions for the Post-Newtonian near zone expansions. This is a nice way of supplementing the PNA schemes by a prescription for solving the Poisson equations (11) which extends the validity of the PNA schemes beyond their usual limits: non-asymptotic flatness and divergencies due to the systematic use of "instantaneous potentials." But there remain problems associated with this method: the use of an outgoing-wave condition instead of a no-incoming radiation one, and the possibility of having higher order terms matching back to a low order. Moreover, Walker and Will (1980) pointed out a flaw in Burke's (1971) paper as well as in the corresponding section of Misner, Thorne and Wheeler (1973). Work aimed at meeting this criticism has been undertaken by Kates (1980a, 1980b). However the consistency of the whole scheme has never been checked and little attention, if any, has been given to time-even post-Newtonian terms, as well as to the role of non-linearities. In 1974 Demianski and Grishchuk applied a technique of asymptotic matching to the problem of the motion of a black hole around an object of comparable mass. Nearly at the same time D'Eath (1975a,b) treated with a similar technique but with more completeness the problems of a test-rotating black hole in a background space time and of a binary black hole system (PNA order $1/c^2$). More recently Kates gave a general discussion of the use of asymptotic matching techniques in General Relativity (Kates 1981) and devised a combination of Burke's and D'Eath's types of approaches when dealing with the motion of a binary system containing possibly compact objects (Kates 1980a,b). (see also Vilenkin and Fomin 1978).

THE POST-MINKOWSKIAN APPROXIMATION stayed dormant for a long while and was revived by Bertotti (1956) (first approximation). The second approximation was first tackled by Bertotti and Plebanski (1960). Other developments are due to Havas (1957), Kerr (1959), Lavas and Goldberg (1962), Kühnel (1963), Stephani (1964), and Schmutzer (1966). Then a new approach to the post-linear formalism was devised by Thorne and Kovács (1975), Crowley and Thorne (1977), Kovács and Thorne (1977) and applied by them to the calculation of bremsstrahlung during (possibly fast) small angle gravitational scattering. More recently a pioneering work of Rosenblum (1978, 1981) has spurred a detailed investigation of the Post-Minkowskian method beyond the linear order: Westpfahl and Göller (1979) have computed some post-linear (second Post-Minkowskian: 2PMA) equations of motion after using some ad hoc regularization (later these equations of motion have been applied by them to the small angle gravitational scattering case: work reported by K. Westpfahl at the Ringberg workshop, 1981, of which I know no published reference). Bel, Damour, Deruelle, Ibañez and Martin (1981) have investigated the formal consistency of the use of point masses and the constraints on the regularization procedures, calculated explicitly a 2PMA gravitational field as well as 2PMA equations of motion and transformed these equations into an ordinary differential system (see section 11). The behaviour at infinity of this 2PMA gravitational field has been worked out by Deruelle (1982). Finally Damour (1982) has investigated the 3PMA and worked out explicitly the 3 PMA equations of motion in Newtonian-like form neglecting terms of order c^{-6}(see section 12) (the necessity of including the contribution from the 3 PMA in a radiation reaction calculation had been first pointed out by Eddington (1924)).

THE POST-NEWTONIAN APPROXIMATION was further investigated in a remarkable work of Lorentz and Droste (1917). (This work seems to have been completely forgotten and is never qoted. I found it serendipitously in Lorentz's Collected Papers while writing up these lectures. A great historical surprise, expounded in Section 13, was contained in this work, like a genie in a bottle.) The work of Levi Civita (1937a, 1937b, 1950) clarified the hypotheses used in the PNAs and stressed the importance of proving the "effacing principle" (or "Cheshire cat principle"). Then came the classical papers of Einstein, Infeld and Hoffmann (1938), Fock (1939) and his school (Fock, 1959), and Papetrou (1951). Further, more complete (and/or more accurate),investigations are due to: the Polish school (see Infeld and Plebanski 1960), Peres (1959, 1960), Carmeli (1964, 1965), Chandrasekhar and his collaborators (see Chandrasekhar 1965 as well as many subsequent papers in the Astrophysical Journal), to Synge (1970), the Japanese school (Ohta et. al. 1973, 1974; Okamura et. al. 1973), Spyrou (1975) and Anderson and Decanio (1975). More recent works are quoted in the next paragraph.

RADIATION DAMPING in General Relativity was first investigated by Eddington (1924). In his supplementary note n° 8, he derived the loss of energy of a spinning rod by a direct near-zone radiation damping approach and not, as Einstein (1918), by a wave-zone energy flux computation (the two results agree though). He pointed out that the physical mechanism responsible for this damping was the effect discussed by Laplace: "If gravitation is not propagated instantaneously, the lag may cause tangential components of the force to occur, so that there will be a couple presumably opposing the rotation...We now know that the first order effect which Laplace expected is compensated; but the loss of energy (1)(the "quadrupole formula") is actually the residual Laplace effect...". However, he concluded that the agreement with the quadrupole flux formula (1.1) was validly demonstrated only for systems which are not gravitationally bound ("linearized theory") and that cubically non-linear terms should be taken into account in the study of gravitationally bound systems, so that in "the problem of the double star...we cannot be sure that even the sign of (1) is correct." This last doubt was increased by many later works: Hu (1947, see also 1982) found an energy gain. Infeld and Scheidegger believed in the absence of any damping (Infeld and Scheidegger 1953, 1955). This view was criticized by Goldberg (1955). In 1957 Havas discussed the contribution to radiation damping coming from the first Post-Minkowskian approximation. The result of a more complete investigation, of an improved Post-Newtonian type, by Peres (1959 a,b,1960) was, in the case of circular orbits, a damping (energy loss) in agreement with the "quadrupole formula" (however, and contrary to what Walker and Will (1980) seem to believe, this agreement is partly fortuitous; a scrutiny of the results of Peres (1960) shows that they are not entirely correct and would not agree with the "quadrupole formula" for elliptic orbits). The first attempt, and the only one until the work presented in these lectures, to work out complete equations of motion up to and including radiation reaction is due to the remarkable work of Carmeli (1964, 1965). The results however are not in agreement with the results presented below. Neither is the "quadrupole" energy loss recovered even for circular orbits. A scrutiny of the work of Carmeli shows that this disagreement is due both: to problems linked with the occurrence of divergent integrals (of infrared and ultraviolet type)and other meaningless quantities which are "regularized," and, to problems linked with the method used for solving Einstein's equations. In 1965 Smith and Havas found antidamping, which is not surprising because they were using only a first Post-Minkowskian approximation whereas three iterations (as done in the works of Peres and Carmeli) were necessary. In 1969 appeared the work of Infeld and Michalska-Trautman which investigated only time-odd effects: they obtained a result which agrees with the "quadrupole formula" for circular orbits, but this agreement would not be preserved in the elliptic case (contrary to what is generally believed). In the same year, Burke, using a matching technique described above, introduced a resistive potential and its associated radiation damping force acting on the mass m located at z:

$$F^i = -\frac{2}{5} c^{-5} m z^k Q^{(5)}_{ik} \quad , \tag{18}$$

where Q_{ik} is the quadrupole moment of the system:

$$Q_{ik} = \Sigma \quad Gm(z^i z^k - \frac{1}{3} z^2 f^{ik}) \ . \tag{19}$$

The validity of (18) was proved by Thorne (1969) in the special case of slight perturbations of a spherical star. The proof by Burke (1971) for the general case was flawed (Walker and Will 1980). Therefore, as in the case of Planck's heuristic derivation of the electromagnetic damping force (5) (which came after the partly incorrect direct calculation by Lorentz), the main reason that can be invoked in favour of the general validity of (18) is its ability, if one assumes the existence of "good" conservation laws at lower orders, to cause secular losses of the energy and of the angular momentum in the near-zone that agree with the corresponding (quadrupole) fluxes in the wave zone. However it is interesting to note that our final result (section 15) will contain a damping force different from (18) but still in agreement with the "quadrupole" secular losses. In fact the coordinate freedom of General Relativity not only prevents one from comparing directly two partial results (like two "damping forces") but, in fact, makes meaningless any partial result: only a complete determination of all the terms of the equations of motion together with a knowledge of the gravitational field has any operational meaning in General Relativity.

The first direct calculation, "à la Lorentz," of a radiation reaction force which obtained a result in agreement with the "quadrupole" losses is due to Chandrasekhar and Esposito (1970). This work was, however,as was pointed out by Ehlers, Rosenblum, Goldberg and Havas (1976), blemished by some mathematical inconsistencies (divergent integrals). This type of approach (extended fluid sources, weak field everywhere, Post-Newtonian Approximation) has been clarified, systematized and made more rigorous by the works of Papapetrou and Linet (1981), McCrea (1981), Kerlick (1980),and Breuer and Rudolph (1981). The last two works use an improved PNA scheme due to Ehlers (1980) which postpones (but does not prevent) the appearance of divergencies. In fact, all the preceding works become mathematically inconsistent at some approximation level (divergent integrals due to the post-Newtonian near zone expansions (11')). On the other hand, the work of Kates (1980) is aimed: at curing these divergencies (by using a Burke-Thorne approach) and at extending the validity of the approach to binary systems containing possibly compact objects (by using asymptotic matching to the "body zone"). However, neither has the overall consistency of Kates' approach ever been checked, nor has the role of time-even post Newtonian terms been clarified. A different approach to the problem of radiation reaction has been introduced by Schutz (1980). This approach has been extended recently to the gravitationally bound case by B. Schutz and Futamase.

Starting from the work of Bel,et.al. (1981), Damour and Deruelle (1981a) computed the Newtonian-like equations of motion of two slowly-moving point masses during a small-angle gravitational scattering (this restriction of validity is due to the lack of the cubically non-linear terms). Their equations of motion include "radiation reaction forces" which imply a net mechanical angular momentum loss which agrees with the quadrupole formula in the small angle scattering limit (the energy loss which depends on terms of order G^3 could not be computed). When neglecting these "radiation reaction forces", they showed (Damour and Deruelle 1981b) that these equations of motion could be deduced from a generalized Lagrangian (for the meaning of this, see Section 13). They studied the symmetries of this generalized Lagrangian and deduced therefrom 10 conservation laws (Damour and Deruelle 1981c) (see Section 14). On the other hand, Linet (1981), starting from the work of Papapetrou and Linet (1981) valid for everywhere-weak field gravitationally bound systems, calculated the time-odd part of the relative acceleration of two extended objects and showed that it could be gauge transformed into Burke's expression (18). However, as said above, the meaning of such a comparison between partial results ("time-odd" only) is a priori unclear because any part of the acceleration can be transformed at will (even into zero) by a suitable coordinate transformation.

As the preceding review has been certainly incomplete and perhaps biased by the author's point of view, the reader is urged to consult: the reviews of Goldberg (1962); Ehlers, Rosenblum, Goldberg and Havas (1976); Ehlers (1980); Walker and Will (1980); Thorne (1980); Cooperstock and Hobill (1982); as well as the proceedings of the third Gregynog relativity workshop (Walker 1979), of the 67th Enrico Fermi School (Ehlers

1979), and the account of the Round Table on the equations of motion (moderator: A. Ashtekar) in these proceedings.

In conclusion, up to 1981:

1) there existed no complete calculations of the equations of motion (and of the gravitational field) of a binary system up to the radiation reaction terms.
2) all the detailed calculations (generally complete only for time-odd terms) were inconsistent at higher orders of approximation, or, even their consistency at low orders had not been checked.
3) most of the detailed calculations were valid only for non-compact objects (weak self fields). The other ones were in want of a firmer footing.

4. "DISCOURS DE LA METHODE"

Before embarking on any details of the method of approximation that will be used, I would like to comment briefly on its connection, or the lack thereof, with what is rigorously known, in a "French mathematical sense", about Einstein's theory. At this point the reader should have a look at the patch-work picture of space-time, which is reproduced in this book. This drawing represents the different domains of validity (or, sometimes, what one thinks or hopes they are) of some of the main approaches used to deal with Einstein's theory and discussed in these proceedings. The best proofs of existence and uniqueness of solutions of Einstein's equations have been obtained in the study of the Cauchy problem (Choquet-Bruhat 1952, see the lectures of J. York in these proceedings) and their domain of validity covers the violin-shaped region in the middle of the drawing: the Cauchy development of initial data given on a space-like hypersurface. However we shall use in the following global harmonic coordinates, boundary conditions in the infinite past (3·10), and we shall solve Einstein's equations by reiterating a flat space Green function (instead of the successively improved curved space Green functions used in the existence proofs). For the moment only partial mathematical results are known which can give us confidence that we are on a right track: Choquet-Bruhat, Christodoulou and Francaviglia 1979, Choquet-Bruhat and Christodoulou (1980), Friedrich (1981, 1982) (see also the contribution of H. Friedrich to these proceedings), and Christodoulou and Schmidt (1980). Nevertheless let us proceed: "Es muss sein!" (Beethoven 1826).

In order to try to meet the requirements listed at the end of Section 1 we shall adopt the following strategy (similar to the one used by Laplace in his study of the shape of a large drop of mercury, see Section 3): to split the problem of the gravitational interaction of two compact bodies in two parts:

a) the internal perturbation schemes (one for each body) where one studies the small perturbation of the internal structure and of the gravitational field inside and outside each body due to the (tidal) influence of its faraway companion,

b) the external perturbation scheme (one for the system) where one studies the (post-post-linear) perturbations of flat space, due to the presence of the two bodies, in a vacuum region outside two 2-surfaces enclosing the bodies.

To fix one's imagination in the case of PSR 1913 + 16 which is very probably constituted of two neutron stars of radii $\sim$ ten kilometers, a million kilometers apart, one can think of the internal regions as extending up to two hundred kilometers away from each pulsar and of the external region as staying at least a hundred kilometers away from each pulsar.

After having so analyzed the problem, we shall take up the following method:

1) Studying, by means of an internal perturbation scheme, the distortion of the gravitational field outside a compact object caused by the influence of its far-away companion (Section 5).

2) Converting, by means of a very general coordinate transformation linking internal and external variables, the information acquired in step 1) into information about the behaviour of the gravitational field in the external region (expressed as a function of the external coordinates and as a functional of a "central world-line" defined by the transformation between internal and external coordinates). This information on the external gravitational field will play the role of boundary conditions for the external perturbation scheme (section 6).

3) Finding a particular solution of the "harmonically relaxed" vacuum external perturbation scheme by means of a well-defined algorithm based on analytic continuation (introduced in Section 7): this solution is a functional of two free world-lines in Minkowski space-time and takes into account the cubic non-linearity of the "harmonic" Einstein equations (Section 8).

4) Showing that this particular solution will be a solution of the complete (non-relaxed) vacuum Einstein equations if and only if the "central" world lines satisfy some equations of motion. This result can be generalized to the quartically non-linear relaxed Einstein equations and yields the required cubically non-linear equations of motion (Section 9). This method of getting the equations of motion, only from the vacuum field equations and the external gravitational field, is a generalization (to higher orders and to a PMA) and a simplification (thanks to analytic continuation) of the approach of Einstein-Infeld-Hoffmann and Kerr sketched in Section 3.

5) Checking that the particular solution of the vacuum Einstein equations now constructed (by implicitly replacing the free world lines of step 3) by solutions of the equations of motion of step 4)) is the unique solution of the cubically non-linear external perturbation scheme which satisfies the boundary conditions of step 2) near each compact object and the Kirchoff "no incoming radiation" conditions (Section 10).

These five steps are therefore sufficient to prove (under several plausible technical assumptions and one plausible physical assumption) that the gravitational field and the equations of motion obtained are effectively the gravitational field outside and the equations of motion of two compact objects. The plausible physical assumption is the set of hypotheses used in step 1). However it is desirable to check explicitly all these plausible assumptions by perfecting the preceding method by a sixth step (that I shall leave to future work):

6) Checking that one can fit an explicit model of a compact object into the previously constructed external gravitational field.

Similarly, we shall leave to future work: the important task of estimating the errors entailed by the approximation method (the method used seems to guarantee that they are finite but one must check their smallness); to check the technical assumptions used (for instance regarding the existence and behaviour of solutions of the equations of motion); to study the behaviour of the gravitational field, constructed in steps 3) and 4), at "infinity."

5. INTERNAL PERTURBATION SCHEME

Assuming the knowledge of the internal structure $T_{ab}^{(0)}$ and of the gravitational field $g_{ab}^{(0)}$ of one isolated body the internal scheme consists in studying the small perturbations thereof caused by the presence of a faraway companion. For simplicity we shall make the following physical hypotheses about the body one considers:

1) the body is compact, i.e., either it is a black hole or its radius $= b \sim G(\text{mass})/c^2$ (this hypothesis can be relaxed to: $(b/d)^5 \ll (v/c)^5$ where d is the distance to the companion and v the orbital velocity (Damour 1981)),

2) the body is non-rotating and spherically symmetric before the interaction (this hypothesis can be relaxed to slow rotation: spin velocity << c (Damour 1982)),

3) the internal perturbations must tend to zero when either m' (the mass of the companion) or d^{-1} tends to zero, and must vary on the same time scale as d, which is much slower than the internal time scale of a compact body: Gm/c^3 (this means physically that we consider only the (tidal) perturbations caused by the companion and exclude the perturbations due to some internal mechanism which could trigger for instance some fast vibrations. Moreover, it must be noted that this "slow internal motion " hypothesis is in no way incompatible with the Post-Minkowskian Approximation scheme which will be used for the external field).

Technically one introduces some internal coordinates X^a with:

$$X^0 = T$$
$$X^1 = R\sin\theta\cos\phi \qquad (1)$$
$$X^2 = R\sin\theta\sin\phi$$
$$X^3 = R\cos\theta \quad .$$

(Beware of notations: in this section R denotes $(X^iX^i)^{1/2}$ and d the distance to the companion, in later sections R will denote the distance to the companion). One looks for a perturbed metric:

$$g_{ab}(X^c) = g_{ab}^{(o)}(X^i,0) + h_{ab}(X^c) + \ldots, \qquad (2)$$

with:

$$g_{ab}^{(o)}dX^a dX^b = -A(c^2R/Gm)dT^2 + B(c^2R/Gm)\,dR^2 + C(c^2R/Gm)R^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (3)$$

where outside the object (Birkhoff's theorem):

$$A = B^{-1} = 1 - 2Gm/c^2R\ , \quad \text{and } C = 1. \qquad (4)$$

We have denoted by m the Schwarzchild mass of the isolated object (a constant). Via the matching of the next section the same constant m will appear in the "boundary conditions" for the external field and, therefore, via the uniqueness result of section 10 the same constant m will appear in the external field and in the equations of motion. The perturbed metric must satisfy Einstein's equations:

$$E_{ab}(g^{(o)} + h + \ldots) = 8\pi\, Gc^{-2}(T_{ab}^{(o)} + s_{ab} + \ldots\ ), \qquad (5)$$

where E_{ab} is the Einstein tensor $(R_{ab} - \frac{1}{2}Rg_{ab})$ and s_{ab} the first order perturbation of the stress-energy tensor · The dots in (2) and (5) mean that we start looking at the first order perturbation only. However, we do need to consider higher order perturbations in order to be consistent with the external scheme. Our strategy consists in studying first the linearized perturbation h_{ab} and to deduce therefrom the functional form of the general non-linear perturbation (eqn (17) below). The information contained in this functional form will be sufficient for our purposes.

A natural tool for studying the first order metric perturbation h_{ab} is the Regge-Wheeler formalism (Regge and Wheeler 1957, Campolattaro and Thorne 1970, Zerilli 1970). It consists in expanding h_{ab} in tensorial spherical harmonics:

$$h_{ab}(X^c) = \sum_{L=0}^{\infty}\ \sum_{M=-L}^{+L}\ \sum_{A=1}^{10} H^L_{M(A)}(\hat{R}, T)\, Y^{M(A)}_{Lab}(\theta,\phi), \qquad (6)$$

where we have introduced the radial variable in units of the small length scale:

$$\hat{R} \equiv \frac{c^2R}{Gm} \quad . \qquad (7)$$

A $L = 0$ term in (6) would correspond to a radial vibration of the body with no change of the gravitational field outside the body (apart from a constant change of the mass which can be incorporated in (4)). Hence such a term would not be caused by the influence of the companion but only by some internal mechanism and therefore by the hypothesis 3) ; we shall not include any $L = 0$ term in (6). Similarly a $L = 1$ term in (6) would correspond to a dipolar vibration of the body with no physical change of the gravitational field outside the body (apart from a constant small Lense-Thirring term which is not present by hypothesis 2)). This has been shown by Campolattaro and Thorne (1970) and Zerilli (1970), who interpreted the coordinate system where the mathematical change of the gravitational field outside the body is zero as a Center of Mass system. As before, in this Center of Mass system the remaining internal perturbations can only be due to some internal mechanism and should be discarded. Therefore, with our hypotheses, there exists a coordinate system such that the series (6) starts only at $L = 2$. We shall in the following use such an internal "Mass Centered" coordinate system.

When $L \geq 2$ we can simplify (6) by using a Regge-Wheeler gauge. In this coordinate system only 6 independent functions of R and T are left for each L, M (instead of 10 in general). Then the L,M part of h_{ab} can be written as:

$$h_{ab}^{LM} = \begin{pmatrix} \frac{\hat{R}-2}{\hat{R}} H_0 & H_1 - h_0 \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & + h_0 \sin\theta \frac{\partial}{\partial\theta} \\ \text{sym.} & \frac{\hat{R}}{\hat{R}-2} H_2 - h_1 \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & + h_1 \sin\theta \frac{\partial}{\partial\theta} \\ \text{sym.} & \text{sym.} & \hat{R}^2 K & 0 \\ \text{sym.} & \text{sym.} & \text{sym.} & \hat{R}^2 \sin^2\theta K \end{pmatrix} Y_L^M(\theta,\phi) \qquad (8)$$

where we have suppressed the indices L, M on the 6 functions $H_0, H_1, H_2, K, h_0, h_1$. These functions satisfy a partial differential system whose right hand side is given by the L,M projection of the perturbed stress-energy tensor s_{ab} and therefore depends explicitly on the internal structure of the body. This differential system is greatly simplified by using our hypothesis 3) which implies that we can in first approximation neglect the time derivatives compared to the space derivatives (this can be checked a posteriori because we shall see that effectively we shall obtain some functions H_0, ... whose length scale is $G\,m/c^2$ but whose time scale is the external time scale). Then we need only to consider stationary perturbations (quasi-stationary tides) (Manasse 1963). It has been shown by Regge-Wheeler (1957) that for vacuum stationary perturbations:

$$H_0 = H_2 \ (= H \text{ say}) \qquad H_1 = 0 \qquad h_1 = 0 \qquad (9)$$

Then one can find in vacuum a decoupled second order differential equation for $H = H_0 = H_2$ for instance (Edelstein and Vishveshwara 1970 , Demianski and Grishchuk 1974):

$$\hat{R}(\hat{R}-2)d^2(H/\hat{R}(\hat{R}-2))/d\hat{R}^2 + 3(2\hat{R}-2)d(H/\hat{R}(\hat{R}-2))/d\hat{R} - (L-2)(L+3)\,H/\hat{R}(\hat{R}-2) = 0. \qquad (10)$$

The general solution of this second order differential equation contains 2 arbitrary constants. For instance, when $L = 2$, one finds for the general quadrupolar H perturbation in vacuum, i.e. outside the body:

$$H = D(\hat{R}(\hat{R}-2) + k\,\hat{R}(\hat{R}-2) \int_{\hat{R}}^{\infty} 5dx/(x^3(x-2)^3)\,). \qquad (11)$$

The dimensionless constant k is a relativistic generalization (Damour 1981) of the second Love number (Love 1909) which was introduced in Section 3. It is, in a sense, a dimensionless measure of the yielding of the object to an external tidal solicitation. It depends on the internal structure of the body (equations of state,...) and can be determined for an ordinary body (not a black hole) by imposing the regularity of the metric perturbation H, K, h_0 at the center of the body and when crossing the surface of the body (see e.g., Thorne and Campolattaro 1967). By our hypothesis 1) we have $\hat{R} \sim 1$ at the radius of the object, therefore as there are no other scales in the problem, k must be of order unity (like the non-relativistic one):

$$k \sim 1 \qquad (12)$$

(More generally for non-necessarily compact objects of dimensionless radius $\hat{b}$, one will have $k \sim \hat{b}^5$ which allows one to justify the remark after hypothesis 1)). In the case of a black hole, k is determined by imposing the regularity of metric perturbation on the future horizon: in this case one finds $k = 0$ (in agreement with D'Eath 1975a). Incidently, one should not conclude from this result that there are no tidal responses of a black hole to an external solicitation: such a non-zero response is contained in the first term of the righthand side of (11): $\hat{R}(\hat{R} - 2)$ which differs from the usual term (in absence of any object): $\hat{R}^2$.

On the other hand the second constant D cannot be determined by internal considerations only but must be obtained by somehow matching out (11) to the yet to be determined "external field." (Manasse 1963, D'Eath 1975). When $\hat{R} \gg 1$, but still $R \ll d$, I.E. in the outer part of the region where we use the internal scheme, (11) becomes:

$$H \sim D\,(\hat{R}^2 - 2\hat{R} + k/\hat{R}^3 + O(1/\hat{R}^4)). \qquad (13)$$

To such a metric perturbation $\sim D\hat{R}^2$ corresponds when $\hat{R} >> 1$ a curvature $\sim Dc^4/G^2m^2$. It is fairly obvious that this must correspond to the curvature of the far field of the companion $\sim Gm'/c^2d^3$, m' being the mass of the companion and d the distance between the two bodies; hence:

$$D \sim \frac{G^3}{c^6} \frac{m^2m'}{d^3} \quad \text{(at lowest order)} \tag{14}$$

(In fact one does not need to appeal to any "obviousness": by imposing only the finiteness of the preceding curvature as $G \to 0$ and by using the results of section 6 and 10 at the lowest order in G one can prove that this curvature is effectively given, in lowest order, by the linearized far field of the companion.). A similar argument for $L > 2$ leads to the introduction of higher orders Love numbers k, and overall coefficients D (for each L there are several of these, corresponding to the different independent metric perturbation functions) with the result that:

$$k_L \sim 1, \tag{15a}$$

$$D_L \sim (\frac{G}{c^2})^{L+1} \frac{m^L m'}{d^{L+1}} \quad \text{(at lowest order).} \tag{15b}$$

At first sight it would seem that the information that we have obtained about internal perturbations: (12) (14)(15) is much too vague to be of any use in the problem of transforming the knowledge of the internal structure of the bodies (now coded in the generalized Love numbers k_L) into a knowledge of the behaviour of the gravitational field "near" but outside the objects in the external approximation scheme (remember from section 3 that it is precisely this kind of knowledge that we need in order to get the equations of motion of the objects by an approach à la Einstein-Infeld-Hoffmann). Moreover we have considered up to now only the lowest order internal perturbations (linearized in h_{ab}, lowest order in the coupling to the companion, neglect of some time derivatives). However, we are going to show that the knowledge so acquired about the functional form of the lowest order internal perturbation can be extended to higher orders (including all kind of non-linearities and inclusion of time derivatives) and, as we said earlier, we shall see later that this information will be sufficient for our purposes.

For convenience let us put c = 1 and denote a product of the type:

$$G^p m^{p-q} (m')^q \text{ with } \underline{\underline{q \geqslant 1}} \text{ by } G'^p, \tag{16}$$

up to now we considered mainly the case $q = 1$, but later we shall include $q > 1$.

Then we can summarize our results (2-15) by:

$$g_{ab}(X^i,T) = g_{ab}^{(0)}(\frac{X^i}{Gm}, 0) + \sum_{p\geqslant 3} G'^p g_{ab}^{(p)}(\frac{X^i}{Gm}, d(T), k) \quad (+\ldots) \tag{17}$$

where the dots denote the higher order terms that we still have to include, and where, as indicated by the somewhat symbolic notation, the functions $g_{ab}^{(p)}$ depend:

- on space only through the "reduced" spatial coordinates: X^i/Gm
- on time only through variables linked with the companion: distance, velocity...
- and, apart from that, depend only on dimensionless numbers of order unity: pure numbers (2,5,...as they appeared in (11))and various Love numbers $k \sim 1$. It is interesting to note at this point that the appearance of the pure numbers is due essentially to Birkhoff's theorem (universality of the functional form of the unperturbed metric outside the body when expressed in reduced variables) and that the appearance of the other numbers of order unity (k) is due to the hypothesis of compactness.

Now we can compute the higher order terms by iteration: we plug (17) in the Einstein equations (5) and "grind", keeping now non-linear terms and time derivatives. Because of the functional form (17) we can now check effectively, a posteriori, that time derivatives generate only higher order terms. All these higher order terms can be considered, together with terms coming from the right hand side of (5), as effective sources for the higher order metric perturbations (the dots in (17)). But as the functional form of (17) is preserved by the"grinding" we can look for higher order metric perturbations with the same functional form as (17). Moreover these source terms are, by construction, multiplied by G'^4 at least. Note that now G'^p can contain m' more than once. The most general higher order metric perturbation will be, at each iteration step, the sum of a particular solution of the inhomogeneous equation of order G'^4 at least, and of the general solution of the homogeneous equation which is the same as the one considered before when dealing with the lowest order perturbations. Therefore the same reasoning as before will allow us to conclude that this general homogeneous solution is of the form (17) in a suitable coordinate system. Finally we end up with the interesting result that the most general fully non-linear internal metric perturbation can be written as (17) where now we can <u>discard the dots.</u>

Before deducing from (17) the general behaviour of the gravitational field outside the body let us comment on one of the most restricted formulations of what was introduced in section 3, without being precisely formulated, under the name of the <u>"principle of effacing internal structure"</u> or <u>"Cheshire cat principle."</u>

The lowest order term, in the metric outside the body, where the internal structure begins to show up, is in the second term of the right hand side of (11) (and in similar terms for the other metric functions): the term which is multiplied by the Love number k whose value depends on the details of the internal structure. As we see from (11) this term can be expanded in inverse powers of $\hat{R}$ when $\hat{R} \gg 1$:

$$H_{\text{dependent}}^{\text{structure}} \sim Dk\,(\hat{R}^{-3} + O\,(\hat{R}^{-4})) \tag{18}$$

Replacing D by (14) and $\hat{R}$ by (7) we get:

$$H_{\text{dependent}}^{\text{structure}} \sim \frac{G^6}{c^{12}}\, k\, \frac{m^5 m'}{d^3 R^3}\,. \tag{19}$$

Therefore when viewed in the external scheme (next section) this term is of extremely high order (G^6). As we shall compute the equations of motion only from the external field we see that the internal structure will show up in the equations of motion of each body as a whole only at the negligible order G^6. Before this order the compact bodies appear, in the external scheme, only through (one central world-line, see next section, and through) one constant parameter (the "grin" of the cat): the mass m. In other words it means that up to and including order G^5 one could replace any two compact bodies (neutron stars with any kind of equation of state,...) by two black holes (Damour 1981). As indicated at the beginning of this section this result can be somewhat generalized by relaxing the hypotheses that were needed to prove it: for instance instead of supposing compactness it is sufficient to require that the radius b of each body is such that $(b/d)^5$ is negligible compared to what we wish to compute (in the case at hand where we are looking for high order radiative effects it means $(b/d)^5 \ll (v/c)^5$ where v is the orbital velocity). However the reader should note that, although the final result about the order of magnitude of the tidal effects would have been the same had we used the Newtonian formulae of section 3, it was necessary to use the preceding relativistic machinery in order to give meaning to and to prove these formulae in the strongly relativistic case of compact objects. The generalization to rotating compact objects is less easy, because if they are very rapidly rotating they will be no longer spherical and their shape, and their quadrupole moment (even at zeroth order) will depend on their internal structure. This is why it has been possible to generalize the preceding results (and the entire approach of the next sections) only in the case of slowly spinning objects (Damour 1982).

At this point, it is convenient, for facilitating the use of our result (17) in the next sections, to change, outside the body, the coordinates X^i,T that we have been using in this section to X'^i, T' with:

$$T' = T$$

$$X'^i = \frac{R - Gm}{R}\, X^i \tag{20}$$

This transformation leaves invariant the functional form of (17), the main difference being now that the zeroth approximation $g^{(0)}_{ab}$ will be, outside the body, the Schwarzchild metric expressed in harmonic coordinates (Fock 1959)(for simplicity we drop the primes):

$$g^{(0)}_{ab}\, dX^a dX^b = -\frac{R - Gm}{R + Gm}\, dT^2 + \frac{R+Gm}{R-Gm}\, dR^2 + (R+Gm)^2\, (d\theta^2 + \sin^2\theta\, d\phi^2\,). \tag{21}$$

In the following we shall find it more convenient to use the "gothic" contravariant metric:

$$\underline{g}^{ab} = g^{1/2} g^{ab} \tag{22}$$

where g is the negative of the determinant of g_{ab}. We have in the quasi-Cartesian coordinates (20):

$$\underline{g}^{ab}_{(0)} = f^{ab} - G^2 m^2 \frac{N^a N^b}{R^2} + \left(1 - \frac{(1+Gm/R)^3}{1-Gm/R}\right) U^a U^b \tag{23}$$

where $N^o = 0$, $N^i = X^i/R$ and $U^o = 1$, $U^i = 0$. (23) can be expanded in powers of Gm/R (when $Gm/R < 1$):

$$\underline{g}^{ab}_{(0)} = f^{ab} - 4\frac{Gm}{R} U^a U^b - \frac{G^2 m^2}{R^2}\,(7U^a U^b + N^a N^b) -$$
$$- 8\,\frac{G^3 m^3}{R^3}\, U^a U^b - 8\,\frac{G^4 m^4}{R^4}\, U^a U^b - \text{and so on.} \tag{24}$$

For later convenience we denote by $S_n^{ab}(U,N)$ the coefficient of $(Gm/R)^n$:

$$S_1^{ab}(U,N) = -4U^a U^b,$$
$$S_2^{ab}(U,\,N) = -7U^a U^b - N^a N^b,$$
$$n \geq 3\,:\quad S_n^{ab}(U,N) = -8U^a U^b, \tag{25}$$

$$\underline{g}^{ab}_{(0)} = f^{ab} + \sum_{n\geq 1} \frac{G^n m^n}{R^n}\, S_n^{ab}(U,N) \tag{26}$$

We have seen above (see eqn (11)) that the lowest order correction $G'^3 g^{(3)}_{ab}\,(X^i/Gm,\, d(T))$ to $g^{(0)}_{ab}$ could be expanded in a series of positive and negative powers of $\hat{R}$. This result can be extended iteratively to all the $g^{(p)}_{ab}$ ($p \geq 3$) of (17) and therefore also to all the $\underline{g}^{ab}_{(p)}$ ($p \geq 3$) of the expansion for $\underline{g}^{ab}$ corresponding to (17) in the new coordinate system (20), hence:

$$p \geq 3,\; G'^p \underline{g}^{ab}_{(p)} = \sum_{n,q} G^p m^{p-q} m'^q\, T^{ab}_{n,p,q}(d(T),k)\left(\frac{Gm}{R}\right)^n \tag{27}$$

Therefore, putting together (17), (26), and (27), we reach the final conclusion that: the coefficient of $1/R^n$ ($n \geq 0$) in the expansion of $\underline{g}^{ab}$ (X^i, T) outside a compact body is:

$$G^n m^n S_n^{ab}(U,N) + \sum_{p\geq 3} \sum_{q\geq 1} G^{n+p} m^{n+p-q} m'^q T_{n,p,q}^{ab}(d(T),k) \ . \tag{28}$$

More simply put: the coefficient of $1/R^n$ ($n \geq 0$) is the same as for an isolated Schwarzchild solution plus corrections due to the tidal interaction of the companion which are smaller by a factor G'^3 at least (i.e. $\sim G^3 m' m^2/d^3$ or $G^3 m'^2 m/d^3$, a very small correction indeed $\sim (v/c)^6$).

In the next section we shall convert this knowledge of the magnitude of the tidal distortion of the gravitational field outside the body in internal coordinates into a similar, but more useful, knowledge in external coordinates.

6. EXTERNAL PERTURBATION SCHEME

In the exterior region, i.e. everywhere except inside two 2-surfaces (two space-time tubes), enclosing each body and of "diameter" much smaller than the distance between the two bodies constituting the binary system, the "gothic" gravitational field:

$$\underline{g}^{ab}(x) = g^{1/2} g^{ab}, \quad (g = -\det g_{ab}), \tag{1}$$

expressed in external coordinates x^a, must satisfy the Einstein (-Grossmann) vacuum field equations:

$$2gE^{ab} = 0 \tag{2a}$$

that is, explicitly:

$$(2gE^{ab} \equiv)\ \underline{g}^{cd}\underline{g}^{ab}{}_{,cd} + \underline{g}^{ab}\underline{g}^{cd}{}_{,cd} - \underline{g}^{ac}\underline{g}^{bd}{}_{,cd} - \underline{g}^{bc}\underline{g}^{ad}{}_{,cd} + \underline{g}^{ab}{}_{,c}\underline{g}^{cd}{}_{,d} - \underline{g}^{ac}{}_{,d}\underline{g}^{bd}{}_{,c} +$$
$$+ \underline{g}_{cd}\underline{g}^{ce}{}_{,f}(\underline{g}^{af}\underline{g}^{bd}{}_{,e} + \underline{g}^{bf}\underline{g}^{ad}{}_{,e}) - \underline{g}_{cd}\underline{g}^{ef}\underline{g}^{ac}{}_{,e}\underline{g}^{bd}{}_{,f} - \frac{1}{2}\underline{g}^{ab}\underline{g}_{cd}\underline{g}^{ce}{}_{,f}\underline{g}^{df}{}_{,e} -$$
$$- \frac{1}{8}(2\underline{g}^{ac}\underline{g}^{bd} - \underline{g}^{ab}\underline{g}^{cd})(2\underline{g}_{eg}\underline{g}_{fh} - \underline{g}_{ef}\underline{g}_{gh})\,\underline{g}^{ef}{}_{,c}\underline{g}^{gh}{}_{,d} = 0, \tag{2b}$$

where $\underline{g}_{ab}$ is the matrix inverse of $\underline{g}^{ab}$(i.e. $g^{-1/2} g_{ab}$). (2) constitutes a system of non-linear, nonhyperbolic, partial differential equations. De Donder and Lanczos found a way of "relaxing" (2) into a diagonal system of non-linear <u>hyperbolic</u> partial differential equations which is better suited to a perturbative treatment (as well as being nearly indispensable for mathematical investigations about the existence

and uniqueness of the solutions of (2), see however the contribution of Y. Choquet to these proceedings). This "relaxing" consists in introducing, as auxiliary equations, the harmonic coordinate conditions:

$$\underline{g}^{ab}{}_{,b} = 0 \tag{3}$$

which, when plugged into (2), yield the "harmonically relaxed" Einstein vacuum equations:

$$\underline{g}^{cd}\underline{g}^{ab}{}_{,cd} + Q^{ab}(\partial\underline{g}, \partial\underline{g}) = 0 \tag{4}$$

where Q^{ab}, which is quadratic in the derivatives of $\underline{g}^{ab}$, is obtained from (2b) by ignoring the 2^d, 3^d, 4^{th} and 5^{th} terms. We shall assume the global existence of harmonic coordinates so that the "external" gravitational field can be considered as fulfilling both eqn (3) and eqn (4) in the exterior region. (4) being a partial differential system we expect to be able to characterize one of its solutions by means of some kind of boundary conditions. More precisely (4) being hyperbolic, one should give, in fact, "initial conditions" (Cauchy data on a space-like hypersurface). However, the exterior region being pierced by two space-time tubes we shall be led to try to characterize a solution of (4) by two kinds of "boundary conditions":

1) some Kirchoff-type boundary conditions at infinite distances in the infinite past (see eqn (3 · 10)),

2) some "boundary conditions" "on" the space-time tubes enclosing the bodies (see below).

Concerning the first type of boundary conditions one should probably, in view of the structure of (4), impose boundary conditions at infinite past (affine) distances along the characteristics of (4), i.e., along the null geodesics of the exact (but unknown!) metric g. However, in our Post-Minkowskian Approximation treatment, we shall introduce an auxiliary flat metric f^{ab} (which at this level means only a diagonal matrix (-1, +1, +1, +1)) and the "gothic"metric"perturbations":

$$h^{ab} = \underline{g}^{ab} - f^{ab} \ . \tag{5}$$

(Beware that, for the ease of notation, we did not put a bar below h^{ab}, so that our future convention for moving indices with the flat metric will yield a "covariant" h_{ab} which is different from the one used in preceding sections: eqn (3.10), (5.2)).

Now we shall assume that the metric that we are looking for satisfies the Kirchoff "no-incoming-radiation" conditions at infinite past (affine) distances along the null geodesics of the auxiliary flat metric. This means that, in the harmonic external coordinate system x^a (introducing auxiliary polar variables $x^o = t, r, \theta, \phi$

constructed in the usual way), rh^{ab} and $rh^{ab}{}_{,c}$ are supposed to be bounded and that:

$$\lim_{\substack{r \to \infty \\ \theta,\phi,\ t + r/c = \text{const.}}} \left((rh^{ab})_{,r} + (rh^{ab})_{,t} \right) = 0 . \tag{6}$$

Concerning the second type of boundary conditions we shall deduce them from the results of the preceding section by transforming internal coordinates X^a of, say, the first body, into external coordinates x^a. In order to deduce necessary conditions on the external field $\underline{g}^{ab}(x)$, we must investigate the most general coordinate transformation $X^a \to x^a$ compatible with the hypotheses 1), 2), and 3) of the beginning of Section 5. We need to further assume that: 4) in the overlap region between the internal and the external scheme the functions $x^a(T, X^i)$ can be expanded in (positive or negative) powers of $R = (X^iX^i)^{1/2}$ as well as in positive powers of the masses and of the small interaction parameter $\sim Gm'/d$, and that: 5) the transformation $X \to x$ reduces to a Poincaré transformation when the interaction vanishes. (These last two assumptions seem very plausible and can probably be justified by some more work as done for proving eqn (5.17)). Then it can be shown that the most general coordinate transformation compatible with these hypotheses is of the type:

$$x^a(T,X^i) = z^a(T) + e^a_i(T)X^i + O(G')\,O(R^2) + O(G'^3)O(1/R) + O(G'^4) \tag{7}$$

where the functions $z^a(T)$ and $e^a_i(T)$ must be such that dz^a/dT, e^a_1, e^a_2, $e_3{}^a$ constitute a Minkowski orthonormal tetrad when the interaction vanishes (i.e. formally $G' \to 0$). (For the meaning of G' see (5.16)). As we see from eqn (7) the meaning of the function $z^a(T)$ is just, at order G'^3, the image in the external coordinate system of the "center" of the internal coordinates: $X^i = 0$. We have seen in Section 5 that the X coordinate system could be considered as "Mass Centered" (this convention can be, evidently, maintained after the transformation (5.20)). In fact, by the way it was defined, it might be better to call it "Field Centered." Therefore the "point": $X^i = 0$ can be called the "Center of Mass" or the "Center of Field." Note that this "point" is defined only by the symmetry of the gravitational field just outside the object and does not need to, and even cannot, in the black hole case, coincide with any material particle. Thence the function $z^a(T)$ defines, in the external coordinate system, a world-line which can be called the "central world-line" of the (first) compact object. From now on, it will be convenient to choose as parameter along this central world-line the Minkowski proper time: s. Let us introduce the Minkoswkian 4-velocity of this "central world line":

$$u^a(s) = \frac{dz^a(s)}{ds} \tag{8}$$

With any x^a we can associate a "contemporary" point: $z^a_C = z^a(s_C)$ such that:

$$f_{ab}\,(x^a - z^a(s_c))\,u^b(s_c) := 0 . \tag{9}$$

Let r_C be:

$$r_c := (f_{ab}(x^a - z^a_c)(x^b - z^b_c))^{1/2}, \tag{10}$$

and n^a_C be:

$$n^a_c := (x^a - z^a{}_c)/r_c . \tag{11}$$

Applying the general coordinate transformation (7) to the final result of the preceding section: eqn (5.28), we conclude that: there exist two constant parameters m, m' and two world-lines $z^a(s)$, $z'^a(s')$ in R^4 (the external coordinate chart) such that the external "gothic" metric corresponding to two compact objects has necessarily the following behaviour near each world-line:

$$\underline{g}^{ab} = -4Gm\left(\frac{u^a u^b}{r}\right)_C - G^2m^2\left(\frac{7u^a u^b + n^a n^b}{r^2}\right)_C - 8G^3m^3\left(\frac{u^a u^b}{r^3}\right)_C + O(G'^2)O\left(\frac{1}{r_c}\right) + O(G'^3)O\left(\frac{1}{r_c^2}\right) + O(r_c^o) + O(G^4), \tag{12}$$

and the corresponding "primed" equation near $z'(s')$. ($(u^au^b/r)_C$ means $u^a_Cu^b_C/r_C \ldots$).

In simpler terms: up to the order G^3 inclusively, the singular behaviour of the metric near the world-line is dominated, at each order G^n, by a Schwarzschild-like behaviour.

This result is another form of what we called before the "Cheshire cat principle": only the mass m appears in (12) for characterizing the behaviour of the external field near the world-line. Although this result is less precise than our preceding statement, eqn (5.19), about the internal field, however it will be very useful later for characterizing uniquely the external field.

To conclude this section, let us gather the necessary conditions that the "gothic metric perturbation" h^{ab} have to satisfy. If we define an effective non-linear source N^{ab} as:

$$N^{ab} = -h^{cd}h^{ab}{}_{,cd} - Q^{ab} \tag{13}$$

(where Q^{ab} was defined in eqn (4)), then we get:

$$f^{cd}h^{ab}{}_{,cd} \ (= \Box h^{ab}) = N^{ab}(h) \qquad (14a)$$

$$h^{ab}{}_{,b} = 0 \qquad (14b)$$

together with the following "boundary conditions":

1) The Kirchoff conditions at Minkowski past null infinity: eqn (6)

2) The third order "Dominant Schwarzschild" condition near each central world-line: eqn (12) and a similar one for z'(s') and m'.

7. INTRODUCTION TO ANALYTIC CONTINUATION

Marcel Riesz (1938, 1949) introduced a very powerful method for solving the wave equation in a flat or curved space-time of any dimension. The basic tool of this method was the process of analytic continuation of functions of a complex variable. This continuation process allowed Riesz to define, to use in rigorous demonstrations and to compute some, otherwise divergent and therefore meaningless, integrals. The technical utility of this method was quickly realized: Gustafson (1945, 1946) applied it in Quantum Field Theory in order to define meaningless integrals; Fremberg (1946) and Riesz (1949) applied it to the Classical Theories of the interaction between electromagnetic or mesonic fields and point particles, again this allowed them to define, and to compute some a priori meaningless quantities: like the self-force of a point electron: Ma (1947) proved the agreement between Riesz's method and another formal regularization process, due to Dirac, Bhabha and Harish-Chandra in giving a meaning to the derivatives of the self electromagnetic field at the position of a point electron: Schwartz (1950) showed that the Riesz kernels could be interpreted as "distributions" depending holomorphically on a parameter; this point of view has been amplified and used systematically, by Guelfand and Chilov (1962), to define new "distributions" by Riesz's method; Havas and Goldberg (1962) pointed out that Riesz's method might be useful in a Post-Minkowskian Approximation to General Relativity, with formal (a priori meaningless) point-like sources, however they used, instead of it, the formal regularization method of Dirac, Bhabha and Harish-Chandra in their study of the first Post-Minkowskian Approximation (1 PMA or linear approximation); later, because of the mathematical difficulties posed by the second Post-Minkowskian Approximation, Schieve, Rosenblum and Havas (1972) considered the simpler case of the interaction between point particles and electromagnetic and mesonic fields they used "Riesz potentials" to define a formal post-linear approximation but they could not carry out the integrations completely even with the simplifying assumptions of small velocities and small oscillatory spatial motion; at the same time, Riesz's idea found renewed applications to Quantum Field Theory under the form of "dimensional regularization"; however, let us point out that even in Quantum Field Theory where

one looks for a method to give a meaning to undefined divergent quantities, one still has to check the consistency of the method when it is used in proving renormalizability or in computing renormalized quantities (this point, which is frequently overlooked in formal calculations, has been stressed, in the context of Classical Renormalization Theory, by Damour (1974, 1975), and, in the context of Quantum Renormalization Theory by Breitenlohner and Maison (1977)); moreover the great technical usefulness of Riesz's method has been further shown by Damour (1974, 1975) who reduced the computation of (linear) self-action terms to mere inspection; more recently a well-defined algorithm, using Riesz's ideas, has been set up, has been checked to define a post-post-linear 3 PMA relaxed gravitational field outside two "point masses" and has been proved to satisfy the complete vacuum Einstein equations (outside two world-lines) if and only if the two world lines satisfied some "equations of motion," themselves computable by means of analytic continuation (Damour 1980).

Let us stress that in all the preceding works analytic continuation has been employed only as a trick for defining a priori meaningless concepts (self-field of a point electron, locally interacting quantum fields, general relativistic point mass) however we are dealing in these lectures with a physically and mathematically well-defined concept (gravitational interaction of two neutron stars of supposedly known internal structure), in other words we do not have the freedom of playing with formal concepts even if we check the consistency of our game. On the contrary, our strategy is going to be the following:

1) We first make use of analytic continuation to define a post-post-post linear (4PMA) "harmonically relaxed" gravitational field which is a functional of two world-lines in R^4 (Section 8) (contrary to what has been asserted by Havas and Goldberg (1962) the possibility of this quartically non-linear definition is far from being a trivial consequence of Riesz's linear method: one must prove already in this first step that the non-linearities do not generate poles at the point of the complex plane where we want to continue the integrals).

2) We have recourse again to analytic continuation to prove that the precedingly constructed "relaxed" gravitational field will fulfill the complete vacuum Einstein equations outside the two world-lines if and only if these world-lines satisfy some "equations of motion" which are, in turn, defined by analytic continuation (section 9).

3) We have recourse again to analytic continuation to compute explicitly these "equations of motion" which contain integrals which seem to be very difficult to calculate by usual methods (Section 12).

4) We check that the precedingly defined solution of the complete vacuum Einstein equations satisfy the "boundary conditions" of Section 6.

5) Finally we prove (modulo some technical assumptions) that there is a unique solution of the vacuum Einstein equations which satisfy these "boundary conditions" (section 10). As we have shown in sections 5 and 6 that these conditions are necessarily satisfied by the gravitational field outside two compact objects, we shall have thereby proved that the precedingly analytic-continuation-defined gravitational field and equations of motion are necessarily the gravitational field outside and the equations of motion of two compact objects.

Before setting up our algorithm let us briefly describe how analytic continuation works and what are the properties that make it such a powerful computational technique. Let a smooth (C^∞) function of a real variable, with compact support, be given: F(x) (the preceding conditions could be relaxed to sufficient differentiability and proper fall off at infinity at the expense of some changes and restrictions in the following results; this is in fact the case in actual use but for simplicity we present the idea in its simplest form). Let A be a complex number. Let us define the following function of A:

$$I_F(A) := \int_0^\infty x^A F(x)\,dx \tag{1}$$

As is well known, the integral (1), i.e. $I_F(A)$, is a priori convergent and therefore defined only when the real part of A satisfies:

$$\mathrm{Re}\,(A) > -1 \tag{2}$$

In other words the complex function I_F of the complex variable A is defined only in half the complex plane. Now we see by formal differentiation of (1) with respect to A that:

$$\frac{dI_F(A)}{dA} = \int_0^\infty x^A \log x\, F(x)\,dx \tag{3}$$

whose right-hand side is convergent, and thus equal to its left-hand side, under the same restriction (2). For the sake of conciseness let us call D_{-1} the domain of the complex A plane defined by eqn (2). We conclude that eqn (1) defines an analytic function I_F of the complex variable in the domain D_{-1}. We are going to show that, although the original definition of I_F was meaningful only in D_{-1}, it is possible to define I_F in a much larger domain. We first write (1) as:

$$I_F(A) = \int_0^1 x^A F(x)\,dx + \int_1^\infty x^A F(x)\,dx. \tag{4}$$

The reason for doing this is that the problems of divergence come from the neighbourhood of the origin. Because of the assumed smoothness of F(x), it is possible to define, for any integer n, a smooth function $G_n(x)$ such that:

$$F(x) = F(0) + x\,F'(0) + \frac{x^2}{2!}F''(0) + \ldots + \frac{x^n}{n!}F^{(n)}(0) + x^{n+1}G_n(x). \tag{5}$$

plugging (5) into (4) and explicitating the integrals of $x^p \cdot (p \le n)$ from 0 to 1 we find:

$$I_F(A) = \frac{F(0)}{A+1} + \frac{F'(0)}{A+2} + \frac{F''(0)}{2!(A+3)} + \frac{F^{(n)}(0)}{n!(A+n+1)} + \\ + \int_0^1 x^{A+n+1}G_n(x)\,dx + \int_1^\infty x^A F(x)\,dx. \tag{6}$$

Eqn (6) has up to now only a meaning in D_{-1}, but we see that the right hand side is an analytic function of A in the larger domain:

$$D^{\odot}_{-n-2} = D_{-n-2} - \{-1, -2, \ldots, -n-1\}, \tag{7}$$

in words: $D^{\odot}_{-n-2}$ is the half of the complex plane at the right of $-n-2$ with the exclusion of the points: $-1, -2, \ldots, -n-1$. So that we can now define $I_F(A)$ in the domain $D^{\odot}_{-n-2}$, and in fact because of the smoothness of F, in the domain:

$$D^{\odot} = D^{\odot}_{-\infty} = C - \{-1,-2,\ldots,-n,\ldots\}, \tag{8}$$

by the formula (6). Moreover we see from the definition (6) that this extended I_F is still an analytic function of the complex variable A with only single poles at $-1, -2, \ldots$. So, in this particular way, we can now give a meaning to the integral (1) even when it is divergent (except for $A = -1, -2, \ldots$): for instance we would find in this way:

$$\int_0^\infty x^{-3/2} e^{-x}\,dx := -2\,\pi^{1/2} \tag{9}$$

However, this way of defining (1) seems to be very particular and a priori one would expect to get a different result if one had used a different trick, e.g. integrating (1) by parts. That this is not so is one of the main advantages of the complex variable approach. Analytic continuation is unique in the following sense: had we used a different trick for extending the definition of I_F, to an analytic function $\tilde{I}_F(A)$ in a domain $\tilde{D}$ larger than D_{-1}, then, necessarily $\tilde{D} \subseteq D^{\odot}$ and $\tilde{I}_F(A) = I_F(A)$, as defined by (6) in $\tilde{D}$. This uniqueness follows from the theorem of "continuation of identity" for (single-valued) analytic functions: Let each of two functions f(A) and g(A) be analytic in a common connected domain D. Let f(A) and g(A) coincide in some portion D' of D (D' may be a subdomain, a segment of a curve, or even only an infinite set of points having a limit point in D). Then f(A) and g(A) coincide throughout D.

Therefore any procedure which allows one to define an analytic fc function which coincides with (1) even only for a small interval of real values of Af A, e.g. 0.33 <A<0.34 will coincide with (6) in all its domain of definition (it is is clear that (6), with variable n, defines the "maximal" analytic continuation of of I_F). The preceding "continuation of identity" theorem is also extremely usefulful (and in fact of constant use in practical calculations) for proving that one can rn recur to familiar integration techniques even when dealing with "divergent" integragrals (e.g.(1) for Re(A) < -1). For instance we know that when Re(A) > -1 we can in integrate (1) by parts, which yields:

$$\int_0^\infty x^A F(x)dx = \left[\frac{x^{A+1}F(x)}{A+1}\right]_0^\infty - \frac{1}{A+1}\int_0^\infty x^{A+1}F'(x)dx. \tag{10}$$

But the "integrated part" or "surface term" vanishes for Re(A)> -1, h, hence:

$$\int_0^\infty x^A F(x)dx = -\frac{1}{A+1}\int_0^\infty x^{A+1}F'(x)dx. \tag{11}$$

We have already seen how the left hand side of (11) could be analytictically continued in $D^{\odot}$ by eqn (6), by the same method we can analytically continue I_F, I_F, (A + 1), and therefore the right hand side of (11), in $D^{\odot}$. As these two analytictic functions coincide for Re(A) >-1, they are identical all over $D^{\odot}$, which means thathat we can always integrate (1) by parts in discarding the "surface terms." Moreover er we have considered here for simplicity a one-dimensional integral but the whole ap approach can be, and actually is, generalized to multiple integrals. Then one can ex extend to analytically continued "divergent" integrals the familiar integration techechniques: linear decomposition of the integrand, decomposition of the domain of integtegration, integration by parts, differentiation with respect to a parameter under tr the integration symbol, change of variable... (however all these operations should nd not be done blindly, but one should always check, from the "continuation of idendentity" theorem, that they can be freely performed).

In spite of the preceding "uniqueness" of the analytic continuanuation, the reader should be warned that this does not mean that there is a unique wayway of regularizing any divergent integral. For instance, if one considers eqn (1) onlynly on the real A axis there is no unique way to find explicit real expressions whiwhich extend (1) below -1. Moreover if one was to start with the integral:

$$\int_0^\infty x^{-3/2}F(x)dx, \tag{12}$$

there would be infinitely many unequivalent ways of giving a meaninning to (12). However, the basic advantage of Riesz' method of analytic continuationion is that it allows one to use all the familiar integration techniques when handandling "divergent" integrals and it is this fact which allows one not only to define an otherwise meaningless quantity but to prove the consistency of the definition and to prove that the analytic continuation defined-quantity satisfies some extra conditions (which in general are not guaranteed by the use of arbitrary "regularization" techniques applied to the formal "divergent" definition of the quantity). There will be many instances of that in the following.

To conclude this section let us define Riesz basic kernel in flat space. Let us consider, on a n dimensional Minkowski space (signature -++...), the following function which depends parametrically on a complex number A:

$$Z_A(x) = \frac{(-f_{ab}x^a x^b)^{\frac{A-n}{2}}}{H_n(A)}, \tag{13a}$$

when x is future directed ($x^o > 0$, $-f_{ab}x^a x^b > 0$), and, otherwise:

$$Z_A(x) = 0, \tag{13b}$$

with the coefficient $H_n(A)$ being:

$$H_n(A) = \pi^{\frac{n-2}{2}} . 2^{A-1} . \Gamma(\tfrac{A}{2}) . \Gamma(\tfrac{A+2-n}{2}). \tag{13c}$$

(Γ denoting the usual Eulerian gamma function).

$H_n(A)$ is infinite for A = 0, -2, -4,... and A = n-2, n-4,..., therefore for these values the function $Z_A(x)$ is zero.

The main properties of Z_A are:

$$\Box Z_{A+2} = -Z_A , \tag{14}$$

$$Z_A * Z_B = Z_{A+B} \tag{15}$$

where * denotes the convolution. If Z_A is considered not as a function but as a distribution then it is an "entire function" of A (i.e. analytic without singularities all over the complex plane) and the value of the distribution Z_A when A = 0 is the n-dimensional Dirac distribution:

$$Z_0 = \delta_{(n)} \tag{16}$$

From (16) and (14), one deduces that the distribution Z_2 (which can be computed as the analytic continuation in A = 2 of the function (13) is (minus) the retarded "Green function" ("elementary kernel") of the n-dimensional space-time. When n =4 one finds:

$$Z_2(x) = +\frac{1}{4\pi}\frac{\delta(x^o - /x/)}{/x/} \tag{17a}$$

$$\Box Z_2 = -\delta_{(4)} \tag{17b}$$

In the following we shall not consider Z_A as a distribution (except Z_2, see below) because we wish, contrary to Riesz and Schwartz, to perform non-linear operations on Z_A (remember that it is impossible in general to define the product of distributions). Therefore we shall consider $Z_A(x)$ as a function over Minkowski space depending analytically on a complex parameter A. This will allow us to consider the integral of some products of some Z_A's and if we can prove that the integral converges for some values of A and can be analytically continued then we shall be able to extend the meaning of the integral beyond its convergence range. At the end of the process we shall try to continue A down to the value zero (so that because of (16) we shall have simple equations satisfied by our A dependent integrals); however, we shall have to prove that this is possible because the non-linearities could create pole singularities in zero. All this will be done only for n = 4 (usual space-time), yet it has been checked by Damour (1980) that one could formally use Z_0 from the beginning, but work in a space-time of dimension:

$$n = 4 - A, \tag{18}$$

and that would yield the same final result (at least at the 3PMA order) as the physical approach: n = 4, Z_A, A ≠ 0. However, I wish to stress that such a formal "dimensionally regularized" calculation has no mathematical meaning and therefore cannot be used to prove anything about the quantities so calculated, contrary to the physical n = 4, A ≠ 0 appraoch.

8. A PARTICULAR SOLUTION OF THE RELAXED EXTERNAL SCHEME

We have seen in section 6, eqn (6.14), that the external "gothic gravitational field" $h^{ab} = \underline{g}^{ab} - f^{ab}$ had to satisfy, in harmonic coordinates, two equations:

$$f^{cd}h^{ab}{}_{,cd} = N^{ab}(h), \tag{1}$$

$$h^{ab}{}_{,b} = 0 \tag{2}$$

and two types of boundary conditions: "Kirchoff" (6.6) and "Dominamt Schwarzschild" (6.12). In this section we shall set up an algorithm which will generate a partic-

ular solution of the "relaxed" vacuum field equation (1) satisfying the two preceding boundary conditions. We shall consider eqn (2) in the next section. We shall try to incorporate the second type of boundary conditions by introducing a fictitious stress-energy tensor (more precisely a contravariant tensor density of weight + 2 corresponding to 16Πg times the usual tensor):

$T_A{}^{ab}$ (x, m, m'; z(s), z'(s'), $\underline{g}$(y)) which is:

- a function of a point x in Minkowski space,
- a function of two constant parameters: m, m',
- a functional of two arbitrary time-like world-lines z(s) and z'(s') in Minkowski space,
- a functional of the, yet to be defined, "gothic metric" $\underline{g}^{cd}(y) = f^{cd} + h^{cd}(y)$
- an analytic function of a complex parameter A.

Moreover, $T_A{}^{ab}(x)$ is such that it vanishes when A = 0 and x is not on any of the world-lines. At the end of the process, we shall analytically continue A down to zero, m and m' will appear as the "Schwarzschild" masses of two compact objects, and, z and z' will appear as the "central world-lines" of two compact objects (see section 6 for the meaning of these concepts).

We define $T_A{}^{ab}$ as:

$$T_A{}^{ab} := \Sigma 16\pi m \int_{-\infty}^{+\infty} ds\, Z_A(x-z)\, u^a u^b\, (\underline{g}(z))^{1/4} (\underline{g}_{cd}(z) u^c u^d)^{-1/2}, \tag{3}$$

where: Σ denotes a summation on the two worldlines z and z' (endowed with their respective parameters m and m'), s is the Minkowski proper-time along z(s), u is the Minkowski 4-velocity: dz/ds ($f_{ab}u^a u^b = -1$), $\underline{g}$ is minus the determinant of the contravariant gothic metric $\underline{g}^{ab}$(it is equal to the usual g), $\underline{g}_{ab}$ is the inverse matrix of $\underline{g}^{ab}$ (or $g^{-1/2} g_{ab}$), and Z_A is the Riesz kernel defined in eqn (7.13) (with n = 4). $T_A{}^{ab}$ must be multiplied by G/c^2: the Newton-Einstein coupling constant between mass and gravitational field. In the sections 8-11 we shall take c = 1 but we shall keep G as the "small" coupling constant which will allow us to define a perturbative solution of Einstein's vacuum equations.

We now consider, instead of (1), the following equation:

$$f^{cd}h^{ab}{}_{,cd} = GT_A^{ab}(f+h) + N^{ab}(h) \tag{4}$$

Then we incorporate the Kirchoff "no-incoming-radiation" condition (6.6) by using the flat space retarded Green function $-Z_2$ (section 7) for transforming (4b) in an integro-differential equation:

$$h^{ab} = -Z_2 * (GT_A^{ab}(f+h) + N^{ab}(h)). \tag{5}$$

In eqn (5) the star $*$ denotes a space-time convolution and $-Z_2$ denotes the retarded "Green function" (7.17) that is: a distribution. In other words $4\Pi Z_2$ T is just the usual retarded integral of T. On the other hand $Z_A(x)$ which appears in $T_A{}^{ab}$ is to be considered as a plain function of x which depends analytically on the complex parameter A. N^{ab} is at least quadratic in h. More precisely we deduce from (6.13) (6.4) and (6.2), in expanding $g_{ab} = (f^{ab}+h^{ab})^{-1}$ in "powers of the matrix h":

$$N^{ab}(h) = \text{II}^{ab}(h,h) + \text{III}^{ab}(h,h,h) + \text{IV}^{ab}(h,h,h,h) + O(h^5), \tag{6}$$

where II, III, IV are respectively quadratically, cubically and quartically non-linear in h. By this we mean for instance: II $\sim h\partial^2 h + \partial h \partial h$ (10 terms), III $\sim h\partial h\partial h$ (21 terms),... The explicit expressions of II and III can be found in the appendix A of Bel, Damour, Deruelle, Ibañez and Martin (1981). By similarly expanding $T^{ab}(f+h)$ in "powers of the matrix h" we can write:

$$T^{ab}(f+h) = T^{ab}(f) + \overline{\text{II}}^{ab}(h) + \overline{\text{III}}^{ab}(h,h) + \overline{\text{IIII}}^{ab}(h,h,h) + O(h^4). \tag{7}$$

Because of this structure of the right hand side of (5) we can now define our iterative Post-Minkowskian Algorithm:

$$h^{ab}_{A(0)} := 0 \qquad (\text{i.e. } g^{ab}_{A(0)} = f^{ab} \tag{8a}$$

$$h^{ab}_{A(n+1)} := -Z_2 * ((GT_A{}^{ab}(f+h_{(n)}))_{(n+1)} + (N^{ab}(h_{(n)}))_{(n+1)}) \tag{8b}$$

where the suffix (n + 1) in the right hand side of (8b) means the following:

- take the preceding iteration, which is a truncated expansion in powers of G of a function-functional:

$$h_{A(n)}(x;z) = Gh_{A1}(x;z) + G^2 h_{A2}(x;z) + \ldots + G^n h_{An}(x;z) \tag{9}$$

- plug it into the non-linearity expansions (6) and (7), and
- keep only the powers of G up to G^{n+1} (included).

In this way $h_{(n+1)}$ is a truncated expansion: (9) + one extra term $O(G^{n+1})$.

Explicitating the algorithm we find for the first coefficients of G^P in (9):

$$h_{A1} = -Z_2 * T(f) \tag{10a}$$

$$h_{A2} = -Z_2 * (\overline{\text{II}}(h_{A1}) + \text{II}(h_{A1}, h_{A1}) \tag{10b}$$

$$h_{A3} = -Z_2 * (\text{II}(h_{A2}) + \text{III}(h_{A1}, h_{A1}) + \text{II}(h_{A1}, h_{A2}) + \text{III}(h_{A1}, h_{A1}, h_{A1})). \tag{10c}$$

For the explicit computation of the "equations of motion" of the binary pulsar it will be necessary and sufficient to compute h_1, h_2 and h_3 (cubic non-linearities). However in order to prove that these "equations of motion" are consequences of the vacuum field equations (Einstein-Infeld-Hoffmann approach) we shall need to consider also h_4 (quartic non-linearities), that is why we have included in (6) and (7) the corresponding terms.

Up to here we have only formally defined an algorithm: (8), we have now to prove that this definition makes sense, that is that the integrals appearing in (8) and (10) are meaningful. In order to see how the algorithm works, and why analytic continuation is so useful, let us first consider the one-body problem (m' =0). Moreover we shall assume from the start that the world-line z(s) is a straight-line in Minkowski space: the justification for this assumption is that, as can be deduced from the arguments of section 9, the "harmonicity condition" eqn (2) implies at each order G^n that the acceleration of the world-line is zero (when m' = 0), therefore in a perturbative sense at least, this acceleration must be taken to be exactly zero if we wish to satisfy the complete Einstein vacuum equations (1) and (2).

In order to be able to consider even the first step of the algorithm (8), i.e. the equation (10a), we must first inquire about the meaning of the fictitious source $T_A(f)$. According to eqn (3) we have, when m' = 0, $\underline{g} = \underline{f}$, and z is a straight line:

$$T_A^{ab}(f) := 16\pi m u^a u^b \int_{-\infty}^{+\infty} ds\, Z_A(x-z)\ ? \tag{11}$$

Using a coordinate system, centered at the point x, with time axis parallel to the straight world line so that $x^\circ - z^\circ = -t = -s$ and $/x^i - z^i/$ = const. = r say (the distance between the point x and the world-line) the integral appearing in (11) is of the form:

$$\int_{-\infty}^{-r} dt \quad (t^2-r^2)^{\frac{A-4}{2}} \quad . \tag{12}$$

This integral converges, and therefore is a priori defined, only if:

$$2 < \text{Re}(A) < 3. \tag{13}$$

The lower bound is due to the behaviour of the integrand near t = -r and the upper bound to its behaviour near t = - ∞. However after computing (12) and plugging it in (11) with the denominator (7.13c) one finds:

$$T_A^{ab}(f) = 16mu^a u^b \frac{\Gamma(\frac{3-A}{2})}{\pi 2^A \Gamma(\frac{A}{2})} r^{A-3} \quad . \tag{14}$$

We see from the explicit expression (14) that although T_A was originally defined only in the vertical strip (13) of the complex plane it can be analytically continued to a meromorphic function defined everywhere apart from simple poles at A = 3,5,7, ... As we have seen in section 7, the process of analytic continuation is unique when possible. Therefore we shall, in fact, define T_A as the maximal analytic continuation of (11) which must necessarily be equal to (14).

Now that the "source" T_A is defined nearly all over the complex plane we can inquire about the meaning of the first step of our algorithm. Plugging (7.17) and (8.14) in (8.10a) we find an integral which is convergent, and therefore a priori defined, only if:

$$0< \mathrm{Re}(A)< 1. \tag{15}$$

However, an explicit computation yields:

$$h_{A1}^{ab} = -4mu^a u^b C_1(A) r^{A-1}, \tag{16}$$

with:

$$C_1(A) = \frac{\Gamma(\frac{1-A}{2})}{\pi^{1/2} 2^A \Gamma(\frac{A+2}{2})} \quad . \tag{17}$$

Therefore h_A can be analytically continued everywhere except for simple poles in A = 1, 3, 5,....Now we can consider the second iteration (10b). We must first inquire about the meaning of $\Pi_A(h_{A1})$ which contains $h_{A1}(z)$. A look at (16) tells us that $h_{A1}(z)$ is defined for Re(A) >1 and A ≠ 3, 5,.... It is then equal to zero (because r = 0 when x = z) and can therefore be analytically continued everywhere. Then we must consider the quadratically non-linear "source terms." We find:

$$\Pi(h_{A1}, h_{A1}) \sim (\partial h_{A1})^2 + h_{A1}\partial^2 h_{A1} \sim r^{2A-4} \tag{18}$$

Plugging (18) into (10b) we find an integral which is convergen when:

$$1/2< \mathrm{Re}(A)< 1. \tag{19}$$

The result of the integration can however be analytically continued everywhere except for (possibly multiple) poles in 1/2, 1, 3, 5,.... This result can be written as:

$$h_{A2}^{ab} = -m^2(7C_2(A)u^a u^b + C_2'(A)n^a n^b)\ r^{2A-2} \ , \tag{20}$$

where n^a denotes the unit radial vector, defined by eqn (6·11) and where the coefficients C_2 and C_2' are, like C_1 in eqn (17), equal to 1 when A = 0. In this simple one-body case it is possible to push the iteration ad infinitum. One finds: that the fictitious source $T_A(f + h)$ is nothing but $T_A(f)$, eqn (14), that the non-linear effective source N_n of the nth iteration leads to convergent integrals if: $1 - 1/n<$ Re(A)< 1 but that the result of the integration can be analytically continued everywhere except for (multiple) poles at: 1/2, 2/3,..., (n-1)/n; 1, 3, 5, ... The nth iterated gravitational field can be written as:

$$h_{A(n)}^{ab} = -4Gmu^a u^b C_1(A) r^{A-1} - G^2 m^2 (7C_2(A)u^a u^b + C_2'(A)n^a n^b) r^{2A-2} - \sum_{p=3}^{n} G^p m^p (8C_p(A)u^a u^b + AC_p'(A)n^a n^b + AC_p''(A)f^{ab}) r^{pA-p}, \tag{21}$$

where all the coefficients C, C', or C" are equal to 1 when A = 0. It is therefore possible to analytically continue down to A = 0 the nth iterated gravitational field; this yields:

$$h_{0(n)}^{ab} = -4Gmu^a u^b r^{-1} - G^2 m^2 (7u^a u^b + n^a n^b) r^{-2} - \sum_{p=3}^{n} 8G^p m^p u^a u^b r^{-p} \ . \tag{22}$$

We recognize in eqn (22) the nth truncated expansion in powers of G of the exact Schwarzchild metric in usual harmonic coordinates: eqns (5.23) and (5.24).

Before proceeding to the much more interesting two-body case let us draw some conclusions from the one-body case:

1) We must supplement our algorithm (8) by the rule of "maximal analytic continuation," that is: any A-dependent quantity which is initially defined only for a small range of the parameter A must be understood as being analytically continued all over the complex plane except for some isolated pole singularities (this rule makes sense because the types of operations involved in (8) generate only poles and no branch points, so that the analytic continuation is unique over C).

2) Although we started with an entire analytic function of A: Z_A, both the linear (integrations) and non-linear operations included in the algorithm have a tendency to generate multiple poles on the real axis. This fact has been sometimes overlooked because of too much confidence in the regularizing power of Riesz' method

(remember that the framework in which Z_A is proved to be regular is the theory of distributions where only certain types of linear operations are considered). Therefore we must check the regularity of the result of our non-linear algorithm near $A = 0$ which is the value that we wish to consider at the end of the algorithm (we do not need to care about pole singularities elsewhere).

3) The last result eqn(22) shows that it is possible, in a perturbative sense at least, to consider the Schwarzschild metric as being "generated" by a (Minkowski) time-like world line. It also shows the fictitious source (3) has the effect that we wanted: to incorporate in the solution of the vacuum equations (1) the "Dominant Schwarzschild" conditions (6.12).

4) If we had tried to set up an algorithm with A = 0 from the beginning ("delta-function-source," "point mass"), then already the second step of the algorithm: (10b) would have been meaningless: both because of $\infty\cdot\delta$-type terms in $\Pi(h_1)$ and because of the appearance of divergent integrals due to the non-linear source $\Pi\ (h_1)$: $\int d^3xr^{-4}$. However, Bel, Damour, Deruelle, Ibañez and Martin (1981) showed that the use of a set of regularizing prescriptions (based on Hadamard's "partie finie" instead of Riesz's analytic continuation) yielded the same result (22) in the one-body case. On the other hand they showed that in the two-body case (at order G^2) the structure of the theory itself demanded to reconsider this set of regularization prescriptions. We shall show below that our analytic continuation approach (which has the further advantage of dealing uniformly with the $\infty.\delta$ and with the divergent integrals problems) not only yields the same G^2 metric as Bel et.al. (1981) but can be extended to the G^3 and the G^4 metric. Moreover the G^2 equations of motion of Westpfahl and Göller (1979) and Bel et. al. (1981) will be proved to be the integrability conditions of the G^3 field equations, and we shall obtain G^3 equations of motion from the integrability conditions of the G^4 field equations.

In the two-body case we must start all over again. Let us consider two sufficiently smooth time-like world-lines z (s) and z'(s') in Minkowski space, and two constant parameters m and m'. According to our algorithm, augmented by the conclusion 1) above, we must first consider the maximal analytic continuation in A of the lowest order fictitious source:

$$T_A^{ab}(f) := \sum_{m,m'} 16\pi m \int_{-\infty}^{+\infty} ds\ u^a(s)u^b(s)\ Z_A(x - z(s)), \tag{23}$$

where the unit tangent vector $u^a = dz^a/ds$ is no longer constant. Let us however assume that $u^a(s)$ (and $u'^a(s')$) has a limit when s(s') tends to minus infinity and, more precisely, that du^a/ds tends to zero as s^{-2}. In other words we assume that the system was unbound in the infinite past. This assumption is not necessary at this stage of our iteration (because one could slightly modify the algorithm so that, in the first iteration, only a finite segment of the world-lines matters), however it is needed for ensuring that the whole sequence of highly non-linear effective sources N^{ab} always lead to infra-red (i.e. at large distances) convergent integrals. On the other hand this assumption seems very plausible on intuitive grounds (a system staying bound for an infinite time would radiate an infinite amount of energy). Moreover it should be possible to check a posteriori this assumption by studying, in the manner of Walker and Will (1979), the infinite past behaviour of the solutions of our final equations of motion which contain "radiation damping" type terms. Granted this assumption the integral (23) converges for $2 < Re(A) < 3$. Moreover, even if contrary to the one-body case, (eqn (14), we cannot compute explicitly (23) in order to analytically continue it, we can nevertheless use the type of reasonings employed in section 7 about similar integrals: (7.1) to prove that (23) can be analytically continued everywhere apart from simple poles at A = 3,5,... (in the following we shall cease to precise the positions and multiplicities of all the poles to concentrate on the possible appearance of a pole at A = 0). Plugging (23) into (10a) leads to an integral which is convergent for $0 < Re(A) < 1$. Moreover , using eqn (7.15) we find the following expression of h_1:

$$h_{A1}^{ab} = -\Sigma\ 16\pi m \int ds\ u^a u^b Z_{A+2}(x-z). \tag{24}$$

Exploiting again the type of arguments used in Section 7 we can prove that h_{A1} can be continued "everywhere"(except some poles but none at A = 0). We now have to face the difficult problem of handling non-linearities: we must plug (24) into (10b) and investigate the existence of the resulting integral and the possibility of continuing it, with respect to A, outside its range of convergence. A way of achieving this has been indicated by Damour (1980). It consists first in generaliaing the formula (16) to <u>curved</u> world-lines. This is achieved by combining some expansion techniques studied in Damour (1974, 1975) with the analytic continuation method. Because of the lack of space the method cannot be explicitated here (full details about this as well as about many other technical results of the following sections will be contained in a paper by myself to be submitted to the Proceedings of the Royal Society, London). The useful result, though, is that h_{A1} admits the following expression <u>near each world-line</u> (here the first for instance):

$$h_{A1} = r_C^{A-1}\cdot F_0(x,A) + F'_0(x,A), \tag{25}$$

where r_c was defined in (6.10) (Minkowski orthogonal distance between the field point x and the first world-line z(s)) and where F_o and F'_o are two functions which are smooth in x (if the world-line is smooth) and analytic in A (no poles at A = 0).

Moreover, the index zero in F_o and F'_o means that these functions are of order one when x is on the world-line. For simplicity we shall denote by F_n any function (sometimes we do not even put primes for distinguishing different functions) which is analytic in A (no poles at A = 0), smooth in x, and of order r_c^n when x approaches the first world-line. Plugging (25) into the quadratically non-linear terms leads to a non-linear effective source which behaves near each world-line as:

$$\mathrm{II}(h_{A1},h_{A1}) = r_c^{2A-6} F_2 + \underline{\underline{r_c^{A-5} F'_2}} + F_0 \,. \tag{26}$$

According to (10b) we must now compute the retarded integral of II. As explained in section 7 we can decompose this integral in two integrals near the two world-lines and one over the exterior region of space-time and continue separately these three integrals. Because of our assumption about the behaviour of the world lines in the infinite past the integral over the exterior region is convergent and analytic in A as soon as Re(A) <1. Therefore possible singularities at A = 0 can come only from the integrals near the world-lines. For studying the analytic behaviour of these integrals it is then sufficient to plug the expression (26) in the retarded integral (10b) and to look for possible singularities in A near A = 0 linked with the singular behaviour in r_c near $r_c = 0$. Using for instance as local space-time coordinates near the first world-line: $x^a \to (s_c, r_c, \Theta_c, \phi_c)$ where where Θ_c and ϕ_c are usual polar coordinates parametrizing the direction n_c^a in the 3-space orthogonal to the world-line, the retarded integral of $\mathrm{II}(h_{A1}, h_{A1})$ takes the form:

$$\int (r_c^{2A-6} \cdot \hat{F}_2 + r_c^{A-5} \cdot \hat{F}'_2 + \hat{F}_0)\, r_c^2 dr_c \, \sin\theta_c \, d\theta_c \, d\phi_c \quad , \tag{27}$$

where, after having made a new expansion over the time variable s_c, the arguments of the new smooth functions $\hat{F}_2$, $\hat{F}'_2$, $\hat{F}_0$ are: a fixed s_c, r_c, Θ_c, ϕ_c . This integral is convergent when Re(A) > 1/2. We can then perform the integration over the angles which gives as possibly singular integrals:

$$4\pi \int_0^R (r_c^{2A-2}((\hat{F}_2/r_c^2)) + r_c^{A-1}((\hat{F}_2'/r_c^2)))\, dr_c \,, \tag{28}$$

R being a finite radius defining a tube around the world-line and the double parentheses denoting an average over the angles. The expression (28) is of the form (7.1) (with $x = r_c$), therefore it can be analytically continued everywhere except for simple poles on the real axis. We must inquire about the location of these poles because we worry about a pole at A = 0. Using the fact that the angle average of a smooth function of $X^a = r_c n_c^a$ is a smooth function of r_c^2 (because odd powers of X^a average to zero) we can now take $x = r_c^2$ as variable, so that (28) is transformed into:

$$\int_0^{R^2} x^{A-3/2} F(x)\,dx + \int_0^{R^2} x^{(A/2)-1} \, F'(x)\,dx. \tag{29}$$

Now it is clear using the results of section 7 (for (7.1) with $x = r_c^2$) that the first integral in (29) may have simple poles at A = 1/2, -1/2, -3/2,... (but none at A = 0, and that the second integral may have simple poles at A = 0, -2, -4, ... Therefore there may be a pole at A = 0, the residue of this pole is 2F'(0) (see section 7) which is proportional to $((\hat{F}'_2))$ and thence to the angle average of the underlined second term of the right hand side of eqn (26): $r_c^{A-5} \cdot F'_2$. However, by looking more precisely at the structure of this non-linear term it is easily checked that $((F'_2)) = 0$, which means that in fact there is no pole at A = 0. In conclusion: the part of h_{A2}(10b) which is the retarded integral of the quadratically non-linear terms is convergent for 1/2 <Re(A)< 1 and can be analytically continued everywhere except for some poles but there is no pole at A = 0. The stress-energy part of h_{A2} (10b) is simpler to handle because according to our conclusion 1) above $h_{A1}(z)$ which appears in $\Pi(h_{A1})$ is to be understood as the maximal analytical continuation of $h_{A1}(z)$. According to eqn (25) $h_{A1}(z)$ is well-defined when Re(A)> 1 and is then equal to $F'_0(z, A)$, this last quantity is analytic in A (no poles at A = 0) and must be replaced in $\Pi(h_1)$. Then the integral over s converges when 2< Re(A)< 3 and, as before, can be continued so as to define its retarded integral. Finally $h_{A2}(x)$ is well-defined as a function of x, a functional of z(s) and z'(s') and as an analytic function of A (no poles at A = 0). In order to proceed to the next iteration we must have some knowledge of the "singular behaviour" of h_{A2} near each world-line. A close scrutiny of the retarded integral (10b) of $\Pi(h_{A1})$ and $\mathrm{II}(h_{A1},h_{A1})$ (see eqn (26): full details will be given elsewhere) allows one to conclude that near the first world-line:

$$h_{A2} = r_c^{2A-4} . F_2 + r_c^{A-3} . F'_2 + F_0 \text{ . (no poles at A=0).} \tag{30}$$

We must now plug (25) and (30) in (10c). As before the first two terms of the right hand side of (10c) (stress-energy terms) are easily dealt with: from (30) $h_{A2}(z) = F_0(z,A)$, from (25) $h^2_{A1}(z) = F_0^2(z,A)$, and the integrations are done as before. On the other hand the cubically non-linear effective sources $\mathrm{II}(h_{A1}, h_{A2})$ and III (h_{A1}, h_{A1}, h_{A1}) are much more complicated. Many "dangerous" terms appear (liable to generate poles at A = 0) but a close scrutiny shows that most of these terms give rise only to false poles (zero residue, as the underlined term in (26)), still a few dangerous terms are left which create real poles of the following type:

$$G^3 h_{A3} \sim \frac{G^3 m^3}{A} (\dot{u}\, F(x) + \ddot{u} G(x)), \tag{31}$$

where $\dot{u}$ is the acceleration (or curvature of the world-line) $\dot{u} = du/ds$ and where $\ddot{u}$ is d^2u/ds^2. However we are going to show in the next section that the harmonicity condition (2) will impose as constraints on the world-lines some equations of motion of the type:

$$\dot{u}^a = GW_1{}^a(z(s);z'(s')) + G^2W_2{}^a(z;z') + G^3W_3{}^a(z;z') + O(G^4), \tag{32}$$

which imply that a term like (31), although coming from the third iteration, is in fact of order G^4. Therefore these terms will create no problems for the G^3 gravitational field and their influence, if any, will show up in the analysis of the G^3 equations of motion which are deduced from the G^4 gravitational field (see next section). Finally, we get a thrice iterated field:

$$h_{A3} = r_c^{3A-7}\cdot F_4 + r_c^{2A-6}\cdot F_4 + r_c^{A-5}F_4 + F_0 + A^{-1}O(G^3\dot{u}) + A^{-1}O(G^3\ddot{u}). \tag{33}$$

In conclusion the process of analytic continuation allowed us to construct a solution of eqn (4), accurate to order G^3 inclusively:

$$h^{ab}_{A(3)}(x;z;z') = Gh^{ab}_{A1}(x;z;z') + G^2h^{ab}_{A2}(x;z;z') + G^3h^{ab}_{A3}(x;z;z'). \tag{34}$$

Due to lack of space we cannot write down here the semi-explicit expression of $h_{A(3)}$ (which is still partly in integral form) which has been used in actual calculations. The solution (34) is a function of the field point x in Minkowski space, a functional of two world-lines (unrestricted apart from $\dot{u} = O(G)$) and is analytic in a complex parameter A except for (usually simple) poles located at some rational points on the real axis but none in a neighbourhood of the origin A = 0. On the other hand we know that the source T_A, when considered as a function of x and not as a distribution over Minkowski space, vanishes when A = 0 (see section 7 particularly eqn (7.16)). Therefore replacing A by zero in (34) generates a solution of eqn (1). Moreover, it appears from eqns (25), (30) and (33) that the most singular terms near the first worldline when A = 0 are, when A ≠ 0 of the type $O(r_c^{nA-n})$ for each order $G^n(n = 1,2,3)$. It is easily checked that these terms contain the nth power of the mass m only (no explicit dependence on m'). Therefore these terms are the same as for the one-body case except that now the world-line is curved. This curvature though can be shown to introduce only less singular terms when $r_c \to 0$. Hence, our preceding result (22) for the straight-line-one-body case shows that for each order G^n the most singular terms in h^{ab}_{On}, near each world-line, are Schwarschild-like: $G^n(m/r_c)^n S^{ab}_n(u_c,n_c)$ (see (5.25-26)). Therefore $h^{ab}_{O(3)}$ satisfies the "Dominant Schwarzschild" conditions (6.12). On the other hand it seems clear that $h^{ab}_{A(3)}$, being constructed with the retarded Green function Z_2, $h^{ab}_{O(3)}$ will satisfy the Kirchoff "no incoming radiation" conditions (6.6) (we leave to future work an explicit check of this property).

In conclusion to this section: by means of an analytic continuation process, we have constructed a particular solution, $h^{ab}_{O(3)}$, of the cubically non-linear relaxed vacuum field equations (1) which satisifes both the "Dominant Schwarschild" conditions near each world-line and the Kirchoff "no-incoming radiation" conditions at past null infinity. In the next section we shall address ourselves to the "stand by" equation (2).

9. EQUATIONS OF MOTION AS INTEGRABILITY CONDITIONS OF THE EXTERNAL SCHEME

In section 8 we constructed a particular solution of:

$$f^{cd}h^{ab}_{\ ,cd} = N^{ab}(h), \tag{1}$$

at least up to the order G^3 inclusively. That solution was a functional of two arbitrary world-lines:

$$h^{ab}(x) = h^{ab}(x;\ z(s),\ z'(s')) \tag{2}$$

Let us define the "harmonicity" of h^{ab} as the following quantity:

$$H^a(x) := h^{ab}{}_{,b} \equiv \mathfrak{g}^{ab}{}_{,b} \tag{3}$$

Our aim is to construct a solution of the complete vacuum Einstein equations (6.2). These equations reduce to the "relaxed" form (1) in harmonic coordinates, that is to say when the "harmonicity" vanishes. Therefore we must investigate when our particular solution h^{ab} satisfy the "harmonicity condition":

$$H^a\ (x;z(s),z'(s')) = 0. \tag{4}$$

As is clear from eqn (4) this condition will in general restrict the still arbitrary world-lines used in the construction of h^{ab}. In fact, we shall see that eqn (4) implies the equations of motion of the two world-lines. An important tool in showing this is the use of the Bianchi identities:

$$E^{ab}{}_{;b} := E^{ab}{}_{,b} + E^{bc}\,\Gamma^a_{bc} + E^{ab}\Gamma^c_{bc} \equiv 0\ . \tag{5}$$

These identities can also be written in function of the covariant derivative of the tensor density of weight +2, $\underline{E}^{ab}$: $=gE^{ab}$: $\underline{E}^{ab}{}_{;b} := \underline{E}^{ab}{}_{,b} + \underline{E}^{bc}\Gamma^a_{bc} - \underline{E}^{ab}\Gamma^c_{bc} \underline{=} 0.$ (6)

(Note the minus sign of the last term). Using eqn (6) and the definition of N^{ab} ((6.2), (6.4) and (6.13)) we get the following identity satisfied by N:

$$N^{ab}{}_{,b} \equiv (\Box h^{bc} - N^{bc})\Gamma^{a}_{bc} - (\Box h^{ab} - N^{ab})\,\Gamma^{c}_{bc} + O(Hh)\ , \tag{7}$$

where the abbreviation O(Hh) denotes some terms which are linear combinations of H and its derivatives with coefficients which are at least linear in h and its derivatives. If we knew only that eqn (1) were fulfilled, we would deduce from (1), (3), and (7) that:

$$f^{cd}H^{a}{}_{,cd} = O(Hh). \tag{8}$$

Eqn (8) is very important in the study of the Cauchy problem, because its consideration is sufficient for proving that if H and $H_{,0}$ vanish on the initial hypersurface (which implies that 4 initial constraint equations must be satisfied by h) then H vanishes everywhere and therefore h is a solution of the full Einstein equations. However in our treatment, which is not of the "initial value" type, eqn (8) is of little help for finding further constraints on h in a useful form. Therfore at this point it is very convenient to recur again to the process of analytic continuation and to remember that the particular solution of (1) was obtained from a perturbative solution of:

$$h^{ab}_{A} = -Z_2 * (GT_{A}{}^{ab} + N^{ab}(h_A)) \tag{9}$$

where $T_A(h_A)$ was defined in eqn (8.3), by analytically continuing A in zero. Because of the nice properties of the analytic continuation, which allows one to treat "divergent" integrals as if they were convergent (as we said in section 7 this should not be done blindly but here it works) one can differentiate the right hand side of (9) under the integration symbol hidden in the convolution symbol:

$$Z_2 * F = \int d^4y\, Z_2(x-y)F(y) = \int d^4y Z_2(y)\, F(x-y). \tag{10}$$

Therefore a derivative of h can be expressed as a convolution with the derivative of the source: GT + N. In particular the definition (3) implies:

$$H^{a}_{A} = -Z_2 * (GT^{ab}_{A\ ,b} + N^{ab}{}_{,b}) \tag{11}$$

Using the identity (7) and the eqn (9) we get:

$$H^{a}_{A} = -Z_2 * (GT^{ab}_{A}{}_{;b} + O(H_A h_A)\ ), \tag{12}$$

where the semi-colon denotes the covariant derivative appropriate to tensor densities of weight + 2 (see eqn (6)) with Christoffel symbols corresponding to the metric h_A. Eqn (12) is much more useful than eqn (8) specially in a perturbative approach because of the appearance of the coupling constant G. Indeed we should have worked, from the beginning of this section, only with truncated G expansions. This has the effect of replacing (12) by its truncated form:

$$H^{a}_{A(n)} = -Z_2 * (\ (GT_{A}{}^{ab}{}_{;b})_{(n)} + (O(H_A h_A))_{(n)}) \ . \tag{13}$$

Now the right hand side of (13) depends at most on $h_{A(n-1)}$. Therefore, imposing the complete solving of the nth iteration ("complete" means including the harmonicity condition: $H_{(n)} = 0$) provides some constraints which depend only on the (n-1)th iterated metric. In order to derive these constraints let us study more precisely $T^{ab}{}_{;b}$. From its definition (8.3) $T_A{}^{ab}$ can be written as:

$$T^{ab}_{A} = \Sigma\ 16\pi \int ds\ M_A(s)u^a u^b Z_A(x-z)\ , \tag{14}$$

where $M_A(s)$ is an "effective" variable mass which reduces to m in absence of interaction and whose value can be read off (8.3) and (8.7). From (14) one deduces by integration by parts:

$$T_{A}{}^{ab}{}_{,b} = \Sigma\ 16\pi \int ds \frac{d}{ds}(M_A u^a)\ Z_A(x-z). \tag{15}$$

Moreover we have seen in the preceding section that, near each world-line, $h_{A(n)}$ $(n\leq 3)$ could be written as:

$$h^{ab}_{A(n)}(x) = \Big(\sum_{p,q,r} r_c^{pA-q} \cdot F^{ab}_{r}(x,A)\Big) + F^{ab}_{0}(x,A), \tag{16}$$

where p,q and r are integers and where $F^{ab}_r(x,A)$ and $F^{ab}_0(x,A)$ are smooth in x and analytic in A (poles at A = 0 appear only at order G^4 or are multiplied at least by a factor $G^3\dot{u}$ or $G^3\ddot{u}$ which will be shown below to be of the same order as G^4.) We call the first term of the right hand side of (16): the "singular part" of $h_{A(n)}$ (denoted $\mathrm{sing}(h_{A(n)})$) and the second term (F_0): the "regular part" of $h_{A(n)}$ (denoted $\mathrm{reg}(h_{A(n)})$). When computing from (16) the Christoffel symbols of the metric f + $h_{A(n)}$ and truncating at order n we find a similar form:

$$\Gamma_{A(n)} = \mathrm{sing}(\Gamma_{A(n)}) + \mathrm{reg}(\Gamma_{A(n)})\ . \tag{17}$$

Moreover it can be checked that:

$$\mathrm{reg}(\Gamma_{A(n)}) := \mathrm{reg}(\ (\Gamma(h_{A(n)})\ )_{(n)}) = (\Gamma(\mathrm{reg}(h_{A(n)})))_{(n)}\ . \tag{18}$$

Taking into account the fact that when A = 0 the function $T_A(x)$ vanishes everywhere except maybe on the world-lines but that when $A \neq 0$ $T_A(x)$ is singular near each world-line so that the distribution T_A tends to a Dirac distribution on each world-line, we see that the first term of the right hand side of (13) will contribute terms O(A) (i.e. vanishing with A) except for the parts of the Z_2 integral coming from an arbitrary small neighbourhood of the world-lines. Hence it will be sufficient to use the near-world-line behaviour (17). Writing (13) for the order n + 1 we get:

$$H^a_{A(n+1)} = \Sigma -GZ_2 * (T^{ab}_{A(n),b} + T^{bc}_{A(n)} \mathrm{reg}(\Gamma^{\ a}_{A(n)bc}) - T^{ab}_{A(n)} \mathrm{reg}(\Gamma^{\ c}_{A(n)bc})) +$$
$$+ \Sigma \ - GZ_2 * (T^{bc}_{A(n)} \ \mathrm{sing}(\Gamma^{\ a}_{A(n)bc}) - T^{ab}_{A(n)} \ \mathrm{sing}(\Gamma^{\ c}_{A(n)bc})) -$$
$$- Z_2 * (O(H_{A(n)} h_{A(n)})) + O(A) + O(G^{n+2}) \quad . \tag{19}$$

Here we have replaced, for simplicity, the "truncation suffix" (n + 1) which should be added to the right hand side of (19) by an extra $O(G^{n+2})$. The first three terms of (19) can be dealt with in the framework of distribution theory because reg (Γ) is a smooth function of x and T_A becomes a Dirac distribution when A = 0. Using eqn (15), replacing $M(\dot{s})$ by its definition (8.3) and (8.7), and taking into account eqn (18) it can be shown that the first three terms of (19) give the following contribution to $H^a_{A(n+1)}$:

$$\Sigma -16\pi G \int ds \ M \ (B^a_{(n)} - u^a \ \mathrm{reg}(g_{bc})_{(n)} B^{\ b}_{(n)} u^c (\mathrm{reg}(g_{de})_{(n)} u^d u^e)^{-1}) \ Z_2(x-z) +$$
$$O(A) + O(G^{n+2}) \tag{20}$$

where $B^a_{(n)}$ is an abbreviation for:

$$B^a_{(n)} := \frac{du^a}{ds} + (f^a_b + f_{bc} u^a u^c) \ u^d u^e \mathrm{reg} \ (\Gamma^{\ \ b}_{0(n)de})(z). \tag{21}$$

Because of eqn (18) and because by definition (eqn (16)) a "singular term," together with its derivatives, vanishes on the world-line when the real part of A is large enough we can write for any function $G_A(x)$ of the type (16):

$$\mathrm{reg}\ (G_0)(z) = \underset{A\,=\,0}{\text{Analytic Continuation}} \ (G_A(z)). \tag{22}$$

Therefore $B_{(n)}^{\ a}$ can be written as:

$$B^a_{(n)} = \underset{A\,=\,0}{\text{An.Cont.}} \ (\dot{u}^a + (f^a_b + u^a u_b) \ u^c u^d \ \Gamma^{\ b}_{cd} \ (h_{A(n)})(z)), \tag{23}$$

where, for simplicity, u_b denotes $f_{bc}u^c$ and not $g_{bc}u^c$.

On the other hand the fourth and the fifth terms of eqn (19) are much more complicated to handle. The mathematical reason for this complication is that they cannot be treated in the framework of distribution theory (where they would yield meaningless quantities of the type: $\infty \cdot \delta$). The physical reason for this complication is that these terms "correspond" to the "self-action" of a compact body. I use the word "correspond" and not "represent" because in the approach taken here, the equations of motion are deduced from the <u>vacuum</u> field equations and the <u>vacuum</u> field in an <u>improved</u> Einstein-Infel-Hoffmann's way. The improvement consists in that, instead of dealing with the <u>surface integral</u> of a very complicated non-linear expression (which would be proportional to $N^{ab}(h_{(n)})_{(n+1)}$) we are now dealing with the retarded <u>volume integral</u> of a much simpler expression (19). This drastic simplification comes from the fact that the divergence of $N^{ab}_{(n+1)}$ is much simpler than $N^{ab}_{(n+1)}$ itself ("Bianchi" identity (7)). However it has been possible to take advantage of this simplification only because of our mathematically well-defined use of analytic continuation (intuitively we are somehow using Gauss' theorem, but it would be meaningless to use it directly on the surface integrals because of the singular behaviour in $r_c = 0$). At the same time our use of a "fictitious" stress energy tensor makes the calculations, which deal in fact only with quantities in vacuum, similar to the familiar extended body calculations, and in this sense the incriminated terms correspond somehow with the "self-action" of the body (we put quotation marks because this concept is not defined in general, and, a fortiori in the case of compact bodies).

In order to analyze these terms we must work out explicitly the expression for the "singular" behaviour of $\Gamma_{A(n)}$ near the first world-line, as well as a similar expression for the behaviour of T_A and we must investigate the resulting integrals. Using arguments similar to the one used in the preceding section we end up with integrals of the following type:

$$A \int_0^R x^{pA-q} F(x) dx \ . \tag{24}$$

Pay attention to the fact that the appearance of a factor A does not mean necessarily that these terms are O(A) in the sense used above. Indeed as was discussed in Section 7 the integral contained in eqn (24) is defined by analytic continuation and can have a simple pole in A = 0, in which case (24) gives a non zero contribution to (19) when A = 0 (and this case is not at all exceptional because the first three terms of (19) can be written as (24) and have been shown to contribute (20)). At this point, in order to include also the contribution of the sixth and last term of (19): O(Hh), it is simpler to proceed iteratively.

When n = 0, which means considering the "harmonicity" up to first order: $H_{(1)}$, only the first term of (19) is left and we reach the result:

$$H^a_{O(1)} = \Sigma - 16\pi G \int ds\ m\ B^a_{(0)} Z_2(x-z) \ , \tag{25}$$

where $B^a_{(0)}$ is nothing but the Minkowski acceleration: du^a/ds. Therefore the first order metric $h_{O(1)}$ will satisfy the harmonicity condition (4), and thence the first order complete vacuum field equations, <u>exactly</u> if and only if each world-line is such that:

$$B^a_{(0)} = \dot{u}^a = 0. \tag{26}$$

However as we are solving Einstein equations in a perturbative manner in any case, there is no need to satisfy the harmonicity condition exactly (which would force the world-lines to be straight because of (26)). It is sufficient to require that the harmonicity condition is satisfied at the same order as the field equations:

$$H^a_{O(1)} = O(G^2), \tag{27}$$

so that the world-lines are only required to satisfy:

$$B^a_{(0)} = \dot{u}^a = O(G). \tag{28}$$

As discussed in Bel, Damour, Deruelle, Ibañez and Martin (1981), eqn (28) does not mean that the world-lines have to be globally close to straight-lines but only that they are solutions of a system of equation of the type written in eqn (8.32).

Assuming that, what we shall call the zeroth order equations of motion (28) are satisfied, we can now consider n = 1 i.e. the harmonicity condition up to second order: $H_{(2)}$. In this case the last term of eqn (19) is $O(H_{(1)}h_{(1)})$ which is, by (27), $O(G^3)$ and therefore negligible at this order. Moreover in this case it is not difficult to prove that the 4th and 5th terms of (19) contribute terms of the type (24) where the integral has no pole in A = 0 which means that these self-action terms vanish when A = 0. Then we are left with:

$$H^a_{O(2)} = \Sigma - 16\pi G \int ds M_{(1)} (B^a_{(1)} - u^a \mathrm{reg}(g_{(1)bc}) B^b_{(1)} u^c (\mathrm{reg}(g_{(1)de}) u^d u^e)^{-1}) \times \\ \times Z_2(x-z) \tag{29}$$

However, by its definition (21) $f_{bc} B^b_{(1)} u^c = 0$, thence the second term of (29) is at least $O(G^2 B_{(1)})$, i.e., because of (28), $O(G^3)$ which is negligible. For the same reason $M_{(1)}$ can be replaced by $M_{(0)} = m$ in the first term of (29). So that we are

left with:

$$H^a_{O(2)} = \Sigma - 16\pi G \int ds\ m\ B^a_{(1)} Z_2(x-z) + O(G^3) \ . \tag{30}$$

Hence the second order metric $h_{O(2)}$ will satisfy the harmonicity conditions, and therefore the full Einstein equations, at third order if and only if each world-line satisfies the "first order equations of motion":

$$B^a_{(1)} = O(G^2). \tag{31}$$

A result equivalent to (31) has been first obtained by Bel et.al.(1981) by direct computation of $H_{(2)}$.

Assuming that the first order equations of motion (31) are satisfied we can now consider n = 2, i.e. the harmonicity condition up to third order. For the same reason as before the last term of (19): O(Hh) is negligible. On the other hand the 4th and 5th terms contribute terms of the type (24) where the integrals do have a pole in A = 0. However the residue of this pole is easily shown to be at least O(G) and therefore the "self-action terms" contribute still negligibly at this order. Now one should beware of the fact that: "self-action terms" means just: "the 4th and 5th terms in eqn (19)". In eqn (16) the "regular" term contains in fact the gravitational analogue of the Lorentz-Schott-Dirac self-electromagnetic field responsible for the "radiation reaction" force (3.6). Therefore our "self-action terms" correspond only to the Newtonian-like self forces which vanish because of the Action and Reaction principle. Then we are left with (20) for n = 2. As before it can be simplified so that we can write:

$$H^a_{O(3)} = \Sigma - 16\pi G \int ds\ m\ B^a_{(2)} Z_2(x-z) + O(G^4). \tag{32}$$

Hence, the third order metric $h_{O(3)}$ which was constructed in the preceding section, will satisfy the harmonicity conditions, and therefore the full vacuum Einstein equations, at fourth order, if and only if each world-line satisfies the "second order equations of motion":

$$B^a_{(2)} = O(G^3). \tag{33}$$

This result was first obtained by Damour (1980) and used to prove that the equations of motion of Westpfahl and Göller (1979) and of Bel et.al. (1981), obtained by formal regularization of meaningless quantities, were effectively the correct integrability conditions of the third order vacuum field equations. Let us emphasize once more that the process of analytic continuation, as employed here, is not another formal regularization procedure but only a tool for finding and computing well-defined quantities. A look back at eqn (21) or at eqn (23) shows that a knowledge of the analytically continued second order metric $h_{A(2)}$ is sufficient to compute the second order equations of motion (33). On the other hand simple dimensional considerations (dating back to Eddington (1924)) show that we need to include terms of order G^3 in the equations of motion in order to get all the terms up to the "radiation reaction" terms inclusively. We can then hope that the precedingly constructed third order metric $h_{A(3)}$ will be sufficient to compute the third order equations of motion. That this is so can be proven by a generalization of the preceding arguments. Starting from the third order metric $h_{0(3)}$ which is an approximate solution of the Einstein vacuum equations if (33) is satisfied we ask the following question: is it possible to find a fourth order metric $h_{(4)}$ which fulfills both eqn (1) with the quartically non-linear terms of eqn (8.6) and eqn (4)? The answer is: this is possible if and only if the world-lines appearing in $h_{(4)}$ satisfy some third order equations of motion (therefore the third order equations of motion can be considered as the integrability conditions of the fourth order vacuum field equations). Because of the appearance of poles at A = 0 of order $G^3\dot{u} \sim G^4$ coming from the third iteration as well as other poles coming from the fourth iteration we must modify our method for constructing $h_{(4)}$. When this is done we find that we must still consider eqn (19). The "self-action" terms are now becoming really intricate and liable to contribute to the final equations of motion. However a scrutiny of these "self action" terms show that their contribution to the third order equations of motion is $O(G^3 v_r^2)$ where v_r^2 denotes the square of the relative velocity of the two world-lines (or geometrically the square of the space-time angle between the world-lines). v_r^2 is a Poincaré invariant quantity ($\sim f_{ab}(u^a-u'^a)(u^b-u'^b)$) which we shall assume to be always small. An example, among the simplest, of such a $G^3 v_r^2$ term is: $G^2 m^2 d^3 u^a/ds^3$ (where $du^a/ds = O(G)$ can be replaced by its value deduced from the equation of motion (31)). Note that we are here making a Slow-Motion hypothesis without abandoning the Poincaré invariance of our Post-Minkowskian Approach.

The other terms of (19) are dealt with as before and we are left with the following third order equations of motion:

$$B^a_{(3)} = O(G^3 v_r^2) + O(G^4). \tag{34}$$

(As we said already, the full proofs of our results will be spelled out in an article to be submitted to the Proceedings of the Royal Society (London)).

We shall see below that the accuracy of these equations of motion is sufficient for getting all the terms smaller than or equal to the "radiation reaction" which will be shown to be $O(G^2 v_r^3 + G^3 v_r)$.

The physical content of eqn (34) will be analyzed in sections 11 - 15; let us just remark here that, because of eqns (18), (21) and (23), eqn (34) means that, with the accuracy indicated, each world-line is a geodesic of the A = 0 analytically continued metric g_A or, equivalently, a geodesic of each (world-line dependent) metric reg (g_0) (the unfamiliar $f_{bc}u^a u^c$ term in eqns (21) and (23) is due to the use of the Minkowski parametrization: z(s)). On the other hand, in the case of slowly spinning compact bodies, Damour (1982) has deduced from the fourth order harmonicity condition two types of integrability conditions:

1) some orbital equations of motion which are a Papapetrou (1951) - like generalization of the preceding equations (34),
2) some spin propagation equations which had been obtained previously by a formal use of Infeld-Plebanski "good delta functions" (Damour 1978, equivalent spin propagation equations had also been obtained by other quantum or classical heuristic arguments: Barker and O'Connel (1975), Hari Dass and Radhakrishnan(1975,)Börner, Ehlers and Rudolph(1975).)

10. UNIQUENESS OF THE SOLUTION

In section 8 we constructed a particular solution, up to third order, of the "relaxed" vacuum field equations. This solution was a functional of two arbitrary world lines:

$$h^{ab}_{0(3)}(x;z(s),z'(s')) = \underset{A=0}{\text{Anal.cont.}}\ h^{ab}_{A(3)}\ (x;\ z(s),z'(s')). \tag{1}$$

In section 9 we showed that $h^{ab}_{0(3)}$ was a solution of the full vacuum field equations if the world-lines were restricted to satisfy the second order equations of motion:

$$B^a_{(2)} := \frac{du^a}{ds} - W^a_{(2)}(z(s),z'(s')) = O(G^3),\ (2) \tag{2}$$

where $W_{(2)}$ denotes (see (9.23) with $u_b := f_{bc}u^c$):

$$W^a_{(2)} := -\underset{A=0}{\text{An.cont.}}\ (f^a_b + u^a u_b)u^c u^d\ \Gamma^b_{cd}(h_{A(2)})(z). \tag{3}$$

Hence (1) together with (2) determines one particular solution of the full vacuum field equations up to third order: $h_{0(3)}$. On the other hand what we want is the solution of the full vacuum field equations which is the gravitational field outside two compact objects: h_c. We have proved in section 6 that h_c satisfies necessarily two types of boundary conditions:

- "3DS": the "third order Dominant Schwarzschild" conditions (6.12)
- "K": the "Kirchoff no-incoming-radiation" conditions (6.6)

We have checked in section 8 that the particular solution $h_{O(3)}$ satisfied both types of boundary conditions: "3DS" and "K". Therefore if we prove that there is a, geometrically and physically, unique solution of the vacuum Einstein equations, up to third order inclusively, which satisfies both 3DS and K, it will follow that $h_{O(3)} = h_c + O(G^4)$ (modulo a diffeomorphism), i.e. that $h_{O(3)}$ is the gravitational field outside two compact objects (truncated at order G^4). For simplicity, and by lack of space, we shall prove this result of uniqueness only at second order. The result, though, is valid at third order but its demonstration is more involved. As usual we proceed by iteration and start immediately at second order (the uniqueness of the first order being proved along the same line).

Given a uniquely determined first order gravitational field $h_{(1)}$ let us look for the most general second order field $h_{(2)}$ which satisfies:

- the vacuum Einstein equations, which can always be written in harmonic coordinates:

$$\Box h^{ab}_{(2)} = N^{ab}_{(2)}(h_{(1)}) \tag{4}$$

$$h^{ab}_{(2),b} = 0 \text{ (or } O(G^3)) \tag{5}$$

- the second order Dominant Schwarzschild conditions: i.e. there exist two world-lines $z(s)$ and $z'(s')$ such that, near each world-line:

$$h^{ab}_{(2)} = -4Gm\left(\frac{u^a u^b}{r}\right)_c - G^2m^2\left(\frac{7u^a u^b + n^a n^b}{r^2}\right)_c + O(r_c^0) + O(G'^2)O\left(\frac{1}{r_c}\right) + O(G^3) , \tag{6}$$

- and the Kirchoff conditions : $rh^{ab}_{(2)}$ and $rh^{ab}_{(2),c}$ are bounded and:

$$\lim_{\substack{r\to\infty \\ \theta,\phi,t+r/c=\text{ const.}}} \left((rh^{ab}_{(2)})_{,r} + (rh^{ab}_{(2)})_{,t}\right) = 0 . \tag{7}$$

In section 8 we have constructed a particular solution $h_{O(2)}$ of (4). Therefore the most general solution of (4) (where $h_{(1)}$ is known from the preceding iteration) is:

$$h^{ab}_{(2)} = h^{ab}_{O(2)} + k^{ab}, \tag{8}$$

where k^{ab} satisfies:

$$\Box k^{ab} = 0. \tag{9}$$

Moreover, as $h_{O(2)}$ satisfied the Kirchoff conditions, k must satisfy them as well:

$$\lim_{\substack{r\to\infty \\ t+rc^{-1}\text{const.}}} \left((rk^{ab})_{,r} + (rk^{ab})_{,t}\right) = 0. \tag{10}$$

and, as $h_{O(2)}$ was constructed so as to satisfy the second order Dominant Schwarzschild conditions (6) (see section 8), k must satisfy near each world-line:

$$k^{ab} = O(G'^2)O(1/r_c) + O(r_c^0) + O(G^3) . \tag{11}$$

Let us write down what was called by Hadamard the Fundamental Identity (valid for two arbitrary functions):

$$Z(x)\,\Box k^{ab}(x) - k^{ab}(x)\,\Box Z(x) \equiv (f^{cd}(Zk^{ab}_{,d} - k^{ab}Z_{,d}))_{,c} . \tag{12}$$

Let us replace $Z(x)$ by $Z_2(x_0 - x)$, for a given x_0 (see (7.17) for the definition of Z_2) and let us integrate both sides of the identity (12) on a region of space-time which is outside two (thin) tubes containing the two world-lines and inside one (thick) tube whose diameter will be allowed to go to infinity. By (9), the first term of the left hand side of (12) vanishes and by (7.17) the second term is equal to $k^{ab}(x_0)$ (assuming that x_0 is in the region of integration). Using Gauss' theorem the right hand side of (12) is reduced to three surface integrals. To start with they are 3-surface integrals but, because $Z_2(x_0 - x)$ is zero everywhere except on the past Minkowski light cone with vertex x_0, they are in fact 2 - surface integrals. On one hand the Kirchoff condition (10) is precisely the condition which ensures that the integral over the thick tube (in fact over a large 2-surface on the past light cone of x_0) tends to zero when the radius of the thick tube tends to infinity. On the other hand the other boundary conditions (11) ensure that the integral over each thin tube tends to a world-line integral of $Z_2(x_0 - z(s))$ when the radius of the tube tends to zero. Hence, suppressing now the index zero to x, we get the following necessary expression for k:

$$k^{ab}(x) = \Sigma -16\pi G^2 \int ds\, K^{ab}(s)\, Z_2(x-z(s)), \tag{13}$$

where $K^{ab} = K^{ba}$ is an undetermined function of s. However we must still fulfill eqn (5). Using the fact that, by construction, $h^{ab}_{O(2)}$ satisfies eqn (9.30), we conclude that k^{ab} must satisfy:

$$k^{ab}_{,b} + \Sigma -16\pi G \int dsm(\dot{u}^a - GW^a_1) Z_2(x-z) = O(G^3). \tag{14}$$

Eqn (14) together with eqn (13) is very constraining for k^{ab}. One finds that K^{ab} must be of the form:

$$K^{ab}(s) = Ku^a u^a + O(G), \tag{15}$$

where K must be constant. Therefore k^{ab} represents only a small $(O(G^2))$ constant readjustment of the monopole term: $h^{ab}_{O(1)}$ (eqn(8.24)). However the symbol $O(G'^2)$ in (11), (6) (and (6.12)) meant in fact that this term was not only $O(G^2)$ but also caused by the interaction and therefore was to vanish in the case of vanishing interaction, thence the constant K must in fact vanish and we are left with:

$$k^{ab} = O(G^3), \tag{16}$$

or, in other words, that the most general solution of (4-7) is necessarily equal to the particular solution $h_{O(2)}$ (modulo G^3). A direct consequence of this uniqueness is that the second order metric of Bel et.al. (1981), which has been obtained by some formal regularizing rules,as the field due to two "point masses",and which can be checked to be a solution of the complete vacuum Einstein equations satisfying "K" and "2DS," is necessarily equal to $h_{O(2)}$ and therefore to the metric (up to second order) outside two compact objects.

It is possible, with more work, to generalize this uniqueness to the third order metric, although several new features show up. On one hand the uniqueness is true only in a geometrical sense, that is to say modulo a coordinate transformation (leaving invariant the harmonicity). On the other hand the "3DS" conditions (6.12) leave the possibility of a $O(G'^2)$ shift of the world-lines which however changes only the functional form of $h_{(3)}(x;z(s))$ but not its numerical value. Therefore the physical content both of the third order metric and of the second order equations of motion is uniquely determined. Moreover it is always possible to normalize the functional form of $h_{(3)}(x;z)$ to $h_{O(3)}(x;z)$ which amounts only to a choice of the "central world-line" in the external scheme (see section 6).

In consequence the physical content of the third order eqnations of motion, which, as shown in the preceding section are the integrability conditions of the fourth order metric but which can be expressed only in function of the third order metric, is uniquely determined.

In conclusion, the particular solution $h_{O(3)}$ is necessarily equal to the metric outside two compact objects (modulo G^4) and the third order equations of motion (9.34) are the equations of a particular image in the external coordinates of the "central world-line" defined in the internal coordinates in sections 5 and 6.

11. PREDICTIVE POINCARE INVARIANT EQUATIONS OF MOTION

The equations of motion obtained in section 9 do not constitute a system of ordinary differential equations because the acceleration du/ds of each world-line is given as a complicated functional of the two worldlines. More precisely, because of our systematic use of "retarded potentials," this functional depends only on the parts of the two world-lines which are in the (Minkowski) past of the point Z whose acceleration $du/ds = d^2z/ds^2$ one computes. This kind of equation is known in the literature as "retarded-functional differential system." Up to the very recent work of Eder (1982) there were no theorems of existence and uniqueness for the solutions of these systems. Under certain technical conditions, on the structure of the system and using "initial data" for the two world-lines in the infinite past, Eder proves the existence and uniqueness of the solution up to a finite time. This result increases our confidence in the framework of "predictive relativistic mechanics" (for a review see Bel and Fustero (1976) and references therein). In this approach it is assumed (see however the recent result of Bel (1982) on "spontaneous predictivisation") that one can describe the evolution of two world-lines satisfying a retarded functional-system:

$$\frac{dz^a}{ds} = u^a \; , \; \frac{du^a}{ds} = W^a\,(s;z(r);z'(r'))\; ,$$

$$\frac{dz'^a}{ds'} = u'^a \; , \; \frac{du'^a}{ds'} = W'^a\,(s';z'(r');z(r))\; , \tag{1}$$

(where, in the first line, r is restricted to be smaller than s and r' smaller than ŝ' the proper time of the intersection of z'(s') with the past light cone of z(s)), by an ordinary differential system of the type:

$$\frac{dz^a(p)}{dp} = u^a \; , \; \frac{du^a(p)}{dp} = X^a\,(z(p),u(p),z'(p),u'(p))$$

$$\frac{dz'^a(p)}{dp} = u'^a \; , \; \frac{du'^a(p)}{dp} = X'^a\,(z'(p),u'(p),z(p),u(p)). \tag{2}$$

The functions X^a must fulfill the following requirements:

1) covariance under the Poincaré group (Poincaré 1905),
2) orthogonality: $f_{ab}X^a u^b = 0$,
3) invariance under world-line "sliding" (Droz-Vincent 1969, 1970):

$$u'^b X^a_{,z'b} + X'^b X^a_{,u'b} = 0. \tag{3}$$

Under these conditions the system (2) can be changed in each frame of reference into a Newtonian type system (12 equations using $t = z^o = z'^o$ as parameter instead of the 16 equations (2)) while still preserving the invariance under the Poincaré group (Bel 1970, 1971). Such a system (2) is called a predictive Poincaré invariant system (its general solution depends only on 12 essential parameters). The consideration of such a system is especially useful in setting up an Hamiltonian formalism for the dynamics of two particles (see Bel and Fustero 1976, Bel and Martin 1980).

When the functionals W are in fact functions of z(s), u(s) and of $\hat{z}' = z'(\hat{s}')$, where $\hat{s}'$ is the proper time of the intersection of the second world-line with the past directed light cone of vertex z(s),

$$W^a(s\ ;\ z\,(r);\ z'(r'),\ r<s,\ r'<\hat{s}') = W^a(z(s),u(s),\ z'(\hat{s}'),\ u'(\hat{s}')), \tag{4}$$

it is possible to set up an iterative algorithm for explicitly constructing the functions X^a, given the W^a (Bel, Salas and Sanchez 1973) (Bel et.al.. 1981). This algorithm is most easily obtained by solving perturbatively the following integro-functional equation (Hirondel 1974):

$$X^a(z,u,z',u') = R'(\hat{p}')W^a(z,u,z',u') - \int_{\hat{p}'}^{0} dp'R'(p')(X'^b X^a_{,u'b}), \tag{5}$$

where $\hat{p}' = -(z-z').u' - (((z-z').u')^2 + (z-z')^2)^{1/2}$ (Minkowski products and squares), and where $R'(p')$ is a "shift" operator which acts on a function of z,u,z',u' by replacing z' by $z' + p'u'$. In practical applications W is always multiplied by a "small" coupling constant, G in our case:

$$W^a = GW_1^a + G^2W_2^a + O(G^3), \tag{6}$$

therefore eqn (5) can be, and has been, used to generate W^a as a truncated expansion in powers of G:

$$X^a = GX_1^a + G^2X_2^a + O(G^3). \tag{7}$$

In the gravitational case, X_1 has been first obtained by Portilla (1979), together with a corresponding first order Hamiltonian formalism.

X_2 has been worked out by Bel, Damour, Deruelle, Ibañez and Martin (1981). We refer to the latter article for the setting up of the algorithm and for the explicit computation of X_2 starting from the W_2 which is derived in the same article (that W_2 being equivalent to the equations of motion of Westpfahl and Göller (1979).

In equation (6) we did not write the third order term G^3W_3, derived in section 9 because, contrary to the first two approximations, it does not seem easy (maybe it is not even possible) to obtain for W_3 an expression having the simplified functional form of eqn (4), and thence to construct X_3 by the algorithm deduced from eqn (5). In the next section we shall resort to a different approach for transforming retarded-functional equations of motion (1) into ordinary differential equations.

12. NEWTONIAN-LIKE EQUATIONS OF MOTION

In order to compare theory and observations one must extract some explicit physical predictions from the equations of orbital motion of two compact objects derived in section 9. One possible way of achieving this is to transform them into a predictive Poincaré invariant system (see section 11). This approach has many theoretical advantages: manifest Poincaré invariance, applicability to fast objects ($v \sim c$), possibility of defining an associated Hamiltonian formulation (references in section 11). However it has several practical drawbacks: it may not be applicable in its usual form to the third order equations of motion and it leads quickly to heavy calculations. Therefore in this section we shall restrict ourselves to the case of slowly moving objects ($v \ll c$). This assumption, on one hand will simplify very much the calculations by allowing us to use truncated expansions in powers of c^{-1} (while the Poincaré invariant approach is somewhat equivalent to resuming infinite power series of c^{-1}), and on the other hand will be sufficient for dealing with the kind of binary systems we have in mind (like the Hulse-Taylor pulsar PSR 1913 + 16).

By Newtonian-like equations of motion up to order c^{-n} (also called (n/2) Post-Newtonian equations of motion) we mean the following: in a given Lorentz coordinate system $x^a = (x^o = ct, x^i)$ $(i = 1,2,3)$, using $t = c^{-1}z^o = c^{-1}z'^o$ as parameter for each world-line one can write:

$$\frac{dz^i(t)}{dt} = v^i\ ,\quad \frac{dv^i(t)}{dt} = \sum_{p=0}^{n} c^{-p}\, A_p^i(z(t),v(t),z'(t),v'(t)). \tag{1}$$

Because the Einstein coupling constant between mass and gravity is in fact Gc^{-2} and because the acceleration contains two differentiations with respect to $x^o = ct$, a "retarded equation of motion up to order G^n" (as considered in sections 9 and 11) neglects terms or order G^{n+1}, which means in the preceding Newtonian sense, that it neglects terms of order c^{-2n} (nth Post-Newtonian terms). This formal rule for playing with the orders in G and c^{-1} is especially useful when dealing with gravitationally bound systems where the virial theorem: $Gc^{-2}m/R \sim c^{-2}v^2$ allows us to replace the Einstein coupling constant Gc^{-2} by c^{-2} when counting the Post-Newtonian order.

Consequently we see that the cubically non-linear equations of motion of section 9 ($n = 3$) neglect terms of order c^{-6} (3 Post-Newtonian terms) that is why it was consistent to neglect terms of order $G^3 v_r^2/c^2$ in section 9 and that it will be sufficient in this section to truncate all the c^{-1} expansions at order c^{-6} so that our final equations of motion will be accurate up to order c^{-5} inclusively ($2^1/2$ Post-Newtonian accuracy).

There are at least three different methods for obtaining Newtonian-like equations of motion up to order c^{-n}:

1) to compute, when possible, the predictive Poincaré invariant equations of motion. To write them down in a given Lorentz frame in their 3 + 1 form which yields: $dv/dt = F(z(t), c^{-1}v(t), z'(t), c^{-1}v'(t))$. At last to expand all the velocity dependent quantities up to order c^{-n}.

2) to start from the retarded system: $\dot{u} = W$, and to use, when possible, the formula of Lagrange (1770) for expanding all the functions of retarded quantities: $\hat{z}'$, $\hat{u}'$, ... in powers of c^{-1}. However Lagrange's expansion formula introduces successive derivatives of $z(t)$, but an iterative use of the lower order Newtonian-like equations of motion allows one to reduce the order of differentiation to one, in the right hand side (Kerner 1965).

3) to start from eqn (9.34) (together with 9.23)) which means that, when neglecting $O(G^4) + O(G^3 v_r^2)$ each world-line is a geodesic of the analytically continued metric g^A_{ab} for $A = 0$. To expand $g^A_{ab}(x)$ in powers of c^{-1} (near zone expansion). To plug this near zone expansion in the geodesic equation and to truncate the result at order c^{-n} inclusively.

When starting from the G^2 retarded equations of motion all three methods are possible and the first and the third have been used, and checked to be equivalent, by Damour and Deruelle (1981a). On the other hand the G^3 retarded equations of motion have a more intricate functional dependence on the world-lines and it has been possible to use only the third method (Damour 1982).

Despite the frightfully complicated structure of the third Post-Minkowskian metric (8.10) it has been found possible to separate many contributions to the equations of motion (9.34) which are, in their exact form, already of order $G^3 v_r^2$ (where v_r^2 is a Lorentz invariant squared relative velocity, see section 9), and which therefore can only contribute terms of order c^{-6}, at least, to the Post-Newtonian equations of motion. Even after this drastic simplification we are still left with many complicated expressions. Although some of them are still in (analytic continuation defined) integral form it is remarkable that all the integrals appearing in the final c^{-1} expanded expression can be carried out. By lack of space, it is impossible to even outline the calculation; I woud like to stress however, that the integrals coming from the cubic non-linearities in the metric were

computed thanks to a "new" process of analytic continuation. By "new" I mean here that this process is theoretically independent from the("old")process used above for defining the metric (section 8) and deriving the retarded equations of motion (section 9). Here analytic continuation is employed only as a technical tool for carrying out complicated integrals (which are convergent or are defined by the "old" process of analytic continuation). Finally, owing to the use of this "new" process, all the integrals can be calculated in function of the following formula defined by the "new" process of analytic continuation with respect to two complex parameters B and C :

$$\frac{1}{4\pi} \int d^3x /x-z/^B /x-z'/^C = \frac{\pi^{1/2}}{4} \frac{\Gamma(\frac{B+3}{2})\Gamma(\frac{C+3}{2})\Gamma(-\frac{B+C+3}{2})}{\Gamma(-\frac{B}{2})\Gamma(-\frac{C}{2})\Gamma(\frac{B+C+6}{2})} /z-z'/^{B+C+3}. \tag{2}$$

Denoting the velocity dz^i/dt by v^i and the acceleration dv^i/dt by a^i, the final result is of the following form:

$$a^i = A^i_0(z-z') + c^{-2}A^i_2(z-z',v,v') + c^{-4}A^i_4(z-z',v,v',S,S') +$$
$$+ c^{-5}A^i_5(z-z',v-v') + O(c^{-6}), \tag{3}$$

where all the quantities appearing in the right hand side are to be taken at the same time t (in a given Lorentz frame) as the lefthand side $a(t)$. This feature is compatible, though not in a manifest way, with the c^{-6} approximate Poincaré invariance of eqn(3), see Currie (1966), Hill (1967), Bel (1970) and the references in section 11. Note however that all the terms of (3) are manifestly invariant under time translations and space translations. The last term, which embodies, as we shall prove below, the "Laplace-Eddington effect" (or "radiation reaction effect"), is moreover invariant under Galileo transformations (Galilei 1638). All the terms of (3) are also invariant under space rotations. Note that A_4, but not A_5, depends on the spins S_{ik} and S'_{ik} of the two objects (Damour 1982). For the sake of simplicity we shall not consider here these spin dependent terms (see Damour 1978, 1982) which do not modify the final results.

Let us denote the instantaneous coordinate distance between the two worldlines by:

$$R := /z^i(t)-z'^i(t)/ := ((z-z')^2)^{1/2} , \tag{4}$$

where, for simplicity, we denote an Euclidean square: $f_{ik}b^ib^k (i,k = 1,2,3)$ by b^2 and a Euclidean scalar product: $f_{ik}a^ib^k$ by (ab). Let us denote the instantaneous unit radial vector from the second to the first world-line by:

$$N^i := R^{-1}(z^i(t) - z'^i(t)), \tag{5}$$

and the instantaneous relative velocity by:

$$V^i := v^i(t) - v'^i(t). \tag{6}$$

Then we can write down the following explicit expressions for the different terms in eqn(3) (m and m' being respectively the two masses):

$$A_o^i = -Gm'R^{-2}N^i\ , \tag{7}$$

$$A_2^i = Gm'R^{-2}(\ N^i(-v^2-2v'^2+4(vv')+\tfrac{3}{2}(Nv')^2+5(Gm/R)+4(Gm'/R))+(v^i-v'^i)(4(Nv)-3(Nv'))), \tag{8}$$

$$A_4^i = B_4^i + C_4^i\ , \tag{9}$$

with :

$$\begin{aligned} B_4^i = Gm'R^{-2}(N^i(&-2v'^4+4v'^2(vv')-2(vv')^2+\tfrac{3}{2}v^2(Nv')^2+\tfrac{9}{2}v'^2(Nv')^2-6(vv')(Nv')^2-\tfrac{15}{8}(Nv')^4+\\ &+(Gm/R)(-\tfrac{15}{4}v^2+\tfrac{5}{4}v'^2-\tfrac{5}{2}(vv')+\tfrac{39}{2}(Nv)^2-39(Nv)(Nv')+\tfrac{17}{2}(Nv')^2)+\\ &+(Gm'/R)(4v'^2-8(vv')+2(Nv)^2-4(Nv)(Nv')-6(Nv')^2))+\\ &+(v^i-v'^i)(v^2(Nv')+4v'^2(Nv)-5v'^2(Nv')-4(vv')(Nv)+\\ &+4(vv')(Nv')-6(Nv)(Nv')^2+\tfrac{9}{2}(Nv')^3+\\ &+(Gm/R)(-\tfrac{63}{4}(Nv)+\tfrac{55}{4}(Nv'))+(Gm'/R)(-2(Nv)-2(Nv')))), \end{aligned} \tag{10}$$

$$C_4^i = G^3m'R^{-4}\ N^i\ (-\tfrac{57}{4}\ m^2 - 9\ m'^2 - \tfrac{69}{2}\ mm'), \tag{11}$$

$$\begin{aligned} A_5^i = \tfrac{4}{5}\ G^2\ mm'R^{-3}(\ &V^i(-V^2 + 2(Gm/R) - 8(Gm'/R))\ +\\ &+N^i(NV)(3V^2 - 6(Gm/R) + \tfrac{52}{3}(Gm'/R))). \end{aligned} \tag{12}$$

A_o was first obtained by Newton (1687), A_2 was first obtained by Lorentz and Droste (1917) (weak field extended sources, see next section) and by Einstein, Infeld and Hoffmann(1938), B_4 and the G^2 part of A_5 were first obtained by Damour and Deruelle (1981a), a result equivalent to $A_5 - A_5'$ was obtained by Linet (1981) (weak field extended sources), finally C_4 and the complete A_5 were first obtained by Damour (1982). The remainder $O(c^{-6})$ of eqn (3) has contributions coming from the 1 PMA, 2 PMA, and 3 PMA (after an expansion in 1/c) and from the 4 PMA and $G^3\ v_r^2$ terms which are already of order c^{-6}. We leave to future work a more rigorous study of these errors.

13. ELECTROMAGNETIC ANALOGY AND POST-POST-NEWTONIAN GENERALIZED LAGRANGIAN

The well-known (time dependent) least action principle of classical point mechanics:

$$\delta \int_{t_1}^{t_2} dt L(z(t),v(t)) = 0\ , \tag{0a}$$

where the "Lagrangian" L is given by

$$L(z,v) = \Sigma\ (\ \tfrac{1}{2}mv^2) - U\ , \tag{0b}$$

(U= potential energy) was in fact discovered by Hamilton (1835). (On the other hand "Hamilton's equations" were first written down by Lagrange! Moreover Euler and Lagrange knew the time independent least action principle, named after Maupertuis). This "Lagrangian" formalism (0a)(0b) has undergone, after the discovery of relativistic interactions, an interesting evolution which is generally ignored and which has caused many mistakes. In order to clarify our later use of a "generalized" gravitational Lagrangian we shall first discuss the case of the retarded electromagnetic interaction between point charges which provides a simple analogue to the gravitational case.

As we have seen in section 3, Lorentz (1892) and Planck (1897) discovered the existence of a "radiation reaction" "self-force" (eqn (3.5)) acting on a charge e (with position z^i, velocity v^i, acceleration a^i):

$$F^i = \frac{2}{3}\frac{e^2}{c^3}\frac{da^i}{dt} + O(\frac{1}{c^5}) \tag{1}$$

However the expression (1) is just a small piece of the total force acting on the charge. The main part of the total force comes from the retarded field of the companion charge (e', for simplicity's sake we restrict our attention to a binary electric system). Expanding, by means of Lagrange's formula (see section 12), the retarded action of the companion, using the lower order equations of motion for reducing the order of differentiation (Kerner 1965) and adding the expression (1) (augmented with its relativistic corrections see eqn (3.6)) one obtains the equations of motion of the first charge in the following form:

$$\begin{aligned} ma^i = F_0^i(z-z') &+ c^{-2}F_2^i(z-z',v,v') + c^{-3}F_3^i(z-z',v-v')\ +\\ &+ c^{-4}F_4^i(z-z',v,v') + c^{-5}F_5^i(z-z',v,v') + O(c^{-6}). \end{aligned} \tag{2}$$

The first term in the right hand side of eqn (2) is the familiar "instantaneous" Coulomb force:

$$F_0^i = ee'R^{-2}N^i, \tag{3}$$

where we use the notations of the preceding section. The third term is equal to:

$$F_3^i = \frac{2}{3}e^2e'\left(\frac{e}{m} - \frac{e'}{m'}\right) R^{-3}(v^i - 3N^i(NV)\ ). \tag{4}$$

It can also be written as (Darwin 1920):

$$F_3^i = \frac{2}{3}\, e\, \frac{d}{dt}\left(\frac{e}{m}F_0^i + \frac{e'}{m'}F_0'^i\right). \tag{5}$$

The expression in the right hand side of eqn (5) contains two terms: the first one is easily recognized as the contribution due to the Lorentz-Planck force (1) and the second one comes from the c^{-3} term in the Lagrange expansion of the retarded force due to the companion. If one computes the power "dissipated" by $c^{-3}F_3$, the result is:

$$F_3^i v^i + F_3'^i v'^i = -\frac{2}{3c^3}(ea^i + e'a'^i)^2 + \frac{1}{c^3}\frac{d}{dt}Q_3 + O\left(\frac{1}{c^5}\right), \tag{6}$$

where dQ_3/dt is an "exact time derivative." The first term in the right hand side of eqn (6) is identical with the familiar "dipole" energy loss calculated from the Poynting flux in the wave zone. It is often believed that this fact alone (appearance of the "dipole" energy loss) is sufficient: 1) for considering $c^{-3}F_3$ as the "total radiation reaction" force (modulo errors $O(c^{-5})$), 2) for concluding that the Coulombian energy of the system will be dissipated, in an average sense at least, according to the "dipole formula" and, 3) for deriving, from the usual conservation laws of mechanics (energy, linear momentum, angular momentum), the secular kinematical effects (shrinkage of the binary orbit,..)due to this "radiation reaction." However these conclusions can be reached only if one proves two lemmas:

Lemma 1: there exist well defined conservation laws at the post-Coulombian level ($F_0 + c^{-2}F_2$) which guarantee the absence of any secular effects similar to or interfering with the ones caused by the "radiation reaction" $c^{-3}F_3$,

Lemma 2: the quantity Q_3, whose "exact time derivative" appears in eqn (6) is a univalued function of the instantaneous state (positions and velocities) of the system.

In the present case it is easy to check directly the validity of Lemma 2 from eqn (5) but we have mentioned it explicitly because in some derivations Q_3 (or its analogue) is only obtained as an integral over the whole 3-space of some "field energy". This makes the corresponding "balance" equation (6) useless because such an integral is in general a functional of the entire past history of the two particles and could contain contributions of the type $\int_{\infty}^{t}(ea^i + e'a'^i)^2 dt$ which would modify completely the meaning of eqn (6). It is also essential to prove the validity of Lemma 1 because the often quoted argument that $F_0 + c^{-2}F_2$ is "time-even" is completely insufficient for ensuring the existence of "good" post-Coulombian conservation laws which are needed for separating the effects due to the "radiation reaction."

The first proof of Lemma 1, when neglecting terms $O(c^{-4})$ in eqn (2), is due to Darwin (1920). His proof was based on the discovery that the post-Coulombian equations of motion ($ma = F_0 + c^{-2}F_2$) could be deduced from a Lagrangian:

$$L(z,v) = L_0(z,v) + c^{-2}L_2(z,v) \tag{7}$$

where L_0 is the usual Lagrangian:

$$L_0 = \Sigma\left(\frac{1}{2}mv^2 - \frac{1}{2}\frac{ee'}{R}\right), \tag{8}$$

(Σ denotes a summation over the two particles) and where:

$$L_2 = \Sigma\left(\frac{1}{8}mv^4 + \frac{ee'}{4R}((vv') + (Nv)(Nv'))\ \right). \tag{9}$$

Because the Lagrangian (7) is manifestly invariant under time translations, space translations and space rotations, Darwin deduced from it some post-Coulombian conservation laws corresponding to the familiar conservation laws of energy, linear momentum and angular momentum. These conservation laws are violated by the term $c^{-3}F_3$. These results allow one to separate from lower order corrections and therefore to investigate meaningfully the effects due to the term $c^{-3}F_3$ which is seen now,and only now,to merit its name of "radiation reaction" (when neglecting terms of order c^{-4}).

However, when the two charges are alike or more generally when:

$$\frac{e}{m} = \frac{e'}{m'}, \tag{10}$$

the "radiation reaction" $c^{-3}F_3$(eqn 4) vanishes. The condition (10) implies also the vanishing of the electric "dipole" energy flux (in the wave zone) and of the "magnetic dipole" energy flux. Therefore the first non zero contribution to the wave zone energy flux is the "quadrupole" one:

$$\text{Energy flux} = -\frac{1}{20}c^{-5}(\dddot{Q}_{ik})^2, \tag{11}$$

where $Q_{ik} = \Sigma\, e(z^i z^k - \frac{1}{3} z^2 \delta^{ik})$. This situation is especially interesting for us because it is analogue to the gravitational case. When condition (10) is satisfied, and because of the result (11), one is led to conjecture that the role of "radiation reaction" will be played by the term $c^{-5}F_5$ in eqn (2). However in order to prove this conjecture, we must prove Lemma 1 at the post-post Coulombian level. We can try to do so by extending the result of Darwin to the next order.

Indeed it has been argued (Landau and Lifshitz 1976, §65) that the possibility of having a Lagrangian description of a system of charges is directly related to the non-existence (at some approximation) of radiation of electromagnetic waves because these waves represent the true degrees of freedom of the electromagnetic field (which cannot be accounted for by the usual degrees of freedom of a mechanical system). Landau and Lifshitz argued therefore that when condition (10) is satisfied there should exist a post-post-Coulombian Lagrangian and they even computed this Lagrangian, from the work of Smorodinski and Golubenkov (1956), in the Problem 2 of §75. However this result cannot be correct because it has been proven, on general grounds, by Martin and Sanz (1979) that the dynamics of a system of particles interacting via any of the usual classical fields (electromagnetic, gravitational,...) could never admit a Lagrangian description when terms of order c^{-4} are included in the equations of motion. The solution of this dilemma is that the Lagrangian of Smorodinski and Golubenkov is effectively incorrect but that there exists a "generalized" Lagrangian, or "higher order Lagrangian", namely a function $L(z,v,a)$ of the instantaneous positions, velocities and <u>accelerations</u> of the two particles such that the solutions of the post-post-Coulombian equations of motion ($ma = F_0 + c^{-2}F_2 + c^{-4}F_4$) extremize, when working up to order c^{-4} inclusively, the following action:

$$S = \int_{t_1}^{t_2} dt\, L\left(z(t), \frac{dz(t)}{dt}, \frac{d^2z(t)}{dt}, z'(t), \frac{dz'(t)}{dt}, \frac{d^2z'(t)}{dt}\right), \tag{12}$$

with given values of z, dz/dt, z', dz'/dt in t_1 and t_2. The correct generalized Lagrangian is easily obtained by proceeding as in §65 of Landau and Lifschitz (1976). One starts from the Lagrangian of one charge in an external electromagnetic potential $A^a = (V, A^i)$:

$$L_1 = -mc^2(1 - v^2/c^2)^{1/2} - eV + c^{-1}eA^i v^i. \tag{13}$$

Then one replaces V and A^i by the Lagrange expansions of the retarded potentials of the second charge. One introduces separate notations for the time differentiation when it acts on the kinematical variables of the first particle only (leaving aside the variables of the second particle) and vice versa (for instance:

$Q(z(t), z'(t))_{,1} = v^i Q_{,z^i}$, $Q(z,z')_{,2} = v'^i Q_{,z'^i}$, $dQ/dt = Q_{,1} + Q_{,2}$). Then by discarding some total time derivative one can transform L_1 into an expression L'_1 which is symmetrical under the exchange of the two particles. From this last expression it is trivial to guess a generalized Lagrangian for the <u>system</u> of the two particles:

$$L(z,v,a) = L_0 + c^{-2}L_2 + c^{-4}L_4, \tag{14}$$

where L_0 and L_2 are the same as before and where:

$$\begin{aligned} L_4(z,v,a) = {} & mv^6/16 + m'v'^6/16 - \\ & - \frac{ee'}{8}\big(R^{-1}(v^2v'^2 - 2(vv')^2 + 3(Nv)^2(Nv')^2 - (Nv)^2v'^2 - (Nv')^2v^2) + \\ & + 2(Nv)(va') - 2(Nv')(v'a) + (Na)(v'^2 - (Nv')^2) - (Na')(v^2 - (Nv)^2) + \\ & + 3R(aa') - R(Na)(Na')\big). \end{aligned} \tag{15}$$

(The interaction terms in L_4 come from: $-ee'(\frac{1}{2}(R(vv'))_{,12} + \frac{1}{24}(R^3)_{,1122})$).
Then one can, a posteriori, see that the "Lagrangian" of Smorodinski and Golubenkov is obtained from (15) by replacing the accelerations by their Coulombian values. Such a replacement is incorrect in a generalized Lagrangian because in the action principle (12) $z(t)$ must be freely varied. Indeed the Euler-Lagrange equations of (12) with (14-15) are:

$$L_{,z} - \frac{d}{dt}L_{,v} + \frac{d^2}{dt^2}L_{,a} = 0, \tag{16}$$

and are not equivalent to the Euler-Lagrange equations of $L(z,v,m^{-1}F_0(z))$. The eqn (16) leads a priori to a fourth order differential system, however the coefficients of $\ddot{a}$ and $\ddot{a}'$ are of order c^{-4} and, now that we have varied $z(t)$, we can correctly replace the accelerations in the equations of motion by their Coulombian (or, when necessary, post-Coulombian) values. This gives the post-post Coulombian equations of motion of eqn (2) (with (10) and the neglecting of $O(c^{-5})$). Now that we have succeeded in deducing the post-post-Coulombian dynamics from a generalized Lagrangian which is manifestly invariant under time and space translations and space rotations, we can make use of Noether's theorem (see next section) to obtain some "good" post-post Coulombian conservation laws which allow us to prove the validity of Lemma 1 (at order c^{-4}) and therefore to prove that $c^{-5}F_5$ can be rightly called "the radiation reaction" (the existence of a "balance" equation of type (6) at order c^{-5}, and Lemma 2 are straightforwardly checked; see Deruelle 1982). In the case of half retarded - half advanced electromagnetic interaction there are no "time-odd" terms and the Lagrangian approach can be generalized to any order of approximation (Kerner 1965).

Going back now to the gravitational case, we come, as announced in section 3, to the very remarkable work of Lorentz and Droste (1917). Indeed it is well known that the post-Newtonian equations of motion derived by Droste and DeSitter (DeSitter 1916) are incorrect, as well as the later result of Levi-Civita (1937); however, I found serendipitously that Lorentz and Droste (1917) not only derived correctly the post-Newtonian equations of motion (generally thought of as having been obtained in 1938 by Einstein, Infeld and Hoffmann, and by Eddington and Clark), but also have proved that these equations could be deduced from a post-Newtonian Lagrangian (generally attributed to Fichtenholz 1950):

$$L(z,v) = L_0(z,v) + c^{-2}L_2(z,v), \tag{17}$$

with:

$$L_0 = \Sigma\left(\frac{1}{2}mv^2 + \frac{1}{2}\frac{Gmm'}{R}\right), \tag{18}$$

$$L_2 = \Sigma\left(\frac{1}{8}mv^4 + \frac{Gmm'}{R}\left(\frac{3}{2}v^2 - \frac{7}{4}(vv') - \frac{1}{4}(Nv)(Nv') - \frac{1}{2}\frac{Gm}{R}\right)\right). \tag{19}$$

On the other hand we have seen in the preceding section that the equations of motion of two non-rotating compact objects were of the form:

$$a^i = A^i_o(z-z') + c^{-2}A^i_2(z-z',v,v') + c^{-4}A^i_4(z-z',v,v') + c^{-5}A^i_5(z-z',v-v') + O(c^{-6}). \tag{20}$$

Therefore, although the Lorentz-Droste-Fichtenholz Lagrangian (17) allows us to define post-Newtonian conservation laws, these laws are not sufficient for proving the preceding Lemma 1 at the post-post-Newtonian level, that is, for proving, as expected from the analogy with the electromagnetic case when (10) is satisfied, that $c^{-5}A_5{}^i$ is "the radiation reaction." In 1974, Ohta, Okamura, Kimura and Hiida, computed a post-post-Newtonian Lagrangian $\hat{L}(z,v) = L_0 + c^{-2}L_2 + c^{-4}L_4$. However, according to the precedingly quoted work of Martin and Sauz, this result cannot be correct. Effectively it has been proved by Damour and Deruelle (1981b), that when neglecting in the equations of motion (20) the terms of order G^3 and/or the terms of order c^{-5} (see eqns (12.7)(12.8)(12.10)) the resulting equations could be deduced from a generalized Lagrangian:

$$L(z,v,a) = L_0(z,v)+c^{-2}L_2(z,v)+c^{-4}L_4(z,v,a). \tag{21}$$

This result has been extended by Damour (1982) to the complete post-post Newtonian equations of motion (including order G^3, neglecting order c^{-5}), even with the inclusion of the spin-orbit interaction (for slowly spinning compact objects). For simplicity we shall not discuss here the spin dependent terms which bring in new subtleties, but we shall give the explicit expression of $L_4(z,v,a)$ (L_o and L_2 being the same as before: (18) and (19)):

$$L_4(z,v,a) = M_4 + N_4\,, \tag{22}$$

$$\begin{aligned} M_4(z,v,a) = {} & \Sigma\left(\frac{1}{16}mv^6\right) + \\ & + \Sigma\, Gmm'R^{-1}\Big(\frac{7}{8}v^4 + \frac{15}{16}v^2v'^2 - 2v^2(vv') + \frac{1}{8}(vv')^2 - \frac{7}{8}(Nv)^2v'^2 + \\ & \qquad + \frac{3}{4}(Nv)(Nv')(vv') + \frac{3}{16}(Nv)^2(Nv')^2\Big) + \\ & + \Sigma\, G^2m^2m'R^{-2}\Big(\frac{1}{4}v^2 + \frac{7}{4}v'^2 - \frac{7}{4}(vv') + \frac{7}{2}(Nv)^2 + \frac{1}{2}(Nv')^2 - \frac{7}{2}(Nv)(Nv')\Big) + \\ & + \Sigma\, Gmm'\Big((Na)\Big(\frac{7}{8}v'^2 - \frac{1}{8}(Nv')^2\Big) - \frac{7}{4}(v'a)(Nv')\Big)\,, \end{aligned} \tag{23}$$

$$N_4(z) = \frac{G^3mm'}{R^3}\left(\frac{1}{2}m^2 + \frac{1}{2}m'^2 + \frac{19}{4}mm'\right). \tag{24}$$

The generalized Lagrangian (21-24) has been obtained by guess-work contrary to the preceding generalized electromagnetic Lagrangian (14-15) which was nearly straightforwardly derived from (13)(there is though some amount of guesswork in the last steps leading to (14)). On the other hand the fact that one succeeded in deriving the complicated equations of motion (20)(neglecting $O(c^{-5})$)(see eqns (12.7) to (12.11)) from a variational principle is a very stringent check on the correctness of these equations of motion: indeed one wrong coefficient in these equations is often sufficient for ruling out the existence of a Lagrangian (generalized or not). There is though a remarkable exception to this rule: the coefficient of the last term of eqn (12.11): -69/2 could have any numerical value and would still allow a Lagrangian deduction. In fact, having obtained -69/2 after a very long calculation divided into sixteen simpler contributions, I was worried because Damour and Deruelle (1981b) had conjectured a different coefficient: -36. However this conjecture was based on the assumption that the only incorrectness in the Lagrangian of Ohta et.al. (1974) was, as in the case of Smorodinski and Golubenkov, due to the replacing of the accelerations by their Newtonian values. However a close scrutiny of their work revealed that the preceding coefficient was related to their U^{TT} (see their §5) itself expressed as the sum of three integrals. A reevaluation of these three integrals I_1, I_2, I_3 by means of the "new" process of analytic continuation (section 12) showed that I_2:

$$I_2 = \frac{G^3}{2\pi}m^2m'^2\int d^3x\; r^{-1}_{\ ,i}\; r'^{-1}_{\ ,k}(\log(r+r'+R))_{,z^k z'^i} \tag{25}$$

was equal to $-1G^3m^2m'^2/R^3$ instead of their result: $-(3/2)G^3m^2m'^2/R^3$. Taking into account this correction, the preceding conjecture, with some more work due to the use of different coordinate systems, then provided a confirmation of the litigious coefficien -69/2 and thereby of the last coefficient of eqn (24): 19/4.

The post-post-Newtonian conservation laws implicitly contained in the generalized Lagrangian (21-24) are derived in the next section.

14. POST-POST-NEWTONIAN CONSERVED QUANTITIES

We have seen in the preceding section that in order to meaningfully identify the term $c^{-5}A_5$ with a "radiation reaction", one had to prove the "Lemma 1," namely that the post-post-Newtonian (2 PN in brief) equations of motion ($a = A_0 + c^{-2}A_2 + c^{-4}A_4$) admitted some "good" conservation laws. We use the word "prove" because, on one hand the "time-even" character of these equations is not sufficient to imply the existence of such conservation laws (counter-examples can be constructed) and, on the other hand, the fact that the complete system, matter plus gravitational field, admits some global conservation laws in integral form is not sufficient to imply that the field-functional conserved quantities are functions of the instantaneous state of the matter only (see the comments after Lemma 2 in the preceding section). However we have seen in section 13 that the 2 PN equations of motion could be deduced from the generalized Lagrangian L(z,v,a) (13.21-24). Therefore if L(z,v,a) admits some symmetries we shall be able to construct some corresponding conserved quantities by using the general theorem of Noether (1918). In fact L(z,v,a) is manifestly invariant under time translations, space translations and space rotations. From these symmetries, Damour and Deruelle (1981c) have constructed quantities (7 integrals of the motion):

$$E(z,v,a) := -L + \Sigma\ (p_i v^i + q_i a^i)\ , \tag{1}$$

$$P_i(z,v,a) := \Sigma\ p_i\ , \tag{2}$$

$$J_{ik}(z,v,a) := \Sigma\ (z_i p_k - z_k p_i + v_i q_k - v_k q_i)\ , \tag{3}$$

where p and q denote:

$$p_i := L_{,v^i} - \frac{d}{dt} L_{,a^i}\ , \qquad q_i := L_{,a^i}\ . \tag{4}$$

For the details of the proof of the constancy of (1 - 3) see Damour and Deruelle (1981c)(note that in their notations one should take $\alpha = 0$ and $\beta = -1/2$ in order to recover exactly L(z,v,a)).

On the other hand L(z,v,a) admits a further symmetry which is not manifest (indeed it is quite hidden!) and which is only approximate. This symmetry is linked to the pure Lorentz transformations (boosts) and I would like to comment more on this symmetry because the literature is nearly nonexistent (and sometimes incorrect) on this subject. Let us first go back to the usual Lagrangian of classical mechanics:

$$L(z,v) = \Sigma\ (\tfrac{1}{2}\ mv^2) - U(z-z') \tag{5}$$

Let us consider an infinitesimal Galileo transformation acting on space-time (b^i being an infinitesimal velocity):

$$\delta x^i = b^i t \qquad , \qquad \delta t = 0 \tag{6}$$

This transformation acts on the motions z(t) in the following way:

$$\delta z^i = b^i t \qquad , \qquad \delta v^i = b^i \tag{7}$$

Hence we find the variation of the Lagrangian:

$$\delta L \equiv \Sigma\ mv^i \delta v^i \equiv \frac{d}{dt}(b^i\ \Sigma\ mz^i). \tag{8}$$

The Lagrangian is changed only by an exact time derivative and this result constitutes a symmetry of the Lagrangian which implies both a symmetry of the dynamics (Galilean relativity) and the existence of a new vectorial integral of the motion. Indeed the general "integration by parts" identity:

$$\delta L(z,v) = \frac{d}{dt}\ (\Sigma\ L_{,v^i}\delta z^i) + \Sigma\ (L_{,z^i} - \frac{d}{dt}\ L_{,v^i})\delta z^i, \tag{9}$$

implies, when the Euler-Lagrange equations are satisfied, an equation for δL , which, when (7) is satisfied, can be reapproached from the identity (8). This leads to the famous "center of mass integral".

$$K^i = \Sigma\ (mz^i) - t\ \Sigma\ (mv^i). \tag{10}$$

A similar, though more complicated, result is valid in the case of interest: generalized Lagrangian L(z,v,a) and Lorentz boosts:

$$\delta x^i = b^i t \qquad , \qquad \delta t = c^{-2} b^i x^i. \tag{11}$$

A first technical delicacy lies in the action of the boosts (11) on the motions z(t) because of the joint change in space and time. Then the main difficulty lies in proving that there exists a univalued function of z,v,a, whose time derivative is identically equal to δL (modulo O(c-6)). This is done in the precedingly quoted work. From this result one deduces, on one hand a check on the approximate Poincaré invariance of the 2PN equations of motion and on the other hand the existence of a vectorial integral:

$$K^i(z,v,a) := G^i - t.P^i \tag{12}$$

where P^i is (2) and where:

$$G^i(z,v) := \Sigma\,(Mz^i) - \frac{7}{4}\,\frac{Gmm'}{c^4}\,((Nv)+(Nv'))(v^i - v'^i), \tag{13a}$$

$$M := m + c^{-2}\left(\frac{1}{2}\,mv^2 - \frac{1}{2}\,\frac{Gmm'}{R}\right) + c^{-4}\,\frac{3}{8}\,mv^4 +$$

$$+\ c^{-4}\,\frac{Gmm'}{R}\left(\frac{19}{8}v^2 - \frac{7}{8}v'^2 - \frac{7}{4}(vv') - \frac{1}{8}(Nv)^2 - \frac{1}{4}(Nv)(Nv') + \frac{1}{8}(Nv')^2 - \frac{5}{4}\,\frac{Gm}{R} + \frac{7}{4}\,\frac{Gm'}{R}\right). \tag{13b}$$

We have stressed the deduction of this further 2 PN integral from the boost-symmetry of the Lagrangian because usually, even for the simpler 1 PN case, K is obtained by a direct calculation starting from the constancy of P and not from a Noetherian approach.

Let us add that the preceding 2 PN integrals E,P,J,K have been defined as functions of z,v, and a, but that once they have been constructed one can correctly replace a by its Newtonian value thereby defining some integrals E^{2PN}, P^{2PN}, J^{2PN}, K^{2PN} functions of z and v only. Finally we wish to point out that we have refrained from calling these ten integrals: energy, linear momentum, angular momentum and center of mass integrals respectively. The reason is that we shall need in the next section the conservation laws associated with these integrals in order to separate the kinematical effects caused by the "radiation reaction", the dynamical meaning of E,P,J,K will be of no interest to us. Similarly the link, if any, of E for instance with the A.D.M. or the Bondi mass is not at all relevant.

15. THE LAPLACE-EDDINGTON EFFECT AND THE BINARY PULSAR

In section 12 we obtained the "second and a half post-Newtonian" (in brief "2 1/2 PN") equations of motion of two non-rotating compact bodies (for slowly rotating compact bodies see Damour 1982):

$$a^i := \frac{d^2 z^i}{dt^2} = A_0^i(z-z') + c^{-2}A_2^i(z-z',v,v') + c^{-4}A_4^i(z-z',v,v') + c^{-5}A_5^i(z-z',v-v') + O(c^{-6}). \tag{1}$$

The first term in the right hand side of (1) is the familiar Newtonian acceleration and all the other terms are relativistic corrections due to the non-linear hyperbolic structure of Einstein's equations. However, we have proved in section 14 the validity of the "Lemma 1" of section 13, namely the existence of ten conservation laws at the 2PN level. We can therefore take advantage of the corresponding 2 PN integrals for investigating the secular effects caused by the $c^{-5}A_5$ term in (1). Let us denote any of the precedingly defined "integral of the 2 PN notion," that is a

function of the instantaneous kinematical state of the binary system by: $C^{2PN}(z(t), v(t), z'(t), v'(t))$ (we have seen that we can always, in a perturbative sense at least, remove the dependence on the instantaneous accelerations). Then the time derivative of $C^{2PN}(z,v)$ during the actual motion (including now the c^{-5} correction) is:

$$\frac{dC^{2PN}(z,v)}{dt} = \Sigma\,\left(C^{2PN}_{,zi}v^i + C^{2PN}_{,vi}(A_0^i + c^{-2}A_2^i + c^{-4}A_4^i + c^{-5}A_5^i + O(c^{-6}))\right) \tag{2}$$

Now, because C^{2PN} is constant during a 2PN motion all the terms of the right hand side of eqn (2), except the c^{-5} term, reduce to $O(c^{-6})$ (they do not reduce exactly to zero on one hand because the replacement of the accelerations by their Newtonian values introduce errors of order c^{-6}, and on the other hand because the constancy of the integral K is, like the corresponding boost symmetry of the generalized Lagrangian, valid only modulo errors of order c^{-6}). Then we are left with:

$$\frac{dC^{2PN}(z,v)}{dt} = \Sigma c^{-5}\,A_5^i\,C^{2PN}_{,vi} + O(c^{-6}) \tag{3}$$

An equivalent result can also be obtained directly from the Noetherian approach of the preceding section (Damour and Deruelle 1981c). Eqn (3) can be further simplified by taking into account the fact that the function of z,v: $C^{2PN}(z,v)$ is easily seen, from its definition (14.1), (14.2), (14.3) or (14.12), to be given by a truncated expansion in powers of c^{-1}:

$$C^{2PN}(z,v) = C^N(z,v) + c^{-2}C_2(z,v) + c^{-4}C_4(z,v) + O(c^{-6}) \tag{4}$$

where the first term $C^N(z,v)$ is in fact the familiar Newtonian Noetherian integral corresponding to the space-time symmetry used for deriving C^{2PN} from L(z,v,a) (Newtonian integrals of "energy", "linear momentum," "angular momentum" and "center of mass motion": eqn (14.10)). Plugging eqn (4) into eqn (3) leads to the simplified expression:

$$\frac{dC^{2PN}(z,v)}{dt} = \Sigma c^{-5}\,A_5^i\,C^N_{,vi} + O(c^{-6}) \tag{5}$$

We can further transform eqn (5) by taking into account that the explicit expression given for A_5 in eqn (12.12):

$$A_5^i = \frac{4}{5}\,G^2mm'R^{-3}\left(V^i(-V^2 + 2(Gm/R) - 8(Gm'/R)) + N^i(NV)(3V^2 - 6(Gm/R) + \frac{52}{3}(Gm'/R))\right), \tag{6}$$

(where $V^i = v^i - v'^i$, $RN^i = z^i - z'^i$, $N^2 = 1$) can also be written as (Damour 1982):

$$A^i_5 = \frac{3}{5} z^k Q^{(5)}_{ik} + 2v^k I^{(4)}_{ik} + \frac{10}{3} a^i I^{(3)} + \frac{1}{5} I^{(5)}_{iss} - J^{(4)}_{iss} + O(c^{-2}) , \tag{7}$$

with:

$$I_{ik} := \Sigma\, Gmz^i z^k \ , \ I := I_{ss} \ , \ Q_{ik} := I_{ik} - \frac{1}{3} If_{ik} \ ,$$

$$I_{iss} := \Sigma\, Gmz^2 z^i \ , \ J_{iss} := \Sigma\, Gmz^2 v^i \ , \ Q^{(n)} := d^n Q/dt^n . \tag{8}$$

Replacing eqn (7) into eqn (5) leads to the following results for the time derivative of the 2PN integrals E^{2PN}, P_i^{2PN}, J_{ik}^{2PN}, $K_i^{2PN} = G_i^{2PN} - tP_i^{2PN}$ defined in section 14:

$$\frac{dE^{2PN}(z,v)}{dt} = -c^{-5} \frac{dE^5(z,v)}{dt} - \frac{1}{5Gc^5} Q^{(3)}_{ik} Q^{(3)}_{ik} + O(c^{-6}) , \tag{9}$$

$$\frac{dP_i^{2PN}(z,v)}{dt} = -c^{-5} \frac{dP_i^5(z,v)}{dt} + 0 + O(c^{-6}) , \tag{10}$$

$$\frac{dJ_{ik}^{2PN}(z,v)}{dt} = -c^{-5} \frac{dJ^5_{ik}(z,v)}{dt} - \frac{2}{5Gc^5} (Q^{(2)}_{is} Q^{(3)}_{ks} - Q^{(2)}_{ks} Q^{(3)}_{is}) + O(c^{-6}) , \tag{11}$$

$$\frac{d(G_i^{2PN}(z,v) - tP_i^{2PN}(z,v))}{dt} = -c^{-5} \frac{d(G_i^5(z,v) - tP_i^5(z,v))}{dt} + 0 + O(c^{-6}) \tag{12}$$

As indicated by the notation the quantities E^5, P^5, J^5, and G^5 appearing in the right hand side of the preceding equations are some univalued functions of the instantaneous state of the binary system: $z(t)$, $z'(t)$, $v(t)$, $v'(t)$ (higher derivatives having been reduced to z and v only). These functions are straightforwardly obtained from (5) and (7), and their explicit expression is not important; what is important is their existence. The equations (9-12) are the gravitational analogues of the electromagnetic "balance" equation (13.6) supplemented by the consequences of Lemma 1 and Lemma 2 of the same section (remember that Lemma 1 was dealing with the existence of pre-radiation reaction conserved quantities (C^{2PN} here) and Lemma 2 with the functional nature of the balance-violating quantities ($-dC^5/dt$ here)). We can further simplify eqn (9-12) by introducing some new functions of the instantaneous state of the system:

$$C^{2\ 1/2PN}(z,v) := C^{2PN}(z,v) + c^{-5} C^5(z,v). \tag{13}$$

Then $P^{2\frac{1}{2}PN}$ and $K^{2\frac{1}{2}PN}$ are constant (modulo c^{-6}) and, $dE^{2\frac{1}{2}PN}/dt$ and $dJ^{2\frac{1}{2}PN}/dt$ precisely balance the "quadrupole" fluxes of energy and angular momentum in the wave zone (the second terms in the right hand sides of eqns (9) and (11), see the lectures of K. Thorne. Such an agreement between the decrease of a well-defined function

of the instantaneous state of the system and a loss due to an outward flux in the wave zone is a result which had never been proved before. All the previous "proofs" of this "balance" were incomplete mainly because they could never control the functional nature of the "energy of the system," for instance. This requirement regarding the functional nature of the energy is generally overlooked; however it is essential because if the "energy" is not a function of the instantaneous kinematical state of the binary system but contains some terms, even of the small order c^{-5}, which are functionals of the past history of the system, then the "energy balance" equation is meaningless and says nothing about the secular kinematical behaviour of the system (see the comments after Lemma 2 of section 13). Moreover the precise knowledge of the variables appearing in $C^{2^{1/2}PN}$ is essential in the case of rotating objects, because we need a "balance" between the orbital energy and the gravitational energy flux, when many previous derivations could only give a "balance" between the "total" energy of the system (including the kinetic spin energy of the bodies) and the "total" energy flux in the wave zone (including the fluxes of electromagnetic waves, relativistic particles,...). Such a "total balance" is useless in the case of the Hulse-Taylor pulsar where the losses of orbital and spin energy are of the same order of magnitude. However it has been possible to extend the preceding method to the case of slowly rotating compact bodies (slowly, meaning for instance $v_{spin} \sim v_{orbit}$, a relation satisfied by the Hulse-Taylor pulsar) with, as result, "balance" equations of the same form as above: (eqn (9-12)), but where the quantities C^{2PN} have only a very small (c^{-4}) and extremely slowly variable (c^{-6}) spin-dependent contribution (Damour 1982). This last "mainly orbital balance" can then be meaningfully applied to the Hulse-Taylor system.

However the result (9-13) is not quite sufficient for reaching our goal which is to investigate the secular kinematical effects implied by the relativistic corrections up to order c^{-5}. One must start from the $2^1/2$ PN equations of motion (1) and study directly their solutions. This study can be carried out by, first investigating the solutions of the 2PN equations of motion (with the help of the 2PN integrals) and then by applying to these solutions Lagrange's method of "variation of arbitrary constants" (which means more than the preceding eqns (9 - 12)). The detailed calculations and results will be published elsewhere, we shall only give here the formula for the time of the Nth periastron passage:

$$t_N = t_0 + PN + \frac{1}{2} P\dot{P} N^2, \tag{14}$$

with,

$$\dot{P} = -\frac{3}{2} P (E^{2PN})^{-1} \left(\left(\frac{dE^{2PN}}{dt}\right)\right), \tag{15}$$

where P is a parameter obtained by fitting (14) to the observations and where the double parentheses denote a time average. From eqn (9) it is seen that eqn(15) can be expressed in function of the time average of the "quadrupole" energy loss (1.1). Replacing the explicit expression of this time average (Peters and Mathews 1963) leads to the formula proposed previously by Esposito and Harrison (1975) and Wagoner (1975) (see also Blandford and Teukolsky (1976) for a clarification of the observational meaning of a "period derivative"). Thanks to the superb work of Taylor and his collaborators this formula, which embodies the Laplace-Eddington effect (see section 3) has been compared with the observed secular acceleration of the binary pulsar PSR 1913 + 16 (Taylor, Fowler and McCulloch (1979), Taylor and Weisberg (1982) and Taylor (May 1982, private communication)):

$$\dot{P}_{\text{observed}} = (-2.40 \pm 0.20) \text{ X } 10^{-12}. \tag{16}$$

Assuming that the binary pulsar is constituted of two neutron stars and that the system is "clean" (for discussions of the high plausibility of these assumptions see Srinivasan and van den Heuvel (1982), and Taylor and Weisberg(1982)) one can use the equations of motion and the metric of two compact objects at PN order for analyzing the pulse arrival times (see the lectures of D. Eardley). This leads to a determination of the parameters of the system and thereby, using formula (15), to a predicted value of the secular acceleration:

$$\dot{P}_{\text{predicted}} = (-2.403 \pm 0.005) \text{ X } 10^{-12} . \tag{17}$$

The 8% agreement between (16) and (17) provides an impressive confirmation of the existence of the Laplace-Eddington effect. Remembering that we derived formulae (14-15) after a long theoretical path where we made an essential use of the following features of Einstein's equations: 1)their hyperbolicity ("retarded potentials"), 2) the quartic non-linearity of the weak-field expansion of the vacuum equations, 3) the infinite non-linearity of these equations (when dealing with the strong field regions), and knowing that investigations of alternative theories of gravity (Will and Eardley 1977, Will 1977, Weisberg and Taylor 1981) have led to quantitatively and qualitatively very different predictions, we conclude that the work reported in these lectures, in conjunction with Taylor's observations, provides the most sensitive available confirmation of the non-linear hyperbolic structure of Einstein's theory (and therefore, also an indirect confirmation of the existence of gravitational radiation).

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge helpful discussions with: L. Bel, B. Carter, Y. Choquet-Bruhat, D. Christodoulou, N. Deruelle, B. Jones, A. Jourdanney, A. Lichnerowicz, D. Maison, A. Papapetrou, B. Schmidt, K.S. Thorne, and J. York.

I would like to thank C.H. Liebow for her suggestions on wording and her courage in typing a difficult manuscript.

REFERENCES

Abraham, M. (1903) Ann.d.Phys. 10, 105 and 156.
Abraham, M. (1904) Ann;d;Phys. 14, 236.
Anderson, J.L. and Decanio, T.C. (1975) Gen. Rel.Grav. 6, 197.
Arago, F.D., Oeuvres, t II, P. 596, quoted by R. Barthalot: "L'Observatoire de Paris," Thèse de 3ème cycle, 21 avril 1982, Paris I.
Beethoven, L. van (1826) Heading to the last movement of his last composition (16th Quartet, op. 135): "The resolution made with great difficulty. Must it be? It must be! It must be! (Es muss sein!)".
Bel, L. (1970) Ann.I.H.P. 12,307.
Bel, L. (1971) Ann.I.H.P. 14, 189.
Bel, L. (1982) C.R.Acad. Sci. Paris, série I. 294, 463.
Bel, L., Damour, T., Deruelle, N., Ibañez, J., and Martin, J. (1981) Gen.Rel. Grav. 13, 963.
Bel, L. and Fustero, X. (1976) Ann.I.H.P., A25, 411, and references therein.
Bel, L. and Martin, J. (1980) Ann.I.H.P., A33, 409 and (1981) A34, 231.
Bel, L.,Salas, A., and Sanchez, J.M. (1973) Phys. Rev.D7, 1099.
Bertotti, B. (1956) Nuov.Cim. 4, 898.
Bertotti, B., and Plebanski, J. (1960) Ann. Phys. (N.Y.), 11, 169.
Blandford, R., and Teukolsky, S. (1976) Ap.J. 205, 580.
Breitenlohner, P. and Maison, D.(1977)Commun.math.Phys. 52, 11, 39, and 55.
Breuer, R. and Rudolph, E. (1981) Gen. Rel. Grav. 13, 777.
Brillouin, M. (1922) CR.R.Acad.Sci.Paris 175, 1008. 1009
Burke, W.L. (1969) unpublished Ph.D.thesis, California Institute of Technology.
Burke, W.L. (1971) J.Math.Phys.12, 401.
Campolattaro, A., and Thorne, K.S. (1970) Ap.J. 159, 847.
Carmeli, M. (1964) Phys.Lett.11, 24.
Carmeli, M. (1965) Nuov.Cim. 37, 842.
Chandrasekhar, S. (1965)Ap.J. 142, 1488.
Chandrasekhar, S., and Esposito, F.P. (1970) Ap.J. 160, 153.
Choquet-Bruhat, Y. (1952) Acta Matematica 88, 141.
Choquet-Bruhat, Y., and Christodoulou, D. (1980) "Cauchy problem at past infinity" to appear in the issue of "Advances in Mathematics," proceedings of the Symposium in honor of I.E.Segal.
Choquet-Bruhat, Y., Christodoulou, D. and Francaviglia, M. (1979) ANN.I.H.P. A31,399.
Christodoulou, D., and Schmidt, B. (1979) Commun. Math. Phys. 68, 275.
Cooperstock, F.I. and Hobill, D.W. (1982) Gen.Rel.Grav.14, 361.
Crowley, R.J., and Thorne, K.S. (1977) Ap.J. 215, 624.
Currie, D.G.(1966) Phys.Rev.142, 817.
Damour,T. (1974) "Théorie classique de la renormalisation," Thèse de 3ème cycle, Paris (unpublished).
Damour, T. (1975) Nuov.Cim. 26B, 157.

Damour, T. (1978) "Note on the Spin Precession Effect in a Relativistic Binary System," p.547 in "Physics and Astrophysics of Neutron Stars and Black Holes," Varenna (1975), course 45, R. Giacconi and R. Ruffini ed., North Holland, Amsterdam.
Damour, T. (1980) C.R.Acad.Sci.Paris, 291, série A, 227.
Damour, T. (1981) "Le problème des N corps en relativité générale" in "Comptes rendus des journées relativistes 1981, Institut Fourier, Grenoble.
Damour, T. (1982) C.R.Acad.Sci.Paris, 294, série II, 1355.
Damour, T., and Deruelle, N. (1981a)Phys.Lett.87A, 81.
Damour, T. and Deruelle, N. (1981b) C.R.Acad.Sci.Paris 293, série II, 537.
Damour, T., and Deruelle, N. (1981c)C.R.Acad.Sci.Paris 293, série II, 877.
Darwin, C.G. (1920) Phil.Mag. 39, 537.
D'Eath, P.D. (1975a) Phys.Rev.D11, 1387.
D'Eath, P.D. (1975b) Phys.Rev.D12, 2183.
Demianski, M., and Grishchuk, L.P. (1974) Gen.Rel.Grav. 5, 673.
Deruelle, N. (1982) "Sur les équations du mouvement et le rayonnement gravitationnel d'un système binaire en relativité générale." Thèse de doctorat d'Etat, Paris (unpublished).
DeSitter, W. (1916) Mon.Not.R.A.S. 76, 699 and 77, 155.
Droste, J. (1916) Versl.K.Akad.Wet.Amsterdam 25, 460.
Droz-Vincent, P. (1969) Lett.Nuov.Cim. 1, 839.
Droz-Vincent, P. (1970) Phys.Scripta 2, 129.
Eddington, A.S. (1924) "The Mathematical Theory of Relativity," Cambridge at the University Press (second edition).
Eddington, A.S., and Clark, G.L. (1938) Proc. Roy. Soc. London, A166, 465.
Edelstein, L.A. and Vishveshwara, C.V. (1970) Phys.Rev.D1, 3514.
Eder, E. (1982) "Existence, uniqueness and iterative construction of motions of charged particles with retarded interactions" preprint Max Planck Institut, April 1982, submitted to Commun. Math. Phys.
Ehlers, J. (1979) Editor: "Isolated Gravitating Systems in General Relativity," Varenna 1976, course 67, North Holland, Amsterdam.
Ehlers, J. (1980) Ann.N.Y.Acad.Sci. 336, 279.
Ehlers, J., Rosenblum, A., Goldberg, J.N. and Havas, P. (1976), Ap.J. 208, L77.
Einstein, A. (1916) Sitzber.PReuss.Akad.Wiss.(Berlin), 688.
Einstein, A. (1918) Sitzber.Preuss.Akad.Wiss.(Berlin), 154.
Einstein, A., and Grommer, J., (1927) Sitzber.Preuss.Akad.Wiss(Berlin) 2 and 235.
Einstein, A., Infeld,L.(1940) Ann.Math. 41, 455.
Einstein,A., Infeld, L. (1949) Can.J.Math.1, 209.
Einstein, A., Infeld,L., and Hoffmann(1938) Ann.Math.39, 65.
Esposito, L.W. and Harrison, E.R. (1975) Ap.J.(Letters), 196, L1.
Fichtenholz, I.G.(1950) Zh.Eksp.Teor.Fiz.20 824.
Fock, V.A. (1939) J.Phys. (Moscow) 1, 81.
Fock, V.A. (1959) "Theory of Space, Time and Gravitation," Pergamon, London.
Fremberg, N.E. (1946) Proc.Roy.Soc. (London) A 188, 18.
Friedrich, H.(1981) Proc.Roy.Soc.Lond. A375, 169 and 378, 401.
Friedrich, H. (1982) Proc.Roy.Soc.Lond.A381, 361.
Galilei, G. (1638) "Discorsi e dimostrazioni matematiche intorno a due nove scienze attenanti alla meccanica e i movimenti locali", Leyde.
Goldberg, J.N.(1955) Phys.Rev.99, 1873.
Goldberg, J.N.(1962) "The Equations of Motion" in "Gravitation: an introduction to current research," L.Witten ed., Wiley, New York.
Guelfand, I.M., and Chilov, G.E. (1962) "Les Distributions," Dunod, Paris.
Gustafson, T. (1945) Kgl.Fys. Sallsk. i Lund Forhandl., 15, n° 28.
Gustafson, T. (1946) Kgl.Fys.Sallsk.i Lund. Forhandl., 16, n° 2.
Hamilton, R.W.(1835)Phil.Trans.Roy.Soc.London, part I, 95.
Havas, P. (1957) Phys.Rev.108, 1351.
Havas, P., and Goldberg, J.N.(1962) Phys.Rev.128, 398.
Hill, R.N.(1967) J.Math.Phys. 8, 201.
Hirondel, D. (1974) J.Math.Phys.15, 1689.
Hu,N. (1947) Proc.Roy.Irish Acad. 51A, 87.
Hu,N.(1982), p.717 in "Proceedings of the Second Marcel Grossmann Meeting on General Relativity," R.Ruffini ed., North Holland, Amsterdam.

Infeld,L. (1954) Acta Phys.Polon. 13, 187.
Infeld, L. (1957) Rev.Mod.Phys. 29, 398.
Infeld,L., and Michalska-Trautman, R. (1969) Ann.Phys.(N.Y.) 55, 561.
Infeld,L., and Plebanski,J. (1960) "Motion and Relativity," Pergamon, London.
Infeld,L., and Scheidegger, A.E.(1951) Can.J.Math. 3, 195.
Infeld,L., and Schild,A., (1949) Rev.Mod.Phys.21, 408.
Kates,R.E.(1980a) Phys.Rev.D22,1853.
Kates,R.E.(1980b) Phys.Rev.D22, 1871.
Kates,R.E.(1981) Ann.Phys.(N.Y.) 132, 1.
Kerlick, G.D.(1980) Gen.Rel.Grav.12,467 and 521.
Kerner, E.H.(1965) J.Math Phys. 6, 1218.
Kerr, R.P. (1959) Nuov.Cim. 13, 469, 492 and 673.
Kerr, R.P. (1960) Nuov. Cim. 16, 26.
Kovács,S. and Thorne, K.S. (1977) Ap.J. 217, 252.
Kühnel, A. (1963) Acta Phys.Polon. 24, 399.
Lagrange, L. (1770). "Nouvelle méthode pour résoudre les équations littérales par le moyen des séries," Mémoires de l'Académie Royale des Sciences et Belles Lettres de Berlin, tome 24. See also E.T. Whittaker and G.N Watson, "A course of modern analysis," 4th ed., Cambridge U.P. (1978), p.132.
Landau,L.D., and Lifshitz,E.M. (1941) "Teoriya Polya," Nauka,Moscow.
Landau,L.D., and Lifshitz,E.M. (1976) "Teoria dei Campi," Editori Riuniti, Roma.
Laplace,P.S.(1798-1825) "Mécanique Céleste," Courcier,Paris. Second part: book 10, chapter 7 ("Laplace effect"), and, second supplement to the book 10 ("Asymptotic matching").
Levi-Civita, T. (1937) Am.J.Math.59, 9 and 225.
Levi-Civita, T. (1950) "Le problème des N corps en relativité générale," Mémorial des Sciences Mathématiques 116, Gauthier-Villars, Paris.
Linet, B. (1981) C.R.Acad. Sci. Paris 292,série II, 1425.
Lorentz,H.A.(1892) Arch.néerl. 25, 363 (see §120). Reprinted in the Collected Papers of H.A.Lorentz, vol.2, p.164, The Hague, Nijhoff (1936).
Lorentz,H.A. (1902) Arch. néerl. 7, 299 (see §3). Reprinted in: Collected Papers, vol. 3, p. 73. and (1903) Encycl.d.math.wiss.V,14,§ 20.
Lorentz, H.A. (1909) "The Theory of Electrons;" Teubner, Leipzig.
Lorentz, H.A., and Droste, J. (1917) Versl.K.Akad.Wet.Amsterdam, 26, 392 and 649. Reprinted in the Collected Papers of H.A.Lorentz, vol. 5, p.330, The Hague, Nijhoff (1937).
Love, A.E.H. (1909) Proc.Roy.Soc.London A82, 73.
Ma, S.T. (1947) Phys.Rev. 71, 787.
Manasse, F.K. (1963) J.Math.Phys. 4, 746.
Martin, J., and Sanz, J.L.(1979) J.Math.Phys. 20, 26.
McCrea, J.D.(1981) Gen.Rel.Grav.13, 397.
Misner, C.W., Thorne, K.S., and Wheeler, J.A.(1973), "Gravitation," Freeman, San Francisco.
Newton, I. (1687) "Philosophiae Naturalis Principia Mathematica", Streater, London.
Noether, E. (1918) Nachr. Ges. Wiss. Göttingen, Math.Phys.Kl., p.235.
Ohta, T., Okamura, H., Kimura, T., and Hiida, K. (1973) Prog.Theor.Phys. 50, 492.
Ohta, T., Okamura, H., Kimura, T., and Hiida, K. (1974) Prog.Theor.Phys. 51, 1220.
Okamura, H., Ohta, T., Kimura, T., and Hiida, K. (1973) Prog. Theor.Phys. 50, 2066.
Papapetrou, A. (1951) Proc.Phys.Soc.Lond. A 64, 57.
Papapetrou, A. (1951) Proc.Roy.Soc.Lond. A 209, 248.
Papapetrou, A., and Linet, B. (1981) Gen.Rel.Grav. 13, 335.
Peres, A. (1959) Nuov.Cim. 11, 617 and 644, and 13, 437.
Peres, A. (1960) Nuov.Cim. 15, 351.
Peters, P.C., and Mathews, J. (1963) Phys.Rev. 131, 435.
Planck, M. (1897) Ann.d.Phys.60, 577.
Poincaré, H. (1905) C.R.Acad.Sci.Paris, 140, 1504, 5 June 1905.
Portilla,M. (1979) J.Phys.A 12, 1075.
Regge, T., and Wheeler, J.A. (1957) Phys.Rev. 108, 1063.
Riesz,M.(1938) Bull.Société Math.France, 66.
Riesz,M. (1949) Acta Mathematica 81, 1.
Rosenblum, A. (1978) Phys.Rev.Lett.41, 1003.

Rosenblum, A. (1981) Phys.Lett.81A, 1.
Scheidegger, A.E. (1953) Rev.Mod.Phys. 25, 451.
Scheidegger, A.E. (1955) Phys.Rev. 99, 1883.
Schieve, W.C., Rosenblum, A., and Havas, P. (1972) Phys.Rev.D 6, 1501.
Schmutzer, E., (1966) Ann.d.Phys. 17, 107.
Schott, G.A. (1912) "Electromagnetic Radiation," Cambridge U.P.
Schott, G.A. (1915) Phil. Mag. 29, 49.
Schutz, B.F. (1980) Phys.Rev.D 22, 249.
Schwartz, L. (1950) "Théorie des Distributions" Publications de l'Institut de Mathématique de l'Université de Strasbourg.
Smith, S.F., and Havas, P. (1965) Phys.Rev. 138, B495.
Smorodinsky, Ya.A., and Golubenkov, V.N. (1956), Zh. Eksp. Teor.Fiz., 31, 330.
Spyrou, N., (1975) Ap.J. 197, 725.
Srinivasan, G., and van den Heuvel, E.P.J. (1982) Astron. Astrophys. 108, 143.
Stephani, H., (1964) Acta Phys.Polon. 26, 1045.
Synge, J.L., (1970) Proc.Roy.Irish Acad. A 69, 11.
Taylor, J.H., Fowler, L.A., and McCulloch, P.M. (1979) Nature 277, 437.
Taylor, J.H., and Weisberg, J.M. (1982) Ap.J.253, 908.
Thorne, K.S.(1969) AP.J. 158, 997.
Thorne, K.S. (1980) Rev.Mod.Phys. 52, 285 and 299.
Thorne, K.S., and Campolattaro,A(1967)ApJ.149, 591 and errata, Ap.J. 152, 673.
Thorne, K.S. and Kovács, S. (1975) Ap.J. 200, 245.
Vilenkin, A.V., and Fomin, P.I. (1978) Nuov. Cim. 45, 59.
Wagoner, R. (1975) Ap.J.(Letters), 196, L63.
Walker, M. (1979) Editor: "Proceedings of the Third Gregynog Relativity Workshop on Gravitational Radiation Rheory," Max Planck Institute publication, Munich.
Walker, M., and Will, C.M. (1979) Phys.Rev. D 19, 3483 and 3495.
Walker, M., and Will, C.M. (1980) Ap.J. (Letters) 242, L129.
Weisberg, J.M., and Taylor, J.H. (1981) Gen. Rel. Grav.13, 1.
Westpfahl, K., and Göller, M. (1979) Lett. Nuov. Cim. 26, 573.
Weyl, H. (1921) "Raum, Zeit, Materie," 4th edition, §36, Berlin.
Will, C.M. (1977) Ap.J. 214, 826.
Will, C.M., and Eardley, D.M. (1977) Ap.J.(Letters) 212, L91.
Zerilli, F.J. (1970) Phys.Rev.D 2, 2141.