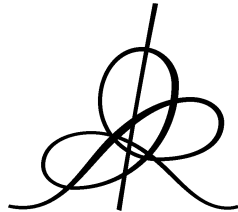


ON THE DETERMINATION OF THE LAST STABLE ORBIT FOR
CIRCULAR GENERAL RELATIVISTIC BINARIES AT THE
THIRD POST-NEWTONIAN APPROXIMATION

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On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation

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We discuss the analytical determination of the location of the Last Stable Orbit (LSO) in circular general relativistic orbits of two point masses. We use several different “resummation methods” (including new ones) based on the consideration of gauge-invariant functions, and compare the results they give at the third post-Newtonian (3PN) approximation of general relativity. Our treatment is based on the 3PN Hamiltonian of Jaranowski and Schäfer. One of the new methods we introduce is based on the consideration of the (invariant) function linking the angular momentum and the angular frequency. We also generalize the “effective one-body” approach of Buonanno and Damour by introducing a non-minimal (i.e. “non-geodesic”) effective dynamics at the 3PN level. We find that the location of the LSO sensitively depends on the (currently unknown) value of the dimensionless quantity ω_{static} which parametrizes a certain regularization ambiguity of the 3PN dynamics. We find, however, that all the analytical methods we use numerically agree between themselves if the value of this parameter is $\omega_{\text{static}} \simeq -9$. This suggests that the correct value of ω_{static} is near -9 (the precise value $\omega_{\text{static}}^* \equiv -\frac{47}{3} + \frac{41}{64}\pi^2 = -9.3439\dots$ seems to play a special role). If this is the case, we then show how to further improve the analytical determination of various LSO quantities by using a “Shanks” transformation to accelerate the convergence of the successive (already resummed) PN estimates.

I. INTRODUCTION

The study of the late dynamical evolution of binaries made of compact objects (neutron stars or black holes) is important because such systems are the most promising candidate sources for interferometric gravitational-wave detectors such as LIGO and VIRGO. In particular, the global structure of the gravitational waveform emitted by such a binary sensitively depends on the frequency at which the system’s orbital evolution changes from a gravitational-radiation-driven inspiral phase to a plunge phase followed by coalescence [1,2].

In the test-mass limit ($\mu \ll M$) the orbital dynamics is that of a test particle (of mass μ) moving in a Schwarzschild background (of mass M). A very important qualitative feature of circular orbits in such a background is the existence of a Last Stable Orbit (LSO) located at the (area) radius $R_{\text{LSO}} = 6GM/c^2$. When considering the effect of gravitational-radiation reaction, one expects the test-particle motion to change abruptly near R_{LSO} from a slow inspiral to a fast plunge. By analogy, one believes that the motion of a compact binary made of comparable masses, m_1 and m_2 , will exhibit a similar transition from inspiral to plunge, with the location of the transition being mainly determined by the existence of a Last Stable Orbit in the *conservative* part of the two-body Hamiltonian.

Several authors have tried to estimate the location of the LSO in comparable-mass (compact) binaries. An early analytical estimate was made by Clark and Eardley [3] using the first post-Newtonian (1PN) Hamiltonian. Some authors [4–6] tried to use initial value formalisms to locate the LSO. However, the initial value approaches used in these works assume a conformally flat metric, and therefore do not correctly incorporate the well known second post-Newtonian (2PN) dynamics [7–9]. In this work we shall take the view that a correct incorporation of all 2PN effects is a necessary (if maybe not sufficient) prerequisite for an accurate determination of the location of the LSO. Previous treatments that used the full 2PN dynamics to try to analytically determine the location of the LSO include Refs. [10–12,1,13]. The aim of the present work is to extend these PN-based analytic determinations of the LSO to the third post-Newtonian (3PN) level. The conservative part of the 3PN Hamiltonian for two point masses has been obtained in 1998 by Jaranowski and Schäfer [14], though with some remaining ambiguity due to the need to regularize the divergent integrals entailed by the use of point-like sources. The determination of the 3PN dynamics has been recently completed by deriving the Hamiltonian in a non-mass-centered frame, and by fixing a certain momentum-dependent

regularization ambiguity [15]. Recently, we have extracted from the 3PN Hamiltonian of Ref. [14] all its *dynamical invariants*, i.e. all the functions linking dynamical quantities which do not depend on the choice of coordinates in phase-space [16]. In this paper, we shall use several of these 3PN invariants to determine the location of the LSO. Some of the methods of LSO determination that we shall use below generalize previous works [1,13], but others are new.

II. METHODS DIRECTLY BASED ON DYNAMICAL INVARIANTS

Before embarking on the discussion of the methods we shall use here to extract the LSO from PN expansions, let us stress that the basic theme underlying our endeavours is the following: Our problem is to extract some semi-non-perturbative information from (badly convergent) perturbation expansions. We shall do that by using several types of “resummation methods”. The basic idea of all resummation methods is simply the following: to complete the information contained in the first few terms of a perturbative expansion $f(z) = c_0 + c_1 z + \dots + c_n z^n + \mathcal{O}(z^{n+1})$ by injecting some non-perturbative information about the global behaviour (if possible in the complex plane) of the exact function $f(z)$. The amount of global information one has about the function $f(z)$ determines the best type of resummation method to use. For instance, if the only information at our disposal is that the function $f(z)$ is (probably) meromorphic in the complex z -plane, then the best, all-purpose method is to use Padé approximants. If we knew more about the location of the singularities of $f(z)$ in the complex plane one might contemplate to use other methods (e.g. change of independent variable, Borel transform, ...).

In this paper, we shall use two distinct classes of methods for extracting the (invariant) location of the LSO from the knowledge of the PN-expanded dynamics. The first class of methods (which was introduced in Ref. [1]) is discussed in the present Section. The second class will be discussed in the next Section. The first class of methods is based on the combination of three ideas:

- (i) to work only with invariant functions;
- (ii) to make a maximal use of the known, exact functional form of invariant functions in the test-mass limit ($\nu \equiv m_1 m_2 / (m_1 + m_2)^2 \rightarrow 0$) and to assume some structural stability when the parameter ν is turned on;
- (iii) to use Padé approximants to represent the invariant functions which are (because of (ii)) expected to be meromorphic functions of their argument.

This method was applied in Ref. [1] to the two invariant functions which play an essential role in the gravitational-damping-driven inspiral of a binary system: the binding energy $E(x)$ of a circular orbit, and the gravitational-wave flux $F(x)$ emitted by a circular orbit, both being considered as functions of the (dimensionless) invariant parameter

$$x \equiv \left(\frac{GM\omega}{c^3} \right)^{2/3}, \quad (2.1)$$

where $M \equiv m_1 + m_2$ denotes the total mass of the binary, and ω the orbital angular frequency along a circular orbit.

Before introducing some variations on this method, let us motivate the interest of using the three ideas (i)–(iii) above. First, we recall that the PN expansions of non-invariant, i.e. gauge-dependent, functions can have (and do have, in some gauges) worse convergence properties than the PN expansions of invariant functions. For instance, the PN expansion of the gravitational-wave flux, from a test mass in circular orbit, say F_{TM} , considered as a function of the harmonic-coordinate parameter $\gamma \equiv GM/(c^2 r_{\text{harmonic}})$, is such that the 1PN “correction” to the leading “quadrupole” result becomes fractionally larger than 100% (while being negative!) for a radius r_{harmonic} larger than the LSO (which is at $r_{\text{LSO}} = 5GM/c^2$ in harmonic coordinates). More precisely, if one formally writes down the expansion of $F_{\text{TM}}(\gamma)$, in powers of γ , near the LSO (i.e. for γ near $1/5$) one gets a series numerically proportional to: $1 - 1.74(5\gamma) + 1.12(5\gamma)^{3/2} + 1.29(5\gamma)^2 + \dots$, i.e. a series whose first terms do not exhibit any convergence near the LSO. By contrast, the flux F_{TM} expanded in terms of the invariant parameter x , Eq. (2.1), ($x_{\text{LSO}} = 1/6$ in the test-mass limit) has a more reasonable expansion proportional to $1 - 0.619(6x) + 0.855(6x)^{3/2} - 0.137(6x)^2 + \dots$. Still, it is clear that one needs some resummation technique for summing a series as slowly converging as $F_{\text{TM}}(x)$. It was shown in great detail in Ref. [1], by making use both of the known analytical results on high-order (5.5PN) terms in the post-Newtonian expansion of the test-mass flux function $F_{\text{TM}}(x)$ [17], and of the existence of a pole-like blow up of $F_{\text{TM}}(x)$ at $x = x_{\text{light ring}} = 1/3$, that one could considerably speed up the convergence of the straightforward (Taylor-like) PN series, by replacing it by a suitably defined sequence of Padé approximants (see, notably, Fig. 3 in [1]).

To further illustrate the idea (ii), and the need for an acceleration of convergence, let us recall the treatment of the energy function $E(x)$ introduced in [1]. In this paper we follow [16] in defining the dimensionless energy function

$$E \equiv \frac{\mathcal{E}^R - Mc^2}{\mu c^2}, \quad (2.2)$$

where \mathcal{E}^R is the total (“relativistic”) energy of the binary system (including the rest-mass contribution), and where

$$M \equiv m_1 + m_2, \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}, \quad \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \quad (2.3)$$

For notational simplicity we shall henceforth use units such that $c = 1$. Note that the energy function used in [1] was $E^{\text{DIS}} \equiv (\mathcal{E}^R - M)/M = \nu E^{\text{here}}$.

Because the argument x is, from its definition (2.1), of formal order $\mathcal{O}(c^{-2})$, the knowledge of the dynamics at, say, the n th post-Newtonian (n PN) order entails the knowledge of the expansion of the ratio $E(x)/x$ up to x^n :

$$E(x; \nu) = -\frac{1}{2} x \left[1 + E_1(\nu) x + E_2(\nu) x^2 + \dots + E_n(\nu) x^n + \mathcal{O}(x^{n+1}) \right]. \quad (2.4)$$

[The term $-\frac{1}{2} x$ corresponds to the Newtonian binding energy $\mathcal{E}^{\text{NR}} = -\frac{1}{2} \mu v^2$. Then the term $E_1(\nu) x$, for instance, represents the fractional 1PN effects, etc.] The symmetric mass ratio ν , Eq. (2.3), enters the expansion coefficients $E_n(\nu)$ as a parameter. At present, only $E_1(\nu)$, $E_2(\nu)$ and $E_3(\nu)$ are known (with some ambiguity for the 3PN coefficient $E_3(\nu)$). They were written down in [16] and will be repeated below. On the other hand, in the test-mass limit $\nu \rightarrow 0$, the coefficients $E_n(0)$ are known (in principle) for any n . What is more, the exact expression of $E(x; \nu = 0)$ is known:

$$E(x; \nu = 0) = \frac{1 - 2x}{\sqrt{1 - 3x}} - 1. \quad (2.5)$$

It is known (see, e.g., [1]) that the location of the LSO corresponds just to the minimum of the function $E(x)$ ($(dE(x)/dx)_{x_{\text{LSO}}} = 0$). Therefore, it would seem that the most straightforward way of locating the LSO is to consider the successive “Taylor approximants” of $E(x)$, say $E_{T_n}(x)$ (defined for each integer n as the R.H.S. of Eq. (2.4), without the $\mathcal{O}(x^{n+1})$ error term), and to solve the equations $dE_{T_n}(x)/dx = 0$. Let us see what this gives in the test-mass limit where the Taylor expansion of Eq. (2.5) is known:

$$E(x; \nu = 0) = -\frac{1}{2} x \left(1 - \frac{3}{4} x - \frac{27}{8} x^2 - \frac{675}{64} x^3 - \frac{3969}{128} x^4 - \dots \right). \quad (2.6)$$

The successive “Taylor” estimates of x_{LSO} , or better $x_{\text{LSO}}^{T_n}/x_{\text{LSO}}^{\text{exact}} \equiv 6 x_{\text{LSO}}^{T_n}$, are found to be (in the test-mass case):

$$6 x^{T_1} = 4; \quad 6 x^{T_2} = 1.49284; \quad 6 x^{T_3} = 1.17565; \quad 6 x^{T_4} = 1.07680. \quad (2.7)$$

[To avoid confusion note that we use here the convention that ‘ T_n ’ corresponds to the n PN approximation, i.e. a $(v/c)^{2n}$ -accurate result, while in Ref. [1] ‘ T_n ’ referred to $(v/c)^n$ -accuracy, i.e. to the $\frac{n}{2}$ PN approximation.] From Eq. (2.1) the corresponding values of the orbital frequency at the LSO, scaled to the exact value, $\hat{\omega} \equiv \omega/\omega^{\text{exact}} = (6x)^{3/2}$, read

$$\hat{\omega}^{T_1} = 8; \quad \hat{\omega}^{T_2} = 1.82398; \quad \hat{\omega}^{T_3} = 1.27472; \quad \hat{\omega}^{T_4} = 1.11739. \quad (2.8)$$

As the value of the frequency at the LSO is the most important observable one wishes to know (for data analysis purposes), one should reject any method which does not have the prospect of determining it to, say, better than about 10%. The test-mass results¹ (2.8) suggest that, if the dynamics is known only up to the 3PN level, straightforward Taylor approximants of the energy function $E(x)$ do not converge fast enough to determine satisfactorily the location of the LSO. [The 4PN level might barely suffice, but it seems anyway excluded that one will be able to analytically derive the 4PN dynamics.]

This preliminary discussion motivates the necessity of boosting up the convergence of the series (2.4) or (2.6). This is here that the ideas (ii) and (iii) enter. The exact test-mass result (2.5) suggests (under the assumption that the $\nu \neq 0$ case represents a structurally stable deformation of the $\nu = 0$ limit) that the function $E(x; \nu)$, for $\nu \neq 0$, will have a branch-cut singularity (at some point $x_0 = \frac{1}{3} + \mathcal{O}(\nu)$) in the complex x plane. It is, a priori, much better to work with functions which are *meromorphic* in the complex plane, because we can then make use of Padé approximants,

¹The logic here is to use the known convergence properties of the $\nu = 0$ limit to estimate the convergence when $\nu \neq 0$. It is, indeed, unlikely that turning on ν will drastically *improve* the convergence properties.

which are efficient tools for accurately representing meromorphic functions. This suggests to work with some (to be defined) new invariant energy function that is known to be meromorphic in the test-mass limit.

Looking at Eq. (2.5) one is tempted to consider the square of the function $1 + E$. However, Ref. [1] remarked that the usual test-mass limit definition of this function, namely ($m_2 \ll m_1$ being the test-mass orbiting m_1)

$$1 + E \equiv \frac{\mathcal{E}_2^R}{m_2} \quad (2.9)$$

with

$$\mathcal{E}_{\text{tot}}^R = m_1 + \mathcal{E}_2^R + \mathcal{O}(m_2^2) = m_1 - \frac{p_1^\mu p_{2\mu}}{m_1} + \mathcal{O}(m_2^2), \quad (2.10)$$

looks unnaturally asymmetric with respect to the labels 1 and 2. [Essentially, this asymmetry comes from the fact, hidden in the symmetric definition (2.2), that \mathcal{E}_2^R represents the sum of the rest-mass energy of m_2 alone and of the binding energy of the binary system.] Ref. [1] therefore suggested to introduce the more symmetric function

$$\varphi(s) \equiv \frac{s - m_1^2 - m_2^2}{2 m_1 m_2} \equiv \frac{(\mathcal{E}^R)^2 - m_1^2 - m_2^2}{2 m_1 m_2} \quad (2.11)$$

of the Mandelstam invariant $s \equiv -(p_1^\mu + p_2^\mu)^2 \equiv (\mathcal{E}^R)^2$. Indeed, in the test-mass limit

$$\varphi(s) \equiv -\frac{p_1^\mu p_{2\mu}}{m_1 m_2} \simeq \frac{\mathcal{E}_2^R}{m_2} = 1 + E = \frac{1 - 2x}{\sqrt{1 - 3x}}. \quad (2.12)$$

It is then natural to define the function $e(x)$ (or rather $1 + e(x)$), by setting

$$1 + e(x) \equiv (\varphi(s(x)))^2 = \left(\frac{(\mathcal{E}^R)^2 - m_1^2 - m_2^2}{2 m_1 m_2} \right)^2. \quad (2.13)$$

Then, $1 + e(x)$, being equal to $(1 - 2x)^2/(1 - 3x)$ in the test-mass limit, i.e. (after subtraction of the trivial constant 1)

$$e(x; \nu = 0) = -x \frac{1 - 4x}{1 - 3x}, \quad (2.14)$$

one expects the function $e(x; \nu)$ to be meromorphic in x when $\nu \neq 0$ (with a pole located at some $x_{\text{pole}} = \frac{1}{3} + \mathcal{O}(\nu)$).

This finally leads to the following “ P -approximant”-improved method for locating the LSO: starting from the Taylor approximants (T -approximants) of the original $E(x)$ function, Eq. (2.4), compute first the corresponding T -approximants of the new $e(x)$ function, say

$$e(x) = -x [1 + e_1(\nu) x + e_2(\nu) x^2 + e_3(\nu) x^3 + \mathcal{O}(x^4)]. \quad (2.15)$$

Then construct a sequence of Padés of the Taylor-expanded $e(x)$:

$$e_{P_n}(x) \equiv P_\ell^k [T_n [e(x)]], \quad (2.16)$$

where $k + \ell = n$. Here k and ℓ are the degrees of the polynomials $N_k(x)$ and $D_\ell(x)$ entering the Padé $P_\ell^k(x) = N_k(x)/D_\ell(x)$. It is known that, generically, the Padé improvements are best when one is near the “diagonal”, i.e. when $|k - \ell|$ is “small” compared to k and ℓ . When dealing with a function $f(x)$ that is expected to have a *pole* at some $x_0 \neq 0$, one imposes the constraint $\ell > 0$. At 1PN order this uniquely fixes the values of k and ℓ , namely $k = 0$ and $\ell = 1$. At 2PN order the Padé closest to the diagonal is that with $k = 1$ and $\ell = 1$. At 3PN order there are two possible Padés near the diagonal, namely $k = 1$ and $\ell = 2$, or $k = 2$ and $\ell = 1$. In this work we shall use the latter one (P_1^2) because we found it to be more robust under variations of the coefficients of the Taylor expansion of the Padéed function. (One aspect of this robustness is that the existence of a real pole in this Padé is always ensured, while this is not the case for the other 3PN possibility: P_2^1 .)

Note also that Padés are originally defined only for series of the regular type $\sigma(x) = c_0 + c_1 x + \dots$ with $c_0 \neq 0$. When dealing with a function of the type $f_p(x) = x^p \sigma(x)$ with some relative integer p , we shall, *by convention*, define any Padé of $f_p(x)$ as being $(k + \ell = n)$

$$P_\ell^k [T_n[f_p(x)]] \equiv x^p P_\ell^k [T_n[x^{-p} f_p(x)]] . \quad (2.17)$$

The “ e_{P_n} -estimate” of the location of the LSO is then defined as the value x_{P_n} at which $e_{P_n}(x)$ reaches a minimum. [It is easily seen that $e(x)$ follows the variations of $E(x)$, and in particular that it reaches a minimum at the same place as $E(x)$.]

We spent some time explaining in detail on one example our general Padé-improved-test-mass-limit-motivated approach because we are going to extend it to several other invariant functions. In this (and the next) section, we present our various methodologies. The 3PN results obtained by them will be presented in a later section.

The introduction of the function $e(x)$ has two defects. First, it is not unique because we do not know for sure whether the function $(\varphi(s))^2$, Eq. (2.13), is a “better” invariant than $(1 + E)^2 = (1 + (\sqrt{s} - M)/\mu)^2$. Second, the Padeing of $e(x)$ starts giving meaningful results only at the 2PN level. Indeed the 1PN expansion of $e(x)$, in the test-mass limit, $e(x; \nu = 0) = -x(1 - x + \mathcal{O}(x^2))$, yields a Padé $e_{P_1}(x; \nu = 0) = -x/(1 + x)$ which contains no pole on the positive real axis, and which formally predicts an LSO (minimum of $e_{P_1}(x)$) located at $x = +\infty$.

In this paper, we propose to consider another invariant function, which is more uniquely defined, and which gives sensible results already at the 1PN level. Let us consider the reduced angular momentum

$$j \equiv \frac{\mathcal{J}}{\mu GM} = \frac{\mathcal{J}}{Gm_1 m_2} , \quad (2.18)$$

where \mathcal{J} denotes the total angular momentum of the system. In the test-mass limit the invariant function giving the (dimensionless) quantity j in terms of the quantity x , Eq. (2.1), reads

$$j(x; \nu = 0) = \frac{1}{\sqrt{x(1 - 3x)}} . \quad (2.19)$$

This motivates the consideration of the squared (reduced) angular momentum $j^2(x)$ which is expected, when $\nu \neq 0$, to be a meromorphic function of x , with a pole at the “light ring” $x_{\text{pole}} = \frac{1}{3} + \mathcal{O}(\nu)$. Therefore we propose to work with the Padeed form of j^2 :

$$j_{P_n}^2(x; \nu) \equiv P_\ell^k [T_n[j^2(x; \nu)]] , \quad (2.20)$$

with $k + \ell = n$, and the choice of $\ell > 0$ discussed above. (We use in Eq. (2.20) the convention (2.17), i.e. the factor x^{-1} in $j^2(x)$ is factored before taking a Padé.) Note that, in the test-mass limit, if we knew only the 1PN approximation to the function $j^2(x)$, i.e. $j^2(x; \nu = 0) = x^{-1}(1 + 3x + \mathcal{O}(x^2))$, the procedure (2.20) would reconstruct the *exact* result: $j_{P_1}^2(x; \nu = 0) = P_1^0 [T_1[j^2(x; \nu = 0)]] = [x(1 - 3x)]^{-1}$.

It is important to note that the perturbative information contained in the PN expansion of the function $j(x)$ (or equivalently $j^2(x)$) is totally equivalent (at any PN accuracy) to the information contained in, either the original energy function $E(x)$, or the new one $e(x)$, Eq. (2.13). Indeed, the generic Hamiltonian equation $\dot{\theta}_i = \omega_i = \partial H / \partial I_i$ in action-angle variables (I_i, θ_i) yields

$$\omega_{\text{circular}} = \frac{d\mathcal{E}^R}{d\mathcal{J}} = \frac{1}{GM} \frac{dE}{dj} , \quad (2.21)$$

which implies the identity

$$\frac{dE(x)}{dx} = x^{3/2} \frac{dj(x)}{dx} . \quad (2.22)$$

The identity (2.22) proves the assertion just made about the identical information content in $E(x)$ and $j(x)$. It also proves several interesting facts. First, the location of the minimum of (the exact) $j(x)$ coincides with that of the minimum of (the exact) $E(x)$ (both of them equivalently defining the LSO). Second, the existence of a branch cut singularity $\propto (x_0 - x)^{-1/2}$ in either $j(x)$ or $E(x)$ necessarily implies the presence of a similar singularity $\propto (x_0 - x)^{-1/2}$ (at the same location x_0) in the other function ($E(x)$ or $j(x)$, respectively). This can be viewed as a confirmation of our generic assumption of structural stability. However, this argument also shows the ambiguities present when trying to work with the energy function. Indeed, if we assume that, near x_0 , $j(x)$ can be expanded as $\varphi(x)(x_0 - x)^{-1/2}$, where $\varphi(x)$ is a smooth function, one finds from Eq. (2.22) that $E(x) = \psi(x)(x_0 - x)^{-1/2} + c$, with some unknown constant c . The lack of knowledge of the constant c (which a priori depends on the parameter ν) implies that we do not know which function $(E - c(\nu))$ we would square to get a meromorphic function with a simple pole. The same reasoning shows another defect of the proposal to consider the (new) function $e(x)$. Indeed, when $E(x) \rightarrow \infty$ the

leading term in $\epsilon(x)$, Eq. (2.13), will be $\epsilon(x) \sim \frac{1}{4}\nu^2 E^4(x)$, which will have a double pole $\propto \nu^2(x_0 - x)^{-2}$, if $j^2(x)$ has a simple pole $\propto (x_0 - x)^{-1}$. For all these reasons, we consider that the “ j -method”, Eq. (2.20), appears as the best way of locating the LSO, within the class of methods dealt with in this section.

To conclude this section, let us, however, mention another invariant function one might wish to consider. This function is the fourth power of the dimensionless periastron parameter $1+k = \Phi/(2\pi)$ considered as a function of the reduced angular momentum. Indeed, in the test-mass limit, and for circular orbits, one knows that [9,18,16]

$$(1+k)^4 = \left(1 - \frac{12}{j^2}\right)^{-1}. \quad (2.23)$$

We recall that $j^2 = 12$ is the location of the LSO. Therefore, if we define $K \equiv (1+k)^4 = (\Phi/(2\pi))^4$ and $y \equiv 1/j^2$, we might consider

$$K_{P_n}(y; \nu) \equiv P_\ell^k [T_n[K(y; \nu)]], \quad (2.24)$$

with $k + \ell = n$, and our canonical choice of $\ell > 0$. Then, we can take the pole of $K_{P_n}(y; \nu)$ as estimate of the value of $1/j^2$ at the LSO when $\nu \neq 0$. We consider, however, that the j -method (or any energy method for that matter) has a better chance of accurately locating the LSO because it incorporates not only the information that something special (a minimum) takes place at the LSO, but also the information that the location of this minimum is a corollary of the presence of a blow up ($j^2 \rightarrow \infty$, $E \rightarrow \infty$) at a point, x_{pole} , further away on the real axis. Therefore, even if the location of x_{pole} is not known very accurately, one can hope that x_{LSO} will be more robustly determined.

III. EFFECTIVE ONE-BODY METHOD

In this section, we shall turn to a rather different method, though one which also makes use of the three ideas (i)–(iii) listed at the beginning of the previous section. This new method incorporates a fourth idea, which has been recently put forward by Buonanno and Damour [13]. This fourth idea consists in mapping (through the use of invariant functions) the real two-body problem we are interested in (two masses m_1, m_2 orbiting around each other) onto an “effective one-body problem” (one mass m_0 moving in some background metric, $g_{\alpha\beta}^{\text{effective}}(x^\gamma)$). At the 2PN level, Ref. [13] has shown the possibility of mapping the real two-body problem onto geodesic motion in some spherically symmetric metric $g_{\alpha\beta}^{\text{effective}}(x^\gamma; \nu)$. It was found that, when $\nu \neq 0$, $g_{\alpha\beta}^{\text{effective}}(x^\gamma; \nu)$ is a smooth deformation of the Schwarzschild metric. The “mapping rules” between the two problems were motivated by quantum considerations:

- (i) the adiabatic invariants $I_i = \oint p_i dq_i$ (which are quantized in units of \hbar) were identified in the two problems, and
- (ii) the energies are mapped through a function $\mathcal{E}_{\text{effective}} = f[\mathcal{E}_{\text{real}}]$ which is, a priori, arbitrary, and which is determined in the process of matching the two problems.

In other words, the idea is to determine a metric $g_{\alpha\beta}^{\text{effective}}$ such that the “energy levels” $\mathcal{E}_{\text{effective}}[I_i]$ (i.e. the Hamiltonian expressed in action variables, or “Delaunay Hamiltonian”) of the bound states of a particle moving in $g_{\alpha\beta}^{\text{effective}}$, are in one-to-one correspondence with the two-body bound states:

$$\mathcal{E}_{\text{effective}}[I_i] = f[\mathcal{E}_{\text{real}}[I_i]]. \quad (3.1)$$

The unknowns of the problem are the numerical coefficients entering a spherically symmetric metric

$$ds_{\text{eff}}^2 = -A(R_{\text{eff}}) dt_{\text{eff}}^2 + \frac{D(R_{\text{eff}})}{A(R_{\text{eff}})} dR_{\text{eff}}^2 + R_{\text{eff}}^2 (d\theta_{\text{eff}}^2 + \sin^2 \theta_{\text{eff}} d\varphi_{\text{eff}}^2), \quad (3.2)$$

namely,

$$A(R) = 1 + a_1 \frac{GM_0}{R} + a_2 \left(\frac{GM_0}{R}\right)^2 + a_3 \left(\frac{GM_0}{R}\right)^3 + a_4 \left(\frac{GM_0}{R}\right)^4 + \dots, \quad (3.3a)$$

$$D(R) = 1 + d_1 \frac{GM_0}{R} + d_2 \left(\frac{GM_0}{R}\right)^2 + d_3 \left(\frac{GM_0}{R}\right)^3 + \dots, \quad (3.3b)$$

and the coefficients entering the energy-map f (here written for the “non-relativistic” energies $\mathcal{E}_{\text{eff}}^{\text{NR}} \equiv \mathcal{E}_{\text{eff}} - m_0$, $\mathcal{E}_{\text{real}}^{\text{NR}} \equiv \mathcal{E}_{\text{real}} - M$):

$$\frac{\mathcal{E}_{\text{eff}}^{\text{NR}}}{m_0} = \frac{\mathcal{E}_{\text{real}}^{\text{NR}}}{\mu} \left[1 + \alpha_1 \frac{\mathcal{E}_{\text{real}}^{\text{NR}}}{\mu} + \alpha_2 \left(\frac{\mathcal{E}_{\text{real}}^{\text{NR}}}{\mu} \right)^2 + \alpha_3 \left(\frac{\mathcal{E}_{\text{real}}^{\text{NR}}}{\mu} \right)^3 + \dots \right]. \quad (3.4)$$

As discussed in Ref. [13] it is natural to require that the effective mass m_0 be exactly equal to the usual non-relativistic effective mass $\mu \equiv m_1 m_2 / (m_1 + m_2)$. One can also (by convention) choose the mass M_0 entering the effective metric to be $M_0 \equiv M = m_1 + m_2$. With these choices the Newtonian limit tells us that the first coefficient in $A(R) = -g_{00}$ is simply $a_1 = -2$. The 1PN level then involves the coefficients a_2 , d_1 , and α_1 , while the 2PN and 3PN levels involve (a_3, d_2, α_2) and (a_4, d_3, α_3) , respectively. In other words, at each PN level, we only have three arbitrary coefficients to play with. This seems to be quite a small number of degrees of freedom, compared to the many possible coefficients which can enter the PN-expansion of the Delaunay Hamiltonian. [The Delaunay Hamiltonian was determined at the 2PN level in Ref. [9], and at the 3PN level in Ref. [16].] In order to clarify the number of independent equations to be satisfied when mapping the real problem onto the effective one, let us consider a generic² PN-expanded Hamiltonian, with the symbolic structure

$$\hat{H}_{n\text{PN}}^{\text{NR}}(\mathbf{q}, \mathbf{p}) = p^{2(n+1)} + \frac{1}{q} [p^{2n} + p^{2n-2}(np)^2 + \dots + (np)^{2n}] + \frac{1}{q^2} [p^{2(n-1)} + \dots + (np)^{2(n-1)}] + \dots + \frac{1}{q^{n+1}}. \quad (3.5)$$

Here, we consider the reduced Hamiltonian $\hat{H}^{\text{NR}} \equiv H^{\text{NR}}/\mu$, in the center-of-mass frame, as a function of the reduced canonical variables $\mathbf{p} \equiv \mathbf{p}_1/\mu = -\mathbf{p}_2/\mu$, $\mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/(GM)$ ((np) denotes $\mathbf{n} \cdot \mathbf{p}$ with $\mathbf{n} \equiv \mathbf{q}/q$). See Ref. [16] for details. Note that we follow here [13] in denoting by \mathbf{q} , \mathbf{p} the original (ADM-like) coordinates. We have suppressed all coefficients in Eq. (3.5) to display the structure. What is important for our present purpose is the total number of coefficients in the n PN-level Hamiltonian (3.5). This is easily checked to be:

$$C_H(n) = \frac{(n+1)(n+2)}{2} + 1. \quad (3.6)$$

As explained in Ref. [13], one way (and the only explicit one) to map the real Hamiltonian (3.5) onto an effective Hamiltonian, while keeping the action variables invariant, is to apply a canonical transformation, with generating function $\tilde{G}(q, p') = q^i p'_i + G(q, p')$. The most generic generating function that we need to consider has the symbolic structure (at the n PN level)

$$G_{n\text{PN}}(\mathbf{q}, \mathbf{p}) = (\mathbf{q} \cdot \mathbf{p}) \left\{ p^{2n} + \frac{1}{q} [p^{2(n-1)} + \dots + (np)^{2(n-1)}] + \dots + \frac{1}{q^n} \right\}. \quad (3.7)$$

Correspondingly to the pure \mathbf{p} -dependence of the leading kinetic term in (3.5) we have written here the leading term in $G_{n\text{PN}}$ as $\propto (qp) p^{2n}$. [It is easily shown that any term of the form $(qp) p^{2(n-k)} (np)^{2k}$ must have a vanishing coefficient.] The number of arbitrary coefficients in the n PN-level generating function (3.7) is easily seen to be

$$C_G(n) = C_H(n-1) = \frac{n(n+1)}{2} + 1. \quad (3.8)$$

Finally, the difference $\Delta(n)$ between the number of equations to satisfy, and the number of unknowns (including the 3 basic parameters (a_{n+1}, d_n, α_n) appearing in the effective metric and the energy-map) reads, at the n PN level

$$\Delta(n) = C_H(n) - C_G(n) - 3 = n - 2. \quad (3.9)$$

In particular: $\Delta(1) = -1$, which means that requiring a 1PN matching leaves one degree of freedom unrestricted. [This freedom was used in [13] to impose the natural condition $d_1 = 0$, i.e. that the linearized effective metric coincides with Schwarzschild.] Then $\Delta(2) = 0$, which means that there will (barring any degeneracy) be a unique solution at the 2PN level. [This was indeed the result of [13].] But $\Delta(3) = +1$, which means that, at the 3PN level, there is one more equation to satisfy than the number of free parameters. Then the situation would get worse and worse at higher PN levels.

By explicitly performing the matching between the canonically-transformed Hamiltonian and (modulo the energy map (3.4)) the effective Hamiltonian of a point particle moving in some $g_{\mu\nu}^{\text{eff}}$, we have established (details will be

²We use the information that the leading kinetic terms in a PN Hamiltonian are given by the expansion of the free Hamiltonian $\sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_2^2 + m_2^2}$, so that, at the n PN level, they are $\propto \mathbf{p}^{2(1+n)}$ without dependence on $\mathbf{n} \equiv \mathbf{q}/q$.

given below) that, if we follow Ref. [13] in imposing the natural condition $d_1 = 0$ at the 1PN level, there are, indeed, $C_H(3) = 11$ linearly *independent* equations, for $C_G(3) + 3 = 10$ unknowns, to be satisfied at the 3PN level. [No miracle occurred!]

At face value, this looks like bad news for the idea of the “effective one-body approach”. However, we think that there are acceptable ways to rescue this approach. A first cure would be to take advantage of the fact that the total number of equations at the three first PN levels (1, 2, and 3) is exactly equal to the number of unknowns (in other words $\Delta(1) + \Delta(2) + \Delta(3) = -1 + 0 + 1 = 0$). Therefore if we relax the (not really necessary) constraint $d_1 = 0$, there will be a unique 3PN-accurate effective metric $g_{\mu\nu}^{\text{eff}}$ (and a unique energy mapping (3.4)) satisfying the necessary constraints. We have verified that this is true and, for completeness, we give this unique solution in Appendix A. But, we do not want to take this solution too seriously for the following reasons: (i) it does not look natural to have to wait to know the 3PN Hamiltonian to determine the 1PN and 2PN effective metrics; (ii) this solution looks more complicated than the 3PN Hamiltonian itself; and (iii) this trick is not expected to be sufficient to ensure the existence of solutions at higher PN levels. (Indeed, $\Delta(n) = n - 2$ continues to increase.)

We propose therefore to consider a second (more radical, and simpler) way to cure the problem. Indeed, we have to face the fact that there is (probably) nothing deep in the effective-one-body approach. After all, it is just a somewhat artificial way of mapping the complicated two-body dynamics on a simpler one-body dynamics. There is no reason to assume that the one-body dynamics can, to all orders, be considered as equivalent to a simple geodesic motion. Let us recall that, in quantum mechanics³, geodesic motion means a simple Klein-Gordon Lagrangian

$$\mathcal{L}_{\text{eff}} = -\sqrt{g_{\text{eff}}} ((\nabla\varphi)^2 + m_0^2 \varphi^2) \quad \text{with} \quad (\nabla\varphi)^2 = g_{\text{eff}}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi. \quad (3.10)$$

It is well-known that effective actions generally include, at higher orders, some higher-derivative terms: for instance of the type $(\square_g \varphi)^2$, $(\nabla\varphi)^4$, etc. Coming back to the classical limit, i.e. to the Hamilton-Jacobi equation (obtained by considering that $\varphi(x) = \exp(iS(x)/\hbar)$ with $\hbar \rightarrow 0$), we should correspondingly expect that, at higher orders, the effective one-body “Hamilton-Jacobi” equation be of the generalized form

$$0 = m_0^2 + g_{\text{eff}}^{\alpha\beta}(x) p_\alpha p_\beta + A^{\alpha\beta\gamma\delta}(x) p_\alpha p_\beta p_\gamma p_\delta + \dots \quad (3.11)$$

with $p_\alpha = \partial S(x)/\partial x^\alpha$. If we were to use a Lagrangian formulation, the general form (3.11) would correspond to an action $S = -m_0 \int ds_{\text{eff}} [1 + A_{\alpha\beta\gamma\delta}(x) u^\alpha u^\beta u^\gamma u^\delta + \dots]$ with $u^\alpha = dx^\alpha/ds_{\text{eff}}$, i.e. to a general (perturbative) Finsler structure.

If we use perturbatively the lowest-order “on shell” condition (i.e. $m_0^2 + g_{\text{eff}}^{\alpha\beta} p_\alpha p_\beta \simeq 0$) we can (for instance) eliminate the presence of the energy $\mathcal{E}_{\text{eff}} = -p_0$ in the quartic (and higher) terms in (3.11). In other words, we can restrict ourselves to considering purely spatial higher-order tensors $A^{\alpha\beta\gamma\delta}(x) = A^{ijkl}(x)$, etc. Dimensional analysis shows that the quartic terms ($\mathcal{O}(\mathbf{p}^4)$) in Eq. (3.11) which can enter at the 3PN level must have a q -dependence of the type $A^{ijkl}(x) \sim q^{-2}$. Finally, we are led to consider, at the 3PN accuracy, (after solving Eq. (3.11) with respect to the effective energy $\mathcal{E}_{\text{eff}} = -p_0$) a generalized effective Hamiltonian of the form

$$\widehat{H}_{\text{eff}}^{\text{R}}(\mathbf{q}', \mathbf{p}') = \sqrt{A(q') \left[1 + \mathbf{p}'^2 + \left(\frac{A(q')}{D(q')} - 1 \right) (\mathbf{n}' \cdot \mathbf{p}')^2 + \frac{1}{q'^2} (z_1 (\mathbf{p}'^2)^2 + z_2 \mathbf{p}'^2 (\mathbf{n}' \cdot \mathbf{p}')^2 + z_3 (\mathbf{n}' \cdot \mathbf{p}')^4) \right]}, \quad (3.12)$$

where the quartic (in \mathbf{p}') terms come from the $A^{\alpha\beta\gamma\delta}$ coupling and modify the normal “geodesic” Hamiltonian $\sqrt{-g_{00}^{\text{eff}}(1 + g_{\text{eff}}^{ij} p'_i p'_j)}$. In anticipation of the need to transform (via a canonical transformation) the original coordinates (\mathbf{q}, \mathbf{p}) of the real (reduced) Hamiltonian into the coordinates of the effective dynamics, we have denoted the latter by $(\mathbf{q}', \mathbf{p}')$.

This procedure introduces three new arbitrary degrees of freedom at the 3PN level: z_1 , z_2 , and z_3 . It is clear that it now becomes possible to map the real dynamics on the generalized effective dynamics (3.12). This becomes, in fact, possible in many ways. As we are primarily interested in (quasi-)circular orbits we shall find convenient to consider only the simple case where $z_1 = z_2 = 0$, i.e. to use only $z_3 \neq 0$ as new degree of freedom. [This degree of freedom then disappears in the discussion of circular orbits, which can then be considered as following essentially from a “geodesic” dynamics.]

An important feature of our proposal (3.11) is that it is clearly general enough to allow for the existence of solutions at arbitrary PN orders. For instance, at 4PN we would have the freedom to introduce either arbitrary sextic terms,

³We recall that the origin of the effective-one-body approach lies in the quantum electrodynamics of two-charge systems, see [19].

$q'^{-2} [p'^6 + p'^4 (n' p')^2 + \dots]$, or a modification of the quartic terms: $q'^{-3} [p'^4 + \dots]$. This is more freedom than is needed to compensate for $\Delta(4) = 4 - 2 = +2$, Eq. (3.9). We can also clearly arrange things so that circular orbits always follow from a “geodesic” dynamics. The only somewhat unsatisfactory feature of the proposal (3.11) is that we cannot offer any principle for determining a priori the structure of the “post geodesic” terms. In particular, it would have been nice to say that (as is often the case in effective actions) the additional terms are somehow generated by the leading term, i.e. by the effective metric. We have in mind here relations of the type:

$$A_{\alpha\beta\gamma\delta} = \lambda_1 R_{\alpha\beta} R_{\gamma\delta} + \lambda_2 \nabla_{\alpha\beta} R_{\gamma\delta} + \lambda_3 \nabla_{\alpha\beta\gamma\delta} R + \dots,$$

where $R_{\alpha\beta}$ is the Ricci tensor of $g_{\alpha\beta}^{\text{eff}}$. However, we have checked that such “geometrical” tensors do not yield possible 3PN corrections, but only much smaller contributions (starting at 5PN).

We shall give in the next section the details of the computation of the coefficients entering $A(q')$, $D(q')$, the generating function $G_{3\text{PN}}$, the energy-mapping function and z_3 . For the time being, let us only stress some conceptual points. First, it is remarkable that the *unique* solution of the 3PN real \rightleftharpoons effective matching problem leads to the simple value

$$\alpha_3 = 0, \quad (3.13)$$

for the 3PN parameter entering the energy-mapping (3.4). Ref. [13] had already found that $\alpha_2 = 0$ at 2PN, and $\alpha_1 = \nu/2$ at 1PN. These values correspond exactly to

$$\frac{\mathcal{E}_{\text{eff}}^R}{m_0} = \varphi(s_{\text{real}}) = \frac{(\mathcal{E}_{\text{real}}^R)^2 - m_1^2 - m_2^2}{2m_1 m_2}. \quad (3.14)$$

We find remarkable that the simplest, symmetric function of the Mandelstam invariant $s_{\text{real}} = (\mathcal{E}_{\text{real}}^R)^2$ which reduces to \mathcal{E}_0/m_0 in the test-mass limit turns out to define the unique energy-map needed to link the real 3PN dynamics to the effective dynamics. We interpret this as a good sign for our generalized dynamics (3.11). [By contrast, the other proposal of relaxing the natural constraint $d_1 = 0$ leads to a very complicated energy-map with $\alpha_1 \neq \nu/2$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, see Appendix A.]

Let us now consider the consequences of the effective-one-body approach for the determination of the LSO. For the case of circular orbits the effective-one-body approach boils down to saying that the real energy is the following function of the effective-one,

$$\mathcal{E}_{\text{real}}^R = M \sqrt{1 + 2\nu \frac{\mathcal{E}_{\text{eff}}^R - \mu}{\mu}} \quad (3.15)$$

(as obtained by inverting Eq. (3.14)) and that the effective energy along circular orbits is the square-root of the minimum value of a certain “effective radial potential”:

$$\frac{\mathcal{E}_{\text{eff}}^R}{\mu} = \sqrt{[W_j(q')]_{\text{min}}}, \quad (3.16)$$

where $W_j(q')$ is obtained from (the square of) Eq. (3.12) by setting $\mathbf{n}' \cdot \mathbf{p}' = 0$ (and $\mathbf{p}'^2 = (\mathbf{n}' \times \mathbf{p}')^2 + (\mathbf{n}' \cdot \mathbf{p}')^2 = j^2/q'^2$)

$$W_j(q') = A(q') \left(1 + \frac{j^2}{q'^2} + z_1 \frac{j^4}{q'^6} \right). \quad (3.17)$$

As said above, we shall assume (as is always possible) that $z_1 = 0$, so that the effective potential has the usual “Schwarzschild-like” value $-g_{00}(q') (1 + j^2/q'^2)$. The value of the metric coefficient $A(q') = -g_{00}(q')$ is obtained, at the 3PN level, as a perturbative expansion in $1/q' = GM/R_{\text{eff}}$:

$$A(q') = 1 - \frac{2}{q'} + \frac{2\nu}{q'^3} + \frac{a_4(\nu)}{q'^4} + \mathcal{O}\left(\frac{1}{q'^5}\right). \quad (3.18)$$

Note that, finally, in this approach the entire effect of the 3PN dynamics (for circular orbits) is contained in the sole coefficient $a_4(\nu)$ (whose value will be discussed in the next section).

In Ref. [13] the metric coefficient $A(q')$ was used (at the 2PN level) as a simple truncated Taylor expansion: $A_{2\text{PN}}(q') = 1 - 2/q' + 2\nu/q'^3$. This simple-minded approach is not adequate for dealing with the 3PN level. Indeed, we shall see in next section that the coefficient $a_4(\nu)$ is positive and can be relatively large. In keeping with the spirit

of the present work where we systematically try to use adequate resummation methods to improve the convergence of PN expansions, we shall *define* our effective-one-body radial potential, at the n PN accuracy (expressed in terms of the convenient variable $u \equiv 1/q'$)

$$W_j^{P_n}(u) = A_{P_n}(u) (1 + j^2 u^2) \quad (3.19)$$

by using a suitable Padé approximant of Eq. (3.18):

$$A_{P_n}(u) \equiv P_\ell^k [T_{n+1}[A(u)]] , \quad (3.20)$$

with $k + \ell = n + 1$ (because it is q'^{-n-1} which corresponds to the n PN level) and, now the constraint $k > 0$ (instead of $\ell > 0$ as above), because we want to factor a zero of $A(u)$ (and no longer a pole). The Padé improvement of $A(u)$ is really needed (and makes a difference) only at the 3PN level. We have found that the most robust Padés (under variation of the Taylor coefficients) are defined by taking $k = 1$ and $\ell = n$.

Summarizing the present method: The effective radius q' of circular orbits is obtained as a function of the reduced angular momentum j by looking for the minimum of the radial potential (3.19), where $u \equiv 1/q'$ and where the Padé-improved function A is given by Eq. (3.20), with $k = 1$ and $\ell = n$. For each value of j above some threshold j_{\min} , $W_j(u)$ admits a (unique) minimum $u_*(j)$. From this one then determines the effective-energy (3.16), and then the real one (3.15), namely

$$\mathcal{E}_{\text{real}}^R(j) = M \sqrt{1 + 2\nu \left[\sqrt{W_j(u_*(j))} - 1 \right]} . \quad (3.21)$$

The real circular orbital frequency corresponding to j is then given by using the identity (2.21). This yields

$$GM\omega_{\text{real}}(j) = \frac{j u_*^2(j) \sqrt{A_{P_n}(u_*(j))}}{\sqrt{1 + j^2 u_*^2(j)} \sqrt{1 + 2\nu \left[\sqrt{W_j(u_*(j))} - 1 \right]}} . \quad (3.22)$$

Note that in this approach $j \equiv j_{\text{real}} \equiv j_{\text{effective}}$. Finally, the LSO is invariantly defined as the minimum value of j , $j_{\text{LSO}} = j_{\min}$, for which $W_j(u)$ admits a local minimum. When $j < j_{\text{LSO}}$, $W_j(u)$ has no local minimum and there are no (stable or unstable) circular orbits (see, e.g., Fig. 1 of Ref. [13]). If one is only interested in locating the LSO (as a function of ν) it suffices to look for the existence of an inflection point of $W_j(u)$, i.e. to solve the simultaneous equations $\partial W_j(u)/\partial u = 0$ and $\partial^2 W_j(u)/\partial u^2 = 0$.

IV. RESULTS

Let us now give the details of the application, at the 3PN level, of the methods explained above. We follow the order of presentation given in the previous two sections. In what follows, we use the results of Ref. [16] for the dynamical invariants of the 3PN two-body Hamiltonian. We recall that the 3PN Hamiltonian derived in Ref. [14] contained two ambiguous parameters ω_{static} and ω_{kinetic} (these ambiguities arise because of the need to regularize badly divergent integrals [14,20,16]), and that all the dynamical invariants involve only the combination $\sigma(\nu) \equiv \omega_{\text{static}}\nu + \omega_{\text{kinetic}}\nu^2$ [16].

Recently the ‘kinetic’ ambiguity parameter ω_{kinetic} was uniquely determined [15] (see also [21]) to be $\omega_{\text{kinetic}} = 41/24$, so that $\sigma(\nu) = \omega_{\text{static}}\nu + \frac{41}{24}\nu^2$ and the remaining 3PN ambiguity is embodied in the product $\omega_{\text{static}}\nu$. As discussed in the Appendix A of [16] one expects (both by judging from the other coefficients, and by looking at some of the sources of ambiguity) that ω_{static} varies in the range

$$-10 \leq \omega_{\text{static}} \leq +10. \quad (4.1)$$

A. ϵ -method

For self-containedness let us quote the results obtained in our previous paper [16] for the link between the energy and the x -variable, Eq. (2.1). The original energy function E , Eq. (2.2), admits the PN expansion (2.4) with coefficients

$$E_1(\nu) = -\frac{1}{12}(9 + \nu), \quad (4.2a)$$

$$E_2(\nu) = -\frac{1}{24}(81 - 57\nu + \nu^2), \quad (4.2b)$$

$$E_3(\nu) = -\frac{10}{3}(w_1(\nu) - \omega_{\text{static}} \nu), \quad (4.2c)$$

where

$$w_1(\nu) \equiv \frac{405}{128} + \frac{1}{64} \left(41\pi^2 - \frac{6889}{6} \right) \nu + \frac{31}{64} \nu^2 + \frac{7}{3456} \nu^3. \quad (4.3)$$

Correspondingly to this expansion, the new energy function e , Eq. (2.13), admits the expansion (2.15) with coefficients

$$e_1(\nu) = -\left(1 + \frac{1}{3}\nu \right), \quad (4.4a)$$

$$e_2(\nu) = -\left(3 - \frac{35}{12}\nu \right), \quad (4.4b)$$

$$e_3(\nu) = -\frac{10}{3}(w_2(\nu) - \omega_{\text{static}} \nu), \quad (4.4c)$$

where

$$w_2(\nu) \equiv \frac{27}{10} + \frac{1}{16} \left(\frac{41}{4}\pi^2 - \frac{4309}{15} \right) \nu + \frac{103}{120} \nu^2 - \frac{1}{270} \nu^3. \quad (4.5)$$

The 2PN and 3PN Padés of $e(x)$ are given by (see Ref. [1] for the 2PN case)

$$e_{P_2}(x) \equiv P_1^1 [T_2[e(x)]] = -x \frac{1 + \frac{1}{3}\nu - (4 - \frac{9}{4}\nu + \frac{1}{9}\nu^2)x}{1 + \frac{1}{3}\nu - (3 - \frac{35}{12}\nu)x}, \quad (4.6a)$$

$$e_{P_3}(x) \equiv P_1^2 [T_3[e(x)]] = -x \frac{1 - (1 + \frac{1}{3}\nu + w_3(\nu))x - (3 - \frac{35}{12}\nu - (1 + \frac{1}{3}\nu)w_3(\nu))x^2}{1 - w_3(\nu)x}, \quad (4.6b)$$

where

$$w_3(\nu) \equiv \frac{40}{36 - 35\nu}(w_2(\nu) - \omega_{\text{static}} \nu). \quad (4.7)$$

The corresponding x -location of the e -LSO (minimum of $e(x)$) can be written down analytically only at the 2PN level [1]:

$$6 x_{\text{LSO}}^{e_{P_2}}(\nu) = \frac{1 + \frac{1}{3}\nu}{1 - \frac{35}{36}\nu} \left(2 - \frac{1 + \frac{1}{3}\nu}{\sqrt{1 - \frac{9}{16}\nu + \frac{1}{36}\nu^2}} \right). \quad (4.8)$$

Note that $6 x_{\text{LSO}}^{e_{P_2}}(\frac{1}{4}) = 1.1916$, which means that the e_{P_2} -predicted value of the orbital frequency at the LSO differs from the ‘‘Schwarzschild value’’, $GM\omega_{\text{LSO}}^{\text{Schw}} = (x_{\text{LSO}}^{\text{Schw}})^{3/2}$ with $x_{\text{LSO}}^{\text{Schw}} = 1/6$, by the factor

$$\widehat{\omega}_{\text{LSO}} \equiv \frac{\omega_{\text{LSO}}}{\omega_{\text{LSO}}^{\text{Schw}}} = (6 x_{\text{LSO}})^{3/2}, \quad (4.9)$$

which is about 1.3007 in the present case. We will use $\widehat{\omega}_{\text{LSO}}$ as our main tracer of the ‘‘observable’’ location of the LSO. It is important to note from the start that (as emphasized in [1]) the e -method predicts (at 2PN) that the orbital frequency at the LSO is *larger* than the ‘‘Schwarzschild value’’. The corresponding results, at 3PN, are

TABLE I. Equal-mass ($\nu = 1/4$) binary-system LSO parameters obtained by means of different methods. The reduced binding energy E_{LSO} , Eq. (2.2), and the reduced angular momentum j_{LSO} , Eq. (2.18), are divided by their ‘‘Schwarzschild values’’: $|E_{\text{LSO}}^{\text{Schw}}| = 1 - \frac{1}{3}\sqrt{8}$ and $j_{\text{LSO}}^{\text{Schw}} = \sqrt{12}$; the dimensionless orbital frequency $\hat{\omega}_{\text{LSO}}$ is defined in Eq. (4.9). The line denoted by ‘BD’ reports the 2PN results obtained in Ref. [13].

| method | $E_{\text{LSO}}/ E_{\text{LSO}}^{\text{Schw}} $ | | | | | $j_{\text{LSO}}/j_{\text{LSO}}^{\text{Schw}}$ | | | | | $\hat{\omega}_{\text{LSO}}$ | | | | |
|------------------|---|--------|--------------------------|--------|--------|---|-------|--------------------------|-------|-------|-----------------------------|-------|--------------------------|-------|-------|
| | 1PN | 2PN | 3PN | | | 1PN | 2PN | 3PN | | | 1PN | 2PN | 3PN | | |
| | | | ω_{static} | | | | | ω_{static} | | | | | ω_{static} | | |
| | | | -10 | 0 | 10 | | | -10 | 0 | 10 | | | -10 | 0 | 10 |
| e-method | – | -1.142 | -1.015 | -1.369 | -1.636 | – | 0.956 | 0.994 | 0.904 | 0.857 | – | 1.301 | 0.998 | 1.969 | 2.870 |
| <i>j</i> -method | -0.973 | -1.091 | -1.039 | -1.322 | -1.842 | 1.014 | 0.970 | 0.986 | 0.913 | 0.833 | 0.960 | 1.173 | 1.060 | 1.774 | 3.915 |
| eff. method | -1.007 | -1.048 | -1.042 | -1.168 | -1.212 | 1.000 | 0.983 | 0.985 | 0.946 | 0.934 | 1.015 | 1.075 | 1.064 | 1.297 | 1.383 |
| BD, Ref. [13] | -1.007 | -1.050 | – | – | – | 1.000 | 0.983 | – | – | – | 1.015 | 1.079 | – | – | – |
| <i>k</i> -method | – | – | – | – | – | 1.000 | 0.980 | 0.974 | 0.955 | 0.936 | – | – | – | – | – |

exhibited in Table I. Let us only note here that the tendency to get $\hat{\omega}_{\text{LSO}} > 1$ seems confirmed, at the 3PN level, rather independently of the value of the ambiguity parameter ω_{static} .

Once the value of $x_{\text{LSO}}(\nu)$ is determined (analytically or numerically) one can compute the corresponding value of the (real) reduced binding energy E , Eq. (2.2). It is obtained by solving Eq. (2.13) in terms of $\mathcal{E}^R \equiv M + \mu E$. The solution is explicitly given by

$$E(x) = \frac{1}{\nu} \left[\sqrt{1 + 2\nu \left(\sqrt{1 + \epsilon(x)} - 1 \right)} - 1 \right]. \quad (4.10)$$

Then, knowing $E(x)$ we can also compute the value of the reduced angular momentum j by integrating the identity (2.22). Integrating Eq. (2.22) by parts yields

$$j(x) = -2x^{-1/2} \frac{dE(x)}{dx} + 2 \int_0^x d\bar{x} \bar{x}^{-1/2} \frac{d^2E(\bar{x})}{d\bar{x}^2}, \quad (4.11)$$

where we have incorporated the information that $j(x) \sim x^{-1/2}$ when $x \rightarrow 0$ (i.e. in the limit of very wide circular orbits, described by Newtonian dynamics). By applying this result to $x = x_{\text{LSO}}(\nu)$, one finally gets the value of $j_{\text{LSO}}(\nu)$. The results so obtained by the ϵ -method are shown in Table I.

B. *j*-method

In this approach the basic PN expansion we consider is that of $1/j^2(x)$. It reads (cf. Eq. (5.11) in Ref. [16])

$$\frac{1}{j^2(x)} = x \left[1 - \frac{1}{3}(9 + \nu)x + \frac{25}{4}\nu x^2 - \frac{16}{3}(w_4(\nu) - \omega_{\text{static}}\nu)x^3 \right], \quad (4.12)$$

where

$$w_4(\nu) \equiv \frac{1}{64} \left(41\pi^2 - \frac{5269}{6} \right) \nu + \frac{61}{64}\nu^2 - \frac{1}{432}\nu^3. \quad (4.13)$$

From Eq. (4.12) one gets

$$j^2(x) = \frac{1}{x} \left[1 + \frac{1}{3}(9 + \nu)x + \frac{1}{36}(36 - \nu)(9 - 4\nu)x^2 + \frac{16}{3}(w_5(\nu) - \omega_{\text{static}}\nu)x^3 \right], \quad (4.14)$$

where

$$w_5(\nu) \equiv \frac{81}{16} + \frac{1}{64} \left(41\pi^2 - \frac{7321}{6} \right) \nu + \frac{23}{64}\nu^2 + \frac{1}{216}\nu^3. \quad (4.15)$$

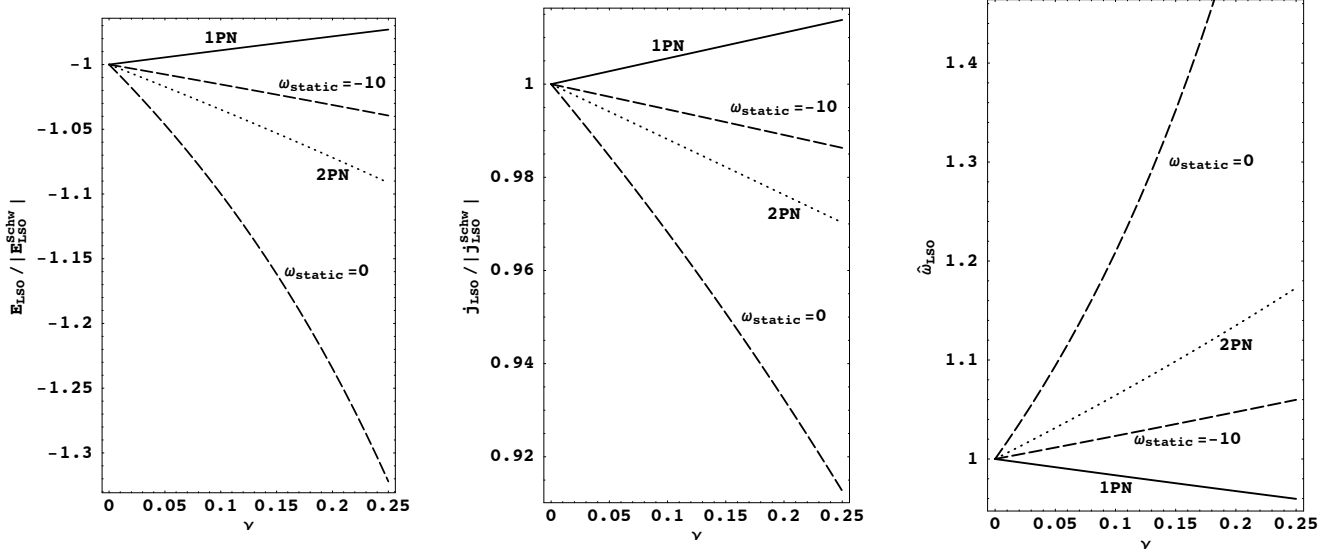


FIG. 1. Binary-system LSO parameters obtained by means of the j -method as functions of the symmetric mass-ratio ν . The reduced binding energy E_{LSO} , Eq. (2.2), and the reduced angular momentum j_{LSO} , Eq. (2.18), are divided by their “Schwarzschild values”: $|E_{\text{LSO}}^{\text{Schw}}| = 1 - \frac{1}{3}\sqrt{8}$ and $j_{\text{LSO}}^{\text{Schw}} = \sqrt{12}$; the dimensionless orbital frequency $\hat{\omega}_{\text{LSO}}$ is defined in Eq. (4.9). The lines shown in the plots correspond to different PN orders of approximation: 1PN (solid), 2PN (dotted), and 3PN (dashed).

As explained above we construct the following sequence of near-diagonal Padés of $j^2(x)$:

$$j_{P_1}^2(x) \equiv P_1^0 [T_1[j^2(x)]] = \frac{1}{x(1 - (3 + \frac{1}{3}\nu)x)}, \quad (4.16a)$$

$$j_{P_2}^2(x) \equiv P_1^1 [T_2[j^2(x)]] = \frac{1 + \frac{1}{9}\nu + \frac{25}{12}\nu x}{x(1 + \frac{1}{9}\nu - (3 - \frac{17}{12}\nu + \frac{1}{27}\nu^2)x)}, \quad (4.16b)$$

$$j_{P_3}^2(x) \equiv P_1^2 [T_3[j^2(x)]] = \frac{1 + (3 + \frac{1}{3}\nu - w_6(\nu))x + (9 - \frac{17}{4}\nu + \frac{1}{9}\nu^2 - (3 + \frac{1}{3}\nu)w_6(\nu))x^2}{x(1 - w_6(\nu)x)}, \quad (4.16c)$$

where

$$w_6(\nu) \equiv \frac{192}{(36 - \nu)(9 - 4\nu)}(w_5(\nu) - \omega_{\text{static}}\nu). \quad (4.17)$$

The corresponding x_{LSO} (now defined as the location of the minimum of $j(x)$, or, equivalently, $j^2(x)$) can be written down analytically at 1PN and 2PN

$$6x_{\text{LSO}}^{j_{P_1}}(\nu) = \frac{1}{1 + \frac{1}{9}\nu}, \quad (4.18a)$$

$$6x_{\text{LSO}}^{j_{P_2}}(\nu) = \frac{8(9 + \nu)}{25\nu} \left[\frac{2(9 + \nu)}{\sqrt{(36 - \nu)(9 - 4\nu)}} - 1 \right]. \quad (4.18b)$$

Note that while $6x_{\text{LSO}}^{j_{P_1}}(\nu)$ is very slightly smaller than 1, $6x_{\text{LSO}}^{j_{P_2}}(\nu)$ is (like for the e_{P_2} estimate) *larger* than 1. In particular, $6x_{\text{LSO}}^{j_{P_2}}(\frac{1}{4}) = 1.1121$, and the corresponding dimensionless frequency is $\hat{\omega}_{\text{LSO}}^{j_{P_2}}(\frac{1}{4}) = 1.1728$. This tendency to get “larger than Schwarzschild” frequency at the LSO is confirmed by the (numerical) 3PN results which are exhibited in Table I and Fig. 1.

Within the present j -method, once the value of $x_{\text{LSO}}(\nu)$ is determined one can compute the corresponding value of the (real) reduced binding energy E by integrating the identity (2.22). Indeed, one can write

$$E(x) = \int_0^x d\bar{x} \bar{x}^{3/2} \frac{dj(\bar{x})}{d\bar{x}}, \quad (4.19)$$

where one has incorporated the boundary condition that $E(x) \rightarrow 0$ when $x \rightarrow 0$. The results so obtained are shown in Table I and Fig. 1.

C. k -method

For completeness, let us (though it is not among our preferred methods) mention some results obtained by using the k -method, Eq. (2.24). The PN expansion of the function $K(y)$, where $K \equiv (1+k)^4$ and $y \equiv 1/j^2$, reads (cf. Eq. (5.27) in Ref. [16])

$$K(y) = 1 + 12y + 24(6 - \nu)y^2 + 24(w_7(\nu) - \omega_{\text{static}} \nu)y^3, \quad (4.20)$$

where

$$w_7(\nu) \equiv 72 + \left(\frac{41}{64} \pi^2 - \frac{128}{3} \right) \nu + \frac{1}{2} \nu^2. \quad (4.21)$$

As explained above, we construct the following sequence of near-diagonal Padés of $K(y)$:

$$K_{P_1}(y) \equiv P_1^0 [T_1[K(y)]] = \frac{1}{1 - 12y}, \quad (4.22a)$$

$$K_{P_2}(y) \equiv P_1^1 [T_2[K(y)]] = \frac{1 + 2\nu y}{1 - 2(6 - \nu)y}, \quad (4.22b)$$

$$K_{P_3}(y) \equiv P_1^2 [T_3[K(y)]] = \frac{1 + (12 - w_8(\nu))y + 12(2(6 - \nu) - w_8(\nu))y^2}{1 - w_8(\nu)y}, \quad (4.22c)$$

where

$$w_8(\nu) \equiv \frac{1}{6 - \nu} (w_7(\nu) - \omega_{\text{static}} \nu). \quad (4.23)$$

Then we take the poles of K_{P_n} as estimates of the value of $1/j^2$ at the LSO. The results for equal-mass binaries ($\nu = 1/4$) are given in Table I.

D. Effective one-body method

Let us explain in detail how we implemented the effective one-body method. Two methods of implementation were presented in Ref. [13]. We could have used the first one by starting from the 3PN Delaunay Hamiltonian given in Ref. [16]. However, we found finally as convenient (given the existence of good algebraic manipulators) to use the second method, which has the advantage of being more informative. This second method consists of writing explicitly the equations that have to be satisfied by the looked for generating function $G(q, p')$ and solving them. Indeed, we look for a canonical transformation between the original (quasi-ADM) coordinates (q, p) of the real problem (i.e. the phase-space coordinates in which was obtained the *order-reduced* Hamiltonian $H(q, p)$ in [16]), and the “effective” coordinates (q', p') (i.e. the coordinates used in Eq. (3.12)). The effect of the generating function $G(q, p')$ reads

$$q'^i = q^i + \frac{\partial G(q, p')}{\partial p'_i}, \quad p_i = p'_i + \frac{\partial G(q, p')}{\partial q^i}. \quad (4.24)$$

Note that, as is well known, the canonical transformation is defined only in implicit form: q' and p being given as functions of q and p' . But there is, in fact, no need to solve for, e.g., (q', p') as functions of (q, p) . As the basic idea is anyway to identify the numerical value of, say, $H_{\text{eff}}(q', p')$ with the numerical value of some (to be determined) function $f(H_{\text{real}}(q, p))$, we can do this identification by expressing both sides in terms of any set of common variables, say q and p' . Finally we write (using Eq. (3.4))

$$\left[\widehat{H}_{\text{eff}}^{\text{R}}(q'(q, p'), p') \right]^2 = \left\{ 1 + \widehat{H}_{\text{real}}^{\text{NR}}(q, p(q, p')) \left[1 + \alpha_1 \widehat{H}_{\text{real}}^{\text{NR}} + \alpha_2 \left(\widehat{H}_{\text{real}}^{\text{NR}} \right)^2 + \alpha_3 \left(\widehat{H}_{\text{real}}^{\text{NR}} \right)^3 \right] \right\}^2, \quad (4.25)$$

in which the L.H.S. is given by the R.H.S. of Eq. (3.12) (without the square root, because we work with the squared equation), while $\widehat{H}_{\text{real}}^{\text{NR}}(q, p)$ on the R.H.S. is the order-reduced Hamiltonian of [16], obtained from the higher-order 3PN Hamiltonian derived in [14]. Both sides of Eq. (4.25) are written in terms of q and p' by using Eq. (4.24). This procedure has been used at the 2PN level in [13], so that we know the values of $\alpha_1 = \frac{1}{2}\nu$, $\alpha_2 = 0$, the 2PN values of the metric functions $A(q')$ and $D(q')$, as well as the 2PN-accurate generating function $G(q, p')$ (see e.g. Eqs. (6.24)–(6.27) in [13]). The new unknowns entering at the 3PN level are: α_3 , a_4 , Eq. (3.3a), d_3 , Eq. (3.3b), z_1 , z_2 , z_3 , Eq. (3.12), and the 7 arbitrary coefficients c_1, \dots, c_7 entering the generic form of $G_{3\text{PN}}$:

$$G_{3\text{PN}}(q, p') = (\mathbf{q} \cdot \mathbf{p}') \left[c_1 \mathbf{p}'^6 + \frac{1}{q} (c_2 \mathbf{p}'^4 + c_3 \mathbf{p}'^2 (\mathbf{n} \cdot \mathbf{p}')^2 + c_4 (\mathbf{n} \cdot \mathbf{p}')^4) + \frac{1}{q^2} (c_5 \mathbf{p}'^2 + c_6 (\mathbf{n} \cdot \mathbf{p}')^2) + \frac{c_7}{q^3} \right]. \quad (4.26)$$

The basic input for writing these equations is the explicit value of the 3PN-accurate Hamiltonian [14,16,15]

$$\widehat{H}_{\text{real}}^{\text{NR}}(\mathbf{q}, \mathbf{p}) = \widehat{H}_{\text{N}}(\mathbf{q}, \mathbf{p}) + \widehat{H}_{1\text{PN}}(\mathbf{q}, \mathbf{p}) + \widehat{H}_{2\text{PN}}(\mathbf{q}, \mathbf{p}) + \widehat{H}_{3\text{PN}}(\mathbf{q}, \mathbf{p}), \quad (4.27)$$

where

$$\widehat{H}_{\text{N}}(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{q}, \quad (4.28a)$$

$$\widehat{H}_{1\text{PN}}(\mathbf{q}, \mathbf{p}) = \frac{1}{8} (3\nu - 1) (\mathbf{p}^2)^2 - \frac{1}{2} [(3 + \nu) \mathbf{p}^2 + \nu (\mathbf{n} \cdot \mathbf{p})^2] \frac{1}{q} + \frac{1}{2q^2}, \quad (4.28b)$$

$$\begin{aligned} \widehat{H}_{2\text{PN}}(\mathbf{q}, \mathbf{p}) &= \frac{1}{16} (1 - 5\nu + 5\nu^2) (\mathbf{p}^2)^3 + \frac{1}{8} [(5 - 20\nu - 3\nu^2) (\mathbf{p}^2)^2 - 2\nu^2 (\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 - 3\nu^2 (\mathbf{n} \cdot \mathbf{p})^4] \frac{1}{q} \\ &\quad + \frac{1}{2} [(5 + 8\nu) \mathbf{p}^2 + 3\nu (\mathbf{n} \cdot \mathbf{p})^2] \frac{1}{q^2} - \frac{1}{4} (1 + 3\nu) \frac{1}{q^3}, \end{aligned} \quad (4.28c)$$

$$\begin{aligned} \widehat{H}_{3\text{PN}}(\mathbf{q}, \mathbf{p}) &= \frac{1}{128} (-5 + 35\nu - 70\nu^2 + 35\nu^3) (\mathbf{p}^2)^4 \\ &\quad + \frac{1}{16} [(-7 + 42\nu - 53\nu^2 - 5\nu^3) (\mathbf{p}^2)^3 + (2 - 3\nu)\nu^2 (\mathbf{n} \cdot \mathbf{p})^2 (\mathbf{p}^2)^2 + 3(1 - \nu)\nu^2 (\mathbf{n} \cdot \mathbf{p})^4 \mathbf{p}^2 - 5\nu^3 (\mathbf{n} \cdot \mathbf{p})^6] \frac{1}{q} \\ &\quad + \left[\frac{1}{16} (-27 + 136\nu + 109\nu^2) (\mathbf{p}^2)^2 + \frac{1}{16} (17 + 30\nu)\nu (\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 + \frac{1}{12} (5 + 43\nu)\nu (\mathbf{n} \cdot \mathbf{p})^4 \right] \frac{1}{q^2} \\ &\quad + \left\{ \left[-\frac{25}{8} + \left(\frac{1}{64}\pi^2 - \frac{335}{48} \right) \nu - \frac{23}{8} \nu^2 \right] \mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3}{64}\pi^2 - \frac{7}{4}\nu \right) \nu (\mathbf{n} \cdot \mathbf{p})^2 \right\} \frac{1}{q^3} \\ &\quad + \left[\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2 + \omega_{\text{static}} \right) \nu \right] \frac{1}{q^4}. \end{aligned} \quad (4.28d)$$

As explained in Refs. [20,16,15] and at the beginning of the present section the 3PN Hamiltonian contains one dimensionless ambiguity parameter ω_{static} .

When written explicitly, the constraint equation (4.25) (truncated at 3PN accuracy) yields a system of 11 equations for the 10 + 3 unknowns ($\alpha_3, a_4, d_3, c_1, \dots, c_7; z_1, z_2, z_3$) (ω_{static} being assumed to be known). This system can be decomposed into three subsystems. The first subsystem consists of 5 equations:

$$\alpha_3 + 16c_1 = -\nu - 3\nu^2 - 5\nu^3, \quad (4.29a)$$

$$\alpha_3 + 2c_1 - 2c_2 = -\frac{1}{8}\nu^2 - 2\nu^3, \quad (4.29b)$$

$$6c_1 + c_2 - 3c_3 = \frac{17}{16}\nu + \frac{19}{4}\nu^2 + \frac{27}{16}\nu^3, \quad (4.29c)$$

$$3c_3 - 5c_4 = -\frac{3}{2}\nu - \frac{27}{2}\nu^2 - \frac{81}{16}\nu^3, \quad (4.29d)$$

$$c_4 = \frac{3}{2}\nu^2 + \frac{7}{16}\nu^3. \quad (4.29e)$$

The second subsystem contains 3 equations:

$$-3\alpha_3 + 2c_2 - 2c_5 + z_1 = \frac{3}{2}\nu - \frac{9}{4}\nu^2 + \frac{19}{8}\nu^3, \quad (4.30a)$$

$$8c_2 + 6c_3 + 4c_5 - 6c_6 + z_2 = \frac{61}{8}\nu + \frac{11}{2}\nu^2 - \frac{11}{2}\nu^3, \quad (4.30b)$$

$$4c_3 + 10c_4 + 8c_6 + z_3 = -\frac{79}{6}\nu - \frac{55}{3}\nu^2 + \frac{39}{8}\nu^3, \quad (4.30c)$$

and the third one consists also of 3 equations:

$$2\alpha_3 + c_5 - c_7 = \left(\frac{271}{48} + \frac{1}{64}\pi^2\right)\nu + \frac{5}{8}\nu^2 - \frac{5}{8}\nu^3, \quad (4.31a)$$

$$-d_3 + 4c_5 + 6c_6 + 6c_7 = \left(-\frac{35}{8} - \frac{3}{32}\pi^2\right)\nu - \frac{57}{4}\nu^2 + 2\nu^3, \quad (4.31b)$$

$$-2\alpha_3 + a_4 + 2c_7 = \left(\frac{221}{12} - \frac{21}{16}\pi^2 + 2\omega_{\text{static}}\right)\nu + \frac{3}{4}\nu^2 + \frac{1}{4}\nu^3. \quad (4.31c)$$

The first subsystem, Eqs. (4.29), yields 5 linear equations for the 5 unknowns c_1 , c_2 , c_3 , c_4 , and α_3 . It is easily found to have a *unique* solution, namely

$$\alpha_3 = 0, \quad (4.32a)$$

$$c_1 = -\frac{1}{16}(1 + 3\nu + 5\nu^2)\nu, \quad (4.32b)$$

$$c_2 = -\frac{1}{16}(1 + 2\nu - 11\nu^2)\nu, \quad (4.32c)$$

$$c_3 = -\frac{1}{24}(12 + 48\nu + 23\nu^2)\nu, \quad (4.32d)$$

$$c_4 = \frac{1}{16}(24 + 7\nu)\nu^2. \quad (4.32e)$$

As already mentioned above, note the remarkably simple result $\alpha_3 = 0$ (which confirms that the energy map takes the nice form (3.14)). It is also remarkable that the result $\alpha_3 = 0$ holds independently of any assumption about the “quartic” parameters z_1 , z_2 , and z_3 .

The second subsystem (4.30) can be viewed (after inserting the unique solution of the first subsystem) as an overdetermined system for the two unknowns c_5 , c_6 . It is then easily seen that it will admit a solution if and only if the parameters z_1 , z_2 , and z_3 satisfy the following linear constraint:

$$8z_1 + 4z_2 + 3z_3 = 6(4 - 3\nu)\nu. \quad (4.33)$$

This linear constraint forbids us to consider the simplest “geodesic” case where $z_1 = z_2 = z_3 = 0$. We can, however, continue to impose the natural conditions $z_1 = 0 = z_2$ which simplify very much the 3PN effective dynamics of circular orbits. With this choice, the general constraint (4.33) yields

$$z_3 = 2(4 - 3\nu)\nu. \quad (4.34)$$

Having fixed the values of z_1 , z_2 , and z_3 , the system (4.30) uniquely determines c_5 and c_6 to be

$$c_5 = -\frac{1}{16}(13 - 16\nu + 6\nu^2)\nu, \quad (4.35a)$$

$$c_6 = -\frac{1}{48}(115 + 116\nu - 26\nu^2)\nu. \quad (4.35b)$$

Finally, the subsystem (4.31) gives 3 equations for the remaining 3 unknowns c_7 , d_3 , and a_4 . The unique solution of this system reads:

$$c_7 = -\left(\frac{1}{64}\pi^2 + \frac{155}{24}\right)\nu + \frac{3}{8}\nu^2 + \frac{1}{8}\nu^3, \quad (4.36a)$$

$$d_3 = 2(3\nu - 26)\nu, \quad (4.36b)$$

$$a_4 = \left(\frac{94}{3} - \frac{41}{32}\pi^2 + 2\omega_{\text{static}}\right)\nu. \quad (4.36c)$$

Note how simple the structure of the coefficient a_4 , Eq. (4.36c), is. Indeed, the right-hand-sides of all the equations (4.29), (4.30), (4.31), were polynomials of the third degree in ν . Therefore, one would have a priori expected the 3PN coefficient a_4 to have the same structure: $a_4 = a_{41}\nu + a_{42}\nu^2 + a_{43}\nu^3$. It is remarkable that the coefficients of ν^2 and ν^3 happen to vanish in a_4 (such a simplification does not occur in the 3PN-level coefficients appearing in the functions $E(x)$, $e(x)$ and $j^2(x)$ considered above). This simple structure of a_4 can be brought out by defining the following quantity,

$$\omega_{\text{static}}^* \equiv -\frac{47}{3} + \frac{41}{64}\pi^2 = -9.3439\dots, \quad (4.37)$$

in terms of which the value of a_4 can be written as

$$a_4 = 2(\omega_{\text{static}} - \omega_{\text{static}}^*)\nu. \quad (4.38)$$

The presence of already two cancellations in a_4 ($a_{42} = 0 = a_{43}$) suggests that the yet undetermined value of ω_{static} might be precisely ω_{static}^* , so that a_4 , Eq. (4.36c), simply vanishes. We shall see below that this conjecture is indirectly supported by the fact that a numerical value $\omega_{\text{static}} \simeq -9$ is selected by the requirement that the various methods discussed in this paper agree in their predictions for LSO quantities. Note also the remarkable fact that if $\omega_{\text{static}} = \omega_{\text{static}}^*$ all the π^2 terms cancell in all the 3PN-level coefficients, so that they all become rational (as were the 2PN ones). [Stated in reverse, if one could a priori prove that all the 3PN coefficients are rational, this would support the conjecture that $\omega_{\text{static}} = \omega_{\text{static}}^*$, though it would be also compatible with having $\omega_{\text{static}} = \omega_{\text{static}}^* +$ a rational number.]

The coefficient a_4 enters the PN expansion of $A(u) \equiv -g_{00}(q')$ (with $u \equiv 1/q'$):

$$A(u) = 1 - 2u + 2\nu u^3 + a_4(\nu)u^4 + \mathcal{O}(u^5). \quad (4.39)$$

As mentioned above, we improve the behaviour of the PN expansion of $A(u)$ by Padéeing it: $A_{P_n}(u) = P_\ell^k[T_{n+1}[A(u)]]$ with $k + \ell = n + 1$. We impose the constraint $k > 0$ to inject the information that $A(u)$ should qualitatively look like $A_{\text{Schw}}(u) = 1 - 2u$, i.e. have a zero at some $u = \frac{1}{2} + \mathcal{O}(\nu)$. As said above, the most robust (for our purpose) Padés of $A(u)$ are the ones with $k = 1$ and $\ell = n$. Finally, we get the following sequence of Padé-improved A :

$$A_{P_1}(u) \equiv P_1^1[T_2[A(u)]] = 1 - 2u, \quad (4.40a)$$

$$A_{P_2}(u) \equiv P_2^1[T_3[A(u)]] = \frac{1 - (2 - \frac{1}{2}\nu)u}{1 + \frac{1}{2}\nu u + \nu u^2}, \quad (4.40b)$$

$$A_{P_3}(u) \equiv P_3^1[T_4[A(u)]] = \frac{2(4 - \nu) + (a_4(\nu) - 16 + 8\nu)u}{2(4 - \nu) + (a_4(\nu) + 4\nu)u + 2(a_4(\nu) + 4\nu)u^2 + 4(a_4(\nu) + \nu^2)u^3}. \quad (4.40c)$$

To extract the LSO quantities from these Padéed A 's, we must consider the effective radial potential

$$W_j^{P_n}(u) = A_{P_n}(u)(1 + j^2 u^2). \quad (4.41)$$

The value of j for which this radial potential has an inflection point defines $j_{\text{LSO}}(\nu)$; the corresponding value of u being $u_{\text{LSO}} = u_*(j_{\text{LSO}})$. As explained in Eqs. (3.21) and (3.22) above one then deduces the energy and the orbital frequency of the LSO. Note that the 2PN Padéed A_{P_2} that we use here differs from the straightforward Taylor approximant A_{T_2} used in Ref. [13]. However, this difference is essentially negligible (as shown by comparing the lines “eff. method” and “BD” in Table I). The Padé improvement is, however, rather important at 3PN in the case where ω_{static} is significantly larger than ω_{static}^* . Indeed, in this case, Padéeing allows one to tame the effect of a largish (positive) $a_4(\nu)$: $a_4(\frac{1}{4}) \simeq 4.67 + \frac{1}{2}\omega_{\text{static}}$. For instance, when $\nu = \frac{1}{4}$ and $\omega_{\text{static}} \gtrsim -1.2$ the radial potential built from a straight Taylor-approximated $A_{T_4}(u)$ would not give rise to an inflection point continuously connected to the test-mass limit. We attribute this lack of structural stability to the known bad properties of high-order PN expansions, and not to the effective-one-body approach. The (numerical) results obtained by the effective one-body approach are exhibited in Table I.

V. DISCUSSION

Before discussing the meaning of the results obtained above, let us state what we would a priori expect. First, we recall that the study in Ref. [1] has shown that the sequence of Padé approximants of the invariant function $F(v)$, giving the gravitational wave flux in terms of $v \equiv (GM\omega/c^3)^{1/3} = x^{1/2}$, had very good (and very monotonic) convergence properties toward the exact result. [By contrast, the sequence of Taylor approximants was badly convergent, and unstable when $v \lesssim v_{\text{LSO}}^{\text{Schw}} = 0.40825$; see Figs. 3a and 3b of [1].] In our case, as one can meaningfully (at least for the j - and effective-one-body methods) consider the 1PN, 2PN, and 3PN approximations, we would expect that a good resummation technique would ensure that any LSO quantity, say Q_{LSO} , be determined with increasing accuracy, when using higher PN information, and, more precisely, that

$$Q_{\text{LSO}}^P \simeq Q_{\text{LSO}}^X + a(b x_{\text{LSO}})^{n+1}, \quad (5.1)$$

with (hopefully) coefficients a and b small enough to ensure a visible convergence (when $x_{\text{LSO}} \simeq x_{\text{LSO}}^{\text{Schw}} = 1/6$). As a minimum test of improved convergence we hope that $|Q_{3\text{PN}} - Q_{2\text{PN}}|$ would be significantly smaller than $|Q_{2\text{PN}} - Q_{1\text{PN}}|$, i.e. that the addition of the 3PN information would have only slightly refined the previous 2PN estimates of LSO quantities [1,13].

Independently of this expectation, we had also hoped, when starting this investigation, that the LSO quantities might be “robust” under the lack of precise knowledge of a sole ambiguous coefficient (ω_{static}) among many others in $H_{3\text{PN}}$. [Given that the amplitude of this coefficient would have some plausible upper bound; as discussed in the Appendix A of [16].] However, the results exhibited in Table I and Fig. 1 show that, in spite of our use of resummation techniques, the LSO quantities appear to be quite sensitive to the exact value of ω_{static} . A first conclusion of our work is therefore that it is quite important to resolve the problem of static ambiguity, arising at 3PN when using delta functions to represent compact (but extended) objects. Until this problem is unambiguously solved, it will not be clear whether (as proposed in [1,13]) it is possible to trust suitably resummed versions of PN-expanded results.

In the meantime, however, we wish to point out several remarkable features of the dependence of our various results on ω_{static} . In Fig. 2 we plot (for the equal-mass case, $\nu = 1/4$) our various predictions, at the 3PN level, and using various methods, as a function of the 3PN ambiguity parameter ω_{static} . It is quite interesting to note that two a priori independent things happen:

- (i) there is a value of ω_{static} , namely

$$\omega_{\text{static}}^{\text{best}} \simeq -9, \quad (5.2)$$

for which the three different methods give, at 3PN, nearly coincident LSO predictions.

- (ii) For this “best fit” value $\omega_{\text{static}}^{\text{best}}$ the 3PN LSO predictions exhibit the expected convergence property that $|Q_{3\text{PN}} - Q_{2\text{PN}}|$ is significantly smaller than $|Q_{2\text{PN}} - Q_{1\text{PN}}|$ (see Table II below).

We have checked that these two remarkable properties hold for all values of the parameter $\nu \leq \frac{1}{4}$. Actually there are several other ways of selecting the approximate value (5.2), i.e. of understanding why it plays a special role. First, we have seen above that the precise value (4.37) (which is near the “best fit” value (5.2)), played a special role in simplifying not only a_4 but also all the other 3PN coefficients. Second, it seems natural to expect that the true value of ω_{static} will be such that the full Taylor expansions of most of the invariant functions will be smooth deformations of their test-mass limits. A minimum requirement for this property of “structural stability” under the turning-on of the parameter ν seems to be that the Taylor coefficients of the functions $e(x)$, $j^2(x)$, $K(y)$, and $1/A(u)$ do not change sign as ν varies from 0 to 1/4 (we only consider functions with infinitely many non-zero Taylor coefficients in the test-mass limit). [One could actually impose a more restricted bound on the ν -variation of the 3PN coefficients,

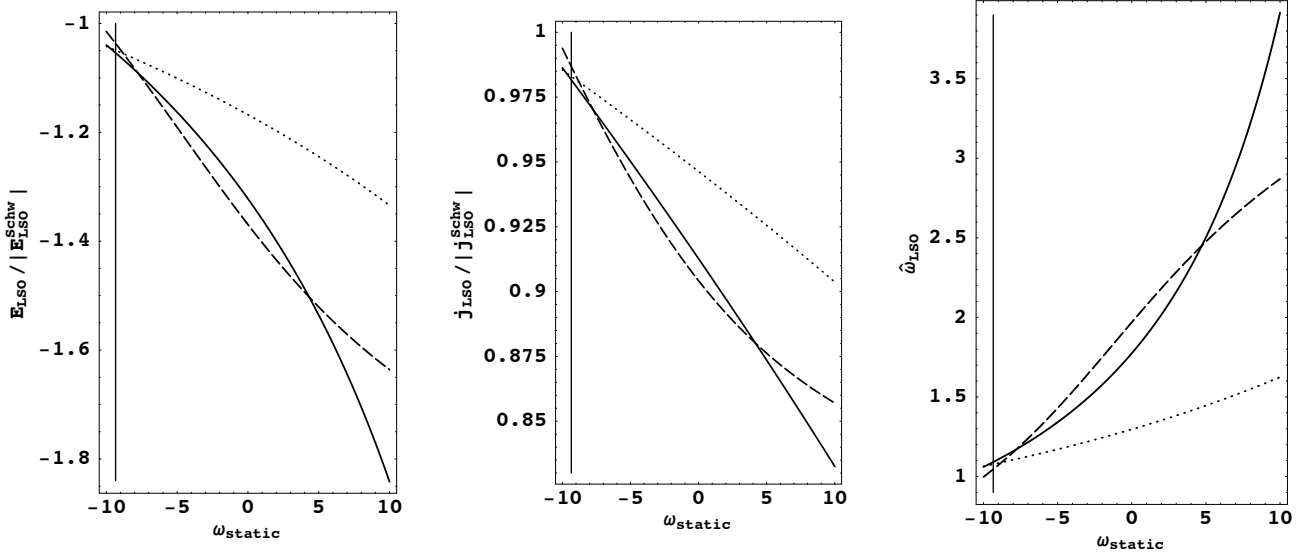


FIG. 2. Equal-mass ($\nu = \frac{1}{4}$) binary systems LSO parameters obtained by means of different methods as functions of the ambiguity parameter ω_{static} . The reduced binding energy E_{LSO} , Eq. (2.2), and the reduced angular momentum j_{LSO} , Eq. (2.18), are divided by their “Schwarzschild values”: $|E_{\text{LSO}}^{\text{Schw}}| = 1 - \frac{1}{3}\sqrt{8}$ and $j_{\text{LSO}}^{\text{Schw}} = \sqrt{12}$; the dimensionless orbital frequency $\hat{\omega}_{\text{LSO}}$ is defined in Eq. (4.9). The lines shown in the plots correspond to different methods: j -method (solid), effective one-body method (dotted), and ϵ -method (dashed). The solid vertical lines correspond to $\omega_{\text{static}} = \omega_{\text{static}}^*$, Eq. (4.37).

especially given the information that the 2PN coefficients are found to vary by a smallish fractional amount.] We find that this minimum requirement is satisfied only if $\omega_{\text{static}} < -0.62$ (the consideration of the expansion of $e(x)$ gives the most stringent bound). Another natural requirement for “structural stability” under $\nu \neq 0$ would be to impose that all the near-diagonal Padés (without restricting oneself, as above, to the most “robust” ones) of the functions above exhibit real poles that are smoothly connected to their test-mass counterparts. This requirement is most stringent when considering $P_2^2(1/A(u))$, and yields the limit $\omega_{\text{static}} < -8.35$. Combining this with the general limits Eq. (4.1) suggests that ω_{static} lies within the small range $-10 < \omega_{\text{static}} < -8.35$.

Let us quote a last way of selecting the value (5.2). It consists in comparing the 3PN Taylor coefficients of the invariant functions that contain only a finite number of terms in the test-mass limit. For instance, consider $A(u)$ and $1/j^2(x)$. In the test-mass limit $A(u) = 1 - 2u$ and $1/j^2(x) = x - 3x^2$. When $\nu \neq 0$ there will appear further powers of u or x with coefficients vanishing with ν . At the 3PN level there is a term $a_4 u^4$ in $A(u)$, and a term $\frac{8}{3}b_4 x^4$ in $1/j^2(x)$. [The factor $8/3$ is introduced to have the same (linear) dependence on ω_{static} in a_4 and b_4 .] These two coefficients read

$$\begin{aligned}
 a_4(\nu) &= \left(\frac{94}{3} - \frac{41}{32}\pi^2 \right) \nu + 2\omega_{\text{static}} \nu \\
 &\approx 18.6879 \nu + 2\omega_{\text{static}} \nu
 \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
 b_4(\nu) &= \left(\frac{5269}{192} - \frac{41}{32}\pi^2 \right) \nu - \frac{61}{32} \nu^2 + \frac{1}{216} \nu^3 + 2\omega_{\text{static}} \nu \\
 &\approx 14.7973 \nu - 1.9063 \nu^2 + 0.00463 \nu^3 + 2\omega_{\text{static}} \nu.
 \end{aligned} \tag{5.4}$$

Let us first note that the terms $\propto \nu^2$ and ν^3 in Eq. (5.4) are numerically nearly negligible. Forgetting about them (i.e. working with $b'_4 \equiv (\frac{5269}{192} - \frac{41}{32}\pi^2) \nu$), we then see, by comparing Eqs. (5.3) and (5.4), that the coefficients $a_4(\nu)$ and $b'_4(\nu)$ are approximately identical. In particular, this means that there will be a small range of values of ω_{static} for which a_4 and b'_4 will be *simultaneously* small. The existence of this range explains why the j - and effective one-body methods can give numerically similar results. We can then make an *analytical* estimate of the ‘best’ value of the

TABLE II. Equal-mass ($\nu = 1/4$) binary systems LSO parameters obtained by means of the j - and effective-one-body methods. The 3PN values of the LSO parameters were calculated for $\omega_{\text{static}} = \omega_{\text{static}}^*$, Eq. (4.37); Eq. (5.8) defines the S -estimates.

| method | $E_{\text{LSO}}/ E_{\text{LSO}}^{\text{Schw}} $ | | | | $j_{\text{LSO}}/j_{\text{LSO}}^{\text{Schw}}$ | | | | $\hat{\omega}_{\text{LSO}}$ | | | |
|-------------|---|--------|--------|---------------|---|-------|-------|---------------|-----------------------------|-------|-------|---------------|
| | 1PN | 2PN | 3PN | S -estimate | 1PN | 2PN | 3PN | S -estimate | 1PN | 2PN | 3PN | S -estimate |
| j -method | -0.973 | -1.091 | -1.054 | -1.063 | 1.014 | 0.970 | 0.982 | 0.979 | 0.960 | 1.173 | 1.091 | 1.114 |
| eff. method | -1.007 | -1.048 | -1.049 | -1.049 | 1.000 | 0.983 | 0.983 | 0.983 | 1.015 | 1.075 | 1.077 | 1.077 |

ambiguity parameter ω_{static} by looking for the value of ω_{static} which *simultaneously* minimizes (in a least square sense) a_4 and b'_4 . It is easily seen that the expression $[a_4(\nu)]^2 + [b'_4(\nu)]^2$ attains its minimal value (as function of ω_{static}) for

$$\omega_{\text{static}}^{\text{min}} = \frac{41}{64}\pi^2 - \frac{11285}{768} \approx -8.37. \quad (5.5)$$

This numerical value is not very far the special values selected by the other arguments discussed above.

Summarizing: several (partially) independent arguments suggest that the true value of ω_{static} lies in the range $-10 \lesssim \omega_{\text{static}} \lesssim -8$. For definiteness, and for the purpose of the following discussion, we shall henceforth assume that the “correct” value of ω_{static} is

$$\omega_{\text{static}} = \omega_{\text{static}}^*. \quad (5.6)$$

In Fig. 2 we have included vertical lines corresponding to $\omega_{\text{static}} = \omega_{\text{static}}^*$, to show visually that Eq. (5.6) is well compatible with our argument based on the convergence of the various methods.

One can also see in Fig. 2 that the curves related to the e - and j -methods have a second intersection point, besides the one around $\omega_{\text{static}}^{\text{best}} \simeq -9$. However, we have checked that the value of ω_{static} at this point strongly depends on the value of the parameter ν . For this reason, and also for the fact that this point does not give an agreement with the “effective” method, we do not take this second intersection point as evidence for a different value of ω_{static} .

Admitting (for the sake of the following argument) Eq. (5.6) we wish to propose a *further* way of improving the accuracy of the predictions of LSO observables. Indeed, if one has at one’s disposal *three* successive approximations, namely the 1PN, 2PN, and 3PN estimates of some quantity Q_{LSO} , one can combine this information to refine the estimate of the (unknown) exact value Q_{LSO}^X . The rationale for this is to assume that the approach to the limit, when the order n of the approximant increases, is approximately described by Eq. (5.1) (i.e. essentially that the inaccuracy of the n th estimate decreases proportionally to the $(n+1)$ th power of some constant $c \equiv b x_{\text{LSO}} < 1$). Then, under this assumption the knowledge of three (successive) approximants, say Q_{n-1} , Q_n , and Q_{n+1} , gives three equations ($Q_m = Q_X + a c^{m+1}$) for the three unknowns (Q_X, a, c). One can solve this system of equations and deduce, in particular, the value of the looked for $n \rightarrow \infty$ limit Q_X in terms of Q_{n-1} , Q_n , and Q_{n+1} . The result defines the so-called “Shanks transformation” [22], namely

$$Q_X \simeq S_n [Q] \equiv \frac{Q_{n+1} Q_{n-1} - Q_n^2}{Q_{n+1} + Q_{n-1} - 2 Q_n}. \quad (5.7)$$

When one disposes of more than three Q_m ’s, the Shanks transformation associates to the original (truncated) sequence (Q_1, Q_2, \dots, Q_N) a shorter, but hopefully faster converging sequence $(S_2 [Q], \dots, S_{N-1} [Q])$. In our case, the Shanks procedure associates to any triplet of LSO quantities $(Q_1, Q_2, Q_3) \equiv (Q_{1\text{PN}}^{\text{LSO}}, Q_{2\text{PN}}^{\text{LSO}}, Q_{3\text{PN}}^{\text{LSO}})$ a single number,

$$Q_S^{\text{LSO}} \equiv \frac{Q_{3\text{PN}}^{\text{LSO}} Q_{1\text{PN}}^{\text{LSO}} - (Q_{2\text{PN}}^{\text{LSO}})^2}{Q_{3\text{PN}}^{\text{LSO}} + Q_{1\text{PN}}^{\text{LSO}} - 2 Q_{2\text{PN}}^{\text{LSO}}}, \quad (5.8)$$

which is a (hopefully better) estimate of the (unknown) exact value Q_{LSO}^X . We shall refer to (5.8) as the S -estimate of Q^{LSO} .

In Table II (see also Fig. 3) we apply this procedure to our two best methods: the j -method and the effective-one-body one, under the assumption (5.6) (which is needed to exhibit a visible convergence among the first three PN approximations).

Given our present (incomplete) knowledge we consider that the S -estimates exhibited in Table II represent our best estimates of LSO observables. To verify the plausibility of these estimates one should resolve the issue of the ambiguous coefficient ω_{static} in the 3PN dynamics. [In principle, this can be done by implementing the matching

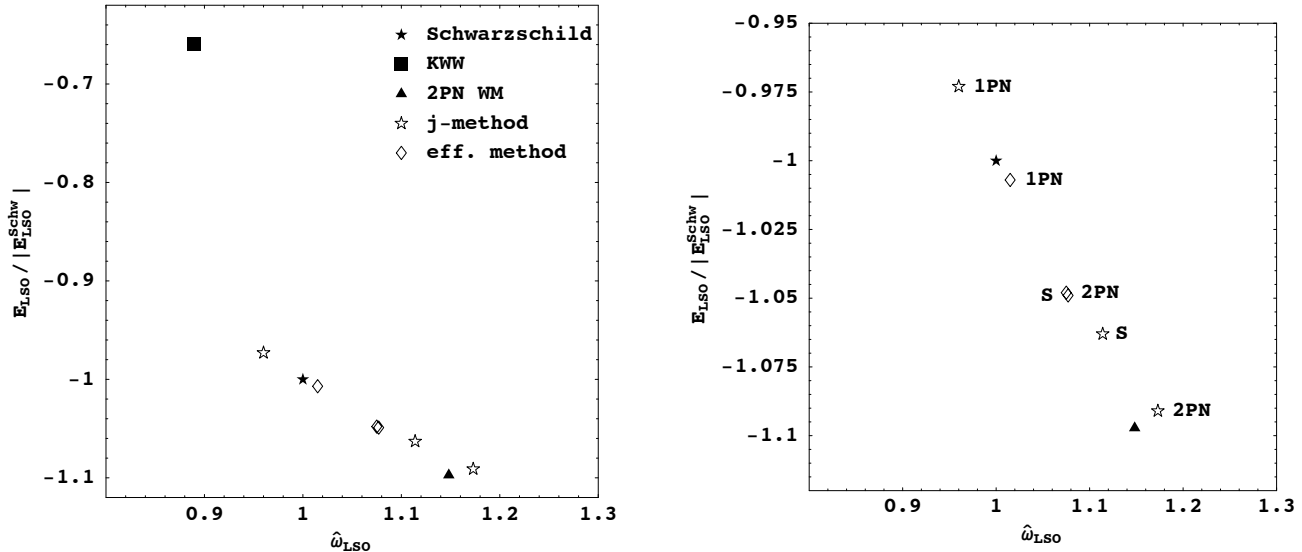


FIG. 3. Equal-mass ($\nu = 1/4$) binary systems reduced binding energy $E_{\text{LSO}}/|E_{\text{LSO}}^{\text{Schw}}|$ versus the dimensionless orbital frequency $\hat{\omega}_{\text{LSO}}$, for different methods discussed in our paper. For j - and effective one-body methods we have plotted the results at the 1PN level, 2PN level, and S -approximants; they are all exhibited in Table II. We have also shown the results obtained in Refs. [8] (labelled by KWW), and by applying the 2PN effective-one-body method to the “Wilson-Mathews” truncation of general relativity (labelled by WM, see Appendix B).

method described in [23] and used there at the 2PN level.] If this resolution approximately confirms the estimate (5.6) the S -estimates will be confirmed. If a very different value of ω_{static} is found, it might still be compatible with a less evidently convergent PN sequence. And hopefully, the S -estimate (5.8) of this new sequence will give an improved 3PN-accurate estimate of LSO observables.

Under the assumption that the S -estimates are accurate, there are several interesting conclusions that we can draw. First, we remark that the final estimates are quite near the 2PN-level predictions of the effective one-body approach, see Fig. 3. Although this may seem disappointing (an enormous, not yet completed, 3PN work leading to a confirmation of 2PN estimates), this would be a scientifically very useful conclusion. Indeed, this would (in our minds at least) establish the soundness of the philosophy advocated in Refs. [1,13] and here, namely that resummation methods can be meaningfully employed to make *analytical* predictions concerning physics near the Last Stable Orbit. This would then also give support to the recent work of Buonanno and Damour [24] in which the 2PN effective one-body Hamiltonian has been used, together with Padé-resummed estimates of gravitational-radiation damping, to study the transition between the inspiral motion and the final plunge of a binary system. [Let us note, in passing, that this work shows that, though it is crucial to have good initial estimates of the LSO quantities defined by the Hamiltonian, the final observable effects linked to the presence of an LSO are blurred by radiation-reaction effects.]

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APPENDIX A: 3PN EFFECTIVE “GEODESIC” ONE-BODY DYNAMICS

We consider here an effective ‘relativistic’ one-body Hamiltonian $\hat{H}_{\text{eff}}^{\text{R}}$ of the simple “geodesic” form

$$\hat{H}_{\text{eff}}^{\text{R}}(\mathbf{q}', \mathbf{p}') = \sqrt{A(q') \left[1 + \mathbf{p}'^2 + \left(\frac{A(q')}{D(q')} - 1 \right) (\mathbf{n}' \cdot \mathbf{p}')^2 \right]}. \quad (\text{A1})$$

The Hamiltonian $\hat{H}_{\text{eff}}^{\text{R}}$ is related to the real ‘non-relativistic’ Hamiltonian $\hat{H}_{\text{real}}^{\text{NR}}$, Eq. (4.27), through the constraint equation

$$\left[\hat{H}_{\text{eff}}^{\text{R}}(q'(q, p'), p') \right]^2 = \left\{ 1 + \hat{H}_{\text{real}}^{\text{NR}}(q, p(q, p')) \left[1 + \alpha_1 \hat{H}_{\text{real}}^{\text{NR}} + \alpha_2 \left(\hat{H}_{\text{real}}^{\text{NR}} \right)^2 + \alpha_3 \left(\hat{H}_{\text{real}}^{\text{NR}} \right)^3 \right] \right\}^2. \quad (\text{A2})$$

As in the text both sides of Eq. (A2) are written in terms of the variables q and p' by means of Eqs. (4.24) with a generating function G of the form

$$G(q, p') = G_{1\text{PN}}(q, p') + G_{2\text{PN}}(q, p') + G_{3\text{PN}}(q, p'), \quad (\text{A3})$$

where

$$G_{1\text{PN}}(q, p') = (\mathbf{q} \cdot \mathbf{p}') \left(g_1 \mathbf{p}'^2 + \frac{g_2}{q} \right), \quad (\text{A4a})$$

$$G_{2\text{PN}}(q, p') = (\mathbf{q} \cdot \mathbf{p}') \left[b_1 \mathbf{p}'^4 + \frac{1}{q} (b_2 \mathbf{p}'^2 + b_3 (\mathbf{n} \cdot \mathbf{p}')^2) + \frac{b_4}{q^4} \right], \quad (\text{A4b})$$

$$G_{3\text{PN}}(q, p') = (\mathbf{q} \cdot \mathbf{p}') \left[c_1 \mathbf{p}'^6 + \frac{1}{q} (c_2 \mathbf{p}'^4 + c_3 \mathbf{p}'^2 (\mathbf{n} \cdot \mathbf{p}')^2 + c_4 (\mathbf{n} \cdot \mathbf{p}')^4) + \frac{1}{q^2} (c_5 \mathbf{p}'^2 + c_6 (\mathbf{n} \cdot \mathbf{p}')^2) + \frac{c_7}{q^3} \right]. \quad (\text{A4c})$$

Written explicitly, the constraint equation (A2) is equivalent to a system of 23 equations for the 23 unknowns: $a_1, \dots, a_4; d_1, d_2, d_3; \alpha_1, \alpha_2, \alpha_3; g_1, g_2; b_1, \dots, b_4; c_1, \dots, c_7$. The *unique* solution to these equations reads

$$a_1 = -2, \quad (\text{A5a})$$

$$a_2 = \frac{(3\nu - 4)\nu}{2(5 - 2\nu)}, \quad (\text{A5b})$$

$$a_3 = \frac{(1600 - 1576\nu + 392\nu^2 - 9\nu^3)\nu}{16(5 - 2\nu)^2}, \quad (\text{A5c})$$

$$a_4 = \frac{(4280 - 3349\nu + 692\nu^2 - 9\nu^3)\nu}{6(5 - 2\nu)^2} - \frac{41\pi^2}{32}\nu + 2\omega_{\text{static}}\nu, \quad (\text{A5d})$$

$$d_1 = \frac{(3\nu - 4)\nu}{2(5 - 2\nu)}, \quad (\text{A5e})$$

$$d_2 = \frac{(-2400 + 1936\nu - 408\nu^2 + 9\nu^3)\nu}{16(5 - 2\nu)^2}, \quad (\text{A5f})$$

$$d_3 = \frac{(-486400 + 703680\nu - 383904\nu^2 + 93704\nu^3 - 8580\nu^4 - 27\nu^5)\nu}{64(5 - 2\nu)^3}, \quad (\text{A5g})$$

$$\alpha_1 = \frac{(4\nu - 3)\nu}{2(5 - 2\nu)}, \quad (\text{A5h})$$

$$\alpha_2 = \frac{(80 - 32\nu + 7\nu^2)(3\nu - 4)\nu}{4(5 - 2\nu)^2}, \quad (\text{A5i})$$

$$\alpha_3 = \frac{(3650 - 3660\nu + 1829\nu^2 - 421\nu^3 + 44\nu^4)(3\nu - 4)\nu}{8(5 - 2\nu)^3}, \quad (\text{A5j})$$

$$g_1 = \frac{(\nu - 6)\nu}{4(5 - 2\nu)}, \quad (\text{A5k})$$

$$g_2 = \frac{5(4 - 2\nu + \nu^2)}{4(5 - 2\nu)}, \quad (\text{A5l})$$

$$b_1 = \frac{(210 - 178\nu + 50\nu^2 - 3\nu^3)\nu}{16(5 - 2\nu)^2}, \quad (\text{A5m})$$

$$b_2 = \frac{(-350 + 419\nu - 149\nu^2 + 13\nu^3)\nu}{8(5 - 2\nu)^2}, \quad (\text{A5n})$$

$$b_3 = \frac{(120 - 81\nu + 38\nu^2 - 6\nu^3)\nu}{8(5 - 2\nu)^2}, \quad (\text{A5o})$$

$$b_4 = \frac{200 + 200\nu - 816\nu^2 + 360\nu^3 - 49\nu^4}{32(5 - 2\nu)^2}, \quad (\text{A5p})$$

$$c_1 = \frac{(5900 - 12700\nu + 8270\nu^2 - 2082\nu^3 + 192\nu^4 - 11\nu^5)\nu}{64(5 - 2\nu)^3}, \quad (\text{A5q})$$

$$c_2 = \frac{(-25850 + 44320\nu - 24876\nu^2 + 5157\nu^3 - 283\nu^4 + 13\nu^5)\nu}{32(5 - 2\nu)^3}, \quad (\text{A5r})$$

$$c_3 = \frac{(-25200 + 21360\nu - 4364\nu^2 - 1077\nu^3 + 408\nu^4 - 47\nu^5)\nu}{96(5 - 2\nu)^3}, \quad (\text{A5s})$$

$$c_4 = \frac{(2160 - 1834\nu + 900\nu^2 - 228\nu^3 + 23\nu^4)\nu^2}{32(5 - 2\nu)^3}, \quad (\text{A5t})$$

$$c_5 = \frac{(247000 - 407080\nu + 225416\nu^2 - 49240\nu^3 + 3730\nu^4 - 151\nu^5)\nu}{128(5 - 2\nu)^3}, \quad (\text{A5u})$$

$$c_6 = \frac{7(10300 - 16360\nu + 7136\nu^2 - 1372\nu^3 + 154\nu^4 - 7\nu^5)\nu}{192(5 - 2\nu)^3}, \quad (\text{A5v})$$

$$c_7 = \frac{(-962800 + 1472880\nu - 813024\nu^2 + 206500\nu^3 - 25110\nu^4 + 1479\nu^5)\nu}{384(5 - 2\nu)^3} - \frac{\pi^2}{64}\nu. \quad (\text{A5w})$$

As said in the text, in view of the complexity of these results, we do not take this possibility seriously. We prefer to it the non-minimal (“non-geodesic”) Hamiltonian given in Sec. IV.

APPENDIX B: 2PN RESULTS FOR THE CONFORMALLY-FLAT TRUNCATION OF GENERAL RELATIVITY

By contrast, let us note that other approximation philosophies are, in our opinion, less reliable to make predictions concerning the LSO. We have in mind here: (i) the use of non-resummed (or only partially resummed) PN expansions, and (ii) the “Wilson-Mathews”-type [25] truncation of Einstein’s theory, in which the spatial metric is taken to be conformally flat. As an example of the first philosophy, let us consider the proposal of Kidder, Will, and Wiseman [10] to partially resum the Damour-Deruelle equations of motion by separating out (and resumming) the “Schwarzschild” ($\nu = 0$) terms. This approach led to the prediction (at 2PN) that the LSO is significantly less bound (when $\nu = 1/4$) than the “Schwarzschild” limit. In terms of orbital frequency at the LSO, Ref. [10] predicts $\hat{\omega}_{\text{LSO}}(1/4) \simeq 0.891 < 1$. This contrasts very much with our 2PN and 3PN estimates above which consistently indicate that $\hat{\omega}_{\text{LSO}}(\nu)$ is larger than one (and that the LSO is more bound than its Schwarzschild limit: $E_{\text{LSO}}/|E_{\text{LSO}}^{\text{Schw}}| < -1$). Independently of

this (biased) argument, we think that both the robustness and the consistency of the “hybrid” approach of [10] are seriously in doubt. Indeed, Refs. [11] and [12] have shown that the hybrid approach was robust neither under a change of formulation (Hamiltonian versus equations-of-motion), nor under a change of coordinate system. Moreover, Ref. [1] has questioned the consistency of this approach by pointing out that the non-resummed “ ν -corrections” represent, in several cases, a very large (larger than 100%) modification of the corresponding ν -independent terms.

Regarding the conformally-flat truncation it was noted by Rieth [26] that this implies significant deviations from the Einstein dynamics already at the 2PN level. We have investigated this question further. In Ref. [16] we gave the invariant functions defined (at 2PN accuracy) by the Wilson-Mathews-type truncation. Applying now our j -method, we find (at 2PN)

$$j_{\text{WM}}^2(x) = \frac{1}{x} \left[1 + \frac{1}{3} (9 + \nu) x + \frac{1}{36} (324 - 333\nu - 50\nu^2) x^2 \right], \quad (\text{B1})$$

whose Padéed form is

$$j_{\text{WMP}_2}^2(x) \equiv P_1^1 [T_2[j_{\text{WM}}^2(x)]] = \frac{1 + \frac{1}{9}\nu + (\frac{15}{4}\nu + \frac{1}{2}\nu^2) x}{x(1 + \frac{1}{9}\nu - (3 - \frac{37}{12}\nu - \frac{25}{54}\nu^2) x)}. \quad (\text{B2})$$

This leads to a prediction for the 2PN x_{LSO} which can be written down analytically

$$6 x_{\text{LSO}}^{j_{\text{WMP}_2}}(\nu) = \frac{8(9 + \nu)}{3(15 + 2\nu)\nu} \left[\frac{2(9 + \nu)}{\sqrt{324 - 333\nu - 50\nu^2}} - 1 \right]. \quad (\text{B3})$$

The corresponding dimensionless orbital frequency, for $\nu = 1/4$, equals $\hat{\omega}_{\text{LSO}} = 1.4378$, the reduced binding energy $E_{\text{LSO}}/|E_{\text{LSO}}^{\text{Schw}}| = -1.2253$, and the reduced angular momentum $j_{\text{LSO}}/j_{\text{LSO}}^{\text{Schw}} = 0.9293$.

We have also studied, at the 2PN level, the effective one-body method for the Wilson-Mathews dynamics. Using the procedure described in Sec. IV D above, imposing at the 1PN level the condition $d_1 = 0$, we have found that the effective-metric function $A_{\text{WM}}(u)$ at the 2PN accuracy reads

$$A_{\text{WM}}(u) = 1 - 2u + a_3(\nu)u^3 + \mathcal{O}(u^4), \quad (\text{B4})$$

where

$$a_3(\nu) = \frac{1}{4}(18 - 5\nu)\nu. \quad (\text{B5})$$

We have improved the behaviour of the 2PN expansion of $A_{\text{WM}}(u)$ by Padéeing it:

$$A_{\text{WMP}_2}(u) \equiv P_2^1 [T_3[A_{\text{WM}}(u)]] = \frac{1 - (2 - \frac{9}{8}\nu + \frac{5}{16}\nu^2) u}{1 + (\frac{9}{8}\nu - \frac{5}{16}\nu^2) u + (\frac{9}{4}\nu - \frac{5}{8}\nu^2) u^2}. \quad (\text{B6})$$

To extract the LSO quantities from this Padéed A , we have considered the inflection point of the effective radial potential $A_{\text{WMP}_2}(u)(1 + j^2u^2)$, which defines the angular momentum $j_{\text{LSO}}(\nu)$ and the location $u_{\text{LSO}}(\nu)$ of the LSO; then using Eqs. (3.21) and (3.22) one calculates the energy and the orbital frequency of the LSO. The results, for $\nu = 1/4$, are: dimensionless orbital frequency $\hat{\omega}_{\text{LSO}} = 1.1482$, reduced binding energy $E_{\text{LSO}}/|E_{\text{LSO}}^{\text{Schw}}| = -1.0972$, and reduced angular momentum $j_{\text{LSO}}/j_{\text{LSO}}^{\text{Schw}} = 0.9647$. Let us also mention that Cook, using another conformally flat approximation [5], obtained, for $\nu = 1/4$, the following LSO parameters: the dimensionless orbital frequency $\hat{\omega}_{\text{LSO}} = 2.528$, the reduced binding energy $E_{\text{LSO}}/|E_{\text{LSO}}^{\text{Schw}}| = -1.579$, and the reduced angular momentum $j_{\text{LSO}}/j_{\text{LSO}}^{\text{Schw}} = 0.8591$. Similar results have been obtained in Ref. [6].

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