Graphical representations of lattice spin models

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Graphical Representations of Lattice Spin Models

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Foreword

The material presented in these notes corresponds to the content of the Cours Péccot given by the author in Collège de France in January 2015, as well as a class given in Princeton from March to May 2015. The aim of these lectures is to provide an introduction to the theory of graphical representations and its applications to so-called spin models. We chose to present a few examples of recent results proved using these representations. Since the theory of graphical representations of spin models has now branched in many directions, we did not attempt to present a complete picture. We rather focus on a few graphical representations. References to more complete books are often included.

The first chapter introduces the notion of lattice spin model. After recalling a few elementary definitions, several examples are discussed. In order to put the results of the next chapters in perspective, a few results and conjectures on lattice spin models are mentioned and discussed. We also introduce non-rigorously what we will call a graphical representation. This chapter should be understood as an introduction and a motivation for what comes next.

As mentioned before, we then sacrifice generality for clarity, and focus on graphical representations of one specific lattice spin model known under the name of Potts model.

The second chapter is devoted to the theory of Bernoulli percolation. Graphical representations are percolation models which usually exhibit long-range dependency. Before diving into the theory of such dependent percolation models, we first take a detour and present the simpler theory of Bernoulli percolation. In this context, we present a few important results which will play an essential role in the more general context of dependent percolation models. This chapter gives us the opportunity to present recent proofs of famous results.

The third chapter introduces our first example of graphical representation, namely the Fortuin-Kasteleyn percolation, also known as the random-cluster model. This model is related to the Potts model. We then explain how the study of the Fortuin-Kasteleyn percolation, which in many aspects is similar to the theory of Bernoulli percolation, enables one to show that Potts models undergo a phase transition on $\mathbb{Z}^d$. In the specific case of $\mathbb{Z}^2$, we also explain how this theory can be used to compute the critical temperature and to prove that the phase transition is sharp (see later for a definition).

The fourth chapter deals with an intimately related graphical representation, known as the loop representation of the *planar* Fortuin-Kasteleyn percolation. We explain how this representation can be used to study the order of the phase transition of the Potts model on $\mathbb{Z}^2$. In particular, we show that the phase transition is continuous when the number of colors q is less or equal to 4, and that it is discontinuous when the number of colors is larger than 3^8 (this constant is obviously not optimized). We also explain how conformal invariance of the model can be proved for the planar Ising model (*i.e.* the Potts model with two colors).

The fifth chapter is devoted to another graphical representation, called the random current representation, which is specific to the Ising model on $\mathbb{Z}^d$. We explain how this representation can be used to prove some correlation inequalities and to show that the phase transition of the (nearest-neighbor ferromagnetic) Ising model is continuous and sharp on $\mathbb{Z}^d$.

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Chapter 1

Lattice spin models

We will work on the lattice $\mathbb{Z}^d = (V(\mathbb{Z}^d), E(\mathbb{Z}^d))$ with $d \geqslant 1$: the vertex-set $V(\mathbb{Z}^d)$, which will be identified with $\mathbb{Z}^d$, is given by

$$\mathbb{Z}^d := \{x = (x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbb{Z}\},$$

and the edge-set is composed of pairs of nearest neighbors, *i.e.*

$$E(\mathbb{Z}^d) := \{\{x, y\} \subset \mathbb{Z}^d : \sum_{i=1}^{d} |x_i - y_i| = 1\}.$$

We will consider finite subgraphs $G = (V(G), E(G))$ of $\mathbb{Z}^d$. For such a graph, ∂G designates the inner (vertex) boundary of G, *i.e.*

$$\partial G := \{x \in V(G) : \text{ there exists } y \notin V(G) \text{ such that } \{x, y\} \in E(\mathbb{Z}^d)\}.$$

We also set $\Lambda_n := [-n, n]^d$ for the box of size n.

The lattice $\mathbb{Z}^2$ is called the *square lattice*; see Fig. 1.1. We will set 0 for the origin $(0, \dots, 0) \in \mathbb{Z}^d$.

1 Definitions

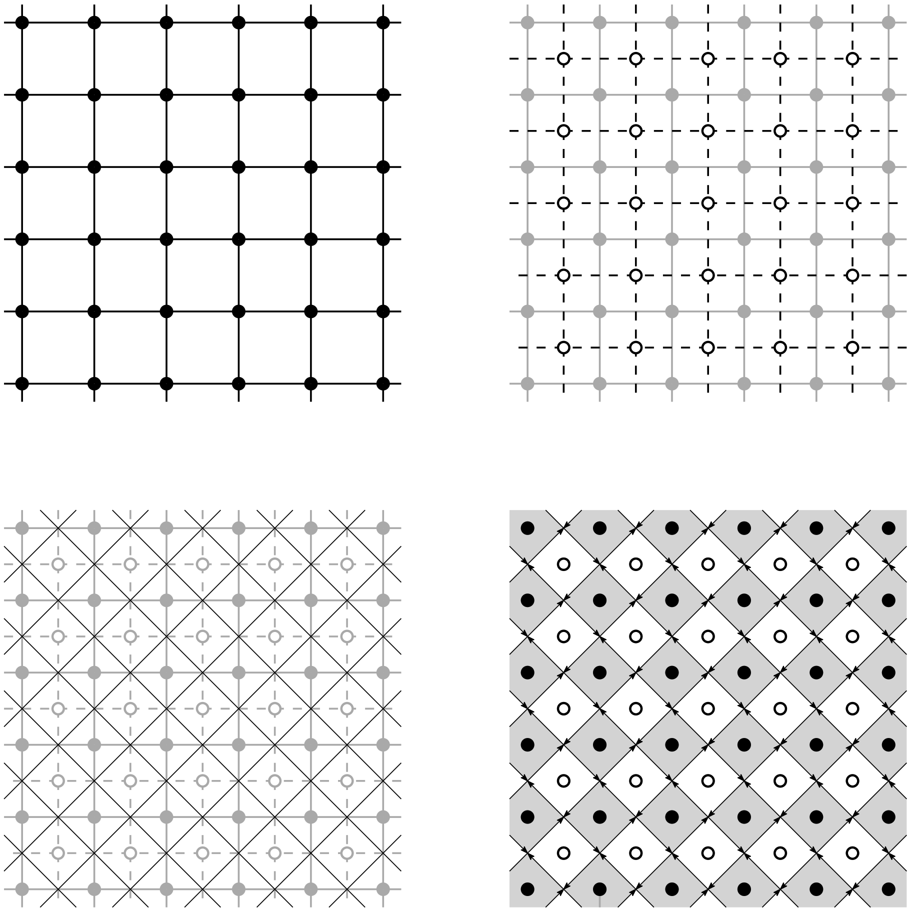
Let ν be a positive integer and G be a finite subgraph of a lattice $\mathbb{Z}^d$. We denote the scalar product on $\mathbb{R}^\nu$ by $\langle x | y \rangle = \sum_{i=1}^{\nu} x_i y_i$. Let us introduce the following formalism.

Spin space. The spin space is an arbitrary subset Ω of $\mathbb{R}^\nu$ (see examples below).

Spin configuration. A spin configuration is an element of $\Omega^{V(G)}$, which will be denoted by $\sigma = (\sigma_x : x \in V(G))$. For every $x \in V(G)$, σ_x is called the *spin at* x.

Boundary condition. A boundary condition is an element of $\Omega^{\mathbb{Z}^d \setminus V(G)}$ which will be denoted by $\tau = (\tau_x : x \in \mathbb{Z}^d \setminus V(G))$.

Figure 1.1. The square lattice (top left), its dual lattice (top right), its medial lattice (bottom left) and a natural orientation on the medial lattice (bottom right).



Hamiltonian. Let $(J_{xy})_{x,y\in\mathbb{Z}^d}$ be a family of ***coupling constants***. We will always assume *invariance under translation*, *i.e.* that $J_{xy}=J(x-y)$ for every $x,y\in\mathbb{Z}^d$. The **Hamiltonian** on G with boundary condition τ is given by

$$H_G^\tau(\sigma):=-\sum_{\{x,y\}\subset V(G)} J_{xy}\langle\sigma_x|\sigma_y\rangle-\sum_{x\in V(G),y\notin V(G)} J_{xy}\langle\sigma_x|\tau_y\rangle.$$

for every spin configuration $\sigma\in\Omega^{V(G)}$. We also define

$$H_G^{\text{free}}(\sigma):=-\sum_{\{x,y\}\subset V(G)} J_{xy}\langle\sigma_x|\sigma_y\rangle.$$

Product measure. Let $\mathrm{d}\sigma_0$ be a measure on Ω. We consider the product measure

$$\mathrm{d}\sigma=\bigotimes_{x\in V(G)}\mathrm{d}\sigma_x$$

on $\Omega^{V(G)}$, where $\mathrm{d}\sigma_x$ are copies of $\mathrm{d}\sigma_0$. Typical examples include the counting measure if Ω is discrete, the Lebesgue measure $\mathrm{d}\lambda$ if $\Omega = \mathbb{R}$, or more generally the Haar measure if Ω is a continuous Lie group.

Gibbs measure. The Gibbs measure on G at inverse temperature β with boundary condition τ, is defined by the formula

$$\mu^{\tau}_{G,\beta}[f] := \frac{\displaystyle\int_{\Omega^{V(G)}} f(\sigma)\exp\left[-\beta H^{\tau}_{G}(\sigma)\right]\mathrm{d}\sigma}{\displaystyle\int_{\Omega^{V(G)}} \exp\left[-\beta H^{\tau}_{G}(\sigma)\right]\mathrm{d}\sigma} \tag{1.1}$$

for every $f : \Omega^{V(G)} \longrightarrow \mathbb{R}$. Similarly, one defines the measure with free boundary condition by replacing τ by free.

Definition 1.1.

When $J_{xy} \geqslant 0$ for every $x, y \in \mathbb{Z}^d$, the model is said to be ***ferromagnetic***. It is said to be *nearest neighbor* if

$$J_{xy} = \begin{cases} J & \text{if } \{x,y\} \in E(\mathbb{Z}^d), \\ 0 & \text{otherwise.} \end{cases}$$

For future reference, we will always fix $J = 1$. In such case, if $E(G)$ is the set of all $\{x,y\} \in E(\mathbb{Z}^d)$ with $x, y \in V(G)$, we find

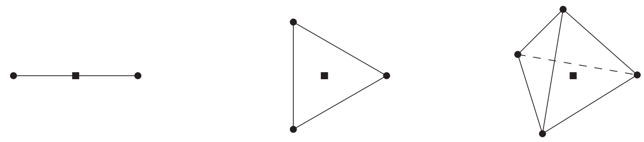
$$H^{\mathrm{free}}_{G}(\sigma) := -\sum_{\{x,y\}\in E(G)} \langle \sigma_x | \sigma_y \rangle.$$

2 Examples of lattice spin models

Let us now describe a few examples of lattice spin models.

Ising model. If $\Omega = \{-1, 1\}$ and $\mathrm{d}\sigma_0$ is the counting measure on Ω, the model is called the Ising model, introduced by Lenz in 1920 [Len 20] and studied in his PhD thesis by Ising [Isi 25]. This model is one of the most classical model of statistical physics.

Figure 1.2. From left to right, $\mathbb{T}_2$, $\mathbb{T}_3$ and $\mathbb{T}_4$.



Potts model. Let $q \in \mathbb{N}^* \setminus \{1\}$. If $\Omega = \mathbb{T}_q$ is a polyhedron in $\mathbb{R}^{q-1}$ (see Fig. 1.2) such that

$$\langle \sigma_x | \sigma_y \rangle = \begin{cases} 1 & \text{if } \sigma_x = \sigma_y, \\ -\frac{1}{q-1} & \text{otherwise.} \end{cases}$$

and $d\sigma_0$ is the counting measure on Ω, the model is called the q-**state Potts model**. This model was introduced by Potts [Pot 52] following a suggestion of his adviser Domb. The model has been a laboratory for testing new ideas and developing far-reaching tools. In two dimensions, it exhibits a rich panel of possible critical behaviors depending on the number of colors, and despite the fact that the model is exactly solvable, the mathematical understanding of its phase transition remains restricted to a few cases (namely $q = 2$ and q large). We refer to [Wu 82] for a review on this model, and to the rest of this book for more details.

Clock model. Let $q \in \mathbb{N}^* \setminus \{1\}$. Let $i \in \mathbb{C}$ such that $i^2 = -1$. If

$$\Omega = \mathbb{U}_q = \{x = (x_1, x_2) \in \mathbb{R}^2 : (x_1 + i x_2)^q = 1\}$$

is the set of q-roots of unity, and $d\sigma_0$ is the counting measure, the model is called the q-state clock model. On the one hand, for $q \in \{2, 3\}$, the model corresponds to the Potts model. On the other hand, for $q \geqslant 4$, the model differs from the Potts model since the scalar product between different spins may be different and therefore two pairs of distinct spins do not play symmetric roles anymore.

Spin $O(n)$-model. Let $n \in \mathbb{N}^*$. If

$$\Omega = \mathbb{S}^{n-1} := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : ||x||_2 = \sqrt{x_1^2 + \dots + x_n^2} = 1\}$$

and $d\sigma_0$ is the surface measure on Ω, the model is called the spin $O(n)$-model. This model was introduced by Stanley in [Sta 68]. When $n = 1$, we end up with the Ising model again. The $n = 2$ and $n = 3$ models were introduced slightly before the general case and are called the XY and Heisenberg models respectively. In two dimensions, the $O(2)$-model is also called the planar rotor model.

Discrete Gaussian Free Field. If $\Omega = \mathbb{R}$ and

$$d\sigma_0 = \frac{\exp(-\sigma_0^2/2)}{\sqrt{2\pi}} \, d\lambda(\sigma_0),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$, the model is called the discrete Gaussian Free Field (GFF). When the model is nearest neighbor, the measure in (1.1) is rewritten in terms of

$$d\lambda(\sigma) = \bigotimes_{x \in V(G)} d\lambda(\sigma_x)$$

as follows:

$$\mu_{G,\beta}^{\tau}[f] := \frac{\displaystyle\int_{\Omega^{V(G)}} f(\sigma) \exp\left[-\mathscr{E}_G^{\tau}(\sigma)\right] d\lambda(\sigma)}{\displaystyle\int_{\Omega^{V(G)}} \exp\left[-\mathscr{E}_G^{\tau}(\sigma)\right] d\lambda(\sigma)}$$

for every $f : \Omega^{V(G)} \longrightarrow \mathbb{R}$, where

$$\mathscr{E}_G^{\tau} := \frac{\beta}{2} \sum_{\{x,y\} \in E(G)} (\sigma_x - \sigma_y)^2 + \left(\tfrac{1}{2} - d\beta\right) \sum_{x \in V(G)} \sigma_x^2.$$

This quantity is called the ***Dirichlet energy*** of the graph. The regime $2d\beta = 1$ corresponds to the *mass-less model*, the regime $2d\beta < 1$ is called *massive*, and the model cannot be defined properly in the regime $2d\beta > 1$ due to divergences at infinity.

The ϕ_d^4 lattice model on $\mathbb{Z}^d$. Let a, b be two constants with $b > 0$. If $\Omega = \mathbb{R}$ and

$$d\sigma_0 = \exp(-a\sigma_0^2 - b\sigma_0^4)\, d\lambda(\sigma_0),$$

the model is called the ϕ_d^4 lattice model. This model interpolates between the GFF corresponding to $a = 1/2$ and $b = 0$ (the normalizing constant $1/\sqrt{2\pi}$ is irrelevant here since it corresponds to the same multiplicative constant appearing in both the numerator and the denominator of the ratio defining the Gibbs measure), and the Ising model corresponding to the limit as $a = -2b$ tends to $+\infty$ (indeed in such case the spins tend to be more and more concentrated on values near -1 and 1).

3 Phase transition in Potts and $O(n)$-models

We wish to illustrate that the theory of lattice spin models is both very challenging and very rich. We will not try to be comprehensive and will simply focus on the examples of the Potts and $O(n)$-models. In order to break the symmetry between all the spins, we fix the boundary condition to be $\tau_x = 1$ (here 1 means $(1, 0, \ldots, 0)$ in $\mathbb{R}^\nu$) for every x (we write $\mu^1_{G,\beta}$ for this measure) and we consider the nearest-neighbor ferromagnetic models.

Introduce the *order parameter* and the *inverse correlation length* by the following respective formulæ

$$m^*(\beta) := \liminf_{n\to\infty} |\mu^1_{G,\beta}[\langle\sigma_0|1\rangle]|,$$
$$\tau(\beta) := \liminf_{n\to\infty} -\tfrac{1}{n}\log|\mu^1_{\Lambda_n,\beta}[\langle\sigma_0|1\rangle]|.$$

Also define

$$\tilde\beta_c := \inf\{\beta > 0 : \tau(\beta) = 0\},$$
$$\beta_c := \sup\{\beta > 0 : m^*(\beta) = 0\}.$$

By definition,

- For any $\beta < \tilde\beta_c$, there exists $c = c(\beta) > 0$ such that

  $$|\mu^1_{\Lambda_n,\beta}[\langle\sigma_0|1\rangle]| \leqslant e^{-cn}$$

  for every $n \geqslant 0$ (we say that the model exhibits *exponential decay of correlations*).

- For any $\beta > \beta_c$, $m^*(\beta) > 0$ (we call this phase the *ordered phase*).

Also note that $\tilde\beta_c \leqslant \beta_c$.

In fact, the following behavior is expected.

Conjecture 1.2. For any $\beta \geqslant 0$ and any $n \geqslant 1$, $\mu^1_{\Lambda_n,\beta}[\langle\sigma_0|1\rangle] \geqslant 0$ and the absolute values in the definitions of the order parameter and the inverse correlation length are therefore useless. In addition to this, the two limits should exist (no need for lim inf). Furthermore, for $\beta \in (\tilde\beta_c, \beta_c)$, the quantity $\mu^1_{\Lambda_n,\beta}[\langle\sigma_0|1\rangle]$ should decay algebraically fast in n.

Several scenarii compatible with the previous conjecture can be imagined:

- If $\tilde{\beta}_c = \beta_c = \infty$, the model does not undergo any phase transition.
- If $\tilde{\beta}_c < \beta_c = \infty$, the model undergoes a ***Berezinsky-Kosterlitz-Thouless*** phase transition. This type of phase transition is named after Berezinsky and Kosterlitz-Thouless, who introduced it (non rigorously) for the planar XY-model in two independent papers [Ber 72, KT 73]. In such case, there is no ordering of the model at any inverse temperature β.
- If $\tilde{\beta}_c = \beta_c < \infty$, the model undergoes a *sharp order/disorder* phase transition. In this context, the phase transition is said to be *continuous* if $m^*(\beta_c) = 0$ and *discontinuous* otherwise.
- One may also have $\tilde{\beta}_c < \beta_c < \infty$, as shown in [FS 81] for the planar Clock model with $q \gg 1$ states but this situation is not expected to occur for Potts and spin $O(n)$-models.

Let us record the different known results in the following table.

<table>
<tr><td colspan="2"></td><td>$d = 1$</td><td>$d = 2$</td><td>$d \geqslant 3$</td></tr>
<tr><td colspan="2" rowspan="2">Ising</td><td rowspan="6">**no PT**

trivial for $d = 1$</td><td colspan="2">**Continuous and sharp order/disorder PT**
Sharpness: [AB 87, DCT 15]</td></tr>
<tr><td>*Continuity:* [Ons 44]</td><td>*Continuity:* [ADCS 15]</td></tr>
<tr><td rowspan="3">Potts</td><td>$q \in \{3,4\}$</td><td>*Sharpness:* [BDC 12b]
Continuity: [DCST 15]</td><td>**Discontinuous and sharp order/disorder PT**</td></tr>
<tr><td rowspan="2">$q \geqslant 5$</td><td>*Sharpness:* [BDC 12b]</td><td>*Sharpness: Conjectural for* $d \geqslant 3$</td></tr>
<tr><td colspan="2">*Discontinuity: Conjectural except for*
$q \geqslant 26$ [KS 82] $q \geqslant q_c(d)$ [KS 82], $d \geqslant d_c(q)$ [BCC 06]</td></tr>
<tr><td rowspan="2">$O(n)$</td><td>$n = 2$</td><td>**Bere.-Kost.-Thou. PT** [FS 81]</td><td rowspan="2">[FSS 76]
(sharpness is conjectural)</td></tr>
<tr><td>$n \geqslant 3$</td><td></td><td>*Absence of PT for* $d = 2$ *is conjectural* [Pol 75]</td></tr>
</table>

On the one hand, at high temperature, spin-spin correlations of lattice spin models can often be proved to decay exponentially fast. On the other hand, a spin model with a discrete spin-space Ω can usually be proved to exhibit order at low temperature, as can be shown using Peierls's argument (see below for more details). On the contrary, this is not the case for continuous spin-space. Indeed, the **Mermin-Wagner** theorem states that a planar spin model for which Ω is a compact continuous connected Lie group does not undergo an order/disorder phase transition (see [MW 66] for the original paper and [ISV 02] for the stronger version mentioned here).

Concerning the question of absence of phase transition for the planar spin $O(n)$-model with $n \geqslant 3$, one of the only known result is due to Kupiainen [Kup 80], who performed a $1/n$ expansion as n tends to infinity. One of the difficulties is the absence (so far) of a tractable graphical representation for the model. The loop $O(n)$-model, introduced in [DMNS 81] provides us with an approximate graphical representation on the hexagonal lattice, for which exponential decay of correlations can be proved at any $\beta > 0$ for $n \gg 1$ [DCPSS 14].

4 Percolation models and graphical representations

For simplicity, we focus on graphical representations of nearest neighbor models (though this notion will be generalized to models with long range interactions in Chapter 5).

Definition 1.3.

A *percolation configuration* $\omega = (\omega_e : e \in E(G))$ is an element of $\{0,1\}^{E(G)}$. If $\omega_e = 1$, the edge e is said to be *open*, otherwise e is said to be *closed*.

The configuration ω can be seen as a subgraph of G with the same set of vertices $V(G)$, and the set of edges given by open edges $\{e \in E(G) : \omega_e = 1\}$.

Definition 1.4.

A percolation model is given by a family of probability measures on percolation configurations on finite subgraphs of $\mathbb{Z}^d$.

We are interested in the connectivity properties of the (random) graph ω. Let us introduce some useful notation. The maximal connected components of ω are called *clusters*. Two vertices x and y are *connected in ω inside $S \subset \mathbb{Z}^d$* if there exists a path of vertices $(v_k)_{0\leqslant k\leqslant K}$ in S such that $v_0 = x$, $v_K = y$, and $\{v_k, v_{k+1}\}$ is open in ω for every $0 \leqslant k < K$. We denote this event by $x \overset{S}{\longleftrightarrow} y$. If $S = G$, we simply drop it from the notation. For $A, B \subset \mathbb{Z}^d$, set $A \overset{S}{\longleftrightarrow} B$ if there exists a vertex of A connected in S to a vertex of B. We also allow ourselves to consider $B = \infty$: in such case, we mean that there exists an infinite open self-avoiding path starting from a vertex in A.

Let us imagine for a moment that we are in the presence of two models: a nearest-neighbor lattice spin model and a percolation model $(\mathbb{P}_G : G \subset \mathbb{Z}^d)$ such that for every $G \subset \mathbb{Z}^d$ and $x, y \in G$,

$$\mu^{\text{free}}_{G,\beta}[\langle \sigma_x | \sigma_y \rangle] = \mathbb{P}_G[x \longleftrightarrow y].$$

In such case, the percolation measure is said to be a ***graphical representation*** of the spin model: spin-spin correlations have been rephrased in terms of connectivity properties of the percolation model. We will see several examples of these graphical representations in the following chapters.

Rephrasing spin-spin correlations in terms of connectivity properties of the associated percolation model is interesting only if the percolation model is itself simpler to study. This will indeed be the case in our examples. For instance, useful correlation inequalities can be obtained via these graphical representations. In addition to this, probabilistic tools may be invoked to analyze the percolation models.

We should mention that a spin lattice model may have more than one graphical representations for instance (we will see several graphical representations of the Ising model). Each graphical representation may have different advantages in different contexts, and choosing the right representation is part of the game.

In order to illustrate that percolation models may be simpler to study, let us focus in the next chapter on the simplest (from the point of view of percolation theory) of such percolation models, namely Bernoulli percolation.

An elementary example of percolation model: Bernoulli percolation

Consider the Bernoulli edge percolation measure $\mathbb{P}_p$ on $\{0,1\}^{E(G)}$ for which each edge of $E(G)$ is declared *open* with probability p and *closed* otherwise, independently for different edges. Note that the definition extends trivially to $\mathbb{Z}^d$. In such case, the σ-algebra of measurable events is the smallest σ-algebra containing the events depending on finitely many edges.

1 Basic properties

Let us first focus on a few basic facts about Bernoulli percolation. Consider the standard (partial) order on $\{0,1\}^{E(\mathbb{Z}^d)}$ given by

$$\omega \leqslant \omega' \text{ if and only if for every } e \in E(\mathbb{Z}^d),\ \omega_e \leqslant \omega'_e.$$

This order induces a notion of increasing function from $\{0,1\}^{E(\mathbb{Z}^d)}$ to $\mathbb{R}$. Also, an event A is said to be *increasing* if its indicator function $\mathbf{1}_A$ defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise} \end{cases}$$

is increasing. Note that A is increasing if for any $\omega \in A$ and $\omega' \geqslant \omega$, we have $\omega' \in A$.

Monotonicity. Let $p \leqslant p'$, then for any increasing event A, we find $\mathbb{P}_p[A] \leqslant \mathbb{P}_{p'}[A]$. In order to prove this fact, observe that Bernoulli percolation measures with different edge-weights p can be defined on the same probability space as follows. Consider $\Omega = [0,1]^{E(\mathbb{Z}^d)}$ and let $(\mathsf{U}_e)_{e\in E(\mathbb{Z}^d)}$ be a family of independent uniform random variables on $[0,1]$. Set $\mathbf{P}$ for the associated measure. For $p \in [0,1]$, introduce

$$\omega_e^{(p)} = \begin{cases} 1 & \text{if } \mathsf{U}_e \leqslant p, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, the law of $\omega^{(p)}$ is $\mathbb{P}_p$ and for any $p \leqslant p'$, $\omega^{(p)} \leqslant \omega^{(p')}$. As a consequence, the law of the family $(\omega^{(p)} : p \in [0,1])$ contains as marginals the different Bernoulli percolation measures with edge parameter in $[0,1]$. The property follows trivially from the fact that $\omega^{(p)} \leqslant \omega^{(p')}$ under the coupling $\mathbf{P}$.

FKG inequality. Let $p \in [0,1]$, for any two increasing functions f and g,

$$\mathbb{P}_p[fg] \geqslant \mathbb{P}_p[f]\mathbb{P}_p[g].$$

As a direct consequence, for two increasing (respectively two decreasing) events A and B,

$$\mathbb{P}_p[A \cap B] \geqslant \mathbb{P}_p[A]\mathbb{P}_p[B].$$

This inequality can easily be proved for increasing functions depending on the states of finitely many edges by induction on the number of edges. The inequality for general increasing functions follows by approximating increasing functions by increasing functions depending on the states of finitely many edges. In the case of Bernoulli percolation, this inequality is in fact called Harris' inequality [Har 60]. The FKG denomination comes from the fact that this inequality was proved by Fortuin, Kasteleyn and Ginibre for a more general class of percolation models (see [FKG 71] or the next chapter for more details).

Russo's formula. For a configuration ω and an edge e, let $\omega^{(e)}$ be the configuration coinciding with ω for edges different from e, and with the edge e open. Similarly, we define $\omega_{(e)}$ to be the configuration coinciding with ω for edges different from e, and with the edge e closed.

Let A be an *increasing* event depending on the states of a finite set of edges E. We say that e is *pivotal for* A if $\omega^{(e)} \in A$ and $\omega_{(e)} \notin A$. Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p[A] = \sum_{e\in E}\mathbb{P}_p[e \text{ is pivotal for } A].$$

There are many proofs of this statement (see [Rus 78] for the original paper). Let us present the following one, which will be useful later in this document. Since

$$\mathbb{P}_p[A] = \sum_{\omega\in\{0,1\}^E} p^{\sum_{f\in E}\omega_f}(1-p)^{\sum_{f\in E}1-\omega_f}\mathbf{1}_A(\omega),$$

we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p[A] &= \sum_{e\in E}\sum_{\omega\in\{0,1\}^E}\left(\tfrac{1}{p}\mathbf{1}_A(\omega^{(e)}) - \tfrac{1}{1-p}\mathbf{1}_A(\omega_{(e)})\right)p^{\sum_{f\in E}\omega_f}(1-p)^{\sum_{f\in E}1-\omega_f}\\ &= \sum_{e\in E}\mathbb{P}_p[\omega^{(e)}\in A] - \mathbb{P}_p[\omega_{(e)}\in A]\\ &= \sum_{e\in E}\mathbb{P}_p[e \text{ is pivotal for } A].\end{aligned}$$

The van den Berg-Kesten inequality. Let A and B be two increasing events. Let $A\circ B$ be the event that A and B *occur disjointly*, meaning that $\omega \in A\circ B$ if there exist two disjoint subsets of edges $E = E(\omega)$ and $F = F(\omega)$ such that any configuration ω' coinciding with ω on E (resp. F) belongs to A (resp. B).

Then,

$$\mathbb{P}_p[A\circ B] \leqslant \mathbb{P}_p[A]\mathbb{P}_p[B].$$

The BK inequality is not very difficult to show, but the proof remains too cumbersome for these notes (see [vdBK 85] for the original paper or [Gri 99] for a modern exposition).

Remark 2.1. We will not need this delicate inequality. In all our applications, we use the BK inequality in a less general context. Assume that there exists a random variable $\mathscr{E}$ taking values in subsets of $E(\mathbb{Z}^d)$, such that

$$\mathbb{P}_p[A \circ B] = \sum_{E \subset E(\mathbb{Z}^d)} \mathbb{P}_p[\omega \in A \cap \{\mathscr{E} = E\}]\mathbb{P}_p[B_E], \tag{2.1}$$

where $\omega \in B_E$ if any configuration coinciding with ω on E belongs to B. If such a formula is available, we immediately deduce from $B_E \subset B$ that

$$\mathbb{P}_p[A \circ B] \leqslant \sum_{E \subset E(\mathbb{Z}^d)} \mathbb{P}_p[\omega \in A_E \cap \{\mathscr{E} = E\}]\mathbb{P}_p[B] = \mathbb{P}_p[A]\mathbb{P}_p[B].$$

While (2.1) can seem strange at first sight, the reader familiar with the so-called Domain Markov property will recognize a rather standard expression corresponding in the case of Bernoulli percolation to conditioning on A, and then using the Markov property for the set of unexplored edges E. Once again, the inequality applies in a more restricted context than the BK inequality since one should find a random variable $\mathscr{E}$, which does not always exist.

2 Phase transition

As a straightforward consequence of the monotonicity property, we deduce the following theorem.

Theorem 2.2.
There exists $p_c = p_c(d) \in [0,1]$ such that

$$\mathbb{P}_p[0 \longleftrightarrow \infty] = \begin{cases} 0 & \text{if } p < p_c(d), \\ \theta(p,d) > 0 & \text{if } p > p_c(d). \end{cases}$$

In other words, there is a phase transition between a regime without infinite cluster and a regime with an infinite cluster. The $p < p_c$ regime is called the ***subcritical regime***, and the $p > p_c$ regime the ***supercritical regime***. When $p = p_c$, one speaks of the ***critical regime***.

We now wish to be slightly more precise and to derive a few properties of percolation in the subcritical and supercritical regimes.

Let $\tau_x : \{0,1\}^{E(\mathbb{Z}^d)} \to \{0,1\}^{E(\mathbb{Z}^d)}$ be the shift by a vector $x \in \mathbb{Z}^d$ defined by

$$\tau_x \omega_{\{a,b\}} := \omega_{\{a+x,b+x\}} \quad , \quad \forall \{a,b\} \in E(\mathbb{Z}^d).$$

Let $\tau_x^{-1}A = \{\omega \in \{0,1\}^{E(\mathbb{Z}^d)} : \tau_x\omega \in A\}$. An event A is *invariant under translations* if for any $x \in \mathbb{Z}^d$, $\tau_x^{-1}A = A$. A measure μ is *invariant under translations* if $\mu(\tau_x A) = \mu(A)$ for any event A.

Theorem 2.3 (Ergodicity of Bernoulli percolation).
The Bernoulli percolation measure on $\mathbb{Z}^d$ with parameter $p \in [0,1]$ is ***ergodic***, *i.e.* that any event A which is invariant under translations satisfies $\mathbb{P}_p[A] \in \{0,1\}$.

Lattice spin models Bernoulli percolation Potts model Random-cluster model Ising model

Proof. Let A be an event which is invariant under translations. Let $\varepsilon > 0$ and choose $n \geqslant 0$ and an event B depending on the edges in Λ_n such that $\mathbb{P}_p[A \Delta B] \leqslant \varepsilon$ (the existence of this event follows from the fact that the σ-algebra of measurable events is the smallest σ-algebra containing events depending on finitely many edges), where $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Let $x \notin \Lambda_{2n}$.

Using the invariance under translation in the second equality and the independence in the third (B and $\tau_x B$ depend on disjoint sets of edges), we deduce that

$$\begin{aligned}\mathbb{P}_p[A] = \mathbb{P}_p[A \cap A] &= \mathbb{P}_p[A \cap \tau_x A] \\ &\leqslant \mathbb{P}_p[B \cap \tau_x B] + 2\varepsilon \\ &= \mathbb{P}_p[B]\mathbb{P}_p[\tau_x B] + 2\varepsilon = \mathbb{P}_p[B]^2 + 2\varepsilon \leqslant \mathbb{P}_p[A]^2 + 4\varepsilon.\end{aligned}$$

By letting ε tend to 0, we deduce that $\mathbb{P}_p[A] \leqslant \mathbb{P}_p[A]^2$ which implies $\mathbb{P}_p[A] \in \{0, 1\}$. □

A corollary of this theorem is that for $p > p_c(d)$,

$$\mathbb{P}_p[\text{there exists an infinite cluster}] = 1.$$

In other words, when the infinite cluster exists with positive probability, it exists in fact almost surely.

The following statement yields that the infinite cluster, when it exists, is unique on $\mathbb{Z}^d$. This argument is very robust and can be applied in a very large context (see for instance Section 4).

Proposition 2.4.

Consider the Bernoulli percolation on $\mathbb{Z}^d$ with parameter $p \in [0, 1]$. Either there is no infinite cluster almost surely, or there exists a *unique* infinite cluster almost surely.

This result was first proved in [AKN 87]. It was later obtained via different types of arguments. The beautiful argument presented in this book is due to **Burton and Keane** [BK 89].

Proof. Let $\mathcal{E}_{\leqslant 1}$, $\mathcal{E}_{<\infty}$ and $\mathcal{E}_{\infty}$ be the events that there is no more than one, finitely many and infinitely many infinite clusters respectively.

Let us start by showing that $\mathbb{P}_p[\mathcal{E}_{<\infty} \setminus \mathcal{E}_{\leqslant 1}] = 0$. Let $\mathcal{F}_n$ be the event that all the infinite clusters intersect Λ_n (there may be none) and choose n large enough that $\mathbb{P}_p[\mathcal{F}_n] \geqslant \frac{1}{2}\mathbb{P}_p[\mathcal{E}_{<\infty}]$. Since $\mathcal{F}_n$ depends on the states of edges outside Λ_n only, we deduce that

$$\mathbb{P}_p[\mathcal{F}_n \cap \{\omega(e) = 1, \forall e \in E(\Lambda_n)\}] \geqslant \tfrac{1}{2} p^{|E(\Lambda_n)|}\, \mathbb{P}_p[\mathcal{E}_{<\infty}].$$

Any configuration in the event on the left contains zero or one infinite connected component since all the vertices in Λ_n are connected. Therefore,

$$\mathbb{P}_p[\mathcal{E}_{\leqslant 1}] \geqslant \tfrac{1}{2} p^{|E(\Lambda_n)|}\, \mathbb{P}_p[\mathcal{E}_{<\infty}].$$

By ergodicity, $\mathbb{P}_p[\mathcal{E}_{<\infty}] = 1$ implies that $\mathbb{P}_p[\mathcal{E}_{\leqslant 1}] = 1$.

We now exclude the possibility of an infinite number of infinite clusters. Consider $n > 0$ large enough that

$$\mathbb{P}_p[C(d) \text{ infinite clusters intersect the box } \Lambda_n] \geqslant \tfrac{1}{2}\mathbb{P}_p[\mathcal{E}_\infty],$$

where $C(d)$ is large enough that among these $C(d)$ clusters, three are intersecting the box Λ_n at vertices which are at distance three of each others at least. By changing the configuration in Λ_N, we deduce that

$$\mathbb{P}_p[\mathrm{CT}_0] \geqslant \tfrac{1}{2}\min\{p, 1-p\}^{|E(\Lambda_N)|}\mathbb{P}_p[\mathcal{E}_\infty], \tag{2.2}$$

where CT_0 is the following event: $\mathbb{Z}^d \setminus \{0\}$ contains three distinct infinite connected components which are connected to 0. We chose $C(d)$ instead of 3 to be certain to be able to do the rewiring inside the box.

A vertex $x \in \mathbb{Z}^d$ is called a *trifurcation* if $\tau_x \mathrm{CT}_0 =: \mathrm{CT}_x$ occurs.

Fix $n \geqslant 1$ and denote the set of trifurcations in Λ_n by T. By invariance under translation, $\mathbb{P}_p[\mathrm{CT}_x] = \mathbb{P}_p[\mathrm{CT}_0]$ and therefore

$$\mathbb{E}_p[|T|] = \mathbb{P}_p[\mathrm{CT}_0] \times |\Lambda_n|.$$

Let us now bound deterministically $|T|$ by $|\partial\Lambda_n|$. Denote the edges in $E(\Lambda_n)$ by $e_1, \dots, e_k$. For each cycle of open edges, turn the edge with smallest index (for some prescribed index) to closed (this is a standard procedure to obtain the minimal spanning tree for instance). We are now in possession of a forest. Remove an edge of the forest with one endpoint not connected in the forest to the boundary, together with the connected component of the endpoint not connected to the boundary. We have now a new forest. Keep doing the previous procedure until there is no such edge anymore. The leafs of the forest are now vertices of the boundary. Since the trifurcations are vertices of degree at least three in this forest, we deduce that $|T|$ is smaller than the number of leafs in the forest, *i.e.* that $|T| \leqslant |\partial\Lambda_n|$. This gives

$$\mathbb{P}_p[\mathrm{CT}_0] = \frac{\mathbb{E}_p[|T|]}{|\Lambda_n|} \leqslant \frac{|\partial\Lambda_n|}{|\Lambda_n|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Combined with (2.2), this implies that $\mathbb{P}_p[\mathcal{E}_\infty] = 0$. The claim follows. □

We now show that the phase transition is sharp. On top of that, we also deduce a so-called mean-field lower bound on the density of the infinite cluster for $p > p_c$.

Theorem 2.5.

For any $d \geqslant 2$,

1. For $p < p_c$, there exists $c = c(p) > 0$ such that for every $n \geqslant 1$,

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leqslant e^{-cn}.$$

2. For $p > p_c$,

$$\mathbb{P}_p[0 \longleftrightarrow \infty] \geqslant \frac{p - p_c}{p(1-p_c)}.$$

This theorem was proved by Aizenman, Barsky [AB 87] and Menshikov [Men 86]. These two proofs are also presented in [Gri 99]. Here, we choose to present a new argument [DCT 15] based on the following crucial quantity.

Let S be a finite set of vertices containing the origin. Given such a set, we denote its edge-boundary by

$$\Delta S = \big\{\{x,y\} \subset E(\mathbb{Z}^d) : x \in S, y \notin S\big\}.$$

For $p \in [0,1]$ and $0 \in S \subset \mathbb{Z}^d$, define

$$\varphi_p(S) := p \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p[0 \xleftrightarrow{S} x]. \tag{2.3}$$

Set

$$\tilde{p}_c = \inf\big\{p \in [0,1] : \exists S \text{ with } \varphi_p(S) < 1\big\}. \tag{2.4}$$

Proof. Let $p < \tilde{p}_c$. By definition, one can fix a finite set S containing the origin, such that $\varphi_p(S) < 1$. Let $L > 0$ such that $S \subset \Lambda_{L-1}$.

Let $k \geqslant 1$ and assume that the event $0 \longleftrightarrow \partial\Lambda_{kL}$ holds. Introduce the random variable

$$\mathscr{C} := \{x \in S : x \xleftrightarrow{S} 0\}$$

corresponding to the cluster of 0 in S. Since $S \cap \partial\Lambda_{kL} = \emptyset$, one can find $\{x,y\} \in \Delta S$ such that the following events occur

- $0 \xleftrightarrow{S} x$,
- $\{x,y\}$ is open,
- $y \xleftrightarrow{\mathscr{C}^c} \partial\Lambda_{kL}$.

Using the union bound, and then a decomposition on the possible realizations of $\mathscr{C}$, we find

$$\begin{aligned}
&\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \\
&\quad\leqslant \sum_{\{x,y\}\in\Delta S} \sum_{C\subset S} \mathbb{P}_p\big[\{0 \xleftrightarrow{S} x\} \cap \{\mathscr{C} = C\} \cap \{\{x,y\} \text{ open}\} \cap \{y \xleftrightarrow{C^c} \partial\Lambda_{kL}\}\big] \\
&\quad\leqslant p \sum_{\{x,y\}\in\Delta S} \sum_{C\subset S} \mathbb{P}_p\big[\{0 \xleftrightarrow{S} x\} \cap \{\mathscr{C} = C\}\big] \mathbb{P}_p\big[y \xleftrightarrow{C^c} \partial\Lambda_{kL}\big] \\
&\quad\leqslant p\Big(\sum_{\{x,y\}\in\Delta S} \sum_{C\subset S} \mathbb{P}_p\big[\{0 \xleftrightarrow{S} x\} \cap \{\mathscr{C} = C\}\big]\Big) \mathbb{P}_p\big[0 \longleftrightarrow \partial\Lambda_{(k-1)L}\big] \\
&\quad\leqslant p\Big(\sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\big[0 \xleftrightarrow{S} x\big]\Big) \mathbb{P}_p\big[0 \longleftrightarrow \partial\Lambda_{(k-1)L}\big] \\
&\quad= \varphi_p(S) \mathbb{P}_p\big[0 \longleftrightarrow \partial\Lambda_{(k-1)L}\big].
\end{aligned}$$

In the second line, we used that $\{y \xleftrightarrow{C^c} \partial\Lambda_{kL}\}$, $\{0 \xleftrightarrow{S} x\} \cap \{\mathscr{C} = C\}$ and $\{\{x,y\} \text{ open}\}$ are independent (they depend on disjoint set of edges). In the third line, we used that since $y \in \Lambda_L$, one finds

$$\mathbb{P}_p[y \xleftrightarrow{C^c} \partial\Lambda_{kL}] \leqslant \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{(k-1)L}].$$

An induction on k gives

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \leqslant \varphi_p(S)^{k-1}.$$

This proves the desired exponential decay.

Remark 2.6. Note that we could have used the BK inequality, but that we only required a weaker form of this inequality, as suggested in the previous section.

Let us now turn to the proof of the second item. Let us start by the following lemma providing a differential inequality valid for every p. Set $\theta_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$.

Lemma 2.7. Let $p \in [0,1]$ and $n \geqslant 1$,

$$\theta_n'(p) \geqslant \frac{1}{p(1-p)} \cdot \inf_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S) \cdot \big(1-\theta_n(p)\big). \tag{2.5}$$

Let us first see how the theorem follows from Lemma 2.7. Integrating the differential inequality (2.5) between $\tilde{p}_c$ and $p > \tilde{p}_c$ implies that for every $n \geqslant 1$,

$$\theta_n(p) \geqslant \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}.$$

By letting n tend to infinity, we obtain the desired lower bound on $\mathbb{P}_p[0 \longleftrightarrow \infty]$. □

Proof (of Lemma 2.7). Recall that $\{x,y\}=e$ is pivotal for the configuration ω and the event $\{0 \longleftrightarrow \partial\Lambda_n\}$ if $\omega_{(e)} \notin \{0 \longleftrightarrow \partial\Lambda_n\}$ and $\omega^{(e)} \in \{0 \longleftrightarrow \partial\Lambda_n\}$. By Russo's formula, we have

$$\begin{aligned}\theta_n'(p) &= \sum_{e\in E(\Lambda_n)} \mathbb{P}_p\big[e \text{ is pivotal for } \{0 \longleftrightarrow \partial\Lambda_n\}\big]\\ &= \frac{1}{1-p} \sum_{e\in E(\Lambda_n)} \mathbb{P}_p\big[e \text{ is pivotal for } \{0 \longleftrightarrow \partial\Lambda_n\}, 0 \not\leftrightarrow \partial\Lambda_n\big].\end{aligned}$$

Define the following random subset of Λ_n:

$$\mathscr{S} := \{x \in \Lambda_n \text{ such that } x \not\leftrightarrow \partial\Lambda_n\}.$$

The boundary of $\mathscr{S}$ corresponds to the outmost blocking surface (which can be obtained by exploring – from the outside – the set of vertices connected to the boundary). When 0 is not connected to $\partial\Lambda_n$, the set $\mathscr{S}$ is always a subset of Λ_n containing the origin. By summing over the possible values for $\mathscr{S}$, we obtain

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \sum_{e\in E(\Lambda_n)} \mathbb{P}_p\big[e \text{ is pivotal for } \{0 \longleftrightarrow \partial\Lambda_n\}, \mathscr{S} = S\big]$$

Observe that on the event $\mathscr{S} = S$, the pivotal edges are the edges $\{x,y\} \in \Delta S$ such that 0 is connected to x in S. This implies that

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\big[0 \stackrel{S}{\longleftrightarrow} x, \mathscr{S} = S\big]. \tag{2.6}$$

The event $\{\mathscr{S} = S\}$ is measurable with respect to edges with (at least) one endpoint outside S and it is therefore independent of $\{0 \stackrel{S}{\longleftrightarrow} x\}$. We obtain

$$\begin{aligned}\theta_n'(p) &= \frac{1}{1-p} \sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\big[0 \stackrel{S}{\longleftrightarrow} x\big]\mathbb{P}_p\big[\mathscr{S} = S\big]\\ &= \frac{1}{p(1-p)} \sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_\beta(S)\mathbb{P}_p\big[\mathscr{S} = S\big] \qquad (2.7)\\ &\geqslant \frac{1}{p(1-p)} \inf_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S) \cdot \big(1-\theta_n(p)\big),\end{aligned}$$

as desired. □

Remark 2.8. The set of parameters p such that there exists a finite set $0 \in S \subset \mathbb{Z}^d$ with $\varphi_p(S) < 1$ is an open subset of $[0,1]$. Since this set is coinciding with $[0, p_c)$, we deduce that $\varphi_{p_c}(\Lambda_n) \geqslant 1$ for any $n \geqslant 1$). As a consequence, the expected size of the cluster of the origin satisfies at p_c,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}[0 \longleftrightarrow x] \geqslant \frac{1}{d} \sum_{n \geqslant 0} \varphi_{p_c}(\Lambda_n) = +\infty.$$

Also, since $\varphi_p(\{0\}) = 2d\,p$, we find $p_c(d) \geqslant 1/2d$.

3 Computation of the critical value on $\mathbb{Z}^2$

Let us focus on planar percolation for a moment. Introduce the dual graph $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. Any edge e of $\mathbb{Z}^2$ intersects a unique edge of $(\mathbb{Z}^2)^*$ that is denoted by e^*. Consider the dual measure $\omega^* \in \{0,1\}^{E((\mathbb{Z}^2)^*)}$ defined by the formula $\omega^*_{e^*} = 1 - \omega_e$. Note that if the law of ω is $\mathbb{P}_p$, then the law of ω^* is a translate by $(\frac{1}{2}, \frac{1}{2})$ of $\mathbb{P}_{1-p}$. This duality is a very specific feature of planar percolation. It enables us to deduce what is the critical point on $\mathbb{Z}^2$.

Theorem 2.9.
On $\mathbb{Z}^2$, $p_c = 1/2$ and $\mathbb{P}_{p_c}[0 \longleftrightarrow \infty] = 0$.

Remark 2.10. Together with the bound of Remark 2.8, we deduce that for any $d \geqslant 2$,

$$\tfrac{1}{2d} \leqslant p_c(d) \leqslant \tfrac{1}{2}.$$

Proof. Let us assume that $p_c > 1/2$. Since a circuit of length n surrounding 0 must intersect Λ_n, the exponential decay provided by Theorem 2.5 for any $p < p_c$ immediately implies that

$$\mathbb{P}_{1/2}[\text{there exists an open circuit surrounding } 0 \text{ of length } n] \leqslant e^{-cn}.$$

The Borel-Cantelli lemma implies that there exist finitely many circuits surrounding the origin. As a consequence, there exists an infinite cluster in the dual configuration ω^*. Since the dual measure is a percolation measure with parameter $1/2$, we deduce that $p_c \leqslant 1/2$ which is contradictory.

Let us now focus on the other bound. Historically, this result was first proved by Harris [Har 60]. We choose to present a beautiful argument due to Zhang [Gri 99, Lemma 11.12] which we isolate in the following lemma (which directly implies the theorem). □

Lemma 2.11 (Zhang's argument). On $\mathbb{Z}^2$, $\mathbb{P}_{1/2}[0 \longleftrightarrow \infty] = 0$.

Proof. Let $\varepsilon < 2^{-8}$. Assume that $\mathbb{P}_{1/2}[0 \longleftrightarrow \infty] > 0$ and choose n large enough that

$$\mathbb{P}_{1/2}[\Lambda_n \longleftrightarrow \infty] > 1 - \varepsilon.$$

The integer n exists since the infinite cluster exists almost surely (therefore the quantity on the left tends to 1 as n tends to infinity).

Let A_{left} (resp. A_{right}, A_{top} and A_{bottom}) be the events that $\{-n\}\times[-n,n]$ (resp. $\{n\}\times[-n,n]$, $[-n,n]\times\{n\}$ and $[-n,n]\times\{-n\}$) are connected to infinity in the complement of Λ_n. By symmetry,

$$\mathbb{P}_{1/2}[A_{\text{left}}\cup A_{\text{right}}]=\mathbb{P}_{1/2}[A_{\text{top}}\cup A_{\text{bottom}}]$$

and

$$\mathbb{P}_{1/2}[A_{\text{left}}]=\mathbb{P}_{1/2}[A_{\text{right}}].$$

We also find that

$$\mathbb{P}_{1/2}[A_{\text{left}}\cup A_{\text{right}}\cup A_{\text{top}}\cup A_{\text{bottom}}]=\mathbb{P}_{1/2}[\Lambda_n\longleftrightarrow\infty]>1-\varepsilon.$$

We can thus invoke the FKG inequality applied twice to obtain that

$$\mathbb{P}_{1/2}[A_{\text{left}}]\geqslant 1-\varepsilon^{1/4}.$$

As a consequence,

$$\mathbb{P}_{1/2}[A_{\text{left}}\cap A_{\text{right}}]\geqslant 1-2\varepsilon^{1/4}.$$

We now use that the dual measure of $\mathbb{P}_{1/2}$ is a translate of $\mathbb{P}_{1/2}$ itself. In particular, let A^*_{top} and A^*_{bottom} be the events that $[-(n+\frac12),n+\frac12]\times\{n+\frac12\}$ and $[-(n-\frac12),n+\frac12]\times\{-(n+\frac12)\}$ are dual-connected to infinity using edges outside $E^*(\Lambda_n)=\{e^*:e\in E(\Lambda_n)\}$. Following the same argument as for the primal model, we find that

$$\mathbb{P}_{1/2}[A^*_{\text{top}}\cap A^*_{\text{bottom}}]\geqslant 1-2\varepsilon^{1/4}.$$

Putting all these facts together, we obtain

$$\mathbb{P}_{1/2}[A_{\text{left}}\cap A_{\text{right}}\cap A^*_{\text{top}}\cap A^*_{\text{bottom}}]>1-4\varepsilon^{1/4}>0.$$

Now, Let B be the event that every dual-edge in $E^*(\Lambda_n)$ is open in ω^*. The events B and $A_{\text{left}}\cap A_{\text{right}}\cap A^*_{\text{top}}\cap A^*_{\text{bottom}}$ depend on disjoint sets of edges. Therefore,

$$\mathbb{P}_{1/2}[B\cap A_{\text{left}}\cap A_{\text{right}}\cap A^*_{\text{top}}\cap A^*_{\text{bottom}}]>0. \tag{2.8}$$

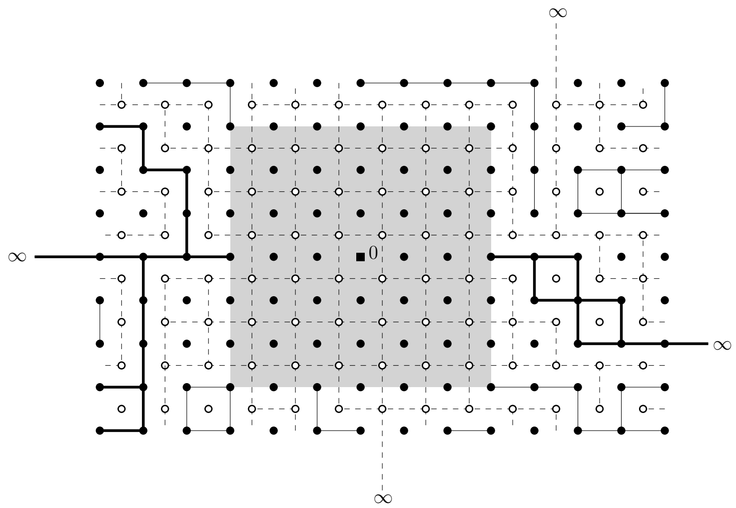
But this last event is contained in the event that there are two disjoint infinite clusters (see Fig. 2.1), which we excluded in Theorem 2.4, thus leading to a contradiction. □

Observe that we just proved that the phase transition is continuous since there is no infinite cluster at $p_c=1/2$. The situation in higher dimension is more complex. The absence of infinite cluster was proved using lace expansion for $d\geqslant 19$ [HS 90] (it was recently improved to $d\geqslant 11$ [FvdH 15]). The techniques involved in the proof may be expected to work until $d\geqslant 6$. This dimension, called the *upper critical dimension* d_c, is the smallest dimension for which the model exhibits a mean-field behavior (in particular $\mathbb{P}_p[0\longleftrightarrow\infty]\asymp(p-p_c)$ for $p\searrow p_c$). For $d\in\{3,4,5\}$, the techniques of [HS 90] will not work and in fact the following conjecture is one of the major open questions in our field.

Conjecture 2.12. For any $d\geqslant 2$, $\mathbb{P}_{p_c}[0\longleftrightarrow\infty]=0$.

Some partial results were obtained in $\mathbb{Z}^3$ in the past decades. For instance, it is known that the probability, at $p_c(3)$, of an infinite cluster in $\mathbb{N}\times\mathbb{Z}^2$ is zero [BGN 91]. Let us also mention that $\mathbb{P}_{p_c(\mathbb{Z}^2\times G)}[0\longleftrightarrow\infty]$ was proved to be equal to 0 on graphs of the form $\mathbb{Z}^2\times G$, where G is finite; see [DST 16].

FIGURE 2.1. In this configuration, two infinite clusters (in bold) coexist. The gray area corresponds to Λ_3.



4 The Russo-Seymour-Welsh theory

The previous section helps us understand what happens away from criticality. Let us now try to discuss the behavior *at* criticality in the planar case. The following result will be our main instrument to study the critical phase. For $a<b$ and $c<d$ integers, let $R:=[a,b]\times[c,d]$ (when a, b, c or d are not integers, an implicit rounding operation is performed), and introduce the events

$$\mathcal{C}_h(R):=\Big\{\{a\}\times[c,d]\overset{R}{\longleftrightarrow}\{b\}\times[c,d]\Big\},$$
$$\mathcal{C}_v(R):=\Big\{[a,b]\times\{c\}\overset{R}{\longleftrightarrow}[a,b]\times\{d\}\Big\}.$$

The events $\mathcal{C}_h^*(R^*)$ and $\mathcal{C}_v^*(R^*)$ are defined in terms of the dual configuration.

Theorem 2.13 (Box crossing property).
Let $\rho>0$, there exists $c=c(\rho)>0$ such that for every $n\geqslant 1$,

$$c\leqslant \mathbb{P}_{1/2}\big[\mathcal{C}_v([0,n]\times[0,\rho n])\big]\leqslant 1-c.$$

The uniform upper bound follows easily from the uniform lower bound and duality since the complement of the event that a rectangle is crossed vertically is the event that the dual rectangle is crossed horizontally in the dual configuration.

This observation also leads to the following

$$\mathbb{P}_{1/2}\big[\mathcal{C}_v([0,n]^2)\big]+\mathbb{P}_{1/2}\big[\mathcal{C}_h^*([-\tfrac12,n+\tfrac12]\times[\tfrac12,n-\tfrac12])\big]=1.$$

Since $[-\frac{1}{2}, n+\frac{1}{2}] \times [\frac{1}{2}, n-\frac{1}{2}]$ is a translate of $[0, n+1] \times [0, n-1]$, which is harder to cross horizontally than $[0, n]^2$, the self-duality implies that

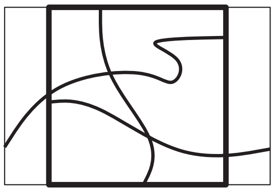
$$\mathbb{P}_{1/2}[\mathscr{C}_v([0,n]^2)] \geqslant \tfrac{1}{2}, \tag{2.9}$$

which leads to the uniform lower bound for any $\rho \leqslant 1$.

Also, as soon as we have to our disposal a uniform lower bound (in n) for some $\rho > 1$ on crossing horizontally rectangles of the form $[0, n] \times [0, \rho n]$, then one can easily combine crossings in different rectangles to obtain a uniform lower bound for any $\rho' > 1$; see Fig. 2.2. Note that combining crossings in squares is much harder. This will in fact be the major obstacle: the main difficulty of this theorem lies in passing from crossing squares with probabilities bounded uniformly from below to crossing rectangles in the hard direction with probabilities bounded uniformly from below. A statement claiming that crossing a rectangle in the hard direction can be expressed in terms of the probability of crossing squares is called a Russo-Seymour-Welsh type theorem. For Bernoulli percolation on the square lattice, such a result was first proved in [Rus 78, SW 78]. Since then, many proofs have been produced, among which [BR 06b, BR 06a, BR 10, Tas 16, Tas 14].

We present a recent proof [BR 06b], which is the shortest one (for the square lattice) we are aware of. We focus on the following proposition, which is sufficient to show the result thanks to the previous observations.

Figure 2.2. The combination of two horizontal crossings of ρn by n rectangles and one vertical crossing of a n by n square creates an horizontal crossing of a $(2\rho-1)n$ by n rectangle. The FKG inequality implies that this happens with probability larger than or equal to $\frac{1}{2}c(\rho)^2$.



Proposition 2.14.
For every $n \geqslant 1$,
$$\mathbb{P}_{1/2}[\mathscr{C}_v([-n,n] \times [-n,2n])] \geqslant \tfrac{1}{128}.$$

Proof. Let $A_n = \mathscr{C}_h([0,n]^2)$ and B_n be the event that there exists an horizontal crossing of $[0,n]^2$ which is connected to $[-n,n] \times \{-n\}$ in $[-n,n]^2$. For a path γ from left to right in $[0,n]^2$, and $\sigma(\gamma)$ the reflection of this path with respect to $\{0\} \times \mathbb{Z}$, define the set $S(\gamma)$ of vertices $x \in [-n,n]^2$ "below" $\gamma \cup \sigma(\gamma)$ (see Fig. 2.3 on the left). Now, on A_n, condition on the highest crossing Γ of $[0,n]^2$. We find that

$$\begin{aligned}
\mathbb{P}_{1/2}[B_n] &\geqslant \sum_{\gamma} \mathbb{P}_{1/2}[B_n \,|\, A_n \cap \{\Gamma = \gamma\}]\, \mathbb{P}_{1/2}[\{\Gamma = \gamma\} \cap A_n] \\
&\geqslant \sum_{\gamma} \mathbb{P}_{1/2}[\gamma \xleftrightarrow{S(\gamma)} [-n,n] \times \{-n\}]\, \mathbb{P}_{1/2}[\{\Gamma = \gamma\} \cap A_n] \\
&\geqslant \frac{1}{4} \sum_{\gamma} \mathbb{P}_{1/2}[\{\Gamma = \gamma\} \cap A_n] = \frac{1}{4}\mathbb{P}_{1/2}[A_n] \geqslant \frac{1}{8}.
\end{aligned}$$

In the third line, to deduce the lower bound $1/4$ we used the facts that conditioned on $A_n \cap \{\Gamma = \gamma\}$, the configuration in $S(\gamma)$ is a Bernoulli percolation of parameter $1/2$ (since $A_n \cap \{\Gamma = \gamma\}$ is measurable with respect to edges on γ or above γ), the symmetry and the fact that the probability of an open path from bottom to top in $S(\gamma)$ is larger than $1/2$ (by (2.9) applied to $[-n,n]^2$). In the last inequality, we used (2.9) applied to $[0,n]^2$.

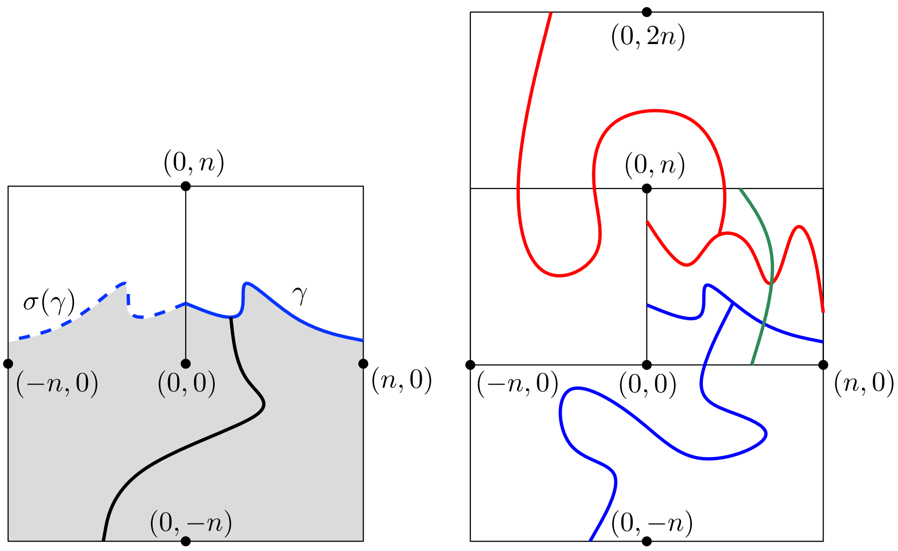
Fig. 2.3 on the right illustrates that $\mathscr{C}_v([-n,n]\times[-n,2n])$ occurs if the three events $\mathscr{C}_v([0,n]^2)$, B_n and $\widetilde{B}_n$ occur, where $\widetilde{B}_n$ is the event that there exists an horizontal crossing of $[0,n]^2$ which is connected to $[-n,n]\times\{2n\}$ in $[-n,n]\times[0,2n]$. By symmetry,

$$\mathbb{P}_{1/2}[\widetilde{B}_n] = \mathbb{P}_{1/2}[B_n] \geqslant \tfrac{1}{8}.$$

The FKG inequality (used in the second inequality) implies that

$$\begin{aligned}\mathbb{P}_{1/2}[\mathscr{C}_v([-n,n]\times[-n,2n])] &\geqslant \mathbb{P}_{1/2}[\mathscr{C}_v([0,n]^2)\cap B_n \cap \widetilde{B}_n]\\ &\geqslant \mathbb{P}_{1/2}[\mathscr{C}_v([0,n]^2)]\mathbb{P}_{1/2}[B_n]\mathbb{P}_{1/2}[\widetilde{B}_n]\\ &\geqslant \frac{1}{128}. \qquad \square\end{aligned}$$

Figure 2.3. **Left.** The set $S(\gamma)$ defined by the boundary of the box, the path γ and its reflection $\sigma(\gamma)$. **Right.** A realization of the event B_n in blue, $\widetilde{B}_n$ in red, and $\mathscr{C}_v([0,n]^2)$ in green.



The box crossing property has many applications, including polynomial bounds for arm events, so-called universal arm-exponents, tightness of interfaces, scaling relations and also absence of infinite connected component at criticality (a fact that we already know from Zhang's argument). We will discuss several of these applications in the more general context of Fortuin-Kasteleyn percolation in the next chapters and we therefore do not spend more time discussing them here.

Remark 2.15 (Conformal invariance).

In fact, crossing probabilities in $[0,n] \times [0,\rho n]$ converge to explicit limits as n tends to infinity. More generally, crossing probabilities in topological rectangles were proved to be conformally invariant by Smirnov in [Smi 01]. We refer to the reviews [BDC 13, Wer 09] for more references.

We now focus on a *dependent* percolation model, called the Fortuin-Kasteleyn percolation, which is directly related to the Potts model. We will see that the theory is more complicated. In particular, the results corresponding to the statements described in this chapter were derived rigorously only very recently.

CHAPTER 3

THE RANDOM-CLUSTER REPRESENTATION OF THE POTTS MODEL

The ***random-cluster model***, also called ***Fortuin-Kasteleyn percolation***, is a dependent percolation model on $\mathbb{Z}^d$ which is intimately related to the Potts model. This chapter presents the theory of this dependent percolation model, and its applications to the understanding of the Potts model. The organization follows the same lines as the previous section: we first define the model, then explain how to derive a few basic properties of the model. Finally, we study the phase transition of the model.

1 Definition of the random-cluster model

Let G be a finite subgraph of $\mathbb{Z}^d$. ***Boundary conditions*** ξ are given by a partition $P_1 \sqcup \cdots \sqcup P_k$ of ∂G, where recall that

$$\partial G := \{x \in G : \exists y \notin G, \{x,y\} \in E(\mathbb{Z}^d)\}.$$

Two vertices are *wired in* ξ if they belong to the same P_i. The graph obtained from the configuration ω by identifying the wired vertices together is denoted by ω^ξ [(1)]. Boundary conditions should be understood informally as encoding how sites are connected outside of G. Let $o(\omega)$ and $c(\omega)$ denote the number of open and closed edges of ω and $k(\omega^\xi)$ the number of (maximal) connected components of the graph ω^ξ.

Definition 3.1.

The probability measure $\phi^\xi_{p,q,G}$ of the random-cluster model on G with *edge-weight* $p \in [0,1]$, *cluster-weight* $q > 0$ and boundary conditions ξ is defined by

$$\phi^\xi_{p,q,G}[\{\omega\}] := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^\xi)}}{Z^\xi_{p,q,G}}$$

for every configuration $\omega \in \{0,1\}^{E(G)}$. The constant $Z^\xi_{p,q,G}$ is a normalizing constant, referred to as the *partition function*, defined in such a way that the sum over all configurations equals 1. From now on, $\phi^\xi_{p,q,G}$ denotes both the measure and the expectation with respect to this measure.

[(1)] Formally, ω^ξ can be seen as the graph $(\widetilde{V},\tilde{E})$, where $\widetilde{V}$ is the vertex set $V(G)$ quotiented by the equivalence relation $x\mathscr{R}y$ if x and y are in the same P_i, and $\tilde{E}$ is the image of the open edges of ω by the canonical projection from $V(G)$ to $\widetilde{V}$. We will not really use this formal definition.

Remark 3.2. For $q=1$, the random-cluster model is simply Bernoulli percolation.

Let us list a few boundary conditions that will come back later in the book.

Wired boundary conditions. They are specified by the fact that all the vertices on the boundary are pairwise wired (the partition is equal to $\{\partial G\}$). The random-cluster measure with wired boundary conditions on G is denoted by $\phi^1_{p,q,G}$.

Free boundary conditions. They are specified by no wiring between vertices on the boundary (the partition is composed of singletons only). The random-cluster measure with free boundary conditions on G is denoted by $\phi^0_{p,q,G}$.

Dobrushin boundary conditions. The interest of these boundary conditions will become apparent in the next chapter. Let (Ω,a,b) be a discrete simply connected domain Ω (*i.e.* that the graph Ω and its complement are connected) with two vertices a and b on its boundary $\partial\Omega$. We call such a triplet a Dobrushin domain. Since Ω is simply connected, $\partial\Omega$ is separated into two boundary arcs denoted by ∂_{ab} and ∂_{ba} (the first one goes from a to b when going counterclockwise around the boundary, while the second goes from b to a). The Dobrushin boundary conditions are defined to be free on ∂_{ab} and wired on ∂_{ba}. In other words, the partition is composed of ∂_{ba} together with singletons. We refer to Fig. 4.8 for an illustration. These arcs are referred to as the *free arc* and the *wired arc*, respectively. The measure associated to these boundary conditions will be denoted by $\phi^{a,b}_{p,q,\Omega}$.

Boundary conditions induced by a configuration outside G. For a configuration ξ on $E(\mathbb{Z}^2)\setminus E(G)$, the boundary conditions *induced* by ξ are defined by the partition $P_1\sqcup\cdots\sqcup P_k$, where x and y are in the same P_i if and only if there exists an open path in ξ connecting x and y. From now on, we identify the boundary conditions induced by ξ with the configuration itself, and we denote the random-cluster measure with these boundary conditions by $\phi^{\xi}_{p,q,G}$.

2 Coupling between the Potts model and the random-cluster model

Let us now describe the connection between the random-cluster model and the Potts model. Consider an integer $q\geqslant 2$ and let G be a finite graph. Assume that a configuration $\omega\in\{0,1\}^{E(G)}$ is given. One can deduce a spin configuration $\sigma\in\mathbb{T}_q^{E(G)}$ by assigning independently to each cluster $\mathscr{C}$ of ω a spin $\sigma_{\mathscr{C}}$ of $\mathbb{T}_q$ among the q possible spins, each with probability $1/q$, except for the clusters intersecting the boundary ∂G which are automatically associated to the spin 1. We then define σ_x to be equal to $\sigma_{\mathscr{C}}$ for every $x\in\mathscr{C}$. Note that all the sites in the same cluster receive the same spin.

Proposition 3.3 (Coupling Potts – Random-Cluster model).

Fix an integer $q\geqslant 2$. Consider $p\in(0,1)$ and $G\subset\mathbb{Z}^d$ a finite graph. If the configuration ω is distributed according to a random-cluster measure with parameters (p,q) and wired boundary conditions, then the spin configuration σ is distributed according to a q-state Potts measure with inverse temperature $\beta=-\frac{q-1}{q}\ln(1-p)$ and boundary conditions 1.

Proof. Consider a finite graph G, and let $p \in (0,1)$. Let Ω be the space of pairs (ω,σ) with $\omega \in \{0,1\}^{E(G)}$ and $\sigma \in \mathbb{T}_q^{V(G)}$, with the property that for any edge $e = \{x,y\}$, $\omega_e = 1$ implies $\sigma_x = \sigma_y$.

Consider a measure $\mathbf{P}$ on Ω, where ω is a random-cluster configuration with free boundary conditions and σ is the corresponding spin configuration constructed as explained above. Then, for (ω,σ), we have:

$$\begin{aligned}\mathbf{P}[(\omega,\sigma)] &= \frac{1}{Z^1_{p,q,G}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^1)} \cdot q^{-k(\omega^1)+1}\\ &= \frac{q}{Z^1_{p,q,G}} p^{o(\omega)}(1-p)^{c(\omega)}.\end{aligned}$$

(The additive constant 1 is due to the fact that the spin of the cluster of ω^1 touching the boundary is necessarily 1.)

Now, we construct another measure $\widetilde{\mathbf{P}}$ on Ω as follows. Let $\tilde\sigma$ be a spin-configuration distributed according to a q-state Potts model with boundary conditions 1 and inverse temperature β satisfying $\exp[-\frac{q}{q-1}\beta] = 1-p$. We deduce $\tilde\omega$ from $\tilde\sigma$ by independently opening edges between neighboring vertices with same spins with probability p. By definition, edges between vertices with different spins are automatically closed. Then, for any $(\tilde\omega,\tilde\sigma)$,

$$\widetilde{\mathbf{P}}[(\tilde\omega,\tilde\sigma)] = \frac{\exp[-\frac{q}{q-1}\beta r(\tilde\sigma)]\, p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)-r(\tilde\sigma)}}{Z} = \frac{p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}}{Z},$$

where $r(\tilde\sigma)$ is the number of edges between vertices with different spins, and Z is a normalizing constant.

This implies that $\mathbf{P} = \widetilde{\mathbf{P}}$ and the marginals of $\mathbf{P}$ are the random-cluster model with parameters (p,q) and the q-state Potts model at inverse-temperature β, which is the claim. □

Remark 3.4. One can also check that the coupling works for the random-cluster model and the q-state Potts model both with free boundary conditions. In this case, one assigns random spins to every cluster, even those touching the boundary (for future reference, let us define the corresponding coupling measure by $\widehat{\mathbf{P}}$).

Remark 3.5. The previous correspondence can be rephrased in terms of free energies of the models. Let us consider the Potts and random-cluster models on G with free boundary conditions. Still set $1-p = e^{-\beta q/(q-1)}$. Below, $\delta_{a,b}$ is the Kronecker symbol equal to 1 if $a=b$ and 0 otherwise. We have

$$\begin{aligned}Z_{\mathrm{Potts}}(G,\beta,q) :&= \sum_{\sigma\in\mathbb{T}_q^{V(G)}} \exp\left[-\beta H_G(\sigma)\right],\\ &= e^{\beta|E(G)|} \sum_{\sigma\in\mathbb{T}_q^{V(G)}} \prod_{\{x,y\}\in E(G)} \left(e^{-\beta q/(q-1)} + (1-e^{-\beta q/(q-1)})\delta_{\sigma_x,\sigma_y}\right)\\ &= e^{\beta|E(G)|} \sum_{\omega\in\{0,1\}^{E(G)}} p^{o(\omega)}(1-p)^{c(\omega)} \Big(\sum_{\sigma\in\mathbb{T}_q^{V(G)}} \prod_{\{x,y\}\in E(\omega)} \delta_{\sigma_x,\sigma_y}\Big)\\ &= e^{\beta|E(G)|} \sum_{\omega\in\{0,1\}^{E(G)}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)}\\ &=: e^{\beta|E(G)|}\cdot Z^0_{\mathrm{RCM}}(G,p,q).\end{aligned}$$

In the third line we used the fact that $\prod_{\{x,y\}\in E(\omega)} \delta_{\sigma_x,\sigma_y}$ equals 1 if σ is constant on every cluster of ω, and 0 otherwise. The previous relation leads to

$$f_{\text{Potts}}(\beta,q)+2\beta := \lim_{n\to\infty} -\frac{1}{|\Lambda_n|}\log e^{-\beta|E(\Lambda_n)|}Z_{\text{Potts}}(\Lambda_n,\beta,q) \tag{3.1}$$
$$= 2\lim_{n\to\infty} -\frac{1}{|E(\Lambda_n)|}\log Z^0_{\text{RCM}}(\Lambda_n,p,q) =: 2f_{\text{RCM}}(p,q).$$

In particular, the singular behaviors of the free energies $f_{\text{Potts}}(\beta,q)$ and $f_{\text{RCM}}(p,q)$ of both models are related and one can study the phase transition of one model by studying the phase transition of the other one. Most of the time though, we will use the coupling rather than just the relation between free energies since it provides us with more information on the typical geometry of configurations.

We are in presence of a graphical representation for the Potts model. The coupling provides us with a dictionary between the properties of the Potts and random-cluster models. For instance, one may easily check that

$$\begin{aligned}\mu^1_{G,\beta}[\langle\sigma_x|1\rangle] &= \mathbf{E}\Big[\langle\sigma_x|1\rangle\mathbf{1}_{\{x\xleftrightarrow{\omega}\partial G\}}\Big]+\mathbf{E}\Big[\langle\sigma_x|1\rangle\mathbf{1}_{\{x\xleftrightarrow{\omega}\partial G\}^c}\Big]\\ &= \phi^1_{p,q,G}[x\longleftrightarrow\partial G].\end{aligned}$$

We used that in the first term in the middle, σ_x must be equal to 1 and that in the second, σ_x is uniformly chosen among spins in $\mathbb{T}_q$. Similarly for $x,y\in V(G)$,

$$\begin{aligned}\mu^{\text{free}}_{G,\beta}[\langle\sigma_x|\sigma_y\rangle] &= \widehat{\mathbf{E}}\Big[\langle\sigma_x|\sigma_y\rangle\mathbf{1}_{\{x\xleftrightarrow{\omega}y\}}\Big]+\widehat{\mathbf{E}}\Big[\langle\sigma_x|\sigma_y\rangle\mathbf{1}_{\{x\xleftrightarrow{\omega}y\}^c}\Big]\\ &= \phi^0_{p,q,G}[x\longleftrightarrow y].\end{aligned}$$

As a side remark, note that $\mu^1_{G,\beta}[\langle\sigma_x|1\rangle]$ and $\mu^{\text{free}}_{G,\beta}[\langle\sigma_x|\sigma_y\rangle]$ are non-negative.

3 Basic properties of the random-cluster model

We wish to deduce properties corresponding to those described in Section 1. Note that some of these properties are only available for $q\geqslant 1$. In order not to slow down the pace of the lecture notes, we do not include the proofs of these statements. We refer to [Gri 06] or [DC 13] for a presentation of the different proofs (in [DC 13], the same notation is used).

FKG inequality. Fix $p\in[0,1]$, $q\geqslant 1$, a finite graph G and some boundary conditions ξ. For any two increasing events A and B,

$$\phi^\xi_{p,q,G}[A\cap B] \geqslant \phi^\xi_{p,q,G}[A]\phi^\xi_{p,q,G}[B].$$

The proof of this inequality is more complicated than for Bernoulli percolation. The argument relies on the so-called Holley criterion. We refer to [DC 13, Sections 4.4.1 and 4.4.2] for more details.

Monotonicity. Fix $p \leqslant p'$, $q \geqslant 1$, a finite graph G and some boundary conditions ξ. For any increasing event A,

$$\phi^{\xi}_{p,q,G}[A] \leqslant \phi^{\xi}_{p',q,G}[A].$$

This inequality follows directly from the FKG inequality (see [DC 13, Section 4.4.3] for more details) since

$$\phi^{\xi}_{p',q,G}[A] = \frac{\phi_{p,q,G}[Y\mathbf{1}_A]}{\phi_{p,q,G}[Y]} \overset{\text{(FKG)}}{\geqslant} \phi_{p,q,G}[A],$$

where

$$Y := \left(\frac{p'(1-p)}{p(1-p')}\right)^{o(\omega)}$$

is an increasing function of ω.

Comparison between boundary conditions. Fix $p \in [0,1]$, $q \geqslant 1$ and a finite graph G. For any boundary conditions $\xi \leqslant \psi$ (meaning that two vertices wired in ξ are wired in ψ) and any increasing event A,

$$\phi^{\xi}_{p,q,G}[A] \leqslant \phi^{\psi}_{p,q,G}[A].$$

This comparison between boundary conditions also follows from the FKG inequality (we leave it as an exercise, and refer to [DC 13, Section 4.4.4] for a complete proof). Note that the free and wired boundary conditions are extremal in the following sense: for any increasing event A and any boundary conditions ξ,

$$\phi^{0}_{p,q,G}[A] \leqslant \phi^{\xi}_{p,q,G}[A] \leqslant \phi^{1}_{p,q,G}[A].$$

Differential formula. Let A be an increasing event depending on edges in G only. Let $I_A(e)$ be the influence of the edge e defined by the formula

$$I_A(e) := \phi^{\xi}_{p,q,G}[A \,|\, \omega_e = 1] - \phi^{\xi}_{p,q,G}[A \,|\, \omega_e = 0].$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}[A] = \sum_{e \in E(G)} I_A(e). \tag{3.2}$$

This proof is easily obtained by differentiating (with respect to p) the ratio

$$\phi^{\xi}_{p,q,G}[A] = \frac{\displaystyle\sum_{\omega \in \{0,1\}^{E(G)}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^{\xi})} \mathbf{1}_{\omega \in A}}{\displaystyle\sum_{\omega \in \{0,1\}^{E(G)}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^{\xi})}}.$$

Domain Markov property and absence of the BK inequality. The BK inequality is not satisfied for the random-cluster model. The absence of such a tool renders the whole theory more intricate than Bernoulli percolation's one. The reason for the absence of the BK inequality can be nicely explained through a very important property of the random-cluster model, called the domain Markov property (this property is the analog of the Dobrushin-Lanford-Ruelle property for spin models; see [Geo 11]). Let $\mathscr{F}_E$ be the σ-algebra of events depending on the states of edges in E only. Fix $F \subset G$ two finite subgraphs of $\mathbb{Z}^2$. Let $p \in [0,1]$, $q > 0$ and ξ some boundary conditions on G. For any $\mathscr{F}_{E(F)}$-measurable random variable X,

$$\phi^{\xi}_{p,q,G}[X \mid \omega(e) = \psi(e), \forall e \in E(G) \setminus E(F)](\psi) = \phi^{\psi^\xi}_{p,q,F}[X],$$

where $\psi \in \{0,1\}^{E(G)\setminus E(F)}$. The proof is a straightforward computation.

A consequence of this property is that even though two events depend on disjoint sets of edges, the boundary condition induced by conditioning on one of the events impacts the measure on the set of edges on which the second event depends. Therefore, there are long-range dependencies. In particular, for $q > 1$, the probability that two edges are open will be strictly larger than the product of the probabilities so that the BK inequality is not satisfied.

Another consequence of this property is the following ***finite-energy*** property: for any $q \geqslant 1$, any finite graph G, any $\psi \in \{0,1\}^{E(G)}$ and any boundary conditions ξ on G,

$$\frac{p}{p+(1-p)q} \leqslant \phi^{\xi}_{p,q,G}\big[\omega(f) = 1 \mid \omega(e) = \psi(e), \forall e \in E(G) \setminus \{f\}\big] \leqslant p. \tag{3.3}$$

4 Phase transition and critical point

The definition of an **infinite-volume random-cluster measure** is not direct. Indeed, one cannot count the number of open or closed edges on $\mathbb{Z}^2$ since they could be (and would be) infinite. We thus define infinite-volume measures by taking a sequence of measures on larger and larger boxes Λ_n ($n \geqslant 1$).

Proposition 3.6.

Let $q \geqslant 1$. There exist two (possibly equal) infinite-volume random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$, called the infinite-volume random-cluster measures with free and wired boundary conditions respectively, such that for any event A depending on a finite number of edges,

$$\lim_{n\to\infty} \phi^1_{p,q,\Lambda_n}[A] = \phi^1_{p,q}[A] \quad \text{and} \quad \lim_{n\to\infty} \phi^0_{p,q,\Lambda_n}[A] = \phi^0_{p,q}[A].$$

Proof. We deal with the case of free boundary conditions. Wired boundary conditions are treated similarly. Fix an increasing event A depending on edges in Λ_N only. Applying the Markov property to $F = \Lambda_n$ and $G = \Lambda_{n+1}$, and the comparison between boundary conditions, we find that for any $n \geqslant N$,

$$\phi^0_{p,q,\Lambda_{n+1}}[A] \geqslant \phi^0_{p,q,\Lambda_n}[A].$$

We deduce that $(\phi^0_{p,q,\Lambda_n}[A])_{n\geqslant 0}$ is increasing, and therefore converges to a certain value $P[A]$ as n tends to infinity.

Since the ϕ^0_{p,q,Λ_n}-probability of an event B depending on finitely many edges can be written by inclusion-exclusion as a combination of the ϕ^0_{p,q,Λ_n}-probability of increasing events, taking the same combination defines a natural value $P[B]$ for which

$$\phi^0_{p,q,\Lambda_n}[B] \longrightarrow P[B].$$

The fact that $(\phi^0_{p,q,\Lambda_n})_{n\geqslant 0}$ are probability measures implies that the function P (which is a priori defined on the set of events depending on finitely many edges) can be extended into a probability measure on $\mathscr{F}_{E(\mathbb{Z}^d)}$. We denote this measure by $\phi^0_{p,q}$. □

The measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ play very specific roles in the theory. First, they are invariant under translations and ergodic (we leave this as an exercise). They are extremal infinite-volume measures, in the sense that any infinite-volume measure ϕ (see [DC 13, Definition 4.24]) for the random-cluster model with parameters p and $q \geqslant 1$ satisfies

$$\phi^0_{p,q}[A] \leqslant \phi[A] \leqslant \phi^1_{p,q}[A] \tag{3.4}$$

for increasing events A. Furthermore, an abstract theorem (see [DC 13, Theorem 4.30]) shows that for a fixed $q \geqslant 1$, $\phi^0_{p,q} = \phi^1_{p,q}$ for all but possibly countably many values of p. This immediately implies that for such values of p, there is a unique infinite-volume measure. We are now in a position to discuss the phase transition of the random-cluster model.

Theorem 3.7.

For $q \geqslant 1$ and $d \geqslant 1$, there exists a ***critical point*** $p_c = p_c(q,d) \in [0,1]$ such that:
- For $p < p_c$, any infinite-volume measure has no infinite cluster a.s.
- For $p > p_c$, any infinite-volume measure has an infinite cluster a.s.

Proof. Let A be the event that there is an infinite cluster. Let us define

$$p_c = \inf\{p \in [0,1] : \phi^0_{p,q}[A] > 0\}.$$

Since the event A is increasing, we deduce that $\phi^0_{p,q}[A] > 0$ for any $p > p_c$. Ergodicity implies that

$$\phi^0_{p,q}[A] = 1.$$

Furthermore, (3.4) implies the result for any infinite-volume measure with parameters p and q.

On the other hand, let $p < p_c$. There exists $p' \in (p, p_c)$ such that there is a unique infinite-volume measure at p' (since the set of values of p for which there are more than one infinite-volume measure is at most countable, see [DC 13, Theorem 4.30] again). We deduce that for any infinite-volume measure $\phi_{p,q}$,

$$\phi_{p,q}[A] \leqslant \phi^1_{p,q}[A] \leqslant \phi^1_{p',q}[A] = \phi^0_{p',q}[A] = 0$$

by uniqueness of the measure at $p' < p_c$. □

The previous theorem can be rephrased in terms of the q-state Potts model. In such case, the quantity

$$\beta_c = \beta_c(q,d) := -\frac{q-1}{q} \log\left[1 - p_c(q,d)\right]$$

satisfies

$$\lim_{n\to\infty}\mu^1_{\Lambda_n,\beta}[\langle\sigma_0|1\rangle]=\begin{cases}0 & \text{if } \beta<\beta_c\\ m^*(\beta,q,d)>0 & \text{if } \beta>\beta_c.\end{cases}$$

In other words, the critical inverse-temperature of the Potts model can be defined rigorously (it does not alternate between ordered and disordered phases as it a priori could), and this critical inverse-temperature can be expressed in terms of the critical value of the corresponding random-cluster model.

5 Computation of the critical point in two dimensions

The derivation of the critical point is the next natural step in the study. Unfortunately, we do not expect the value to be special for general values of $d\geqslant 3$. In dimension two, a duality relation enables us to compute the value explicitly. Let us describe non-rigorously this derivation.

Theorem 3.8 (Beffara, Duminil-Copin [BDC 12b]).

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q\geqslant 1$. The critical value $p_c=p_c(q,2)$ satisfies

$$p_c=\frac{\sqrt{q}}{1+\sqrt{q}}.$$

Furthermore, for $p<p_c$, there exists $c=c(q)>0$ such that

$$\phi^1_{p,q,\Lambda_n}[0\longleftrightarrow\partial\Lambda_n]\leqslant\exp[-cn].$$

This result was previously known for $q=1$ [Kes 80], $q=2$ [Ons 44] and $q\geqslant 4$ [HKW 78]. Note that in general (with the exception of $q=2$ on planar graphs, see [Li 12, CDC 13]), the critical point is not expected to satisfy a special equation, even for planar models.

Let us discuss why $\sqrt{q}/(1+\sqrt{q})$ is a very special edge-parameter for the model. Recall that for any $e\in E(\mathbb{Z}^2)$, there exists a unique edge e^* of $E((\mathbb{Z}^2)^*)$ intersecting it in its middle. Let G^* be the graph with edge-set $E(G^*)=\{e^*: e\in E(G)\}$ and vertex-set given by the end-points of $E(G^*)$.

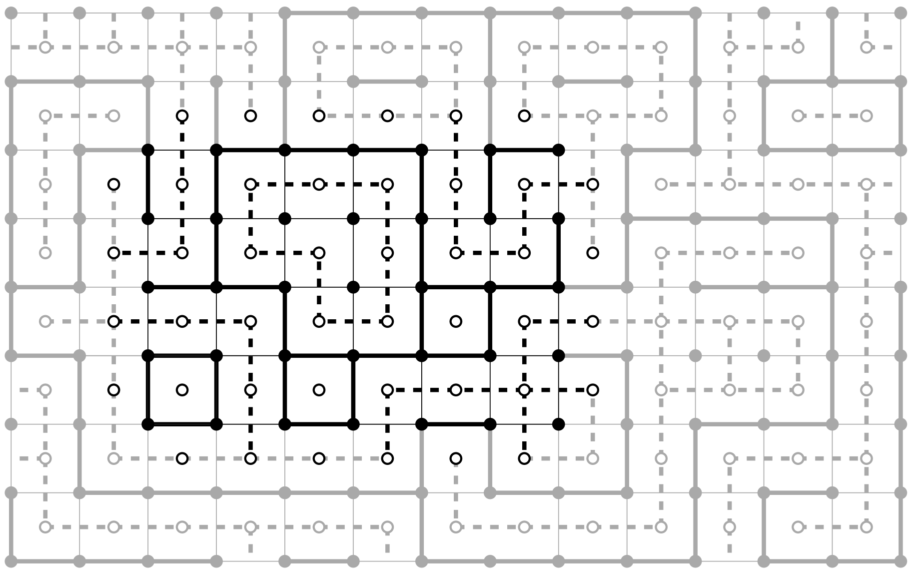
Definition 3.9.

For a configuration $\omega\in\{0,1\}^{E(G)}$, we define the ***dual configuration*** $\omega^*\in\{0,1\}^{E(G^*)}$ by the formula

$$\omega^*(e^*)=1-\omega(e),\quad \forall e\in E(G).$$

A dual-edge e^* is said to be ***dual-open*** if $\omega^*(e^*)=1$ and ***dual-closed*** otherwise. Two sites u and v in G^* are said to be ***dual-connected*** if there is a ***dual-open path***, *i.e.* a path of open dual-edges between u and v.

Figure 3.1. A configuration and its dual configuration. The graphs G and G^* are in black and the boundary conditions ξ and ξ^* are in gray.



Set $p^* = p^*(p,q)$ satisfying

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Proposition 3.10 (Duality for planar boundary conditions).

Let $\xi \in \{0,1\}^{E(\mathbb{Z}^2)\setminus E(G)}$. The dual model of the random-cluster on G with parameters (p,q) and boundary conditions ξ is the random-cluster with parameters (p^*,q) on G^* with boundary conditions induced by ξ^*, where ξ^* is defined by $\xi^*(e^*) = 1-\xi(e)$ for any $e \notin E(G)$.

Proof. The probability of ω^* is $\phi^{\xi}_{p,q,G}[\{\omega\}]$, which is written in terms of $o(\omega), c(\omega)$ and $k(\omega^\xi)$. Simply write these quantities in terms of $o(\omega^*)(=c(\omega))$, $c(\omega^*)(=o(\omega))$ and the number of faces in $(\omega^*)^{\xi^*}$. The proof then follows directly from Euler's formula for planar graphs which enables us to rewrite the number of faces in $(\omega^*)^{\xi^*}$ in terms of $o(\omega^*)$ and $k((\omega^*)^{\xi^*})$. □

We are now in a position to discuss Theorem 3.8. First, fix $q \geqslant 1$. Introduce

$$p_{\mathrm{sd}}(q) := \frac{\sqrt{q}}{1+\sqrt{q}}.$$

This is the unique value of p satisfying $p^*(p,q) = p$.

Second, assume that $p_c < p_{\mathrm{sd}}(q)$, in such case there exists an infinite connected component of open edges at $p_{\mathrm{sd}}(q)$. But the dual model of $\phi^0_{p_{\mathrm{sd}}(q),q}$ is a translate by $(\frac{1}{2},\frac{1}{2})$ of $\phi^1_{p_{\mathrm{sd}}(q),q}$, and therefore there exists an infinite connected component of dual-open edges on $(\mathbb{Z}^2)^*$ as well. These two facts enable us to rerun Zhang's argument (Lemma 2.11) to show that with positive probability, there are more than two infinite connected components for $\phi^0_{p_{\mathrm{sd}}(q),q}$ (the only difference is that one used independence in (2.8), which get replaced by the finite energy property (3.3) in this proof). We reach a contradiction by observing that uniqueness of the infinite cluster (*i.e.* Proposition 2.4) can also be proved for the random-cluster model, since independence was used only when proving that $\mathbb{P}_p[\mathrm{CT}_0] > 0$, and can be replaced by the finite energy property again. Overall, we just proved that $p_c \geqslant p_{\mathrm{sd}}(q)$.

Third, assume that $p_c > p_{\mathrm{sd}}(q)$. If one could prove exponential decay for $p < p_c$, one would reach a contradiction by following the same argument (at $p_{\mathrm{sd}}(q)$ this time) as in Theorem 2.9. It is indeed possible to prove such a result on $\mathbb{Z}^2$ [BDC 12b, BDC 12a] or on planar lattices with sufficient symmetries [DCM 14] but this represents the major difficulty in the proof (which we omit here).

We immediately deduce that the critical inverse-temperature $\beta_c(q,2)$ of the Potts model on $\mathbb{Z}^2$ satisfies

$$\beta_c(q,2) = \frac{q-1}{q}\log(1+\sqrt{q}).$$

In the literature, one often sees that the critical temperature is equal to $\log(1+\sqrt{q})$. This is due to the fact that the Hamiltonian is usually defined by the formula

$$H_{G,\beta}(\sigma) = \sum_{\{x,y\}\in E(G)} \delta_{\sigma_x,\sigma_y},$$

where $\delta_{a,b}$ is the Kronecker symbol equal to 1 if $a = b$ and 0 otherwise. This model is exactly similar to the other one, except that β is multiplied by $\frac{q}{q-1}$.

6 Russo-Seymour-Welsh theory for the critical random-cluster model on $\mathbb{Z}^2$

We now focus on the critical phase. As for Bernoulli percolation, we need a tool to study the geometry of the critical phase. This tool is provided by the following result, whose proof is omitted here (see [DCST 15] for details). Recall that $\mathcal{C}_h(R)$ is the event that there exists an horizontal crossing of the rectangle R from left to right.

Theorem 3.11 (Duminil-Copin, Sidoravicius, Tassion [DCST 15]).
Let $q \geqslant 1$, the following assertions are equivalent :

P1 (Absence of infinite cluster at criticality) $\phi^1_{p_c,q}[0 \longleftrightarrow \infty] = 0$.

P2 (Uniqueness of infinite-volume measure) $\phi^0_{p_c,q} = \phi^1_{p_c,q}$.

P3 (Infinite susceptibility) $\chi^0(p_c,q) := \sum_{x\in\mathbb{Z}^2} \phi^0_{p_c,q}[0 \longleftrightarrow x] = \infty$.

P4 (Sub-exponential decay of correlations for free boundary conditions)

$$\lim_{n\to\infty} \tfrac{1}{n}\log \phi^0_{p_c,q}[0 \longleftrightarrow \partial\Lambda_n] = 0.$$

P5 (RSW) Let $\alpha > 0$. There exists $c = c(\alpha) > 0$ such that for any $n \geqslant 1$ and any boundary conditions ξ,

$$c \leqslant \phi^\xi_{p_c,q,[-n,(\alpha+1)n]\times[-n,2n]}\big[\mathcal{C}_h([0,\alpha n]\times[0,n])\big] \leqslant 1-c.$$

The previous theorem does not show that these conditions are all satisfied, only that they are equivalent. In fact, whether the conditions are satisfied or not depend on the value of q, see Sections 2 and 3 for a more detailed discussion.

The previous result was previously known in a few cases. For $q = 1$ (Bernoulli percolation), **P1–5** follow from Lemma 2.11, Remark 2.8 and Theorem 2.13 (**P2** is trivial). For $q \gg 1$, none of the above properties are satisfied (see [KS 82, LMMS+ 91] and Theorem 4.5 below). For $q = 2$ (which is coupled to the Ising model), all of these properties can be proved to be true using the following results on the Ising model: Onsager proved that the critical Ising measure is unique and that the phase transition is continuous in [Ons 44], thus implying **P1–2**; Properties **P3–4** follow from [Sim 80] (see also Remark 5.15), and **P5** was proved in [DCHN 11].

Property **P5** is the strongest one. Note that we did not only require that crossing probabilities remain bounded away from 0 uniformly in the size n of the rectangle, but also uniformly in boundary conditions. This property is crucial for the potential applications of this theorem. The restriction on boundary conditions being at distance n from the rectangle can be relaxed in the following way: if **P5** holds, then for any $\alpha > 0$ and $\varepsilon > 0$, there exists $c = c(\alpha, \varepsilon) > 0$ such that for every $n \geqslant 1$,

$$c \leqslant \phi^{\xi}_{p_c,q,[-\varepsilon n,(\alpha+\varepsilon)n]\times[-\varepsilon n,(1+\varepsilon)n]}\left[\mathscr{C}_h([0,\alpha n]\times[0,n])\right] \leqslant 1-c.$$

(We leave it as an exercise for the reader.) It is natural to ask why boundary conditions are fixed at distance εn of the rectangle $[0,\alpha n]\times[0,n]$ and not simply on its boundary. It may in fact be the case that **P5** holds but that the crossing probability of the rectangle $[0,\alpha n]\times[0,n]$ with free boundary conditions on its boundary converge to zero as n tends to infinity. Such a phenomenon does not occur for $1 \leqslant q < 4$ as shown in [DCST 15] (for such values of q, the crossing probabilities on rectangles with free boundary conditions directly on the boundary are bounded away from 0 uniformly in n) but is expected to occur for $q = 4$. To avoid this difficulty, we will always work with boundary conditions at "macroscopic distance" from the boundary.

Let us now mention a few applications of this theorem.

Lemma 3.12. Property **P5** implies the existence of $c' > 0$ such that for all $n \geqslant 2$,

$$\phi^0_{p_c,q,\Lambda_{2n}\setminus\Lambda_n}[\text{there exists an open circuit in } \Lambda_{2n}\setminus\Lambda_n \text{ surrounding } \Lambda_n] \geqslant c'.$$

Proof. Consider the four rectangles

$$\begin{aligned}
R_1 &:= [4n/3, 5n/3]\times[-5n/3, 5n/3],\\
R_2 &:= [-5n/3, -4n/3]\times[-5n/3, 5n/3],\\
R_3 &:= [-5n/3, 5n/3]\times[4n/3, 5n/3],\\
R_4 &:= [-5n/3, 5n/3]\times[-5n/3, -4n/3].
\end{aligned}$$

If the intersection of $\mathscr{C}_v(R_1)$, $\mathscr{C}_v(R_2)$, $\mathscr{C}_h(R_3)$ and $\mathscr{C}_h(R_4)$ occurs, then there exists an open circuit in $\Lambda_{2n}\setminus\Lambda_n$ surrounding Λ_n. In particular, the FKG inequality and the comparison between boundary conditions imply that c' can be chosen to be equal to $c(10)^4$. □

Corollary 3.13. If **P5** is satisfied, there exists $\varepsilon > 0$ such that for any $n \geqslant 1$,

$$\frac{\varepsilon}{n} \leqslant \phi^1_{p_c,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leqslant \frac{1}{n^{\varepsilon}}.$$

Proof. The lower bound is straightforward: if $[0,n]^2$ is crossed horizontally, there exists a vertex in $\{n\}\times[0,n]$ connected to distance n. The union bound implies the lower bound directly with $\varepsilon \leqslant c(1)$.

For the upper bound, let k be such that $2^k \leqslant n < 2^{k+1}$. Also define the annuli $A_j = \Lambda_{2^j} \setminus \Lambda_{2^{j-1}-1}$ for $j \geqslant 1$. We have

$$\begin{aligned}\phi^1_{p_c,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] &\leqslant \prod_{j=1}^{k} \phi^1_{p_c,q,\Lambda_n}\Big[\partial\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j} \Big| \bigcap_{i>j}\{\partial\Lambda_{2^{i-1}} \overset{A_i}{\longleftrightarrow} \partial\Lambda_{2^i}\}\Big] \\ &\leqslant \prod_{j=1}^{k} \phi^1_{p_c,q,\Lambda_n}[\partial\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j}].\end{aligned}$$

In the second line, we used the fact that the event upon which we condition depends only on edges outside of Λ_{2^j} together with the domain Markov property and the comparison between boundary conditions.

Now, the complement of $\{\partial\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j}\}$ is the event that there exists a dual-open circuit in A_j^* surrounding the origin. Lemma 3.12 implies that this dual-open circuit exists with probability larger than or equal to $c>0$ independently of $n \geqslant 1$. This leads to

$$\phi^1_{p_c,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leqslant \prod_{j=1}^{k}(1-c) = (1-c)^k \leqslant (1-c)^{\log n/\log 2}.$$

The proof follows by setting $\varepsilon \leqslant -\frac{\log(1-c)}{\log 2}$. □

Remark 3.14. The proof illustrates the need for *bounds which are uniform with respect to boundary conditions*. Indeed, it could be the case that the $\phi^1_{p_c,q}$-probability of an open path from the inner to the outer sides of A_j is bounded away from 1, but conditioning on the existence of paths in each annulus A_i (for $i<j$) could favor open edges drastically, and imply that the probability of the event under consideration is close to 1.

Let us now show how to prove Theorem 3.8 from Theorem 3.11. Historically, the proof of Theorem 3.8 was independent of Theorem 3.11. Nevertheless, there is a neat way of getting Theorem 3.8 from Theorem 3.11 and we therefore present it now. We also refer to Section 3 for a simple proof when $q \in [1,3]$.

Proof (of Theorem 3.8 (sketch)). We already know that $p_c \geqslant p_{sd}$ by Zhang's argument (we omit the proof since it is a trivial adaptation of the proof presented for Bernoulli percolation). We focus on $p_c \leqslant p_{sd}$ and leave the proof of exponential decay aside. Assume that $p_c > p_{sd}$.

Let $\mathscr{A}_n$ be the event that $\Lambda_{2n}\setminus\Lambda_n$ contains an open circuit surrounding Λ_n. Our goal is to show that for $p_c > p > p_{sd}$ and $\varepsilon>0$, there exists $n = n(p,\varepsilon)>0$ such that

$$\phi^0_{p,q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n] \geqslant 1-\varepsilon. \tag{3.5}$$

Indeed, if such an n exists, one may cover the plane with boxes of the form $\tau_{nx}\Lambda_n$ with $x=(x_1,x_2)\in\mathbb{Z}^2$, and declare a box *good* if $\tau_{nx}(\mathscr{A}_n)$ occurs, and *bad* otherwise. The previous assumption on the probability of $\mathscr{A}_n$ implies that the process of good boxes dominates a Bernoulli site percolation on $n\mathbb{Z}^2$ with parameter $p=p(\varepsilon)$ (indeed one may prove that it dominates a 3-dependent percolation, and then use a classical result about domination by product measures [LSS 97]). One can now choose ε small enough that

p is larger than the critical value of site percolation on $\mathbb{Z}^2$. We deduce that there is an infinite path of good boxes, which immediately implies the existence of an infinite path of open edges. Hence, $p \geqslant p_c$, which is in contradiction with the original assumption $p_c > p_{\mathrm{sd}}$.

We therefore need to prove (3.5). We need two results on the random-cluster model. First, let us recall a general theorem which is a direct adaptation of the more general statement of Graham and Grimmett [GG 11, Thm. 5.3].

Theorem 3.15 ([GG 11]).

For any $q \geqslant 1$ there exists a constant $c > 0$ such that, for any $p \in (0,1)$, any finite graph G, any boundary conditions ξ and any increasing event A,

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}[A] \geqslant c\,\phi^{\xi}_{p,q,G}[A]\big(1-\phi^{\xi}_{p,q,G}[A]\big)\log\left(\frac{1}{\max\{I_A(e) : e \in E(G)\}}\right). \tag{3.6}$$

Second, an easy coupling argument which can be found in [GG 11] or [DCM 14] yields that

$$I_{\mathscr{A}_n}(e) \leqslant \phi^1_{p,q}[0 \longleftrightarrow \partial\Lambda_n].$$

If $p_{\mathrm{sd}} < p < p_c$, one can choose n so that $\phi^1_{p,q}[0 \longleftrightarrow \partial\Lambda_n] \leqslant \delta$. Since for any $p' \leqslant p$,

$$\phi^1_{p',q}[0 \longleftrightarrow \partial\Lambda_n] \leqslant \phi^1_{p,q}[0 \longleftrightarrow \partial\Lambda_n] \leqslant \delta,$$

we may integrate (3.6) between p_{sd} and p to obtain

$$\frac{\phi^0_{p,q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n]\big(1-\phi^0_{p_{\mathrm{sd}},q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n]\big)}{\phi^0_{p_{\mathrm{sd}},q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n]\big(1-\phi^0_{p,q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n]\big)} \geqslant \delta^{-c(p-p_{\mathrm{sd}})}$$

which implies that

$$\phi^0_{p,q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n] \geqslant 1 - \frac{\delta^{c(p-p_{\mathrm{sd}})}}{\phi^0_{p_{\mathrm{sd}},q,\Lambda_{2n}\setminus\Lambda_n}[\mathscr{A}_n]}.$$

The proof follows by using Lemma 3.12 and by choosing $\delta = \delta(p,\varepsilon)$ small enough. □

As mentioned before, we deduce that $\beta_c(q) = \frac{q-1}{q}\log(1+\sqrt{q})$. We also deduce that for every $n \geqslant 1$,

$$\frac{\varepsilon}{n} \leqslant \mu^1_{\Lambda_n,\beta_c}[\langle\sigma_0|1\rangle] \leqslant \frac{1}{n^{\varepsilon}}.$$

Chapter 4

Loop representation of the random-cluster model

We now introduce another representation of the Potts model. It is derived directly from the planar random-cluster model. It is not quite a percolation model, but rather a loop model. In this chapter, we restrict our attention to $d = 2$.

1 Loop representation for free boundary conditions

We start by defining the loop configuration $\overline{\omega}$ associated to a percolation configuration ω. Let $(\mathbb{Z}^2)^\diamond$ be the lattice with midpoints of edges of $\mathbb{Z}^2$ as vertices, and edges between nearest neighbors. It is a rotated and rescaled version of the square lattice, see Fig. 1.1.

Let Ω be a finite subgraph of $\mathbb{Z}^2$. Let Ω^* be the subgraph of $(\mathbb{Z}^2)^*$ induced by dual edges bordering faces corresponding to vertices of Ω (it is not quite the same as the notion of dual graph introduced in Section 5). Let $\Omega^\diamond$ be the subgraph of $(\mathbb{Z}^2)^\diamond$ defined by the vertices corresponding to midpoints of edges in $E(\Omega)$, and edges between two vertices of $V(\Omega^\diamond)$ on the same face of Ω.

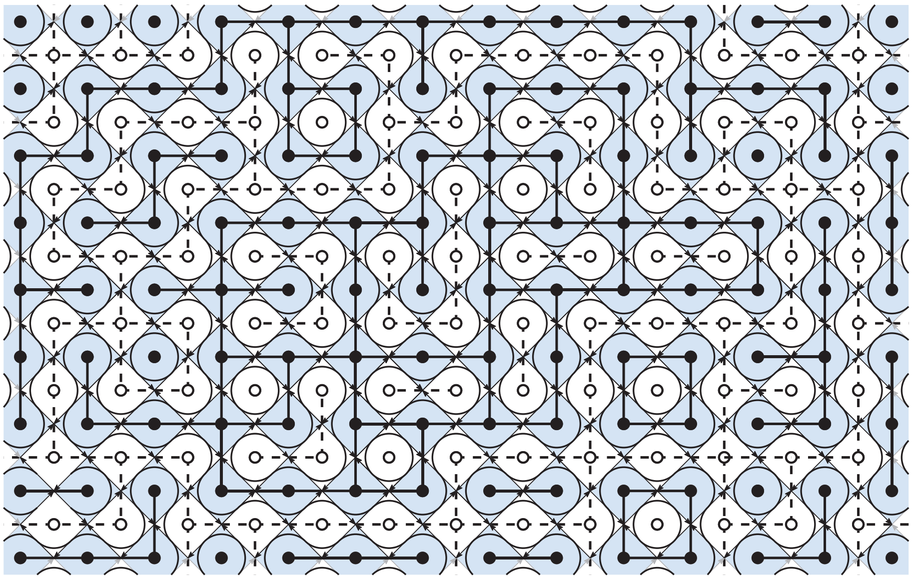
Consider a configuration ω together with its dual configuration ω^*. We draw ω^* in such a way that dual edges between vertices of $\partial\Omega^*$ are open in ω^* (we make such an arbitrary choice since these dual edges have no corresponding edges in $E(\Omega)$).

By definition, through every vertex of the **medial graph** $\Omega^\diamond$ of Ω passes either an open edge of Ω or a dual-open edge of Ω^*. Draw self-avoiding loops on $\Omega^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edge through this vertex, see Fig. 4.1. In the future, the loop configuration associated to ω is denoted by $\overline{\omega}$.

We allow ourselves a slight abuse of notation: below, $\phi^0_{p,q,\Omega}$ denotes the measure on percolation configurations as well as the measure on loop configurations. Fix

$$x = x(p,q) := \frac{p}{\sqrt{q}(1-p)}.$$

FIGURE 4.1. The configurations ω (in bold lines), ω^* (in dashed lines) and $\overline{\omega}$ (in plain lines).



Proposition 4.1.

Let Ω be a connected finite subgraph of $\mathbb{Z}^2$ which complement is connected. Let $p \in (0,1)$ and $q > 0$. For any configuration ω,

$$\phi^0_{p,q,\Omega}[\overline{\omega}] = \frac{x^{o(\omega)}\sqrt{q}^{\ell(\overline{\omega})}}{Z(\Omega,p,q)},$$

where $\ell(\overline{\omega})$ is the number of loops in $\overline{\omega}$ and $Z(\Omega,p,q)$ is a normalizing constant.

Proof. Note that

$$\begin{aligned} p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)} &= (1-p)^{|E(\Omega)|}\sqrt{q}^{|V(\Omega)|}\left(\frac{p}{(1-p)\sqrt{q}}\right)^{o(\omega)}\sqrt{q}^{2k(\omega)+o(\omega)-|V(\Omega)|} \\ &= (1-p)^{|E(\Omega)|}\sqrt{q}^{|V(\Omega)|}x^{o(\omega)}\sqrt{q}^{\ell(\overline{\omega})}. \end{aligned}$$

We used an induction on the number of open edges in order to show that

$$\ell(\overline{\omega}) = 2k(\omega) + o(\omega) - |V(\Omega)|$$

in the second line. The proof follows readily. □

Remark 4.2. In particular, when $p = p_c(q)$, we obtain that $x = 1$ and the probability of a loop configuration is expressed in terms of the number of loops only.

The loop representation of the random-cluster model is a well-known representation. It allows to map the free energies of the Potts and the random-cluster models to the free energy of a solid-on-solid ice-type model. For completeness, we succinctly present the mapping here and we refer to [Bax 89, Chapter 10] for more details on the subject. We focus on the case of a domain Ω with free boundary conditions.

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Consider a configuration $\vec{\omega}$ of arrows on edges of $\Omega^\diamond$ with the constraint that for every vertex of $\Omega^\diamond$, the number of incoming arrows is equal to the number of outgoing arrows. The set of possible configurations is denoted by $\mathrm{A}(\Omega)$. The probability of $\vec{\omega}$ is given by

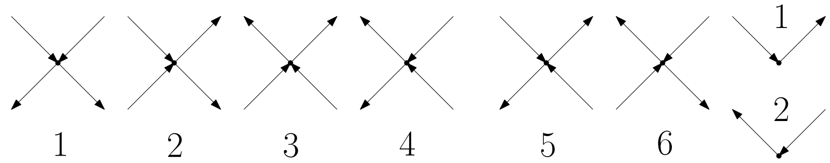
$$\mathrm{P}[\vec{\omega}] = \frac{x_1^{N_1(\vec{\omega})} x_2^{N_2(\vec{\omega})} x_3^{N_3(\vec{\omega})} x_4^{N_4(\vec{\omega})} x_5^{N_5(\vec{\omega})} x_6^{N_6(\vec{\omega})} y_1^{\widetilde{N}_1(\vec{\omega})} y_2^{\widetilde{N}_2(\vec{\omega})}}{Z_{6V}(x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2)},$$

where $x_1,\dots,x_6,y_1,y_2>0$, and $N_1,\dots,N_6,\widetilde{N}_1,\widetilde{N}_2$ correspond to the number of patterns of types 1–6 and boundary patterns of types 1–2; see Fig. 4.2. This model is called the ***Six Vertex model***. Let

$$f_{6\mathrm{V}}(x_1,\dots,x_6) := \lim_{n\to\infty} \frac{1}{|E(\Lambda_n)|} \log Z_{6\mathrm{V}}(\Lambda_n,x_1,\dots,x_6,y_1,y_2).$$

We do not recall the dependency in y_1 and y_2 since it can easily be seen to disappear in the limit $n\to\infty$.

FIGURE 4.2. Different possible patterns for the six vertex model on the medial lattice. The two patterns on the right are boundary patterns.



Proposition 4.3 (Baxter [Bax 89]).

Let G be a finite graph, we have

$$f_{\mathrm{Potts}}(\beta_c,q) = \frac{2\beta_c}{q-1} - \log q + 2 f_{6\mathrm{V}}(1,1,1,1,2\cos(\pi\tilde{\sigma}),2\cos(\pi\tilde{\sigma}))$$

where $\cos(2\pi\tilde{\sigma}) = \sqrt{q}/2$.

Proof. Introduce

$$Z_{\mathrm{RCM}}(\Lambda_n,p,q) := \sum_{\omega\in\{0,1\}^{E(\Lambda_n)}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)},$$

$$Z_{\mathrm{loop}}(\Lambda_n,q) := \sum_{\overline{\omega}\in \mathrm{L}(\Lambda_n)} \sqrt{q}^{\,\ell(\overline{\omega})},$$

$$Z_{6\mathrm{V}}(\Lambda_n,x_1,\dots,x_6,y_1,y_2) := \sum_{\vec{\omega}\in \mathrm{A}(\Lambda_n)} x_1^{N_1(\vec{\omega})} \cdots x_6^{N_6(\vec{\omega})} y_1^{\widetilde{N}_1(\vec{\omega})} y_2^{\widetilde{N}_2(\vec{\omega})}.$$

The correspondence given by Proposition 4.1 leads to

$$Z_{\mathrm{RCM}}(\Omega,p_c(q),q) = (1-p_c)^{|E(\Omega)|} \sqrt{q}^{\,|V(\Omega)|} Z_{\mathrm{loop}}(\Omega,q). \tag{4.1}$$

Each loop of the loop configuration $\overline{\omega}$ can be oriented in two possible orientations.

Since the number of right turns minus the number of left turns of a loop is equal to ±4 depending on its orientation, one can use that $\sqrt{q} = e^{i\tilde{\sigma}2\pi} + e^{-i\tilde{\sigma}2\pi}$ to show that

$$\sqrt{q}^{\ell(\overline{\omega})} = \sum_{\vec{\omega} \text{ corresponding to } \overline{\omega}} \exp\left[i\tilde{\sigma}\tfrac{\pi}{2}(\#\text{ left turns in } \vec{\omega} - \#\text{ right turns in } \vec{\omega})\right].$$

Above, $\vec{\omega}$ *corresponding to* $\overline{\omega}$ means that the former can be obtained from the latter by putting and orientation and forgetting the way loops turn. We deduce that

$$Z_{\text{loop}}(\Lambda_n, q) = Z_{6V}(\Lambda_n, 1, 1, 1, 1, 2\cos(\pi\tilde{\sigma}), 2\cos(\pi\tilde{\sigma}), e^{i\tilde{\sigma}\pi/2}, e^{-i\tilde{\sigma}\pi/2}). \tag{4.2}$$

Indeed, there is only one way to recover the way loops intersect for patterns of type 1–4 and for boundary patterns. For patterns of type 5–6, there are two ways to draw non-intersecting loops, hence the weights corresponding to the two possibilities contributing $e^{-i\tilde{\sigma}\pi}$ and $e^{i\tilde{\sigma}\pi}$ respectively. Equations (4.2), (4.1) and (3.1) lead to the result. □

The main advantage of the six vertex model over the Potts model is that it is exactly solvable. In other words, one may compute the free energy of the model via transfer matrices and the so-called Bethe Ansatz (see [Bax 89] and references therein). As a result, the free energy of the critical Potts model can be computed explicitly. Note that this provides little information on the critical behavior of the Potts model since thermodynamical quantities of the model are expressed in terms of *derivatives* of the free energy and computing the free energy at one point only is not sufficient to access these derivatives. However, Baxter [Bax 71, Bax 73, Bax 78, Bax 89] used this correspondence together with additional unproved assumptions to state the following conjecture.

Conjecture 4.4. Consider the Potts model on the square lattice. For $q \leqslant 4$, the phase transition is continuous, while for $q > 4$ it is discontinuous.

2 Discontinuous phase transition for $q \gg 1$

Let us present a proof of the following result.

Theorem 4.5.

For $q > 3^8$, the phase transition of the random-cluster model is discontinuous, in the sense that properties **P1–5** are not satisfied.

In [LMR 86, LMMS+ 91, KS 82], the Potts and random-cluster models were proved to undergo a discontinuous phase transition at criticality when $q \geqslant 25.72$ via a Pirogov-Sinai type argument. The proof that we present is inspired by these arguments.

Proof. We show that **P1** is not satisfied and then invoke Theorem 4.6 to show **P2 − 5** (one may in fact easily prove the properties directly). Let L be a loop of length n (on the medial lattice) surrounding the origin. Fix a graph Ω containing L and its interior. Choose a vector u (seen as a complex number) among $\frac{1+i}{2}$, $\frac{1-i}{2}$, $\frac{-1-i}{2}$ and $\frac{-1+i}{2}$ for which L contains more than $n/4$ medial edges which are translates of $-iu$.

Ising model | Random-cluster model | Potts model | Bernoulli percolation | Lattice spin models

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We construct a map s_L from the set A_L of loop configurations ϖ on Ω containing L as a loop to the set of loop configurations in three steps (see Figures 4.3–4.7):

Step 1 Remove the loop L.

Step 2 Translate the loops of ϖ which were surrounded by L by u.

Step 3 Fill the "holes" corresponding to faces of $\Omega^\diamond$ intersecting no loop with trivial loops (they are exactly $n/4$ such holes).

The details of why $s_L(\varpi)$ is indeed a loop configuration are left to the reader. We have that

$$\begin{aligned}\phi^0_{p_c,q,\Omega}[s_L(\varpi)] &= \sqrt{q}^{\ell(s_L(\varpi))-\ell(\varpi)}\phi^0_{p_c,q,\Omega}[\varpi]\\ &= \sqrt{q}^{n/4-1}\phi^0_{p_c,q,\Omega}[\varpi].\end{aligned}$$

Since s_L is one-to-one, we deduce that

$$\begin{aligned}\phi^0_{p_c,q,\Omega}[A_L] &= \sum_{\varpi\in A_L}\phi^0_{p_c,q,\Omega}[\varpi]\\ &\leqslant q^{1/2-n/8}\sum_{\varpi\in A_L}\phi^0_{p_c,q,\Omega}[s_L(\varpi)]\\ &\leqslant q^{1/2-n/8}.\end{aligned}$$

The proof follows by letting Ω tend to the full lattice and by noticing that there are less than $n\cdot 4\cdot 3^{n-1}$ loops of length n surrounding the origin, and therefore as soon as $q>3^8$, Borel-Cantelli's lemma implies that there are finitely many loops surrounding the origin almost surely. This implies that there exists an infinite-cluster in the dual model almost surely, which leads to $\phi^1_{p_c,q}[0\longleftrightarrow\infty]>0$. □

Figure 4.3. Consider a loop configuration ϖ containing the loop L (in bold). In this case $u=\frac{1-i}{2}$.

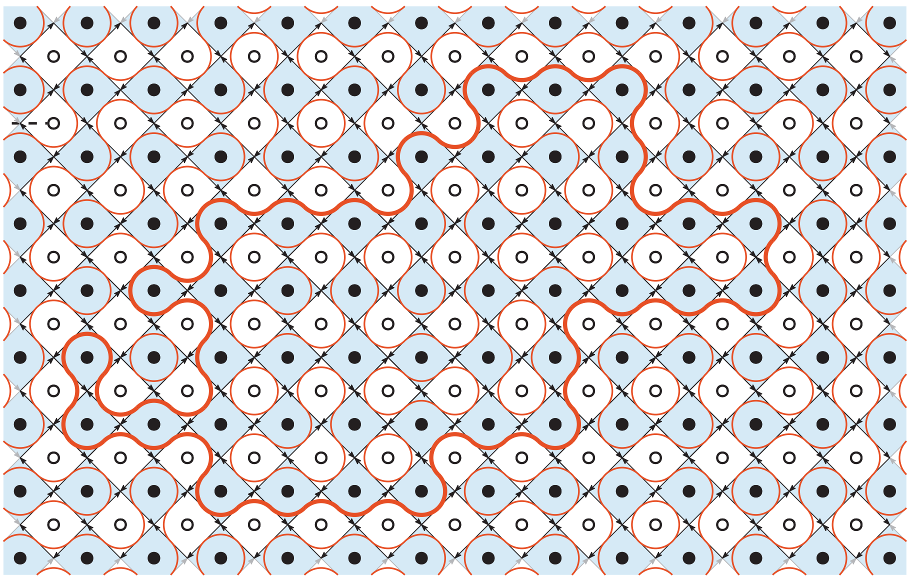


Figure 4.4. (Step 1) Remove the loop L from $\overline{\omega}$.

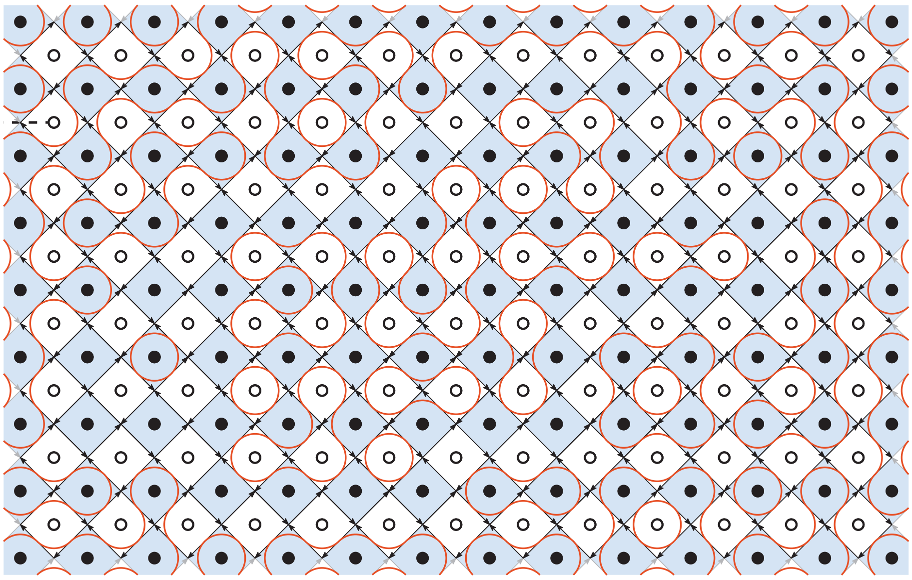


Figure 4.5. (Step 2a) The loops of $\overline{\omega}$ surrounded by L are depicted in bold.

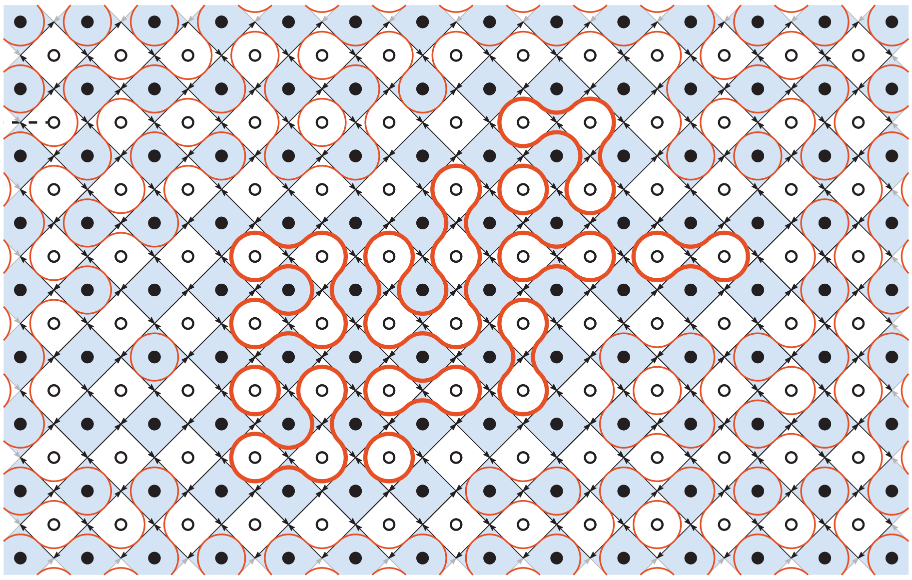


Figure 4.6. (Step 2b) Translate the loops of $\overline{\omega}$ surrounded by L in the south-east direction $\frac{1-i}{2}$.

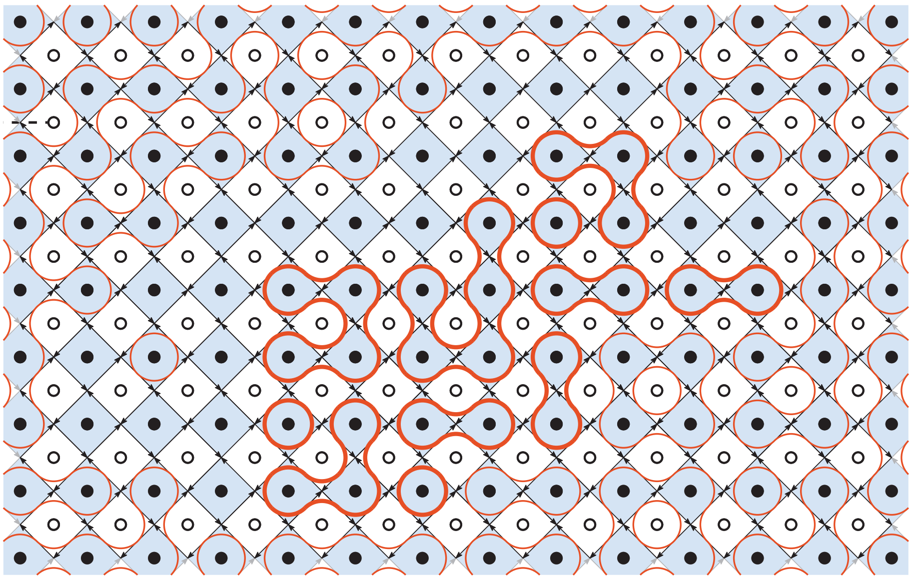
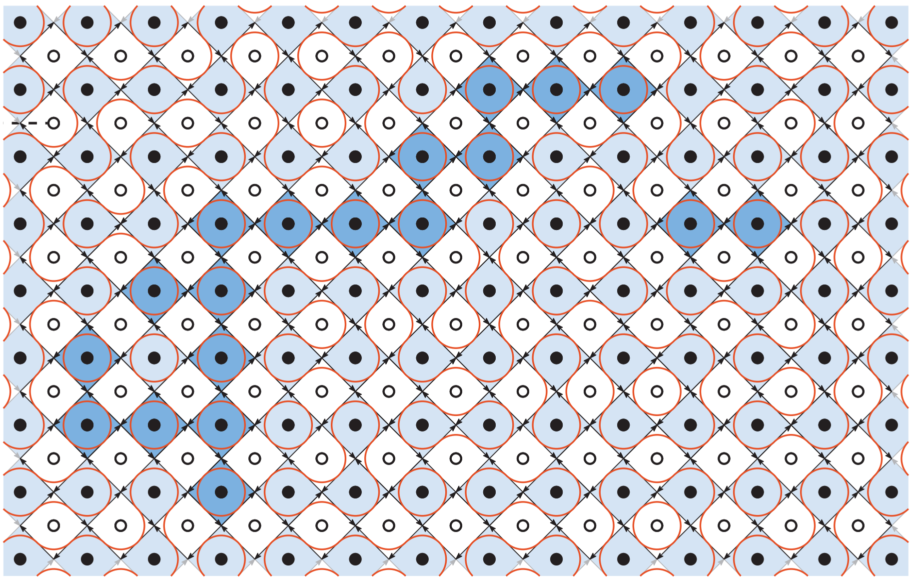


Figure 4.7. (Step 3) Fill the "holes" (depicted in darker blue) with trivial loops.



3 Continuous phase transition for $q \leqslant 4$

The goal of this section is to show the following theorem.

Theorem 4.6 (Duminil-Copin [DC 12]).

Let $1 \leqslant q \leqslant 4$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \phi^{0}_{p_c,q}[0 \longleftrightarrow \partial\Lambda_n] = 0.$$

This theorem, combined with Theorem 3.11, shows that the phase transition is continuous for random-cluster models with $1 \leqslant q \leqslant 4$, and therefore for the Potts models with 2, 3 or 4 colors.

In order to prove this result, we introduce a new tool, called the parafermionic observable.

3.1 Definition of parafermionic observables

The edges of the medial lattice can be oriented in a counter-clockwise way around faces centered on a vertex of $\mathbb{Z}^2$, see Fig. 1.1. Fix a **Dobrushin domain** (Ω, a, b) (recall the definition from Section 1). As suggested by planar duality, we define the Dobrushin boundary conditions by taking the edges (between endpoints) of ∂_{ba} to be open, and the dual-edges of ∂^*_{ab} to be open in ω^* (see Fig. 4.8 for a definition of ∂^*_{ab}).

Define the medial graph $\Omega^\diamond$ of Ω as follows: the vertices are the vertices of $(\mathbb{Z}^2)^\diamond$ at the center of edges of Ω or Ω^*. Let e_a and e_b be the two medial edges separating ∂_{ab} from ∂^*_{ba}. Also set $a^\diamond$ and $b^\diamond$ to be the ending and starting points of e_a and e_b respectively. One may define a loop configuration $\overline{\omega}$ exactly as before, which this times contains loops together with a self-avoiding path going from e_a to e_b, see Figures 4.8–4.10. This curve is called the *exploration path* and is denoted by $\gamma = \gamma(\omega)$. One may easily check that the probability of a loop configuration $\overline{\omega}$ is proportional to $x^{o(\omega)}\sqrt{q}^{\ell(\omega)}$.

FIGURE 4.8. The primal and dual Dobrushin domains associated to a medial Dobrushin domain. Note the position of a and b and the definition of ∂_{ba} and ∂^*_{ab}.

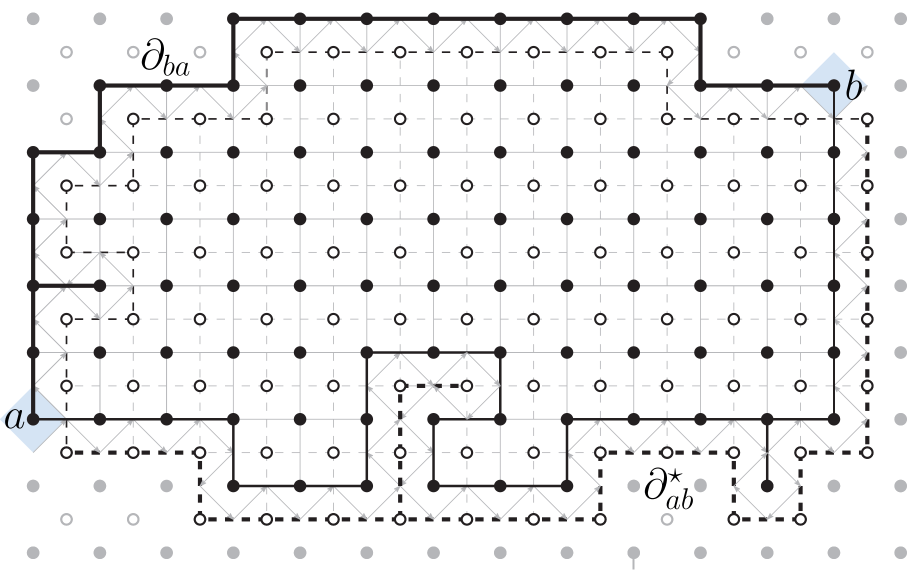


FIGURE 4.9. The configuration ω with its dual configuration ω^*.

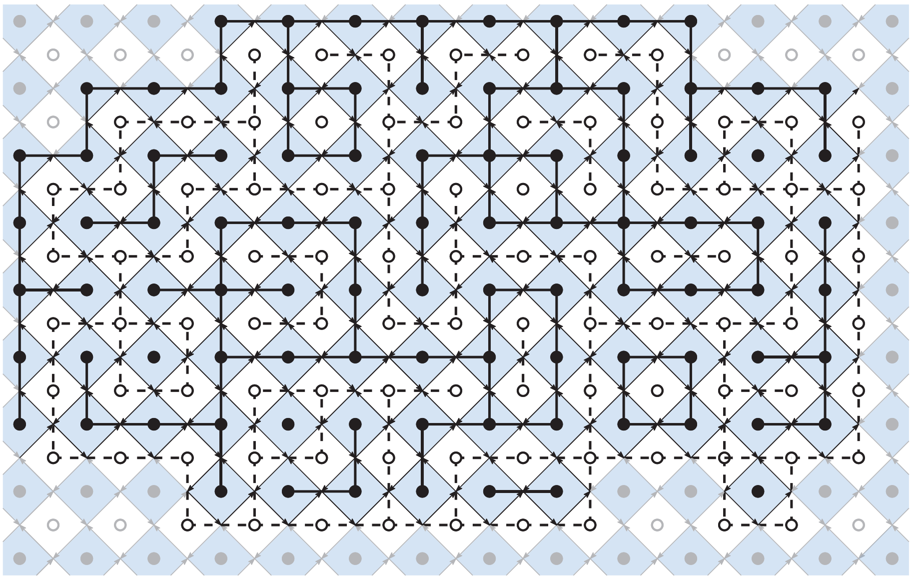
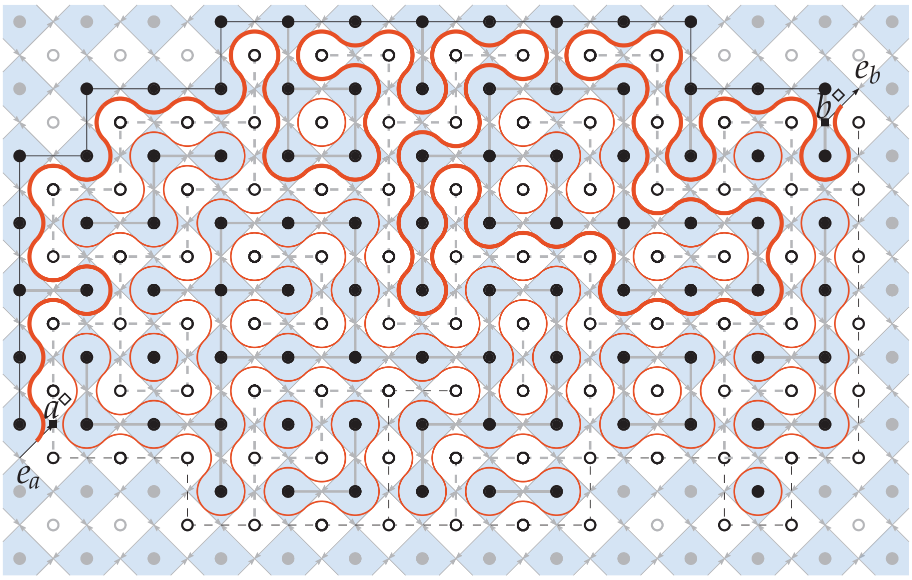


FIGURE 4.10. The loop configuration $\overline{\omega}$ associated to the primal and dual configurations ω and ω^* in the previous picture. The exploration path is drawn in bold, starts at e_a and finishes at e_b. We also depicted $a^\diamond$ and $b^\diamond$.



Remark 4.7. The loops correspond to the interfaces separating clusters from dual clusters, and the exploration path corresponds to the interface between the vertices connected in ω to ∂_{ba} and the dual vertices connected in ω^* to ∂^*_{ab}.

We now wish to introduce a new object, called the ***parafermionic observable***, which will be instrumental in our proof. Before defining it properly, let us recall the notion of winding of a curve on the medial lattice.

Definition 4.8.

The ***winding*** $\mathrm{W}_\Gamma(e,e')$ of a curve Γ (on the medial lattice) between two medial-edges e and e' of the medial graph is the total signed rotation in radians that the (now oriented) curve makes from the mid-point of the edge e to that of the edge e' (see Fig. 4.11). By convention, if Γ does not go through e', we set $\mathrm{W}_\Gamma(e,e')=0$.

The winding can be computed in a very simple way: it corresponds to $\frac{\pi}{2}$ times the number of $\frac{\pi}{2}$-turns on the left minus the number of $\frac{\pi}{2}$-turns on the right.

Definition 4.9 (Smirnov [Smi 10]).

Consider a Dobrushin domain (Ω,a,b). The *(edge) parafermionic observable* $F=F(p,q,\Omega,a,b)$ is defined for any medial edge $e\in E(\Omega^\diamond)$ by

$$F(e) := \phi^{a,b}_{p,q,\Omega}[\mathrm{e}^{\mathrm{i}\sigma \mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}],$$

where γ is the exploration path and σ is given by the relation

$$\sin(\sigma\pi/2)=\sqrt{q}/2.$$

Remark 4.10. Note that σ is real for $q\leqslant 4$, and belongs to $1+i\mathbb{R}$ for $q>4$. For $q\in[0,4]$, σ has the physical interpretation of a spin, which is fractional in general, hence the name parafermionic (fermions have half-integer spins while bosons have integer spins, there are no particles with fractional spin, but the use of such fractional spins at a theoretical level has been very fruitful in physics).

For $q>4$, σ is not real anymore and does not have any physical interpretation. Also note that for $q=2$, $\sigma=1/2$ corresponds to the spin of a fermion. For this reason, we speak of the ***fermionic observable*** in this special case.

Remark 4.11. Similar observables have been used to study other models such as Ising (see below for references) and $O(n)$-models [Smi 06, DCS 12b, BBMDG+ 14, Gla 13].

3.2 Contour integrals of the parafermionic observable

The parafermionic observable satisfies a very specific property at criticality regarding contour integrals. Let (Ω,a,b) be a Dobrushin domain. One may define a dual $(\Omega^\diamond)^*$ of $\Omega^\diamond$ in the following way: the vertex set of $(\Omega^\diamond)^*$ is $V(\Omega)\cup V(\Omega^*)$ and the edges of the dual connect nearest vertices together. We extend the definitions available for other graphs to this context.

Definition 4.12.

A *discrete contour* $\mathscr{C}$ is a finite sequence $z_0 \sim z_1 \sim \cdots \sim z_n = z_0$ in $(\Omega^\diamond)^*$ such that the path $(z_0, \ldots, z_n)$ is edge-avoiding. The discrete contour integral of the parafermionic observable F along $\mathscr{C}$ is defined by

$$\oint_{\mathscr{C}} F(z)\mathrm{d}z := \sum_{i=0}^{n-1}(z_{i+1}-z_i)F\left(\{z_i, z_{i+1}\}^*\right),$$

where the z_i are considered as complex numbers and $\{z_i, z_{i+1}\}^*$ denotes the edge of $\Omega^\diamond$ intersecting $\{z_i, z_{i+1}\}$ in its center.

Theorem 4.13 (Vanishing contour integrals).

Let (Ω, a, b) be a Dobrushin domain, $q > 0$ and $p = p_c$. For any discrete contour $\mathscr{C}$ of (Ω, a, b),

$$\oint_{\mathscr{C}} F(z)\mathrm{d}z = 0.$$

Remark 4.14. The fact that discrete contour integrals vanish seems to correspond to a well-known property of holomorphic functions. Nevertheless, one should be slightly careful when drawing such a parallel: the observable is defined on edges, and should rather be understood as the discretization of a form rather than a function. As a form, the fact that these discrete contour integrals vanish should be interpreted as the discretization of the property of being closed.

The following lemma will be important for the proof of Theorem 4.13.

Lemma 4.15. Let (Ω, a, b) be a Dobrushin domain, $p \in [0,1]$ and $q > 0$. Consider $v \in V(G^\diamond)$ with four incident medial edges A, B, C and D indexed in counter-clockwise order in such a way that A and C are pointing towards v (the two others are pointing away). Then,

$$F(A) - F(C) \;=\; \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}[F(B) - F(D)], \tag{4.3}$$

where $\alpha = \alpha(p,q) \in [0, 2\pi)$ is given by the relation $\mathrm{e}^{\mathrm{i}\alpha(p)} := \dfrac{\mathrm{e}^{\mathrm{i}\sigma\pi/2} + \mathrm{i}x(p)}{\mathrm{e}^{\mathrm{i}\sigma\pi/2}x(p) + \mathrm{i}}$.

Proof (of Theorem 4.13 using Lemma 4.15). When $p = p_c$, $\alpha = 0$ and the relation (4.3) can be understood as the fact that the discrete contour integral along the small lozenge surrounding v is zero. The theorem thus follows by summing the relation (4.3) over vertices of $\Omega^\diamond$ enclosed by $\mathscr{C}$ (in other words faces of $(\Omega^\diamond)^*$ surrounded by $\mathscr{C}$). We use that $\mathscr{C}$ does not surround any boundary point of $\Omega^\diamond$, which follows from the fact that $\Omega^\diamond$ can be seen as a simply connected domain of $\mathbb{R}^2$. In particular, its complement is a connected graph. ☐

Proof (of Lemma 4.15). Assume that $v \in V(\Omega^\diamond)$ corresponds to a vertical edge of Ω. The case of an horizontal edge can be treated in a similar fashion.

Let s be the involution (on the space of configurations) switching the state open or closed of the edge in ω passing through v. Let e be an edge of $\Omega^\diamond$ and let

$$e_\omega \;:=\; \phi^{a,b}_{p,q,\Omega}[\omega]\mathrm{e}^{\mathrm{i}\sigma \mathrm{W}_{\gamma(\omega)}(e,e_b)}\mathbb{1}_{e\in\gamma(\omega)}$$

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be the contribution of the configuration ω to $F(e)$. With this notation, $F(e) = \sum_\omega e_\omega$. Since s is an involution, the following relation holds:

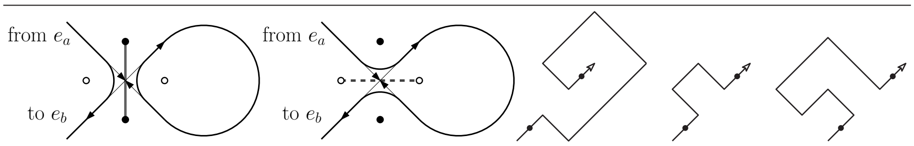
$$F(e) \;=\; \tfrac{1}{2}\sum_{\omega}\big[e_\omega + e_{s(\omega)}\big].$$

To prove (4.3), it is thus sufficient to show that

$$A_\omega + A_{s(\omega)} - C_\omega - C_{s(\omega)} \;=\; \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha(p)}[B_\omega + B_{s(\omega)} - D_\omega - D_{s(\omega)}] \tag{4.4}$$

for any configuration ω.

Figure 4.11. **Left.** The neighborhood of v for two associated configurations ω and $s(\omega)$. **Right.** Three examples for the winding: it is respectively equal to 2π, 0 and 0.



There are three possible cases:

Case 1. No edge incident to v belongs to $\gamma(\omega)$. Then, none of these edges is incident to $\gamma(s(\omega))$ either. For any e incident to v, e_ω and $e_{s(\omega)}$ equal 0 and (4.4) trivially holds.

Case 2. Two edges incident to v belong to $\gamma(\omega)$, see Fig. 4.11. Since $\gamma(\omega)$ and the medial lattice possess a natural orientation, $\gamma(\omega)$ enters through either A or C and leaves through B or D. Assume that $\gamma(\omega)$ enters through the edge A and leaves through the edge D (*i.e.* that the primal edge corresponding to v is open). It is then possible to compute the contributions for ω and $s(\omega)$ of all the edges adjacent to v in terms of A_ω. Indeed,

- Since $s(\omega)$ has one less open edge and one less loop, we find

$$\phi^{a,b}_{p,q,\Omega}[s(\omega)] = \frac{1}{Z}x^{o(s(\omega))}\sqrt{q}^{\ell(s(\omega))} = \frac{1}{Z}x^{o(\omega)-1}\sqrt{q}^{\ell(\omega)-1}$$
$$= \frac{1}{x\sqrt{q}}\phi^{a,b}_{p,q,\Omega}[\omega].$$

- Windings of the curve at B, C and D can be expressed using the winding at A. For instance, $W_{\gamma(\omega)}(B,e_b) = W_{\gamma(\omega)}(A,e_b) - \pi/2$.

The other cases are treated similarly. The contributions are given in the following table.

configuration	A	C	B	D
ω	A_ω	0	0	$\mathrm{e}^{\mathrm{i}\sigma\pi/2}A_\omega$
$s(\omega)$	$\frac{A_\omega}{x\sqrt{q}}$	$\mathrm{e}^{\mathrm{i}\sigma\pi}\frac{A_\omega}{x\sqrt{q}}$	$\mathrm{e}^{-\mathrm{i}\sigma\pi/2}\frac{A_\omega}{x\sqrt{q}}$	$\mathrm{e}^{\mathrm{i}\sigma\pi/2}\frac{A_\omega}{x\sqrt{q}}$

Using the identity $\mathrm{e}^{\mathrm{i}\sigma\pi/2} - \mathrm{e}^{-\mathrm{i}\sigma\pi/2} = \mathrm{i}\sqrt{q}$, we deduce (4.4) by summing (with the right weight) the contributions of all the edges incident to v.

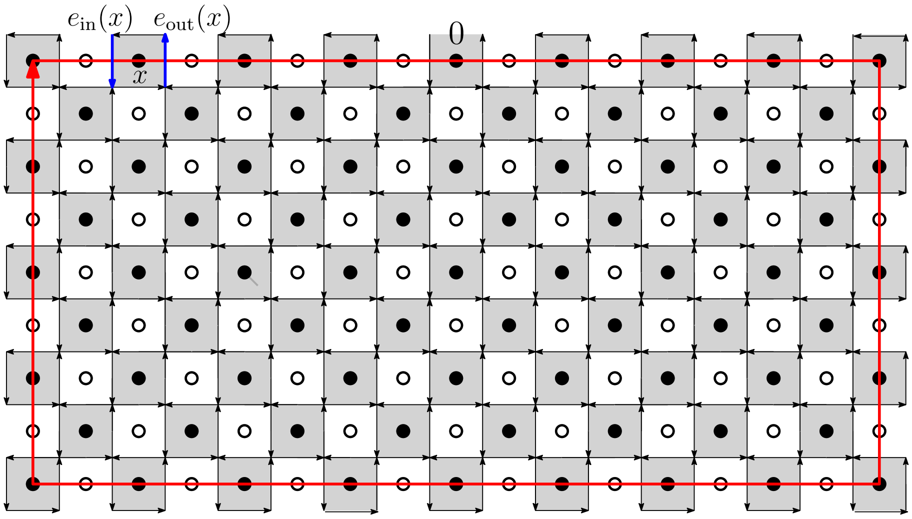
Case 3. The four edges incident to v belong to $\gamma(\omega)$. Then only two of these edges belong to $\gamma(s(\omega))$ and the computation is similar to Case 2 with $s(\omega)$ instead of ω.

In conclusion, (4.4) is always satisfied and the claim is proved. □

3.3 Proof of Theorem 4.6

In this section, the lattices $\mathbb{Z}^2$, $(\mathbb{Z}^2)^*$ and $(\mathbb{Z}^2)^\diamond$ are rotated by an angle of $\pi/4$ and rescaled by a factor of $1/\sqrt{2}$.

FIGURE 4.12. The domain Ω for $n=4$. We also depicted $e_{\mathrm{out}}(x)$ and $e_{\mathrm{in}}(x)$ for a vertex x on ∂_2.



Cluster-weight q **between** 1 **and** 2. We refer to Fig. 4.12. Consider the graph Ω induced by vertices in $[-n,n]\times[-n,0]$ and divide the boundary $\partial\Omega$ into four pieces:

$$\begin{aligned}
\partial_0 &:= \{0=(0,0)\},\\
\partial_1 &:= [0,n-1]\times\{0\},\\
\partial_2 &:= [-(n-1),0]\times\{0\},\\
\partial_3 &:= \partial\Omega\setminus(\{0\}\cup\partial_1\cup\partial_2).
\end{aligned}$$

Note that every vertex $x\in\partial\Omega$ is naturally associated to two medial edges bordering the face associated to it, one pointing towards $\Omega^\diamond\setminus\partial\Omega^\diamond$, and one pointing away from it. We call them $e_{\mathrm{in}}(x)$ and $e_{\mathrm{out}}(x)$ respectively.

We look at the free boundary conditions and consider them as being Dobrushin boundary conditions with $a=b=0$. In such case, the exploration path $\gamma(\omega)$ is the loop passing through $e_a=e_{\mathrm{in}}(0)$ and ending at $e_b=e_{\mathrm{out}}(0)$.

Consider $\mathscr{C}:=\partial\Omega\cup\partial\Omega^*$, Theorem 4.13 gives us

$$\begin{aligned}
0 &= \oint_{\mathscr{C}} F(z)\,\mathrm{d}z\\
&= F(e_{\mathrm{out}}(0))+F(e_{\mathrm{in}}(0))+\sum_{x\in\partial_1} F(e_{\mathrm{out}}(x))+F(e_{\mathrm{in}}(x))+\sum_{x\in\partial_2} F(e_{\mathrm{out}}(x))+F(e_{\mathrm{in}}(x))\\
&\quad +\mathrm{i}\sum_{x\in\partial_3} e_{\mathrm{out}}(x)F(e_{\mathrm{out}}(x))+e_{\mathrm{in}}(x)F(e_{\mathrm{in}}(x)).
\end{aligned}$$

Now, observe that $1 = F(e_{\mathrm{out}}(0)) = e^{-\mathrm{i}\pi\sigma} F(e_{\mathrm{in}}(0))$ so that

$$F(e_{\mathrm{out}}(0)) + F(e_{\mathrm{in}}(0)) = 1 + e^{\mathrm{i}\pi\sigma} = 2\cos(\tfrac{\pi}{2}\sigma)e^{\mathrm{i}\frac{\pi}{2}\sigma}.$$

Furthermore, by gathering the contributions of $x = (x_1, 0) \in \partial_1$ with its symmetric $(-x_1, 0) \in \partial_2$, and by noticing that the loop coming from 0 goes around x if and only if 0 and x are connected, we find that

$$\begin{aligned}
&\sum_{x\in\partial_1} F(e_{\mathrm{out}}(x)) + F(e_{\mathrm{in}}(x)) + \sum_{x\in\partial_2} F(e_{\mathrm{out}}(x)) + F(e_{\mathrm{in}}(x))\\
&\quad = \sum_{x\in\partial_1} F(e_{\mathrm{out}}(x)) + F(e_{\mathrm{in}}(x)) + F(e_{\mathrm{out}}(-x)) + F(e_{\mathrm{in}}(-x))\\
&\quad = \sum_{x\in\partial_1} (e^{2\mathrm{i}\pi\sigma} + e^{\mathrm{i}\pi\sigma} + 1 + e^{-\mathrm{i}\pi\sigma})\phi^0_{p_c,q,\Omega}[0 \longleftrightarrow x]\\
&\quad = \frac{\sin(2\pi\sigma)}{\sin(\frac{\pi}{2}\sigma)} e^{\mathrm{i}\frac{\pi}{2}\sigma} \sum_{x\in\partial_1} \phi^0_{p_c,q,\Omega}[0 \longleftrightarrow x].
\end{aligned}$$

Finally,

$$\begin{aligned}
\Big|\sum_{x\in\partial_3} e_{\mathrm{out}}(x)F(e_{\mathrm{out}}(x)) + e_{\mathrm{in}}(x)F(e_{\mathrm{in}}(x))\Big| &\leqslant 2\sum_{x\in\partial_3} \phi^0_{p_c,q,\Omega}[0 \longleftrightarrow x]\\
&\leqslant 2\sum_{x\in\partial\Lambda_n} \phi^0_{p_c,q}[0 \longleftrightarrow x].
\end{aligned}$$

In the second line, we used the comparison between boundary conditions and the fact that $\partial_3 \subset \partial\Lambda_n$. For $q \in [1,2]$, $\frac{\sin(2\pi\sigma)}{\sin(\frac{\pi}{2}\sigma)} \geqslant 0$ and therefore we obtain

$$\sum_{x\in\partial\Lambda_n} \phi^0_{p_c,q}[0 \longleftrightarrow x] \geqslant \cos(\tfrac{\pi}{2}\sigma) > 0.$$

Summing over n gives us

$$\sum_{x\in\mathbb{Z}^2} \phi^0_{p_c,q}[0 \longleftrightarrow x] = \infty,$$

which implies Theorem 4.6.

Cluster-weight q between 1 and 3. In the previous proof, it is very important that the coefficients in front of $\phi^0_{p_c,q,\Omega}[0 \longleftrightarrow x]$ are negative for boundary vertices $x \notin \partial\Lambda_n$. This property is no longer true for $q \in (2,3]$. Nevertheless, one may "open the domain slightly more" by considering the slit domain C_n obtained by removing from Λ_n the vertices of $\{(0,k) : 1 \leqslant k \leqslant n\}$. By considering $\mathscr{C}$ to be the boundary of the medial graph, the coefficients in front of $\phi^0_{p_c,q,\Omega}[0 \longleftrightarrow x]$ are negative for $x \notin \partial\Lambda_n$ and $q \leqslant 3$ and we may proceed as before.

Cluster-weight q between 1 and 4. The problem is that the previous geometry works only for $q \leqslant 3$. As soon as $q > 3$, problems arise from the fact that it is no longer true that

$$\sum_{x\in\partial\Lambda_n} \phi^0_{p_c,q,C_n}[0 \longleftrightarrow x] \geqslant \mathrm{cst}.$$

(One can for instance use the conformal invariance prediction to guess that the term on the left is tending to 0; see [DCST 15].) For this reason, one needs to “open the domain” even more and consider the following non-planar domain $\mathbb{U}$ (see Fig. 4.13): the vertices are given by $\mathbb{Z}^3$ and the edges by

- $[(x_1,x_2,x_3),(x_1,x_2+1,x_3)]$ for every $x_1,x_2,x_3 \in \mathbb{Z}$,
- $[(x_1,x_2,x_3),(x_1+1,x_2,x_3)]$ for every $x_1,x_2,x_3 \in \mathbb{Z}$ such that $x_1 \neq 0$ or such that $x_1 = 0$ and $x_2 \geqslant 0$,
- $[(0,x_2,x_3),(1,x_2,x_3+1)]$ for every $x_2 < 0$ and $x_3 \in \mathbb{Z}$.

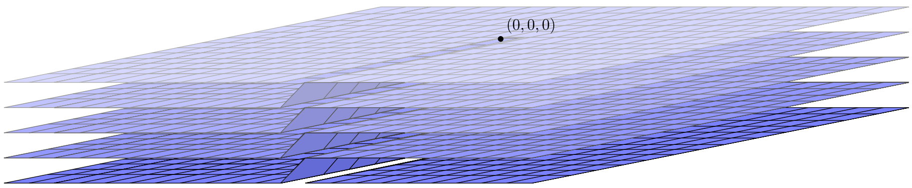
This graph is the universal cover of $\mathbb{Z}^2 \setminus F$, where F is the face centered at $(-\frac{1}{2},-\frac{1}{2})$. It can also be seen as $\mathbb{Z}^2$ with a branching point at $(-\frac{1}{2},-\frac{1}{2})$. All definitions of dual and medial graphs extend to this context.

For $n \geqslant 1$, define

$$U_n := \{(x_1,x_2,x_3) \in \mathbb{U} : |x_1|,|x_2| \leqslant n \text{ and } |x_3| \leqslant n^5\}.$$

The same reasoning as before can be performed in this geometry. The comparison between boundary conditions does not work directly here and some work needs to be done to derive an estimate on the plane. This proof is substantially more complicated and we omit the details here.

Figure 4.13. The graph $\mathbb{U}$.



A proof of $p_c(q) = \sqrt{q}/(1+\sqrt{q})$ based on the observable. Let us present quickly a proof of $p_c(q) = \sqrt{q}/(1+\sqrt{q})$ for $1 \leqslant q \leqslant 2$, which is solely based on the observable F. Let us mention that a similar proof works for any $q \leqslant 3$. For $q > 4$, a proof based on the observable only can be found in [BDCS 12].

We already know by Zhang's argument that $p_c(q) \geqslant \sqrt{q}/(1+\sqrt{q})$ and we therefore focus on the other inequality.

First, observe that for an increasing event A and an edge e, the FKG inequality implies that

$$\begin{aligned} I_A(e) &= \phi^{\xi}_{p,q,G}(A \mid \omega_e = 1) - \phi^{\xi}_{p,q,G}(A \mid \omega_e = 0) \\ &= \frac{\phi^{\xi}_{p,q,G}(\omega^{(e)} \in A,\ \omega_e = 1)}{\phi^{\xi}_{p,q,G}(\omega_e = 1)} - \frac{\phi^{\xi}_{p,q,G}(\omega_{(e)} \in A,\ \omega_e = 0)}{\phi^{\xi}_{p,q,G}(\omega_e = 0)} \\ &\geqslant \phi^{\xi}_{p,q,G}(\omega^{(e)} \in A) - \phi^{\xi}_{p,q,G}(\omega_{(e)} \in A) = \phi^{\xi}_{p,q,G}(e \text{ pivotal for } A). \end{aligned}$$

Therefore, (3.2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}(A) \geqslant \sum_{e \in E(G)} \phi^{\xi}_{p,q,G}(e \text{ pivotal for } A).$$

We will show the same differential inequality as in Lemma 2.7. We adopt the notation of its proof. In particular, $\mathscr{S}=\{z\in\Lambda_n$ not connected to $\partial\Lambda_n\}$. Following the same lines as in the proof of Lemma 2.7 (specifically until (2.6)), we find

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,\Lambda_n}(A)\geqslant\sum_{S\ni 0}\sum_{\{x,y\}\in\Delta S}\phi^{\xi}_{p,q,\Lambda_n}[0\xleftrightarrow{S}x,\mathscr{S}=S].$$

At this point, we cannot use independence of the percolation configuration inside and outside of S. Nevertheless, the domain Markov property implies

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,\Lambda_n}(A)\geqslant\sum_{S\ni 0}\Big(\sum_{\{x,y\}\in\Delta S}\phi^{0}_{p,q,S}[0\xleftrightarrow{S}x]\Big)\,\phi^{\xi}_{p,q,\Lambda_n}[\mathscr{S}=S].$$

Define

$$\varphi_{p,q}(S):=\sum_{\{x,y\}\in\Delta S}\phi^{0}_{p,q,S}[0\xleftrightarrow{S}x].$$

Note that the definition is very similar to the definition for Bernoulli percolation, except that we specify free boundary conditions on S (boundary conditions were irrelevant for Bernoulli percolation).

If there exists $c=c(q)>0$ such that $\varphi_{\sqrt{q}/(1+\sqrt{q}),q}(S)\geqslant c$ for any finite set $S\ni 0$, we will be able to integrate the differential inequality to obtain that for $p\geqslant\sqrt{q}/(1+\sqrt{q})$, $\phi_{p,q}[0\longleftrightarrow\infty]>0$, thus proving the result. Also note that we may restrict our attention to S with connected complement, since otherwise the probability of $\mathscr{S}=S$ would be zero. We therefore focus on the following lemma.

Lemma 4.16. Let $1\leqslant q\leqslant 2$. There exists $c=c(q)>0$ such that for any $S\ni 0$ finite and with connected complement,

$$\varphi_{\sqrt{q}/(1+\sqrt{q}),q}(S)\geqslant c.$$

Proof. We cannot use the same argument as for percolation because of the dependence on free boundary conditions. We therefore invoke the parafermionic observable. Let S be a finite set with connected complement. Let S' be the connected component of S containing 0. Apply the same reasoning as in the proof of Theorem 4.6 with the domain $\Omega=S'\cap[-n,n]\times[0,n]$ instead of $[-n,n]\times[-n,0]$. We deduce that

$$\varphi_{\sqrt{q}/(1+\sqrt{q}),q}(S)\geqslant\sum_{x\in\partial\Omega\cap\partial\Lambda_n}\phi_{\sqrt{q}/(1+\sqrt{q}),q,\Omega}[0\longleftrightarrow x]\geqslant\cos(\tfrac{\pi}{2}\sigma).\qquad\square$$

4 Conformal invariance for the random-cluster model with $q=2$

In this section, we wish to prove that the critical random-cluster model with $q=2$ is conformally invariant. The section is organized as follows. In the next section, we introduce a few notations. Then, we state Smirnov's theorem yielding conformal invariance of the parafermionic observable (which in this case is called *fermionic observable*). The next three sections are devoted to the proof of this theorem. The first of these sections introduces the notion of s-holomorphicity, the next one rephrases Smirnov's theorem into another theorem, which is proved in the last of these three sections. Finally, the last section presents a few applications of this theorem, in particular the conformal invariance of interfaces in the model.

4.1 Notation

Consider a simply connected domain Ω with two marked points a and b on its boundary. We will be interested in finer and finer graphs approximating continuous domains. For $\delta > 0$, we consider the rescaled square lattice $\delta\mathbb{Z}^2$. The definitions of dual and medial Dobrushin domains (see previous sections) extend to this context. Note that the medial lattice $(\delta\mathbb{Z}^2)^\diamond$ has mesh-size $\delta/\sqrt{2}$. Generically, Dobrushin domains on $\delta\mathbb{Z}^2$, $(\delta\mathbb{Z}^2)^\star$ and $(\delta\mathbb{Z}^2)^\diamond$ will be denoted by $(\Omega_\delta, a_\delta, b_\delta)$, $(\Omega_\delta^\star, a_\delta^\star, b_\delta^\star)$ and $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$.

Since we wish to speak of Dobrushin domains approximating a continuous domain with marked points on its boundary, we need to quantify how close a discrete graph is to its continuum counterpart. To this end, we introduce the notion of *Carathéodory convergence*. Consider a Dobrushin domain $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ as a simply connected domain by taking the union of its faces. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane.

Definition 4.17.

Let (Ω, a, b) be a simply connected domain with two marked points on its boundary. Consider a sequence of Dobrushin domains $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. We say that $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ converges to (Ω, a, b) in the Carathéodory sense if

$$f_\delta \longrightarrow f \qquad \text{on any compact subset } K \subset \mathbb{H},$$

where f_δ (resp. f) is the unique conformal map from $\mathbb{H}$ to $\Omega_\delta^\diamond$ (resp. Ω) satisfying $f_\delta(0) = a_\delta^\diamond$, $f_\delta(\infty) = b_\delta^\diamond$ and $f_\delta'(\infty) = 1$ (resp. $f(0) = a$, $f(\infty) = b$ and $f'(\infty) = 1$).

Remark 4.18. Let us mention that this notion of convergence coincides with the Haussdorff convergence in the case of smooth domains. Therefore, one may simply think of the Carathéodory convergence as being a very natural notion of convergence and not bother with details, for sufficiently smooth domains, take as a possible example of a converging sequence the family $(\Omega_\delta, a_\delta, b_\delta)$ where $\Omega_\delta = \Omega \cap (\delta\mathbb{Z}^2)$ and a_δ and b_δ are the closest vertices of Ω_δ to a and b.

We will also be considering sequences of functions on $V(\Omega_\delta)$ for δ going to 0 and we wish to speak of uniform convergence on every compact subset of Ω. In order to do this, we implicitly perform the following operation: for a function f on Ω_δ, choose a diagonal for every (square) face and extend the function to the faces of Ω_δ in a piecewise linear way on the two triangles made of the diagonal and two edges. Since no confusion will be possible, the extension will be denoted by f as well. Constructed like that, the function is not necessarily defined on all of Ω (since the union of faces of Ω_δ may be different from Ω). Nevertheless, we will restrict our attention to sequences of domains Ω_δ tending to Ω in the Carathéodory sense: in such case f_δ is defined on any compact subset of Ω provided that δ is small enough (how small δ must be depends on the compact subset). The same procedure will also be applied to functions defined on $V(\Omega_\delta^\star)$ and $V(\Omega_\delta^\diamond)$.

4.2 Smirnov's theorem

We are now in a position to state Smirnov's theorem. For $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$, define the ***normalized vertex fermionic observable*** *(at criticality)* by

$$f_\delta(v) = f_\delta(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond, p_c, 2, v) := \frac{1}{\sqrt{2e_b}} \begin{cases} \frac{1}{2}\sum_{u\sim v} F_\delta(\{u,v\}) & \text{if } v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond, \\ \frac{2}{2+\sqrt{2}} \sum_{u\sim v} F_\delta(\{u,v\}) & \text{if } v \in \partial\Omega_\delta^\diamond, \end{cases}$$

where F_δ is the edge fermionic observable defined in Definition 4.9, and e_b is seen as a complex number.

Theorem 4.19 (Smirnov [Smi 10]). Fix $q = 2$ and $p = p_c(2) = \sqrt{2}/(1+\sqrt{2})$. Let (Ω, a, b) be a simply connected domain with two marked points on its boundary. Let $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ be a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. Let f_δ be the *normalized vertex fermionic observable at criticality* in $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. We have

$$f_\delta(\cdot) \to \sqrt{\phi'(\cdot)} \quad \text{when } \delta \to 0$$

uniformly on any compact subset of Ω, where ϕ is any conformal map from Ω to the strip $\mathbb{R} \times (0,1)$ mapping a to $-\infty$ and b to ∞.

Remark 4.20. Observe that $\mathrm{Im}(\phi)$ is the harmonic solution of the Dirichlet boundary value problem on Ω with boundary conditions 0 on the boundary arc from a to b, and 1 from b to a. As a consequence, $\sqrt{\phi'}$ is the holomorphic solution of a Riemann-Hilbert boundary value problem on Ω.

4.3 Notion of s-holomorphicity

We introduce the notion of s-**holomorphic** functions, which was developed in [Smi 10, CS 11, CS 12]. The interest of this definition lies in the fact that s-holomorphic solutions of discretizations of certain Riemann-Hilbert boundary value problems converge to their continuous counterparts. This observation is at the heart of the proof of Theorem 4.19, which we provide in the next section.

Consider each edge e of the medial lattice as being oriented via the natural orientation of the medial lattice mentioned earlier. Like this, each edge can be thought of as a complex number. The real line passing through the origin and $\sqrt{e}$ is denoted by $\ell(e)$ (the choice of the square root is irrelevant since we will be looking at projections on lines only). The different lines associated with medial edges on $(\delta\mathbb{Z}^2)^\diamond$ are $e^{i\pi/8}\mathbb{R}$, $e^{-i\pi/8}\mathbb{R}$, $e^{-i3\pi/8}\mathbb{R}$ and $e^{-i5\pi/8}\mathbb{R}$, see Fig. 4.14. For a line ℓ, define

$$P_\ell(x) = \alpha \mathrm{Re}(\overline{\alpha} x) = \tfrac{1}{2}\left(x + \alpha^2 \overline{x}\right),$$

where α is any unit vector collinear to ℓ.

Ising model Random-cluster model Potts model Bernoulli percolation Lattice spin models

Definition 4.21 (Smirnov).

A function $f:\Omega_\delta^\diamond \to \mathbb{C}$ is *s-holomorphic* if for any edge $e=\{u,v\}$ of $\Omega_\delta^\diamond$, we have

$$P_{\ell(e)}[f(u)] = P_{\ell(e)}[f(v)].$$

The notion of s-holomorphicity is stronger than the more classical notion of discrete holomorphic function. Let us briefly discuss this fact before manipulating s-holomorphic functions in more details.

Discrete holomorphic functions distinctively appeared for the first time in the papers [Isa 41, Isa 52] of Isaacs, where he proposed two definitions(1). Both definitions ask for a discrete version of the Cauchy-Riemann equations $\partial_{i\alpha}F = i\partial_\alpha F$ or equivalently that the $\bar z$-derivative is 0. The definition involves the following discretization of the $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$ operator. For convenience, we will consider discrete holomorphic functions on the medial lattice.

For a complex valued function f on $V(\Omega_\delta^\diamond)$, and for $x\in\Omega_\delta\cup\Omega_\delta^\star$, define

$$\bar\partial_\delta f(x) \;=\; \tfrac{1}{2}[f(E)-f(W)] + \tfrac{i}{2}[f(N)-f(S)]$$

where S, E, N and W denote the four vertices of $\Omega_\delta^\diamond$ adjacent to the medial vertex x indexed in the obvious way (N, E, S and W stand for cardinal directions). A function $f:\Omega_\delta^\diamond \to \mathbb{C}$ is called ***discrete holomorphic*** if $\bar\partial_\delta f(x) = 0$ for every $x\in\Omega_\delta\cup\Omega_\delta^\star$. The equation $\bar\partial_\delta f(x)=0$ is called the *discrete Cauchy-Riemann equation* at x.

Proposition 4.22.

Any s-holomorphic function $f:\Omega_\delta^\diamond\to\mathbb{C}$ is discrete holomorphic on $\Omega_\delta^\diamond$.

Proof. Let $f:\Omega_\delta^\diamond\to\mathbb{C}$ be a s-holomorphic function. Let v be a vertex of $\delta\mathbb{Z}^2\cup(\delta\mathbb{Z}^2)^\star$ corresponding to a face of $\Omega_\delta^\diamond$. Assume that $v\in\delta\mathbb{Z}^2$, the case $v\in(\delta\mathbb{Z}^2)^\star$ works similarly. We wish to show that $\bar\partial_\delta f(v)=0$. Let N, W, S and E be the four medial-vertices around v as illustrated in Fig. 4.14, and let us write one relation provided by the s-holomorphicity, for instance

$$P_{e^{-i\pi/8}\mathbb{R}}[f(E)] \;=\; P_{e^{-i\pi/8}\mathbb{R}}[f(S)].$$

Expressed in terms of f and its complex conjugate $\bar f$ only, the previous equality becomes

$$f(E)+e^{-i\pi/4}\overline{f(E)} \;=\; f(S)+e^{-i\pi/4}\overline{f(S)}.$$

Doing the same with the three other relations, we find

$$\begin{aligned} f(S)+ie^{-i\pi/4}\overline{f(S)} &= f(W)+ie^{-i\pi/4}\overline{f(W)},\\ f(W)-e^{-i\pi/4}\overline{f(W)} &= f(N)-e^{-i\pi/4}\overline{f(N)},\\ f(N)-ie^{-i\pi/4}\overline{f(N)} &= f(E)-ie^{-i\pi/4}\overline{f(E)}.\end{aligned}$$

Multiplying the second identity by $-i$, the third by -1, the fourth by i, and then summing the four identities, we obtain

$$0=(1-i)[f(E)-f(W)+if(N)-if(S)] = 2(1-i)\bar\partial_\delta f(v)$$

which is exactly the discrete Cauchy-Riemann equation around v. □

Let us now check that the normalized vertex fermionic observable f_δ is s-holomorphic.

(1) Isaacs called such functions "mono-diffric" functions.

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Proposition 4.23.

Fix $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. The function f_δ is s-holomorphic.

We start by two lemmata.

Lemma 4.24. For an edge $e \in \Omega_\delta^\diamond$, $F_\delta(e)$ belongs to $\sqrt{e_b}\,\ell(e)$.

Proof. The winding $W_{\gamma(\omega)}(e, e_b)$ at an edge e can only take its value in the set $W + 2\pi\mathbb{Z}$ where W is the winding at e of an arbitrary oriented path going from e to e_b. Therefore, the winding weight $\mathrm{e}^{\mathrm{i}W_{\gamma(\omega)}(e,e_b)/2}$ involved in the definition of $F_\delta(e)$ is always equal to $\mathrm{e}^{\mathrm{i}W/2}$ or $-\mathrm{e}^{\mathrm{i}W/2}$, *ergo* $F_\delta(e)$ is proportional to $\mathrm{e}^{\mathrm{i}W/2}$. Since $\mathrm{e}^{\mathrm{i}W/2}$ belongs to $\sqrt{e_b}\,\ell(e)$ for any e, so does $F_\delta(e)$. □

Lemma 4.25. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a Dobrushin domain, $p \in [0,1]$ and $q > 0$.

- For $x \in \partial_{ab}$ and $e \in \partial_{ab}^\diamond$ bordering x,

$$F_\delta(e) \;=\; \exp[\mathrm{i}\tfrac{1}{2} W_{\partial_{ab}^\diamond}(e, e_b)] \cdot \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[x \longleftrightarrow \partial_{ba}].$$

- For $u \in \partial_{ba}^\star$ and $e \in \partial_{ba}^\diamond$ bordering u,

$$F_\delta(e) \;=\; \exp[\mathrm{i}\tfrac{1}{2} W_{\partial_{ba}^\diamond}(e, e_b)] \cdot \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[u \overset{\star}{\longleftrightarrow} \partial_{ab}^\star].$$

Proof. We prove the result for $x \in \partial_{ab}$. The proof for $u \in \partial_{ba}^\star$ follows the same lines. Since $\gamma(\omega)$ is the interface between the open cluster connected to ∂_{ba} and the dual open cluster connected to $\partial_{ab}^\star$, x is connected to ∂_{ba} if and only if e is on the exploration path. Therefore,

$$\phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[x \longleftrightarrow \partial_{ba}] = \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[e \in \gamma].$$

The edge e being on the boundary, the winding of the curve is deterministic and equal to $W_{\partial_{ab}^\diamond}(e, e_b)$, thus we find

$$\begin{aligned}
F(e) &= \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[\mathrm{e}^{\mathrm{i}\frac{1}{2} W_{\partial_{ab}^\diamond}(e,e_b)} \mathbf{1}_{e\in\gamma}] \\
&= \mathrm{e}^{\mathrm{i}\frac{1}{2} W_{\partial_{ab}^\diamond}(e,e_b)} \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[e \in \gamma] \\
&= \mathrm{e}^{\mathrm{i}\frac{1}{2} W_{\partial_{ab}^\diamond}(e,e_b)} \phi_{p,q,\Omega_\delta}^{a_\delta, b_\delta}[x \longleftrightarrow \partial_{ba}].
\end{aligned}$$

□

Proof (of Proposition 4.23). Consider a medial vertex $v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$ first. Four medial vertices are adjacent to v. We index them by NW, NE, SE and SW (the notation refers to cardinal directions). Write $\sigma = 1/2 = 1 - \sigma$. When rewriting (4.3) of Lemma 4.15 by setting $1/2 = 1 - \sigma$, we find

$$\overline{F_\delta(NW)} + \overline{F_\delta(SE)} \;=\; \overline{F_\delta(NE)} + \overline{F_\delta(SW)}$$

and therefore

$$F_\delta(NW) + F_\delta(SE) \;=\; F_\delta(NE) + F_\delta(SW).$$

The previous equation and the definition of the normalized vertex fermionic observable imply

$$\sqrt{2e_b}\, f_\delta(v) := \frac{1}{2}\sum_{u\sim v} F_\delta(\{u,v\}) = F_\delta(NW) + F_\delta(SE) = F_\delta(NE) + F_\delta(SW).$$

Using Lemma 4.24, $F_\delta(NW)/\sqrt{2e_b}$ and $F_\delta(SE)/\sqrt{2e_b}$ belong to $\ell(NW)$ and $\ell(SE)$ (they are in particular orthogonal to each other), so that $F_\delta(NW)/\sqrt{2e_b}$ is the projection of $f_\delta(v)$ on $\ell(NW)$ (and similarly for other edges). Therefore, for a medial edge $e=\{u,v\}$, $F_\delta(e)/\sqrt{2e_b}$ is the projection of $f_\delta(u)$ and $f_\delta(v)$ with respect to $\ell(e)$. A direct consequence is that the two projections are equal, *i.e.* that the normalized vertex fermionic observable is s-holomorphic at v.

Let us now treat the case of $v\in\partial\Omega_\delta^\diamond$. We assume without loss of generality that $v\in\partial_{ab}^\diamond$ and we set x to be the primal-vertex bordered by v. Let e and e' be the two medial edges of $\Omega_\delta^\diamond$ incident to v. Lemma 4.25 implies that

$$\frac{1}{\sqrt{2e_b}}F_\delta(e)=\frac{1}{\sqrt{2e_b}}\exp\left[\tfrac{i}{2}W_{\partial_{ab}^\diamond}(e,e_b)\right]\cdot\phi_{p_c,2,\Omega_\delta}^{a_\delta,b_\delta}[e\in\gamma],$$
$$\frac{1}{\sqrt{2e_b}}F_\delta(e')=\frac{1}{\sqrt{2e_b}}\exp\left[\tfrac{i}{2}W_{\partial_{ab}^\diamond}(e',e_b)\right]\cdot\phi_{p_c,2,\Omega_\delta}^{a_\delta,b_\delta}[e\in\gamma].$$

We deduce that

$$\begin{aligned}P_{\ell(e)}(f_\delta(v))&=\frac{2}{2+\sqrt{2}}\left[P_{\ell(e)}(F_\delta(e)/\sqrt{2e_b})+P_{\ell(e)}(F_\delta(e')/\sqrt{2e_b})\right]\\&=\frac{2}{2+\sqrt{2}}\left[F_\delta(e)/\sqrt{2e_b}+\cos(\tfrac{\pi}{4})F_\delta(e)/\sqrt{2e_b}\right]=F_\delta(e)/\sqrt{2e_b}.\end{aligned}$$

(The normalization $\frac{2}{2+\sqrt{2}}$ was introduced in order to have this property.) If $e=\{u,v\}$, we deduce that

$$P_{\ell(e)}(f_\delta(u))=F_\delta(e)/\sqrt{2e_b}=P_{\ell(e)}(f_\delta(v)),$$

where in the second equality we have used the fact that u belongs to $\Omega_\delta^\diamond\setminus\partial\Omega_\delta^\diamond$ (and we can therefore apply what we proved previously). A similar statement can be proved for e', and we deduce that f_δ is s-holomorphic at v, thus concluding the proof. □

4.4 Discrete primitive of the square of f_δ

Since f_δ is predicted to converge to $\sqrt{\phi'}$, and since $\mathrm{Im}(\phi)$ is the harmonic solution of the Dirichlet boundary value problem with boundary conditions 1 on ∂_{ba} and 0 on ∂_{ab}, it is natural to consider the discrete analog H_δ of $\mathrm{Im}(\int^z f_\delta^2)$ defined below.

Theorem 4.26.

There exists a unique function $H_\delta:\Omega_\delta\cup\Omega_\delta^*\to\mathbb{C}$ such that $H_\delta(b_\delta)=1$ and

$$H_\delta(b)-H_\delta(w)\ =\ \sqrt{2}\delta\ \left|P_{\ell(e)}[f_\delta(x)]\right|^2\ \left(=\ \sqrt{2}\delta\ \left|P_{\ell(e)}[f_\delta(y)]\right|^2\right)$$

for every edge $e=\{x,y\}$ of $\Omega_\delta^\diamond$ bordered by $b\in\Omega_\delta$ and $w\in\Omega_\delta^*$. Furthermore, for two neighboring vertices $b_1,b_2\in\Omega_\delta$, with v being the medial vertex at the center of $\{b_1,b_2\}$,

$$H_\delta(b_1)-H_\delta(b_2)\ =\ \mathrm{Im}\left[f_\delta(v)^2\cdot(b_1-b_2)\right],\qquad(4.6)$$

the same relation holding for vertices of Ω_δ^*.

Remark 4.27. The last relation legitimizes the fact that H_δ can be thought of as a discrete analogue of $\mathrm{Im}\left(\int^z f_\delta^2\right)$.

Proof. The uniqueness of H_δ is straightforward since $\Omega_\delta^\diamond$ is connected: the value of H_δ at $x \in \Omega_\delta \cup \Omega_\delta^*$ is simply the sum of increments along an arbitrary path from b_0 to x.

To obtain the existence, construct the value at some point by summing increments along an arbitrary path from b_0 to this point. The only thing to check is that the first displayed equation in the previous theorem is satisfied for the other edges. Equivalently, it is sufficient to check that the sum of increments does not depend on the path chosen between two points. Since the domain is the union of all the faces of $(\delta\mathbb{Z}^2)^\diamond$ within it, it is sufficient to check it for elementary "square" contours around each medial vertex v (these are the simplest closed contours). Therefore, we need to prove that

$$\left|P_{\ell(ne)}[f_\delta(v)]\right|^2 - \left|P_{\ell(se)}[f_\delta(v)]\right|^2 + \left|P_{\ell(sw)}[f_\delta(v)]\right|^2 - \left|P_{\ell(nw)}[f_\delta(v)]\right|^2 = 0, \qquad (4.7)$$

where nw, ne, se and sw are the four medial edges with end-point v, indexed once again according to cardinal directions. Note that $\ell(ne)$ and $\ell(sw)$ (resp. $\ell(se)$ and $\ell(nw)$) are orthogonal. Hence, (4.7) follows from

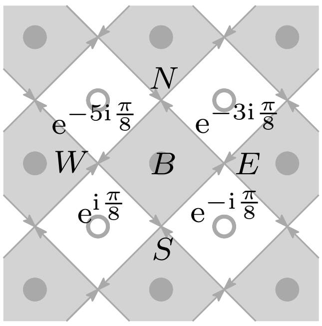
$$\left|P_{\ell(ne)}[f_\delta(v)]\right|^2 + \left|P_{\ell(sw)}[f_\delta(v)]\right|^2 = |f_\delta(v)|^2 = \left|P_{\ell(se)}[f_\delta(v)]\right|^2 + \left|P_{\ell(nw)}[f_\delta(v)]\right|^2.$$

Let us now turn to (4.6). Let $b_1 \sim b_2$ be two neighboring vertices of Ω_δ and v the medial-vertex associated to $\{b_1, b_2\}$. Let w be one of the dual-vertices in Ω_δ^* adjacent to both b_1 and b_2 (there may be only one of them in Ω_δ^* if b_1 and b_2 are on the boundary). Let e_1 and e_2 be the two medial edges bordered by b_1 and w, and b_2 and w respectively. We find

$$\begin{aligned}
H_\delta(b_1) - H_\delta(b_2) &= \sqrt{2}\delta\left[\left|P_{\ell(e_1)}[f_\delta(v)]\right|^2 - \left|P_{\ell(e_2)}[f_\delta(v)]\right|^2\right]\\
&= \frac{1}{2}\left[\left(\sqrt{e_1}f_\delta(v) + \overline{\sqrt{e_1}f_\delta(v)}\right)^2 - \left(\sqrt{e_2}f_\delta(v) + \overline{\sqrt{e_2}f_\delta(v)}\right)^2\right]\\
&= \frac{1}{2}\left[e_1 f_\delta(v)^2 + \overline{e_1 f_\delta(v)^2} + |f_\delta(v)|^2 - e_2 f_\delta(v)^2 - \overline{e_2 f_\delta(v)^2} - |f_\delta(v)|^2\right]\\
&= \frac{1}{2}\left[(e_1 - e_2)f_\delta(v)^2 + \overline{(e_1 - e_2)f_\delta(v)^2}\right]\\
&= \frac{1}{2\mathrm{i}}\left[(b_1 - b_2)f_\delta(v)^2 - \overline{(b_1 - b_2)f_\delta(v)^2}\right] = \mathrm{Im}[f_\delta(v)^2 \cdot (b_1 - b_2)].
\end{aligned}$$

In the second equality, we used the fact that $\frac{\delta}{\sqrt{2}} = |e_1|$ and $\frac{\delta}{\sqrt{2}} = |e_2|$. □

Figure 4.14. The different directions of the lines $\ell(e)$ for medial edges around a black face.



We wish to prove the following convergence result.

Theorem 4.28 (Smirnov [Smi 10]).

Fix $q=2$ and $p=p_c(2)=\sqrt{2}/(1+\sqrt{2})$. Let (Ω,a,b) be a simply connected domain with two marked points on its boundary. Let $(\Omega_\delta^\diamond,a_\delta^\diamond,b_\delta^\diamond)$ be a family of Dobrushin domains converging to (Ω,a,b) in the Carathéodory sense. Let H_δ be defined as before on $(\Omega_\delta^\diamond,a_\delta^\diamond,b_\delta^\diamond)$. We have

$$H_\delta(\cdot)\ \to\ \mathrm{Im}(\phi)\quad \text{when } \delta\to 0$$

uniformly on any compact subset of Ω, where ϕ is any conformal map from Ω to the strip $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞.

As mentioned before, $\mathrm{Im}(\phi)$ is the solution of the Dirichlet Boundary Value Problem with 0 boundary conditions on ∂_{ab} and 1 on ∂_{ba}. Indeed, fix a solution H of this BVP. The function $H\circ\phi^{-1}$ is solution of the Dirichlet problem on $\mathbb{R}\times(0,1)$ with boundary condition 1 on the top and 0 on the bottom. Therefore, $(H\circ\phi^{-1})(z)=\mathrm{Im}(z)$ which leads to $H(z')=\mathrm{Im}(\phi(z'))$ for any $z'\in\Omega$.

The natural strategy to prove Theorem 4.28 would be to prove that H_δ is the discrete solution of a discrete version of this Dirichlet Boundary Value Problem. The problem is that even if the primitive of an holomorphic map is holomorphic and thus harmonic, this is not the case of the primitive of the square of a s-holomorphic map. Nonetheless, H_δ satisfies subharmonic and superharmonic properties, which would ultimately be sufficient for our purpose. More precisely, denote by $H_\delta^\bullet$ and H_δ° the restrictions of $H_\delta:\Omega_\delta\cup\Omega_\delta^*\to\mathbb{C}$ to Ω_δ (black faces) and Ω_δ^* (white faces) respectively. Let $\Delta^\bullet$ and Δ° be the nearest-neighbor discrete Laplacian for functions on Ω_δ and Ω_δ^* respectively.

Proposition 4.29.

If $f_\delta:\Omega_\delta^\diamond\to\mathbb{C}$ is s-holomorphic, then $H_\delta^\bullet$ and H_δ° are respectively subharmonic for $\Delta^\bullet$ on $\Omega_\delta\setminus\partial\Omega_\delta$ and superharmonic for Δ° on $\Omega_\delta^*\setminus\partial\Omega_\delta^*$.

Proof. Let us focus on $H_\delta^\bullet$ (the proof for H_δ° follows the same lines). Let B be a vertex of $\Omega_\delta\setminus\partial\Omega_\delta$. Let N, E, S and W be the four medial-vertices adjacent to B (once again the letters refer to cardinal directions). Also set

$$\begin{aligned}
a&=\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}P_{\ell([ES])}[f_\delta(E)]=\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}P_{\ell([ES])}[f_\delta(S)],\\
b&=\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}P_{\ell([SW])}[f_\delta(S)]=\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}P_{\ell([SW])}[f_\delta(W)],\\
c&=\mathrm{e}^{5\mathrm{i}\frac{\pi}{8}}P_{\ell([WN])}[f_\delta(W)]=\mathrm{e}^{5\mathrm{i}\frac{\pi}{8}}P_{\ell([WN])}[f_\delta(N)],\\
d&=\mathrm{e}^{3\mathrm{i}\frac{\pi}{8}}P_{\ell([NE])}[f_\delta(N)]=\mathrm{e}^{3\mathrm{i}\frac{\pi}{8}}P_{\ell([NE])}[f_\delta(E)].
\end{aligned}$$

Note that a, b, c and d are real. With these definitions, we may rewrite $f_\delta(N)$, $f_\delta(E)$, $f_\delta(S)$ and $f_\delta(W)$ as follows:

$$\begin{aligned}
f_\delta(E)&=\sqrt{2}\mathrm{i}(e^{-3i\pi/8}d+\mathrm{e}^{-\mathrm{i}\pi/8}a),\\
f_\delta(S)&=\sqrt{2}\mathrm{i}(\mathrm{e}^{3\mathrm{i}\pi/8}a-\mathrm{e}^{5\mathrm{i}\pi/8}b),\\
f_\delta(W)&=\sqrt{2}\mathrm{i}(\mathrm{e}^{\mathrm{i}\pi/8}b-\mathrm{e}^{3\mathrm{i}\pi/8}c),\\
f_\delta(N)&=\sqrt{2}\mathrm{i}(\mathrm{e}^{-\mathrm{i}\pi/8}c-\mathrm{e}^{\mathrm{i}\pi/8}d).
\end{aligned}$$

By definition of $\Delta^\bullet$ and (4.6), we find

$$\begin{aligned}\Delta^\bullet H^\bullet_\delta(B) &= \frac{\delta}{4}\mathrm{Im}\big[f_\delta(E)^2 - \mathrm{i}f_\delta(S)^2 - f_\delta(W)^2 + \mathrm{i}f_\delta(N)^2\big]\\ &= -\frac{\delta}{2}\mathrm{Im}\big[(e^{-3i\pi/8}d + \mathrm{e}^{-\mathrm{i}\pi/8}a)^2 + \mathrm{i}(\mathrm{e}^{3\mathrm{i}\pi/8}a - \mathrm{e}^{5\mathrm{i}\pi/8}b)^2\\ &\qquad -(\mathrm{e}^{\mathrm{i}\pi/8}b - \mathrm{e}^{3\mathrm{i}\pi/8}c)^2 - \mathrm{i}(\mathrm{e}^{-\mathrm{i}\pi/8}c - \mathrm{e}^{\mathrm{i}\pi/8}d)^2\big]\\ &= \delta\left[a^2+b^2+c^2+d^2-\sqrt{2}(ab+bc+cd-ad)\right]\end{aligned}$$

On the other hand, let us compute

$$\begin{aligned}|f_\delta(E)-f_\delta(S)|^2 + |f_\delta(W)-f_\delta(N)|^2 &= 2|e^{-3i\pi/8}d + \mathrm{e}^{-\mathrm{i}\pi/8}a - \mathrm{e}^{3\mathrm{i}\pi/8}a + \mathrm{e}^{5\mathrm{i}\pi/8}b|^2\\ &\quad + 2|\mathrm{e}^{\mathrm{i}\pi/8}b - \mathrm{e}^{3\mathrm{i}\pi/8}c - \mathrm{e}^{-\mathrm{i}\pi/8}c + \mathrm{e}^{\mathrm{i}\pi/8}d|^2\\ &= 2(d+\sqrt{2}a-b)^2 + 2(b-\sqrt{2}c+d)^2\\ &= 4(a^2+b^2+c^2+d^2) - 4\sqrt{2}(ab+bc+cd-ad).\end{aligned}$$

In conclusion,

$$4\Delta^\bullet H^\bullet_\delta(B) = \delta|f_\delta(E)-f_\delta(S)|^2 + \delta|f_\delta(W)-f_\delta(N)|^2 \geqslant 0$$

and the claim follows. □

Let us now study the boundary values of H.

Lemma 4.30. The function H_δ equals 0 on ∂^*_{ab} and 1 on ∂_{ba}.

Proof. We first prove that $H^\bullet_\delta$ is constant on ∂_{ba}. Let B and B' be two adjacent consecutive sites of ∂_{ba}, and x the medial-edge in the middle of the edge $\{B,B'\}$. Note that x is on the boundary of $\partial\Omega^\diamond_\delta$. The argument of $f_\delta(x)$ can be obtained easily from Lemma 4.25. In particular, (4.6) implies that $H^\bullet_\delta(B) = H^\bullet_\delta(B')$. Hence, $H^\bullet_\delta$ is constant along the arc. Since $H^\bullet_\delta(b_\delta) = 1$, the result follows readily.

Similarly, H°_δ is constant on the arc ∂^*_{ab}. Moreover, the dual white face $b^*_\delta \in \partial^*_{ab}$ bordering b_δ (see Fig. 4.8) satisfies

$$H^\circ_\delta(b^*_\delta) \;=\; H^\bullet_\delta(b_\delta) - \sqrt{2}\delta|P_{\ell(e_b)}[f(b_\delta)]|^2 \;=\; 1-1 \;=\; 0.$$

In the second equality, we used the normalization hypothesis (recall that $|e_b| = \delta/\sqrt{2}$) and the fact that

$$P_{\ell(e_b)}[f(b_\delta)] = \frac{1}{\sqrt{2e_b}}F_\delta(e_b) = \frac{1}{\sqrt{2e_b}}.$$

Therefore $H^\circ_\delta = 0$ on ∂^*_{ab}. □

Lemma 4.31. For any $r>0$ and $\varepsilon>0$, there exists $\delta_0 = \delta_0(r,\varepsilon) > 0$ such that for any $\delta \leqslant \delta_0$, $|H_\delta(x) - 1| \leqslant \varepsilon$ (resp. $|H(x)-\delta| \leqslant \varepsilon$) for any $x \in \partial_{ab}$ (resp. $x \in \partial^*_{ba}$) at distance larger than r from a and b.

Proof. Observe that for any $v \in \Omega_\delta$ and $e \in E(\Omega_\delta)$ incident to v,

$$|\sqrt{2e_b}P_{\ell(e)}[f_\delta(v)]| = |F_\delta(e)| \leqslant \phi^{a_\delta,b_\delta}_{p_c,2,\Omega_\delta}[e\in\gamma].$$

Let us assume for a moment that e is on the free arc ∂_{ab}. For e to be in γ, there must be an open path going from the vertex $x \in \Omega_\delta$ bordered by e, going to distance r. Let $B_\delta(x,r) := \delta\mathbb{Z}^2 \cap (x+[-r,r]^2)$. Therefore,

$$\begin{aligned}\phi^{a_\delta,b_\delta}_{p_c,2,\Omega_\delta}[e\in\gamma] &\leqslant \phi^{a_\delta,b_\delta}_{p_c,2,\Omega_\delta}[x \longleftrightarrow \partial B_\delta(x,r)]\\ &\leqslant \phi^1_{p_c,2,B_\delta(x,r)}[x \longleftrightarrow \partial B_\delta(x,r)]\\ &= \phi^1_{p_c,2,\Lambda_{r/\delta}}[0 \longleftrightarrow \partial\Lambda_{r/\delta}].\end{aligned}$$

In the last line, we consider the measure on $\mathbb{Z}^2$ again. We used the comparison between boundary conditions in the second line. The result follows from $\phi^1_{p_c,2,\Lambda_{r/\delta}}[0 \longleftrightarrow \partial\Lambda_{r/\delta}]$ tends to 0 (by Theorem 4.6 and Corollary 3.13), as well as from the fact that $|e_b| = \delta/\sqrt{2}$. □

Proof (of Theorem 4.28). Recall that here convergence means uniform convergence on every compact subset of Ω. Since $H^\bullet_\delta$ is sub-harmonic for $\Delta^\bullet$, it is smaller than the $\Delta^\bullet$-harmonic function $\widehat{H}^\bullet_\delta$ with the same boundary conditions. Since $H^\bullet_\delta$ converges to the solution $\mathrm{Im}(\phi)$ of the continuum Dirichlet BVP with boundary condition 0 on ∂_{ab} and 1 on ∂_{ba}, we therefore deduce that

$$\limsup_{\delta\to 0} H^\bullet_\delta \leqslant \mathrm{Im}(\phi).$$

Now, H°_δ is super-harmonic for Δ°, it is thus larger than the Δ°-harmonic function $\widehat{H}^\circ_\delta$ with the same boundary conditions. In particular, $\widehat{H}^\circ_\delta$ converges to $\mathrm{Im}(\phi)$, and therefore

$$\liminf_{\delta\to 0} H^\circ_\delta \geqslant \mathrm{Im}(\phi).$$

But by construction, $H^\circ_\delta(W)$ is smaller than $H^\bullet_\delta(B)$ for any neighbor B of W. Therefore,

$$\mathrm{Im}(\phi) \leqslant \liminf_{\delta\to 0} H^\circ_\delta \leqslant \liminf_{\delta\to 0} H^\bullet_\delta \leqslant \mathrm{Im}(\phi)$$

and the same holds true for the lim sup. Therefore, both $H^\bullet_\delta$ and H°_δ converge to $\mathrm{Im}(\phi)$. □

4.5 From Theorem 4.28 to Theorem 4.19

Proof (of Theorem 4.19). Below, Q denotes a square, and $9Q$ denotes the square of same center, but 9 times bigger. Let $Q \subset \Omega$ such that $9Q \subset \Omega$ (recall the definition of $9Q$ from Theorem 4.32). Since H_δ converges uniformly to the continuous function $\mathrm{Im}(\phi)$, the family H_δ is bounded uniformly in $\delta > 0$. Theorem 4.32 below thus implies that $(f_\delta)_{\delta>0}$ is a precompact family of functions on Q.

Let $(f_{\delta_n})_{n\in\mathbb{N}}$ be a convergent subsequence and denote its limit by f. For two points x and y in Ω, we have:

$$H_{\delta_n}(y_{\delta_n}) - H_{\delta_n}(x_{\delta_n}) = \mathrm{Im}\left(\int_{x_{\delta_n}}^{y_{\delta_n}} f^2_{\delta_n}(z)\,\mathrm{d}z\right)$$

where x_{δ_n} and y_{δ_n} denote the closest points to x and y in Ω_{δ_n}. On the one hand, the convergence of $(f_{\delta_n})_{n\in\mathbb{N}}$ being uniform on any compact subset of Ω, the right hand side converges to $\mathrm{Im}\left(\int_x^y f(z)^2\,\mathrm{d}z\right)$. On the other hand, the left hand side converges to $\mathrm{Im}(\phi(y) - \phi(x))$.

Recall that s-holomorphic functions are discrete holomorphic by Proposition 4.22. Hence, f is holomorphic as limit of the discrete holomorphic functions $(f_{\delta_n})_{n\in\mathbb{N}}$ (we leave this easy property of discrete holomorphic functions as an exercise for the reader). Since both $\mathrm{Im}(\phi(y) - \phi(x))$ and $\mathrm{Im}\left(\int_x^y f(z)^2\,\mathrm{d}z\right)$ are harmonic functions of y, there exists $C \in \mathbb{R}$ such that $\phi(y) - \phi(x) = C + \int_x^y f(z)^2\,\mathrm{d}z$ for every $x, y \in \Omega$. We deduce that f equals $\sqrt{\phi'}$. Since this is true for any convergent subsequence, we find that f_δ tends to $\sqrt{\phi'}$. □

The proof of Theorem 4.19 is now complete subject to the proof of the following theorem, that we sketch here (we refer to [DC 13] for details).

Theorem 4.32 (Precompactness for s-holomorphic functions).

Let $Q \subset \Omega$ such that $9Q \subset \Omega$. Let $(f_\delta)_{\delta>0}$ be a family of s-holomorphic maps on $\Omega_\delta^\diamond$ and $(H_\delta)_{\delta>0}$ be the corresponding functions defined in the previous section. If $(H_\delta)_{\delta>0}$ is uniformly bounded on $9Q$, then $(f_\delta)_{\delta>0}$ is a precompact family of functions on Q.

Proof (sketch). Color the vertices of $(\delta\mathbb{Z}^2)^\diamond$ in black and white in a chessboard way (medial vertices corresponding to vertical primal edges are all colored the same, and the same for horizontal edges). The sets of black and white vertices are denoted by $(\delta\mathbb{Z}^2)^\diamond_\bullet$ and $(\delta\mathbb{Z}^2)^\diamond_\circ$ respectively.

Since f_δ is s-holomorphic, it is also holomorphic and therefore discrete harmonic for the standard Laplacian on $(\delta\mathbb{Z}^2)^\diamond_\bullet$, *i.e.* that

$$\Delta_\delta f_\delta(x) := \frac{1}{4}\sum_{y\sim x}\big[f_\delta(y)-f_\delta(x)\big] = 0$$

for any $x \in (\Omega_\delta)^\diamond_\bullet$, where $\sim$ means that y and x are nearest neighbors in $(\Omega_\delta)^\diamond_\bullet$. Let us denote the restriction of f_δ to $(\Omega_\delta)^\diamond_\bullet$ by $f_\delta^\bullet$.

In such case, there exists $C = C(Q) > 0$ such that for any two neighboring vertices x and y in $Q \cap \delta\mathbb{Z}^2$,

$$|f_\delta^\bullet(x) - f_\delta^\bullet(y)| \leqslant C\delta\ \sup\{|f_\delta^\bullet(z)| : z \in \Omega_\delta\}. \tag{4.10}$$

(We leave this classical inequality as an exercise[(2)].) Imagine for a moment that the family of functions $(f_\delta^\bullet)$ satisfies the following property: there exists $C > 0$ such that for any $\delta > 0$,

$$\delta^2 \sum_{v\in Q_\delta^\diamond} \left|f_\delta^\bullet(v)\right|^2 \leqslant C. \tag{4.11}$$

Then, the equation

$$f_\delta^\bullet(x) \;=\; \sum_{y\in\partial 9Q_\delta^\diamond} f_\delta^\bullet(y) H_{9Q_\delta^\diamond}(x,y),$$

where $H_{9Q_\delta^\diamond}(\cdot,\cdot)$ is the harmonic measure on $9Q_\delta^\diamond$, implies that f_δ is bounded on $Q_\delta^\diamond$. If f_δ is bounded on $Q_\delta^\diamond$, Equation (4.10) combined with Ascoli's theorem imply that the family $(f_\delta^\bullet)_{\delta>0}$ is precompact for the uniform topology on compact subsets of Ω.

Let us now use the s-holomorphicity to deduce that $(f_\delta)_{\delta>0}$ itself is precompact. Let $x \in \Omega_\delta^\diamond \cap (\delta\mathbb{Z}^2)^\diamond_\circ$. Denote the north-east and south-west neighboring vertices of x in $(\delta\mathbb{Z}^2)^\diamond_\bullet$ by y and z. The s-holomorphicity shows that

$$\begin{aligned} f_\delta(x) &= P_{\ell(xy)}(f_\delta(x)) + P_{\ell(xz)}(f_\delta(x)) \\ &= P_{\ell(xy)}(f_\delta(y)) + P_{\ell(xz)}(f_\delta(z)) \\ &= f_\delta(y) + O\big(|f_\delta(z) - f_\delta(y)|\big), \end{aligned} \tag{4.13}$$

where we used the fact that $\ell(xy)$ and $\ell(xz)$ are orthogonal to each others. The previous paragraph implies that we may extract a sub-sequence $(f_{\delta_n}^\bullet)_n$ converging uniformly on every compact subset of Ω when seen as a function of $\Omega_\delta^\diamond \cap (\delta\mathbb{Z}^2)^\diamond_\bullet$. The relation (4.13) implies that (f_{δ_n}) itself converges uniformly on every compact subset of Ω.

[(2)]Simply use the fact that $(f_\delta^\bullet(X_n))_{n\geqslant 0}$ is a martingale whenever (X_n) is a simple random walk on $(\delta\mathbb{Z}^2)^\diamond_\bullet$ stopped on first exiting $(\Omega_\delta)^\diamond_\bullet$.

Therefore, we would be done if we could prove (4.11). Fix $\delta>0$. When jumping over a medial-vertex v, the function H_δ changes by $\delta\mathrm{Re}(f_\delta^2(v))$ or $\delta\mathrm{Im}(f_\delta^2(v))$ depending on the direction (vertical or horizontal), so that

$$\delta^2\sum_{v\in Q_\delta^\diamond}\left|f_\delta(v)\right|^2 = \delta\sum_{x\in Q_\delta}|\nabla H_\delta^\bullet(x)| + \delta\sum_{x\in Q_\delta^*}|\nabla H_\delta^\circ(x)|$$

where $\nabla H_\delta^\bullet(x)=(H_\delta^\bullet(x+\delta)-H_\delta^\bullet(x),H_\delta^\bullet(x+i\delta)-H_\delta^\bullet(x))$, and ∇H_δ° is defined similarly for H_δ°. It follows that it is enough to prove uniform boundedness of the right-hand side in (4.14). We only treat the sum involving $H_\delta^\bullet$, the other sum can be handled similarly.

Write $H_\delta^\bullet=S_\delta+R_\delta$ where S_δ is a harmonic function with same boundary conditions on $\partial 9Q_\delta$ as $H_\delta^\bullet$. In order to prove that the sum of $|\nabla H_\delta^\bullet|$ on Q_δ is bounded by C/δ, we deal separately with $|\nabla S_\delta|$ and $|\nabla R_\delta|$. First,

$$\begin{aligned}\sum_{x\in Q_\delta}\left|\nabla S_\delta(x)\right| &\leqslant \frac{C_1}{\delta^2}\cdot C_2\delta\left(\sup_{x\in 9Q_\delta}|S_\delta(x)|\right)=\frac{C_1}{\delta^2}\cdot C_2\delta\left(\sup_{x\in\partial 9Q_\delta}|S_\delta(x)|\right)\\ &=\frac{C_3}{\delta}\left(\sup_{x\in\partial 9Q_\delta}|H_\delta^\bullet(x)|\right)\leqslant\frac{C_4}{\delta},\end{aligned}$$

where in the first inequality we used derivative estimates like (4.10) for S_δ, in the first equality the maximum principle[3] for S_δ (to show that the supremum is reached on the boundary), and in the second the fact that S_δ and $H_\delta^\bullet$ share the same boundary conditions on $9Q_\delta$. The last inequality comes from the fact that $H_\delta^\bullet$ remains bounded uniformly in δ.

Second, let us treat $|\nabla R_\delta|$. This function is subharmonic since S_δ is harmonic and $H_\delta^\bullet$ is subharmonic (Proposition 4.29). Introduce the Green function $G_{9Q_\delta}(\cdot,y)$ in $9Q_\delta$ with singularity at y defined by the fact that the function has Laplacian equal to -1 at y, and 0 everywhere else in Ω_δ°, and that $G_{9Q_\delta}(\cdot,y)$ equals 0 outside Ω_δ°. Since R_δ equals 0 on the boundary, one may easily check Riesz's formula:

$$R_\delta(x) = \sum_{y\in 9Q_\delta}\Delta R_\delta(y)G_{9Q_\delta}(x,y).$$

We deduce that

$$\nabla R_\delta(x) = \sum_{y\in 9Q_\delta}\Delta R_\delta(y)\nabla_x G_{9Q_\delta}(x,y)$$

Therefore,

$$\begin{aligned}\sum_{x\in Q_\delta}\left|\nabla R_\delta(x)\right| &= \sum_{x\in Q_\delta}\Big|\sum_{y\in 9Q_\delta}\Delta R_\delta(y)\nabla_x G_{9Q_\delta}(x,y)\Big|\\ &\leqslant \sum_{y\in 9Q_\delta}\Delta R_\delta(y)\sum_{x\in Q_\delta}|\nabla_x G_{9Q_\delta}(x,y)|\\ &\leqslant \sum_{y\in 9Q_\delta}\Delta R_\delta(y)\,C_5\delta\sum_{x\in Q_\delta}G_{9Q_\delta}(x,y)\\ &= C_5\delta\sum_{x\in Q_\delta}\sum_{y\in 9Q_\delta}\Delta R_\delta(y)G_{9Q_\delta}(x,y)\\ &= C_5\delta\sum_{x\in Q_\delta}R_\delta(x) = C_6/\delta\end{aligned}$$

[3] We also leave this fact as an easy exercise to the reader: show that for a discrete harmonic function, the maximum and minimum of the function are reached on the boundary of the function.

Lattice spin models | Bernoulli percolation | Potts model | **Random-cluster model** | Ising model

The second line uses the fact that $\Delta R_\delta \geqslant 0$, the third Theorem 4.33 below, the fifth the Riesz's formula again, and the last equality the fact that Q_δ contains of order $1/\delta^2$ sites and the fact that R_δ is bounded uniformly in δ (since H_δ and S_δ are). Thus, $\delta \sum_{x\in Q_\delta} |\nabla H_\delta^\bullet|$ is uniformly bounded. Since the same result holds for H_δ°, we obtain (4.11) and we are done. □

Theorem 4.33.

There exists $C>0$ such that for any $\delta>0$ and $y\in 9Q_\delta$,

$$\sum_{x\in Q_\delta} |\nabla_x G_{9Q_\delta}(x,y)| \leqslant C\delta \sum_{x\in Q_\delta} G_{9Q_\delta}(x,y).$$

We omit the proof of this statement, which can be found in the original article [Smi 10].

4.6 Conformal invariance: further results

Conformal invariance of the fermionic observable can be used to derive conformal invariance of many other quantities of the model. Let us mention a few of them.

Let us start by conformal invariance of interfaces. Conformal field theory leads to the prediction that the exploration path mentioned before converges as $\delta\to 0$ to a random, continuous, non-self-crossing curve from a to b staying in Ω, and which is expected to be conformally invariant in the following sense.

Definition 4.34.

A family of random non-self-crossing continuous curves $\gamma_{(\Omega,a,b)}$, going from a to b and contained in Ω, indexed by simply connected domains with two marked points on the boundary (Ω,a,b) is ***conformally invariant*** if for any (Ω,a,b) and any conformal map $\psi:\Omega\to\mathbb{C}$,

$$\psi(\gamma_{(\Omega,a,b)}) \text{ has the same law as } \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

In words, the random curve obtained by taking the scaling limit of the random-cluster model in $(\psi(\Omega),\psi(a),\psi(b))$ has the same law as the image by ψ of the random curve obtained by taking the scaling limit of the random-cluster model in (Ω,a,b).

In 1999, Schramm proposed a natural candidate for the possible conformally invariant families of continuous non-self-crossing curves. He noticed that interfaces of models further satisfy the *domain Markov property* which, together with the assumption of conformal invariance, determine a one-parameter family of possible random curves. In [Sch 00], he introduced the Stochastic Loewner evolution (SLE for short) which is now known as the **Schramm–Loewner evolution**. For $\varkappa>0$, a domain Ω and two points a and b on its boundary, $SLE(\varkappa)$ is the random Loewner evolution in Ω from a to b with driving process $\sqrt{\varkappa}B_t$, where (B_t) is a standard Brownian motion (we refer to the literature for a precise definition). By construction, the process is conformally invariant, random and fractal. In addition, it is possible to study quite precisely the behavior of SLEs using stochastic calculus and to derive path properties such as the Hausdorff dimension, intersection exponents, etc...

Recently, the interfaces in the critical random-cluster model with $q = 2$ were proved to convergence to a SLE curve. In order to state the result, we need the following notion. Let X be the set of continuous parametrized curves and d be the distance on X defined for $\gamma_1 : I \to \mathbb{C}$ and $\gamma_2 : J \to \mathbb{C}$ by

$$d(\gamma_1,\gamma_2) = \min_{\substack{\varphi_1:[0,1]\to I\\ \varphi_2:[0,1]\to J}} \sup_{t\in[0,1]} |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|,$$

where the minimization is over *increasing bijective functions* φ_1 and φ_2. Note that I and J can be equal to $\mathbb{R}_+ \cup \{\infty\}$. The topology on (X,d) gives rise to a notion of weak convergence for random curves on X.

Theorem 4.35 (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov). (See [CDCH+ 14].) Let Ω be a simply connected domain with two marked points a and b on its boundary. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. The exploration path γ_δ of the *critical* random-cluster model with $q = 2$ with Dobrushin boundary conditions in $(\Omega_\delta, a_\delta, b_\delta)$ converges weakly to SLE(16/3) as $\delta \to 0$.

The strategy of the proof is the following. First, prove that the family (γ_δ) is tight for the weak convergence (in fact one needs to prove a slightly stronger result). The proof of this fact can be found in [KS 12, CDCH 16, DCS 12a]. Second, identify the possible sub-sequential limits using the fermionic observable. More precisely, imagine for a moment that a sub-sequential limit γ can be parametrized by a Loewner chain, and that its driving process is given by (W_t). The fermionic observable may be seen as a martingale for the exploration process, a fact which implies that its limit is a martingale for γ. This martingale property, together with Itô's formula, allows to prove that W_t and $W_t^2 - \varkappa t$ are martingales (where $\varkappa$ equals 16/3 for the FK-Ising model). Lévy's theorem thus implies that $W_t = \sqrt{\varkappa} B_t$. This identifies SLE($\varkappa$) as being the only possible sub-sequential limit, which proves that (γ_δ) converges to SLE($\varkappa$).

The Ising model itself was proved to be conformally invariant in [CS 12]. Since then, conformal invariance of many quantities have been derived, including crossing probabilities [BDCH 14, Izy 15], other interfaces [CDCH+ 14, HK 13]. The energy and spin fields were also proved to be conformally invariant in a series of paper (respectively [HS 13, Hon 10] and [CI 13, CHI 15]). The observable has also been used off criticality, see [BDC 12, DCGP 14].

To conclude, let us mention that convergence to a SLE curve should occur for any $q \leqslant 4$.

Conjecture 4.36 (Schramm). Let Ω be a simply connected domain with two marked points a and b on its boundary. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. The exploration path γ_δ of the *critical* random-cluster model with parameters q and $p = p_c(q) = \sqrt{q}/(1+\sqrt{q})$ with Dobrushin boundary conditions in $(\Omega_\delta, a_\delta, b_\delta)$ converges weakly to SLE($\varkappa$) as $\delta \to 0$, where

$$\varkappa = \frac{8}{\sigma+1} = \frac{4\pi}{\pi - \arccos(\sqrt{q}/2)}.$$

CHAPTER 5

The random current representation of the Ising model

We now focus on a graphical representation of the ferromagnetic Ising model. This alternative perspective on the Ising model's phase transition is driven by the observation that the onset of long range order coincides with a percolation transition in a dual system of currents. This point of view, as an intuitive guide to diagrammatic bounds which under certain conditions provide information on the critical model's scaling limits, was developed in [Aiz 82] and a number of subsequent works.

We start by introducing this representation, called the *random current representation*. We then present three applications of this representation. In this section $A\Delta B := (A\setminus B)\cup(B\setminus A)$ denotes the symmetric difference between A and B.

We do not restrict our attention to the nearest neighbor model and treat general models with interactions $(J_{xy} : x,y \in \mathbb{Z}^d)$ with $J_{xy} \geqslant 0$ for any x,y. In this section, edges are simply pairs $\{x,y\} \subset \mathbb{Z}^d$. The definition of percolation configurations is modified accordingly.

To adopt standard notation, we will write $\mu^+_{G,\beta}$ instead of $\mu^1_{G,\beta}$. Also note that the Ising model may be defined in infinite-volume thanks to the coupling with the random-cluster model[(1)]: consider the infinite-volume random-cluster measure with free boundary conditions and assign to each cluster a spin $+$ or $-$ to obtain a measure that we denote by μ^{free}_β. Similarly, consider the infinite-volume random-cluster measure with wired boundary conditions and assign to each finite cluster a spin $+$ or $-$, and to the infinite cluster (if it exists) a spin $+$, to obtain μ^+_β. We have that $\mu^{\text{free}}_{G,\beta}$ and $\mu^+_{G,\beta}$ converge as $G \nearrow \mathbb{Z}^d$ to μ^{free}_β and μ^+_β respectively.

Finally, let us introduce for $A \subset V(G)$, the notation

$$\sigma_A := \prod_{x\in A} \sigma_x.$$

[(1)]Here a long range random-cluster model, which can be seen as a random-cluster on a complete graph with edge-weight depending on edges.

1 The random current representation

Definition 5.1.

A ***current*** $\mathbf{n}$ on $G \subset \mathbb{Z}^d$ (also called a current configuration) is a function from pairs of points in $V(G)$ to $\mathbb{N} := \{0,1,2,\ldots\}$. A ***source*** of $\mathbf{n} = (\mathbf{n}_{xy} : \{x,y\} \subset G)$ is a vertex x for which $\sum_{y \in G} \mathbf{n}_{xy}$ is odd. The set of sources of $\mathbf{n}$ is denoted by $\partial \mathbf{n}$. The collection of current configurations on G is denoted by Ω_G. Also set

$$w_\beta(\mathbf{n}) = \prod_{\{x,y\} \subset G} \frac{(\beta J_{xy})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

Let us describe the connection between currents and the Ising model. We start with the random current representation for free boundary conditions. For $\beta > 0$, G a finite graph and $A \subset V(G)$, introduce the quantity

$$Z^{\text{free}}(G,\beta,A) = \sum_{\sigma \in \{-1,1\}^G} \sigma_A \prod_{\{x,y\} \subset G} \exp[\beta J_{xy} \sigma_x \sigma_y]. \tag{5.1}$$

Proposition 5.2 (Random current representation for free boundary conditions).

Let $\beta > 0$, G be a finite graph and $A \subset V(G)$, then

$$Z^{\text{free}}(G,\beta,A) = 2^{|V(G)|} \sum_{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = A} w_\beta(\mathbf{n}). \tag{5.2}$$

Proof. Expanding $e^{\beta J_{xy}\sigma_x\sigma_y}$ for each $\{x,y\}$ into

$$e^{\beta J_{xy}\sigma_x\sigma_y} = \sum_{\mathbf{n}_{xy}=0}^{\infty} \frac{(\beta J_{xy}\sigma_x\sigma_y)^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}$$

and substituting this relation in (5.1), one gets

$$Z^{\text{free}}(G,\beta,A) = \sum_{\mathbf{n} \in \Omega_G} w_\beta(\mathbf{n}) \sum_{\sigma \in \{-1,1\}^G} \prod_{x \in G} \sigma_x^{1_{x\in A} + \sum_{y \in G} \mathbf{n}_{xy}}.$$

Now, the symmetry $-\sigma \leftrightarrow \sigma$ implies that

$$\sum_{\sigma \in \{-1,1\}^G} \prod_{x \in G} \sigma_x^{1_{x\in A} + \sum\limits_{y \in G} \mathbf{n}_{xy}} = \begin{cases} 0 & \text{if } \sum_{y \in G} \mathbf{n}_{xy} \text{ is odd for some } x \notin A \\ & \text{or even for some } x \in A, \\ 2^{|V(G)|} & \text{otherwise.} \end{cases}$$

Thus, the definition of a current's source enables one to write

$$Z^{\text{free}}(G,\beta,A) = 2^{|V(G)|} \sum_{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = A} w_\beta(\mathbf{n}). \qquad \square$$

We deduce that for every $A \subset V(G)$,

$$\mu^{\text{free}}_{G,\beta}[\sigma_A] = \frac{Z^{\text{free}}(G,\beta,A)}{Z^{\text{free}}(G,\beta,\varnothing)} = \frac{\sum\limits_{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = A} w_\beta(\mathbf{n})}{\sum\limits_{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \varnothing} w_\beta(\mathbf{n})}. \tag{5.3}$$

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Remark 5.3. Note that this does not completely correspond to a graphical representation as defined in the previous sections, since the spin-spin correlation get rephrased in terms of sum of different types of currents, and cannot be interpreted as the probability of connection for some percolation model. We will see later in this section how to make this representation fit in the framework of graphical representations.

Remark 5.4. Equation (5.3) implies the first **Griffiths' inequality** yielding that for any $A \subset V(G)$,

$$\mu^{\text{free}}_{G,\beta}[\sigma_A] > 0.$$

Let us now turn to the random current representation for $+$ boundary conditions. Introduce an additional vertex $g \notin \mathbb{Z}^d$, to which we refer as the ***ghost vertex***, and set the coupling $J_{xg} = J_{xg}(G)$ between it and vertices $x \in G$ to be $\sum_{y \notin G} J_{xy}$. We also set $J_{gg} = 0$.

Introduce, for $\beta > 0$, G a finite graph and $A \subset V(G)$ the quantity

$$\begin{aligned} Z^+(G,\beta,A) &:= \sum_{\sigma \in \{-1,1\}^G} \sigma_A \prod_{\{x,y\} \subset G} \exp[\beta J_{xy}\sigma_x\sigma_y] \prod_{x \in G, y \notin G} \exp[\beta J_{xy}\sigma_x] \\ &= \sum_{\sigma \in \{-1,1\}^G} \sigma_A \prod_{\{x,y\} \subset G} \exp[\beta J_{xy}\sigma_x\sigma_y] \prod_{x \in G} \exp[\beta J_{xg}\sigma_x]. \end{aligned}$$

A development similar to the above yields the following proposition.

Proposition 5.5 (Random current with + boundary conditions). Let $\beta > 0$, G be a finite graph and $A \subset V(G)$, then

$$Z^+(G,\beta,A) = 2^{|V(G)|} \sum_{\mathbf{n} \in \Omega_{G \cup \{g\}} : \partial \mathbf{n} = A} w_\beta(\mathbf{n}). \tag{5.4}$$

Observe that (5.4) differs from (5.2) only through the fact that the summation is over all currents on $G \cup \{g\}$ instead of G. Also note that J_{xg} depends on G.

2 Application to correlation inequalities

We obtain a percolation configuration from a current configuration as follows. For an integer valued edge function, *i.e.* an element $\mathbf{n} \in \{0,1,2,...\}^{E(G)} =: \Omega_G$, we associate the associated percolation $\widehat{\mathbf{n}} \in \{0,1\}^{E(G)}$ defined by

$$\widehat{\mathbf{n}}_{xy} = \begin{cases} 1 & \text{if } \mathbf{n}_{xy} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we may speak of $x \overset{\widehat{\mathbf{n}}}{\longleftrightarrow} y$ for two vertices x and y.

The following graph-theoretic switching lemma is one of the main tools facilitating the study of this representation. It was originally introduced in [GHS 70]; see also [Aiz 82].

Lemma 5.6 (Switching lemma). For any nested pair of finite sets $G \subset H$, any pair of vertices $x, y \in G$ and any $A \subset V(H)$, and any function $F : \Omega_H \to \mathbb{R}$:

$$\sum_{\substack{\mathbf{n}_1 \in \Omega_G : \partial \mathbf{n}_1 = \{x,y\} \\ \mathbf{n}_2 \in \Omega_H : \partial \mathbf{n}_2 = A}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)$$
$$= \sum_{\substack{\mathbf{n}_1 \in \Omega_G : \partial \mathbf{n}_1 = \varnothing \\ \mathbf{n}_2 \in \Omega_H : \partial \mathbf{n}_2 = A \Delta \{x,y\}}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbf{I}[x \overset{\widehat{\mathbf{n}_1 + \mathbf{n}_2}}{\longleftrightarrow} y \text{ in } G].$$

Proof. In the argument, a current on the subgraph corresponding to G is also viewed as a current on H which vanishes on pairs $\{x, y\}$ not contained in G. The switching is performed within collections of pairs of currents $\{\mathbf{n}_1, \mathbf{n}_2\}$ of a specified value for the sum $\mathbf{m} := \mathbf{n}_1 + \mathbf{n}_2$. It is therefore convenient to take as the summation variables the current pairs $\mathbf{m}$ and $\mathbf{n} = \mathbf{n}_1 \leqslant \mathbf{m}$ (with $\mathbf{n} \leqslant \mathbf{m}$ defined as the natural *partial order* relation). One obtains

$$\sum_{\substack{\mathbf{n}_1 \in \Omega_G : \partial \mathbf{n}_1 = \{x,y\} \\ \mathbf{n}_2 \in \Omega_H : \partial \mathbf{n}_2 = A}} F(\mathbf{n}_1 + \mathbf{n}_2)\, w_\beta(\mathbf{n}_1)\, w_\beta(\mathbf{n}_2) = \sum_{\mathbf{m} \in \Omega_H : \partial \mathbf{m} = A \Delta \{x,y\}} F(\mathbf{m})\, w_\beta(\mathbf{m}) \sum_{\substack{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \{x,y\} \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}},$$

and

$$\sum_{\substack{\mathbf{n}_1 \in \Omega_G : \partial \mathbf{n}_1 = \varnothing \\ \mathbf{n}_2 \in \Omega_H : \partial \mathbf{n}_2 = A \Delta \{x,y\}}} F(\mathbf{n}_1 + \mathbf{n}_2)\, w_\beta(\mathbf{n}_1)\, w_\beta(\mathbf{n}_2) \mathbf{I}[x \overset{\widehat{\mathbf{n}_1 + \mathbf{n}_2}}{\longleftrightarrow} y \text{ in } G]$$
$$= \sum_{\mathbf{m} \in \Omega_H : \partial \mathbf{m} = A \Delta \{x,y\}} F(\mathbf{m})\, w_\beta(\mathbf{m}) \mathbf{I}[x \overset{\widehat{\mathbf{m}}}{\longleftrightarrow} y \text{ in } G] \sum_{\substack{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \varnothing \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}},$$

where $\binom{\mathbf{m}}{\mathbf{n}} = \prod_{\{x,y\} \subset G} \binom{\mathbf{m}_{xy}}{\mathbf{n}_{xy}}$ and where we used the fact that

$$w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) = \prod_{\{x,y\} \subset G \cup \{g\}} \left[\frac{(\beta J_{xy})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!} \right] \left[\frac{(\beta J_{xy})^{\mathbf{m}_{xy}}}{\mathbf{m}_{xy}!} \right] = w_\beta(\mathbf{m}) \binom{\mathbf{m}}{\mathbf{n}}.$$

The claim follows if the relation below is proved for every current $\mathbf{m} \in \Omega_H$:

$$\sum_{\substack{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \{x,y\} \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}} = \mathbf{I}[x \overset{\widehat{\mathbf{m}}}{\longleftrightarrow} y \text{ in } G] \sum_{\substack{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \varnothing \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}}. \tag{5.5}$$

First, assume that x and y are not connected in G by $\widehat{\mathbf{m}}$. The right-hand side is trivially zero. Moreover, there is no current $\mathbf{n}$ on G which is smaller than $\mathbf{m}$ and which connects x to y. The left-hand side is thus 0 and (5.5) is proved in this case.

Let us now assume that x and y are connected in G by $\widehat{\mathbf{m}}$. Associate to $\mathbf{m}$ the graph $\mathscr{M}$ with vertex set G, and $\mathbf{m}_{a,b}$ edges between a and b. For a subgraph $\mathscr{N}$ of $\mathscr{M}$, let $\partial \mathscr{N}$ be the set of vertices belonging to an odd number of edges. Since x and y are connected in G by $\mathbf{m}$, there exists a subgraph $\mathscr{K}$ of $\mathscr{M}$ with $\partial \mathscr{K} = \{x, y\}$.

The involution $\mathscr{N} \mapsto \mathscr{N} \Delta \mathscr{K}$ provides a bijection between the set of subgraphs of $\mathscr{M}$ with $\partial \mathscr{N} = \varnothing$, and the set of subgraphs of $\mathscr{M}$ with $\partial \mathscr{N} = \{x, y\}$. Therefore, these two sets have the same cardinality. Since the summations in

$$\sum_{\substack{\mathbf{n} \in \Omega_G : \partial \mathbf{n} = \varnothing \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}} \quad \text{and} \quad \sum_{\substack{n \in \Omega_G : \partial \mathbf{n} = \{x,y\} \\ \mathbf{n} \leqslant \mathbf{m}}} \binom{\mathbf{m}}{\mathbf{n}}$$

are over currents $\mathbf{n}$ in G, these sums correspond to the cardinality of the two sets mentioned above. In particular, they are equal and the statement follows. □

Corollary 5.7. Let G be a finite graph, $\beta > 0$ and $x \neq y \in G$, then

$$\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]^2 = \frac{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y]}{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}.$$

Proof. The switching lemma gives

$$\begin{aligned}\frac{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y]}{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)} &= \frac{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\{x,y\}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}{\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}\\ &= \left(\frac{\sum_{\partial\mathbf{n}_1=\{x,y\}} w_\beta(\mathbf{n}_1)}{\sum_{\partial\mathbf{n}_1=\emptyset} w_\beta(\mathbf{n}_1)}\right)^2\\ &= \mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]^2. \end{aligned}$$

□

Corollary 5.8 (Simon's and a special case of the second Griffiths' inequalities). Let G be a finite graph, $\beta > 0$, $A \subset G$ and $x \neq y \in G$,

$$\frac{\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]\,\mu^{\text{free}}_{G,\beta}[\sigma_A]}{\mu^{\text{free}}_{G,\beta}[\sigma_{A\Delta\{x,y\}}]} = \frac{\sum_{\partial\mathbf{n}_1=A\Delta\{x,y\},\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y]}{\sum_{\partial\mathbf{n}_1=A\Delta\{x,y\},\partial\mathbf{n}_2=\emptyset} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}. \tag{5.6}$$

As a consequence, for any A we find

$$\mu^{\text{free}}_{G,\beta}[\sigma_A]\,\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y] \leqslant \mu^{\text{free}}_{G,\beta}[\sigma_A\sigma_x\sigma_y]. \tag{5.7}$$

Furthermore for the nearest-neighbor model, we obtain that for any set S disconnecting x from z (meaning that any path from x to z uses a vertex in S),

$$\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_z] \leqslant \sum_{y\in S} \mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]\,\mu^{\text{free}}_{G,\beta}[\sigma_y\sigma_z]. \tag{5.8}$$

The second inequality (5.8) is called Simon's inequality. The inequality (5.7) is a special case of Griffiths' second inequality [Gri 67]:

$$\mu^{\text{free}}_{G,\beta}[\sigma_A]\,\mu^{\text{free}}_{G,\beta}[\sigma_B] \leqslant \mu^{\text{free}}_{G,\beta}[\sigma_A\sigma_B].$$

Proof. Let G be a finite graph, $\beta > 0$, $A \subset V(G)$ and $x \neq y \in G$, the switching lemma in the second line,

$$\frac{\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]\mu^{\text{free}}_{G,\beta}[\sigma_A]}{\mu^{\text{free}}_{G,\beta}[\sigma_{A\Delta\{x,y\}}]} = \frac{\displaystyle\sum_{\partial\mathbf{n}_1=A,\partial\mathbf{n}_2=\{x,y\}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}{\displaystyle\sum_{\partial\mathbf{n}_1=A\Delta\{x,y\},\partial\mathbf{n}_2=\varnothing} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}$$

$$= \frac{\displaystyle\sum_{\partial\mathbf{n}_1=A\Delta\{x,y\},\partial\mathbf{n}_2=\varnothing} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y]}{\displaystyle\sum_{\partial\mathbf{n}_1=A\Delta\{x,y\},\partial\mathbf{n}_2=\varnothing} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}.$$

On the one hand, (5.7) follows by noticing that the right-hand side of (5.6) is always smaller or equal to 1. On the other hand, by setting $A = \{y\}\Delta\{z\}$ (we assume $x \neq y$, otherwise the inequality is trivial), we find

$$\frac{\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_y]\mu^{\text{free}}_{G,\beta}[\sigma_y\sigma_z]}{\mu^{\text{free}}_{G,\beta}[\sigma_x\sigma_z]} = \frac{\displaystyle\sum_{\partial\mathbf{n}_1=\{x,z\},\partial\mathbf{n}_2=\varnothing} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y]}{\displaystyle\sum_{\partial\mathbf{n}_1=\{x,z\},\partial\mathbf{n}_2=\varnothing} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}.$$

We deduce the result by summing on S and noticing that $\widehat{\mathbf{n}_1+\mathbf{n}_2}$ must contain a path from x to z, which necessarily passes through a vertex in S. □

Corollary 5.9. Let G be a finite graph and $J := (J_{xy})_{x,y\in G}$ and $J' := (J'_{xy})_{x,y\in G}$ two families of coupling constants on G. If $0 \leqslant J_{xy} \leqslant J'_{xy}$ for every x, y, then for any $A \subset G$,

$$\mu^{\text{free}}_{G,\beta,J}[\sigma_A] \leqslant \mu^{\text{free}}_{G,\beta,J'}[\sigma_A],$$

where $\mu^{\text{free}}_{G,\beta,J}$ and $\mu^{\text{free}}_{G,\beta,J'}$ denote the Ising measures on G with coupling constants J and J'.

Proof. It is sufficient to treat the case where all coupling constants are the same except for one $\{x,y\} \subset G$. When differentiating with respect to J_{xy}, we find

$$\frac{1}{\beta}\cdot\frac{\mathrm{d}\mu^{\text{free}}_{G,\beta,J}[\sigma_A]}{\mathrm{d}J_{xy}} = \mu^{\text{free}}_{G,\beta,J}[\sigma_A\sigma_x\sigma_y] - \mu^{\text{free}}_{G,\beta,J}[\sigma_A]\mu^{\text{free}}_{G,\beta,J}[\sigma_x\sigma_y] \geqslant 0$$

by the second Griffiths' inequality, thus proving the claim. □

3 A proof of sharpness of the phase transition for Ising models

The following theorem is an equivalent of Theorem 2.5 for the Ising model. It was first proved in [ABF 87] for the Ising model on the d-dimensional hypercubic lattice, but the proof extends to general transitive graphs. Here, we present a recent proof from [DCT 15]. Let $||\cdot||$ be the infinite norm on $\mathbb{R}^d$.

Theorem 5.10.

Consider a translational invariant ferromagnetic Ising model with coupling constants $J=(J_{xy})_{x,y\in\mathbb{Z}^d}$.

1. For $\beta>\beta_c$, $\mu^+_\beta[\sigma_0]\geqslant\sqrt{\frac{\beta^2-\beta_c^2}{\beta^2}}$.
2. For $\beta<\beta_c$, the susceptibility is finite, *i.e.*
$$\sum_{x\in\mathbb{Z}^d}\mu^+_\beta[\sigma_0\sigma_x]<\infty.$$
3. If $(J_{xy})_{x,y\in\mathbb{Z}^d}$ is finite range, then for any $\beta<\beta_c$, there exists $c=c(\beta)>0$ such that
$$\mu^+_\beta[\sigma_0\sigma_x]\leqslant e^{-c\|x\|}\qquad\text{for all } x\in\mathbb{Z}^d.$$

The proof follows closely the proof of exponential decay for Bernoulli percolation. For $\beta>0$ and a finite subset S of $\mathbb{Z}^d$, define

$$\varphi_S(\beta):=\sum_{x\in S}\sum_{y\notin S}\tanh(\beta J_{xy})\mu^{\text{free}}_{S,\beta}[\sigma_0\sigma_x],$$

which bears a resemblance to (2.3). Similarly to (2.4), set

$$\tilde\beta_c:=\sup\{\beta\geqslant 0:\varphi_\beta(S)<1\text{ for some finite }S\subset\mathbb{Z}^d\text{ containing }0\}.$$

The proof of Theorem 2.5 proceeds in two steps. As before, the quantity $\varphi_\beta(S)$ appears naturally in the derivative of a "finite-volume approximation" of $\mu^+_\beta[\sigma_0]$. Roughly speaking (see below for a precise statement), one obtains a finite-volume version of the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_\beta[\sigma_0]^2\geqslant\frac{2}{\beta}\inf_{S\ni 0}\varphi_\beta(S)\cdot\left(1-\mu^+_\beta[\sigma_0]^2\right).$$

This inequality implies the first item of Theorem 5.10 and the inequality $\tilde\beta_c\geqslant\beta_c$. The remaining items follow from an improved Simon's inequality, proved below.

Remark 5.11. Finite susceptibility does not imply exponential decay of correlations for infinite-range models. Hence, the second condition of Theorem 2.5 is not weaker than the third one.

Proof. For $S\subset\Lambda$ two finite subsets of $\mathbb{Z}^d$, introduce $\mu^\Lambda_{S,\beta}$ obtained from $\mu^+_{\Lambda,\beta}$ by setting all the coupling constants J_{xy} with x or y in $\Lambda\setminus S$ to be equal to 0. Note that if $S=\Lambda$, then $\mu^\Lambda_{\Lambda,\beta}=\mu^+_{\Lambda,\beta}$ and for each fixed S, $\mu^\Lambda_{S,\beta}$ tends to $\mu^{\text{free}}_{S,\beta}$ as $\Lambda\nearrow\mathbb{Z}^d$. We adopt some special notation for this proof. For $S\subset\Lambda$, define

$$w_S(\mathbf{n})=w_S(\Lambda,\beta,\mathbf{n}):=\prod_{\{x,y\}\subset S\cup\{g\}}\frac{(\beta J_{xy})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

Introduce

$$Z^\Lambda_{S,\beta}=\sum_{\partial\mathbf{n}=\varnothing}w_S(\mathbf{n}).$$

The same formulas as (5.3) hold trivially in this context. When $S=\Lambda$, we write $w(\mathbf{n})$ instead of $w_\Lambda(\mathbf{n})$, $Z_{\Lambda,\beta}$ instead of $Z^\Lambda_{\Lambda,\beta}$, and $\mu^+_{\Lambda,\beta}$ instead of $\mu^\Lambda_{\Lambda,\beta}$.

Lemma 5.12. Let $\beta > 0$ and $\Lambda \subset \mathbb{Z}^d$ finite. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_\beta[\sigma_0]^2 \geqslant \tfrac{2}{\beta}\inf_{S\ni 0}\varphi_\beta(S)\left(1-\mu^+_\beta[\sigma_0]^2\right).$$

We may integrate between $\tilde\beta_c$ and β, then let $\Lambda \nearrow \mathbb{Z}^d$ to obtain $\beta_c \leqslant \tilde\beta_c$ and the first item of Theorem 5.10. In order to obtain Inequality (5.10), we used a computation similar to one provided in [ABF 87].

Proof (of Lemma 5.12). Let $\beta > 0$ and a finite subset Λ of $\mathbb{Z}^d$. The derivative of $\mu^+_{\Lambda,\beta}[\sigma_0]$ is given by the following formula

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0] = \sum_{\{x,y\}\subset\Lambda\cup\{g\}} J_{xy}\left(\mu^+_{\Lambda,\beta}[\sigma_0\sigma_x\sigma_y] - \mu^+_{\Lambda,\beta}[\sigma_0]\mu^+_{\Lambda,\beta}[\sigma_x\sigma_y]\right),$$

where σ_g is considered as $+1$. Using (5.5) and the switching Lemma, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0] = \frac{1}{Z^2_{\Lambda,\beta}}\sum_{\{x,y\}\subset\Lambda\cup\{g\}}\ \sum_{\substack{\partial\mathbf{n}_1=\{0,g\}\Delta\{x,y\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\not\longleftrightarrow} g].$$

If $\mathbf{n}_1$ and $\mathbf{n}_2$ are two currents such that $\partial\mathbf{n}_1 = \{0,g\}\Delta\{x,y\}$, $\partial\mathbf{n}_2 = \varnothing$ and 0 and g are not connected in $\widehat{\mathbf{n}_1+\mathbf{n}_2}$, then exactly one of these two cases holds: $0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} x$ and $y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g$, or $0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y$ and $x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g$. Since the second case is the same as the first one with x and y permuted, we obtain the following expression,

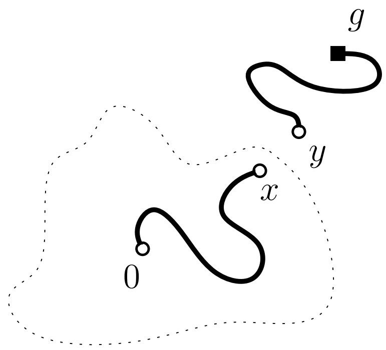
$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0] = \frac{1}{Z^2_{\Lambda,\beta}}\sum_{x\in\Lambda\cup\{g\}}\sum_{y\in\Lambda\cup\{g\}}\delta_{xy}, \tag{5.9}$$

where

$$\delta_{xy} = \sum_{\substack{\partial\mathbf{n}_1=\{0,g\}\Delta\{x,y\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} x, y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g, 0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\not\longleftrightarrow} g]$$

(see Fig. 5.1 and notice the analogy with the event involved in Russo's formula, namely that the edge $\{x,y\}$ is pivotal, in Bernoulli percolation).

FIGURE 5.1. A diagrammatic representation of δ_{xy}: the solid lines represent the currents, and the dotted line the boundary of the cluster of 0 in $\widehat{\mathbf{n}_1+\mathbf{n}_2}$.



We will now manipulate the previous definition of δ_{xy} to try to end up with a sum on two sourceless currents (we need to remove the sources of $\mathbf{n}_1$). Given two currents $\mathbf{n}_1$ and $\mathbf{n}_2$, and $z \in \{0, g\}$, define $\mathscr{S}_z$ to be the set of vertices in $\Lambda \cup \{g\}$ that are NOT connected to z in $\widehat{\mathbf{n}_1 + \mathbf{n}_2}$.

We first remove the sources y and g in the sum. Let us compute δ_{xy} by summing over the different possible values for $\mathscr{S}_0$:

$$\begin{aligned}\delta_{xy} &= \sum_{S \subset \Lambda \cup \{g\}} \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\} \Delta \{x,y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, 0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} x, y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g, 0 \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\not\longleftrightarrow} g] \\ &= \sum_{\substack{S \subset \Lambda \cup \{g\} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\} \Delta \{x,y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g].\end{aligned}$$

When $\mathscr{S}_0 = S$, the two currents $\mathbf{n}_1$ and $\mathbf{n}_2$ vanish on every $\{u, v\}$ with $u \in S$ and $v \notin S$. Thus, for $i = 1, 2$, we can decompose $\mathbf{n}_i$ as

$$\mathbf{n}_i = \mathbf{n}_i^S + \mathbf{n}_i^{\Lambda \setminus S},$$

where $\mathbf{n}_i^A$ denotes the current in $A \subset \mathbb{Z}^d$ with source $\partial \mathbf{n}_i^A = A \cap \partial \mathbf{n}_i$. Using this observation together with the second identity in (5.5), we obtain

$$\delta_{xy} = \sum_{\substack{S \subset \Lambda \cup \{g\} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mu_{S,\beta}^{\Lambda}[\sigma_y] \mathbf{I}[\mathscr{S}_0 = S].$$

Multiplying the expression above by $\mu_{\Lambda,\beta}^{+}[\sigma_0]$, and using Corollary 5.9 (which yields $\mu_{S,\beta}^{\Lambda}[\sigma_0] \leqslant \mu_{\Lambda,\beta}^{+}[\sigma_0]$), we find

$$\begin{aligned}\mu_{\Lambda,\beta}^{+}[\sigma_0] \delta_{xy} &\geqslant \sum_{\substack{S \subset \Lambda \cup \{g\} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mu_{S,\beta}^{\Lambda}[\sigma_y]^2 \, \mathbf{I}[\mathscr{S}_0 = S] \\ &= \sum_{\substack{S \subset \Lambda \cup \{g\} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \Delta \{y,g\} \\ \partial \mathbf{n}_2 = \{y,g\}}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S] \\ &= \sum_{\substack{S \subset \Lambda \cup \{g\} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g] \\ &= \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[y \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g, 0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g],\end{aligned}$$

where in the third line we used the switching lemma. We now sum over the possible values of $\mathscr{S}_g$:

$$\begin{aligned}
\mu^+_{\Lambda,\beta}[\sigma_0]\delta_{xy} &\geqslant \sum_{S\subset\Lambda}\sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g=S, y\overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g, 0\overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g]\\
&= \sum_{\substack{S\subset\Lambda\\ 0,x\in S\\ y\notin S}}\sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g=S]\\
&= \sum_{\substack{S\subset\Lambda\\ 0,x\in S\\ y\notin S}}\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\mathbf{I}[\mathscr{S}_g=S].
\end{aligned}$$

The third line follows from the fact that since $\mathscr{S}_g=S$, the currents $\mathbf{n}_1$ and $\mathbf{n}_2$ can be decomposed as $\mathbf{n}_i=\mathbf{n}_i^S+\mathbf{n}_i^{\Lambda\setminus S}$ as we did before for $\mathscr{S}_0=S$.

By plugging the inequality above in (5.9), we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0]^2 = 2\mu^+_{\Lambda,\beta}[\sigma_0]\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0]&\\
&= \frac{2}{Z^2_{\Lambda,\beta}}\sum_{\substack{S\subset\Lambda\\ 0\in S}}\sum_{\substack{x\in S\\ y\notin S}}\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)J_{xy}\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\mathbf{I}[\mathscr{S}_g=S]\\
&\geqslant \frac{2}{\beta}\cdot\frac{1}{Z^2_{\Lambda,\beta}}\cdot\sum_{\substack{S\subset\Lambda\\ 0\in S}}\varphi_\beta(S)\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g=S] \qquad (5.10)\\
&\geqslant \frac{2}{\beta}\inf_{S\ni 0}\varphi_\beta(S)\cdot\frac{\displaystyle\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\varnothing} w(\mathbf{n}_1)w(\mathbf{n}_2)\big(1-\mathbf{I}[0\overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g]\big)}{\displaystyle\sum_{\partial\mathbf{n}_1=\partial\mathbf{n}_2=\varnothing} w(\mathbf{n}_1)w(\mathbf{n}_2)}\\
&= \frac{2}{\beta}\cdot\inf_{S\ni 0}\varphi_\beta(S)\cdot(1-\mu^+_{\Lambda,\beta}[\sigma_0]^2).
\end{aligned}$$

In the first inequality, we used that $J_{xy}\geqslant\frac{1}{\beta}\tanh(\beta J_{xy})$. In the last line, we used the switching lemma and (5.5) one more time. Recall that we are working in Λ and therefore $y\notin S$ really means $y\in\Lambda\cup\{g\}\setminus S$. Nevertheless in this context the definition of J_{xg} gives us

$$\sum_{y\notin S}J_{xy} := \sum_{y\in\Lambda\setminus S}J_{xy}+J_{xg} = \sum_{y\in\mathbb{Z}^d\setminus S}J_{xy},$$

which enables us to claim that

$$\sum_{x\in S}\sum_{y\notin S}J_{xy}\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x] = \varphi_\beta(S). \qquad \square$$

Remark 5.13. Inequality (5.10) is reminiscent of (2.7). Indeed, one can consider a measure $\mathbf{P}_\beta$ on sourceless currents attributing a probability proportional to $w_\beta(\mathbf{n})$ to the current $\mathbf{n}$. Then, (5.10) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mu^+_{\Lambda,\beta}[\sigma_0]^2 \geqslant \tfrac{2}{\beta}\sum_{S\ni 0}\varphi_\beta(S)\,\mathbf{P}_\beta\otimes\mathbf{P}_\beta(\mathscr{S}_g=S).$$

Interpreted like that, the trace of the sum of two independent sourceless currents plays the role of the percolation configuration. We will see a similar interpretation in the next section.

We turn to the second and third items of the theorem. We need a replacement for the BK inequality used in the case of Bernoulli percolation. The relevant tool for the Ising model will be an improved version of Simon's inequality.

Lemma 5.14 (Improved Simon's inequality). Let S be a subset of $\mathbb{Z}^d$ containing 0. For every $z\notin S$,

$$\mu^+_\beta[\sigma_0\sigma_z]\leqslant\sum_{x\in S}\sum_{y\notin S}\tanh(\beta J_{xy})\,\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\,\mu^+_\beta[\sigma_y\sigma_z].$$

To understand intuitively (5.8), consider for a moment the simple random-walk model. Let $G(x,y)$ be the expected number of visits to y starting from x, and $G_S(x,y)$ the same quantity when counting visits before exiting S. Then, the union bound and the Markov property at the first visit of ∂S leads to $G(0,x)\leqslant\sum_{y\in\partial S}G_S(0,y)G(y,x)$, which is the direct analogue of (5.8). Therefore, it does not come as a surprise that the backbone representation can be used to prove the lemma.

We will need the notion of backbone of a current. Fix two finite subsets $S\subset\Lambda$ of $\mathbb{Z}^d$. Choose an arbitrary order of the oriented edges of the lattice. Consider a current $\mathbf{n}$ on S with $\partial\mathbf{n}=\{x,y\}$. Let $\omega(\mathbf{n})$ be the edge self-avoiding path from x to y passing only through edges e with $\mathbf{n}_e>0$ which is minimal for the lexicographical order on paths induced by the previous ordering on oriented edges. Such an object is called the *backbone* of the current configuration. For a backbone ω with endpoints $\partial\omega=\{x,y\}$, set

$$\rho^\Lambda_S(\omega)=\rho^\Lambda_S(\beta,\omega):=\frac{1}{Z^\Lambda_{S,\beta}}\sum_{\partial\mathbf{n}=\{x,y\}}w_S(\mathbf{n})\mathbf{I}[\omega(\mathbf{n})=\omega].$$

The backbone representation has the following properties (the proofs are straightforward, except for the last item which follows from Corollary 5.9):

1. $\mu^\Lambda_{S,\beta}[\sigma_x\sigma_y]=\sum_{\partial\omega=\{x,y\}}\rho^\Lambda_S(\omega)$.
2. If the backbone ω is the concatenation of two backbones ω_1 and ω_2 (this is denoted by $\omega=\omega_1\circ\omega_2$), then
$$\rho^\Lambda_S(\omega)=\rho^\Lambda_S(\omega_1)\rho^\Lambda_{S\setminus\overline{\omega}_1}(\omega_2),$$
where $\overline{\omega}_1$ is the set of bonds whose states are determined by the fact that ω_1 is an admissible backbone (this includes bonds of ω_1 together with some neighboring bonds).
3. For a backbone ω not using any edge outside $T\subset S$, then
$$\rho^\Lambda_S(\omega)\leqslant\rho^\Lambda_T(\omega).$$

Proof. Fix Λ a finite subset of $\mathbb{Z}^d$ containing S. We consider the backbone representation of the Ising model on Λ defined in the previous section. Let $\omega=(v_k)_{0\leqslant k\leqslant K}$ be a backbone from 0 to z. Since $z\notin S$, one can define the first k such that $v_{k+1}\notin S$. We obtain that the following occur:

- ω goes from 0 to v_k staying in S,
- then ω goes through $\{v_k, v_{k+1}\}$,
- finally ω goes from v_{k+1} to z in Λ_n.

We find

$$\begin{aligned}\mu^+_{\Lambda,\beta}[\sigma_0\sigma_z] &= \sum_{\partial\omega=\{0,z\}} \rho^\Lambda_\Lambda(\omega)\\ &\leqslant \sum_{x\in S}\sum_{y\notin S}\sum_{\partial\omega_1=\{0,x\}}\sum_{\partial\omega_2=\{x,y\}}\sum_{\partial\omega_3=\{y,z\}} \rho^\Lambda_S(\omega_1)\rho^\Lambda_{\{x,y\}}(\omega_2)\rho^\Lambda_{\Lambda\setminus\overline{\omega_1\circ\omega_2}}(\omega_3)\\ &\leqslant \sum_{x\in S}\sum_{y\notin S} \mu^\Lambda_{S,\beta}[\sigma_0\sigma_x]\rho^\Lambda_{\{x,y\}}(\{x,y\})\,\mu^+_{\Lambda,\beta}[\sigma_y\sigma_z].\end{aligned}$$

The first and third lines are based on the first property of backbones. The second line follows from the second and third properties of the backbone as well as Corollary 5.9. The proof follows by taking Λ to infinity and by observing that

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \rho^\Lambda_{\{x,y\}}(\{x,y\}) = \frac{\sinh(\beta J_{xy})}{\cosh(\beta J_{xy})} = \tanh(\beta J_{xy}). \qquad \square$$

We are now in a position to conclude the proof. Let $\beta < \tilde\beta_c$ (recall that $\beta_c \leqslant \tilde\beta_c$ thanks to the first part of the proof). Fix a finite set S such that $\varphi_\beta(S) < 1$. Define

$$\chi_n(\beta) := \max\Big\{\sum_{v\in\Lambda_n} \mu^+_\beta[\sigma_u\sigma_v] : u\in\Lambda_n\Big\}.$$

For $u \in \Lambda_n$, recall that τ_u denotes the translation by u. We find

$$\begin{aligned}\sum_{v\in\Lambda_n}\mu^+_\beta[\sigma_u\sigma_v] &= \sum_{v\in\tau_u S}\mu^+_\beta[\sigma_u\sigma_v] + \sum_{v\in\Lambda_n\setminus\tau_u S}\mu^+_\beta[\sigma_u\sigma_v]\\ &\leqslant |S| + \sum_{v\in\Lambda_n\setminus\tau_u S}\sum_{x\in\tau_u S}\sum_{y\notin\tau_u S}\tanh(\beta J_{xy})\,\mu^{\mathrm{free}}_{S,\beta}[\sigma_u\sigma_x]\,\mu^+_\beta[\sigma_y\sigma_v]\\ &= |S| + \sum_{x\in S}\sum_{y\notin S}\tanh(\beta J_{xy})\,\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\Big(\sum_{v\in\Lambda_n\setminus\tau_u S}\mu^+_\beta[\sigma_y\sigma_v]\Big)\\ &\leqslant |S| + \sum_{x\in S}\sum_{y\notin S}\tanh(\beta J_{xy})\,\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\,\chi_n(\beta) = |S| + \varphi_\beta(S)\chi_n(\beta).\end{aligned}$$

Optimizing on u, we find

$$\chi_n(\beta) \leqslant |S| + \varphi_\beta(S)\chi_n(\beta),$$

thus leading to

$$\chi_n(\beta) \leqslant \frac{|S|}{1-\varphi_\beta(S)}.$$

Letting $n \nearrow \infty$, we obtain the second item.

We finish by the proof of the third item. Let R be the range of the $(J_{xy})_{x,y\in\mathbb{Z}^d}$, and let L be such that $S \subset \Lambda_{L-R}$. Lemma 5.14 implies that for any z with $\|z\| \geqslant n > L$,

$$\mu^+_\beta[\sigma_0\sigma_z] \leqslant \sum_{x\in S}\sum_{y\notin S}\tanh(\beta J_{xy})\,\mu^{\mathrm{free}}_{S,\beta}[\sigma_0\sigma_x]\,\mu^+_\beta[\sigma_y\sigma_z] \leqslant \varphi_\beta(S)\max_{y\in\Lambda_L}\mu^+_\beta[\sigma_y\sigma_z].$$

The proof follows by iterating $\lfloor n/L\rfloor$ times. $\square$

Remark 5.15. Like for Bernoulli percolation, one deduces that $\varphi_{\beta_c}(\Lambda_n) \geqslant 1$ for every $n \geqslant 1$ and therefore

$$\sum_{x\in\mathbb{Z}^d} \mu^+_{\beta_c}[\sigma_0\sigma_x] = \infty.$$

4 Continuity of the phase transition for Ising models

The goal of this section is to prove the following result.

Theorem 5.16 (Aizenman, Duminil-Copin, Sidoravicius [ADCS 15]).

Let $d \geqslant 1$ and consider a ferromagnetic Ising model on $\mathbb{Z}^d$ with coupling constants $(J_{xy})_{x,y\in\mathbb{Z}^d}$. If

$$\lim_{\|x-y\|\to\infty} \mu^{\text{free}}_{\beta_c}[\sigma_x\sigma_y] = 0,$$

then $\mu^+_{\beta_c}[\sigma_0] = 0$.

This theorem yields that as soon as the critical Ising Gibbs measure with free boundary conditions has no long-range order, then so does the $+$ Gibbs measure (in fact in such case $\mu^{\text{free}}_{\beta_c} = \mu^+_{\beta_c}$). This fact is utterly wrong for general Potts models, as illustrated by the *planar* Potts model with $q \gg 1$ colors since in such case the phase transition is discontinuous, meaning $\mu^1_{\beta_c}[\langle\sigma_0|1\rangle] > 0$ and there is exponential decay of the spin-spin correlations for $\mu^{\text{free}}_{\beta_c}$.

Note that for the Ising model, the phase transition may be discontinuous, as shown by the 1D model with $J_{xy} = 1/|x-y|^2$, see [ACCN 88]. Nevertheless, the previous theorem provides us with a sufficient condition to decide that the phase transition is continuous. Let us apply this tool to the case of the nearest-neighbor model using the following known fact. Let $G(x,y)$ be the Green function for the simple random-walk on $\mathbb{Z}^d$, *i.e.* the expected number of visits of y for a simple random-walk starting from x.

Theorem 5.17 (Infrared bound).

Consider the nearest-neighbor Ising model on $\mathbb{Z}^d$, $d \geqslant 3$. For any $\beta \leqslant \beta_c$ and any $x,y\in\mathbb{Z}^d$,

$$\mu^{\text{free}}_{\beta}[\sigma_x\sigma_y] \leqslant \tfrac{1}{2\beta} G(x,y).$$

This statement is called the infrared bound. Note that $G(x,y)$ is the spin-spin correlation for the discrete GFF. Since the ϕ^4_d lattice model interpolates between the Ising and the discrete GFF, it is not so surprising that spin-spin correlations of both models can be compared. One may also use the backbone to interpret the spin-spin correlations as the Green function of a self-repulsive random walk, so that it may be natural to expect that this Green function is smaller that the one of the simple-random walk. The proof of this theorem is based on the so-called reflection-positivity (RP) technique, see Fröhlich, Simon and Spencer [FSS 76]. This technique has many applications in different fields of mathematical physics, we refer to [Bis 09] and references therein for a more comprehensive study of this subject and we now focus on an application of Theorem 5.17. Since the simple random walk is transient on

$\mathbb{Z}^d$ for $d \geqslant 3$, the infrared bound, together with Theorem 5.16, implies the following result.

Corollary 5.18 (Aizenman, Duminil-Copin, Sidoravicius [ADCS 15]). For $d \geqslant 3$, the phase transition of the nearest neighbor ferromagnetic Ising model on $\mathbb{Z}^d$ is continuous.

Recall that the phase transition is also continuous for $d = 2$ by the result of Section 3. The proof of Theorem 5.16 is based on the study of the percolative properties of the infinite volume limit of the random current representation.

Let $\mathrm{P}^{\mathrm{free}}_{G,\beta}$ be the law on currents on G defined by

$$\mathrm{P}^{\mathrm{free}}_{G,\beta}[\mathbf{n}] := \frac{w_\beta(\mathbf{n})\mathbf{I}[\partial\mathbf{n}=\varnothing]}{\sum_{\mathbf{m}\in\Omega_G:\partial\mathbf{m}=\varnothing} w_\beta(\mathbf{m})} \quad , \qquad \forall \mathbf{n}\in\Omega_G \, .$$

The push forward of $\mathrm{P}^{\mathrm{free}}_{G,\beta}$ by $\mathbf{n}\mapsto\widehat{\mathbf{n}}$ is denoted by $\widehat{\mathrm{P}}^{\mathrm{free}}_{G,\beta}$. It is a percolation measure on G. One may also define a law on currents on $G\cup\{g\}$ which induces a measure denoted by $\widehat{\mathrm{P}}^{+}_{G,\beta}$.

Proposition 5.19.

Let $\beta > 0$. There exist two percolation laws $\widehat{\mathrm{P}}^{+}_{\beta}$ and $\widehat{\mathrm{P}}^{\mathrm{free}}_{\beta}$ on $\mathbb{Z}^d$ such that

R1 (Convergence) For any event $\mathscr{A}$ depending on finitely many edges,

$$\lim_{L\to\infty}\widehat{\mathrm{P}}^{+}_{\Lambda_L,\beta}[\mathscr{A}] = \widehat{\mathrm{P}}^{+}_{\beta}[\mathscr{A}] \quad \text{and} \quad \lim_{L\to\infty}\widehat{\mathrm{P}}^{\mathrm{free}}_{\Lambda_L,\beta}[\mathscr{A}] = \widehat{\mathrm{P}}^{\mathrm{free}}_{\beta}[\mathscr{A}].$$

R2 (Invariance under translations) $\widehat{\mathrm{P}}^{+}_{\beta}$ and $\widehat{\mathrm{P}}^{\mathrm{free}}_{\beta}$ are invariant under the shifts τ_x, $x\in\mathbb{Z}^d$.

R3 (Ergodicity) $\widehat{\mathrm{P}}^{+}_{\beta}$ and $\widehat{\mathrm{P}}^{\mathrm{free}}_{\beta}$ are ergodic with respect to the group of shifts $(\tau_x)_{x\in\mathbb{Z}^d}$.

Proof. Except for a minor difference in the very last step, the proof is identical for the $+$ and the free boundary conditions. Let us therefore use the symbol # as a marker for either of the two. In this proof, we denote the set of edges by $\mathscr{P}_2(G)$ (*i.e.* the set of pairs $\{x,y\}\subset G$). Let Ω_G be the set of percolation configurations on G, *i.e.* $\{0,1\}^{\mathscr{P}_2(G)}$.

Proof of R1 (Convergence) To prove convergence of the finite volume probability measures, let us first note that the distribution of the random currents simplifies into a product measure when conditioned on the parity variables $\mathbf{r}(\omega) = (\mathbf{r}_{xy})_{x,y}$, with:

$$\mathbf{r}_{xy}(\omega) := (-1)^{\mathbf{n}_{xy}(\omega)}.$$

The conditional distribution of $\mathbf{n}$, given $\mathbf{r}(\omega)$, is simply the product measure of independent Poisson processes of mean values βJ_{xy} conditioned on the corresponding parity. Thus, for a proof of convergence it suffices to establish convergence of the law of the parity variables $\mathbf{r}(\omega)$.

For a set of edges E, define the events

$$\begin{aligned} \mathscr{C}_E &= \{\, \omega : \mathbf{r}_{xy}(\omega)=1 \quad \forall\{x,y\}\in E \,\}, \\ \mathscr{C}^{(0)}_E &= \{\, \omega : \mathbf{n}_{xy}(\omega)=0 \quad \forall\{x,y\}\in E \,\}. \end{aligned}$$

Let us prove that for any finite subset E of edges of $\mathbb{Z}^d$, $\widehat{\mathrm{P}}^{\#}_{\Lambda_L,\beta}[\mathscr{C}_E]$ converges as L tends to infinity.

To facilitate a unified treatment of the two boundary conditions we denote

$$\Omega^{\#}_L = \begin{cases} \Omega_{\Lambda_L} & \text{for } \# = \text{free}, \\ \Omega_{\Lambda_L \cup \{g\}} & \text{for } \# = +. \end{cases}$$

For L large enough (so that $\Lambda_L \supset E$) we have:

$$\begin{aligned} \widehat{\mathrm{P}}^{\#}_{\Lambda_L,\beta}[\mathscr{C}_E] &= \frac{\sum_{\mathbf{n}\in\Omega^{\#}_L:\partial\mathbf{n}=\varnothing} w_\beta(\mathbf{n})\mathbf{I}[\mathscr{C}_E]}{\sum_{\mathbf{n}\in\Omega^{\#}_L:\partial\mathbf{n}=\varnothing} w_\beta(\mathbf{n})} \\ &= \frac{\sum_{\mathbf{n}\in\Omega^{\#}_L:\partial\mathbf{n}=\varnothing} w_\beta(\mathbf{n})\mathbf{I}[\mathscr{C}^{(0)}_E]}{\sum_{\mathbf{n}\in\Omega^{\#}_L:\partial\mathbf{n}=\varnothing} w_\beta(\mathbf{n})} \prod_{x,y\in E} \cosh(\beta J_{xy}) \\ &= \frac{Z^{\#}(\Lambda_L \setminus E,\beta)}{Z^{\#}(\Lambda_L,\beta)} \prod_{x,y\in E} \cosh(\beta J_{xy}). \end{aligned}$$

Above, $\Lambda_L \setminus E$ designates the graph obtained by removing the edges of E but keeping all the vertices of Λ_L. The above ratio can be expressed in terms of an expectation value of a finite term:

$$\widehat{\mathrm{P}}^{\#}_{\Lambda_L,\beta}[\mathscr{C}_E] = \mu^{\#}_{\Lambda_L,\beta}\left[e^{-\beta K_E}\right] \prod_{x,y\in E} \cosh(\beta J_{xy}) \tag{5.13}$$

with the finite volume collection of energy terms

$$K_E(\omega) := \sum_{x,y\in E} J_{xy}\sigma_x\sigma_y\,.$$

The convergence of the above expression follows now directly from the convergence of measures of the Ising model as L tends to infinity.

The events $\mathscr{C}_E$ with E ranging over finite sets of edges span (by inclusion-exclusion) the algebra of events expressible in terms of finite collections of the binary variables of $\mathbf{r}(\omega)$. This fact, and the above observation that the probability distribution of the random current $\mathbf{n}$ conditioned on $\mathbf{r}(\omega)$ does not depend on L, implies the existence of $\widehat{\mathrm{P}}^{\#}_\beta$.

Proof of R2 (Translation invariance) Fix $x \in \mathbb{Z}^d$. The limit of the probability of the event $\mathscr{C}_E$, where E is a finite set of edges, is the same if the sequence $(\Lambda_L)_{L\geqslant 0}$ is replaced by the sequence $(x+\Lambda_L)_{L\geqslant 0}$. (Simply use (5.13) and the convergence of $\mu^{\#}_{x+\Lambda_L,\beta}$ to $\mu^{\#}_\beta$.) This immediately implies that $\widehat{\mathrm{P}}^{\#}_\beta$ is invariant under translations.

Proof of R3 (Ergodicity) Since every translationally invariant event can be approximated by events depending on a finite number of edges, it is sufficient to prove that for any events A and B depending on a finite number of edges,

$$\lim_{\|x\|_1\to\infty} \widehat{\mathrm{P}}^{\#}_\beta[A\cap\tau_x B] = \widehat{\mathrm{P}}^{\#}_\beta[A]\,\widehat{\mathrm{P}}^{\#}_\beta[B].$$

In view of the conditional independence of $\mathbf{n}$ given the parity variables $\mathbf{r}$, the requirement can be further simplified to the proof that for any two finite sets E and F of edges,

$$\lim_{\|x\|_1\to\infty} \widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_{E\cup(x+F)}] = \widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_E]\,\widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_F]\,.$$

Using the expression (5.13), for x large enough so that $E\cap(x+F)=\varnothing$:

$$\frac{\widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_{E\cup(x+F)}]}{\widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_E]\,\widehat{\mathrm{P}}^{\#}_{\beta}[\mathscr{C}_F]} \;=\; \frac{\mu^{\#}_{\beta}\left[e^{-\beta K_E}\,e^{-\beta K_{x+F}}\right]}{\mu^{\#}_{\beta}\left[e^{-\beta K_E}\right]\,\mu^{\#}_{\beta}\left[e^{-\beta K_F}\right]}.$$

Ergodicity of the random current states can therefore be presented as an implication of the statement that this ratio tends to 1. This condition holds as a consequence of the mixing property of the states $\mu^{\#}_{\beta}$ when restricted to functions which are invariant under global spin flip (*i.e.* that $f(-\sigma)=f(\sigma)$ for every spin configuration σ). For completeness we enclose the statement, which may be part of the folklore among experts, in the next lemma, but we leave the proof as an exercise (one may try to use the random-cluster model or an extension of Griffiths' second inequality). We refer to [ADCS 15] for the proof. □

Lemma 5.20. For any $\beta>0$, the state $\mu^{\#}_{\beta}$ is ergodic and mixing in its restriction to the σ-algebra of even events, *i.e.* that for any pair of functions F,G from spin configurations to $\mathbb{R}$ with $F(-\sigma)=F(\sigma)$ and $G(-\sigma)=G(\sigma)$, the following limit exists and satisfies:

$$\lim_{\|x\|\to\infty} \mu^{\#}_{\beta}[F\times G\circ\tau_x] \;=\; \mu^{\#}_{\beta}[F]\cdot\mu^{\#}_{\beta}[G].$$

Define $\mathbb{P}_{\beta}$ to be the law of $\widehat{\mathbf{n}_1+\mathbf{n}_2}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are two independent currents with laws $\mathrm{P}^{\mathrm{free}}_{\beta}$ and P^{+}_{β}. We also set $\mathbb{E}_{\beta}$ for the expectation with respect to $\mathbb{P}_{\beta}$. Properties **R2** and **R3** of Proposition 5.19 imply immediately that $\mathbb{P}_{\beta}$ is invariant and ergodic with respect to shifts. We now state that there cannot be more than one infinite cluster.

Theorem 5.21.

For any $\beta>0$, there exists at most one infinite cluster $\mathbb{P}_{\beta}$-almost surely (at any $\beta\geqslant 0$).

The proof of this theorem follows from the Burton-Keane argument presented in the proof of Theorem 2.4. The proof requires a few trivial modifications and we refer to [ADCS 15] for details. The main different comes from the fact that we must replace independence in the proof by the fact that conditioned on the state of everything else, one may open the edge e with positive probability. Let us show that this property is true for the trace of the random current.

Lemma 5.22. Consider the map $\widehat{\Phi}_N$ from percolation configurations onto itself opening all edges $\{x,y\}$ in Λ_N with $J_{xy}>0$. Let $N>0$, then there exists $c=c(N,J,\beta)>0$ such that for any event $\mathscr{E}$,

$$\mathbb{P}_{\beta}[\widehat{\Phi}_N(\mathscr{E})] \geqslant c\,\mathbb{P}_{\beta}[\mathscr{E}].$$

Proof. It is sufficient to consider events $\mathscr{E}$ depending on a finite number of edges. Let $\mathbb{P}_{\Lambda_n,\beta}$ be the law of $\widehat{\mathbf{n}_1+\mathbf{n}_2}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are two independent currents with respective laws $\mathrm{P}^{\mathrm{free}}_{\Lambda_n,\beta}$ and $\mathrm{P}^{+}_{\Lambda_n,\beta}$. Property **R1** of Proposition 5.19 shows that $\mathbb{P}_{\Lambda_n,\beta}$ converges weakly to $\mathbb{P}_\beta$. This reduces the proof to showing the existence of $c = c(N,J,\beta) > 0$ on Λ_n, with a value which does not depend on $n > N$.

Consider the transformation Φ_N from pairs of percolation configurations on Λ_n onto itself defined by

$$\Phi_N(\mathbf{n}_1,\mathbf{n}_2)\{x,y\} = \begin{cases} (0,2) & \text{if } (\mathbf{n}_1\{x,y\},\mathbf{n}_2\{x,y\}) = (0,0), \\ & J_{xy} > 0, \text{ and } x,y \in \Lambda_N, \\ (\mathbf{n}_1\{x,y\},\mathbf{n}_2\{x,y\}) & \text{otherwise,} \end{cases}$$

where exceptionally $\mathbf{m}\{x,y\}$ denotes $\mathbf{m}_{\{x,y\}}$ for ease of notation. Fix $\omega \in \{0,1\}^{\Lambda_n}$ and let

$$\Omega_2 = \{(\mathbf{n}_1,\mathbf{n}_2) \in (\Omega_{\Lambda_n})^2 : \widehat{\mathbf{n}_1+\mathbf{n}_2} = \omega\}.$$

The set $\Phi_N(\Omega_2)$ is obtained from Ω_2 by changing the value of the current $\mathbf{n}_2$ on edges $\{x,y\}$ with $\omega_{\{x,y\}} = 0$ from 0 to 2. Therefore,

$$\begin{aligned}
\mathbb{P}_{\Lambda_n,\beta}[\widehat{\Phi}_N(\mathscr{E})] &= \sum_{\omega' \in \widehat{\Phi}(\mathscr{E})} \mathbb{P}_{\Lambda_n,\beta}[\omega'] \\
&= \sum_{\omega \in \mathscr{E}} \frac{1}{\mathrm{Card}[\widehat{\Phi}_N^{-1}(\widehat{\Phi}_N(\omega))]} \mathbb{P}_{\Lambda_n,\beta}[\widehat{\Phi}_N(\omega)] \\
&\geqslant 2^{-|\Lambda_N|^2} \sum_{\omega \in \mathscr{E}} \mathbb{P}_{\Lambda_n,\beta}[\widehat{\Phi}_N(\omega)] \\
&= 2^{-|\Lambda_N|^2} \sum_{\omega \in \mathscr{E}} \mathrm{P}^{\mathrm{free}}_{\Lambda_n,\beta} \otimes \mathrm{P}^{+}_{\Lambda_n,\beta}[\Phi_N(\Omega_2)] \\
&\geqslant 2^{-|\Lambda_N|^2} \sum_{\omega \in \mathscr{E}} \Big(\prod_{\{x,y\} \subset \Lambda_N : \omega_{xy} = 0} \frac{(\beta J_{xy})^2}{2} \Big) \mathrm{P}^{\mathrm{free}}_{\Lambda_n,\beta} \otimes \mathrm{P}^{+}_{\Lambda_n,\beta}[\Omega_2] \\
&\geqslant c \sum_{\omega \in \mathscr{E}} \mathrm{P}^{\mathrm{free}}_{\Lambda_n,\beta} \otimes \mathrm{P}^{+}_{\Lambda_n,\beta}[\Omega_2] = c\, \mathbb{P}_{\Lambda_n,\beta}[\mathscr{E}],
\end{aligned}$$

where $c = c(N,J,\beta) > 0$ does not depend on n. In the first inequality, we used the fact that the number of pre-images of each configuration is smaller than 2 to the power the number of pairs of points in Λ_N (since one has to decide whether edges of Λ_N were open or closed before the transformation). □

Let us continue with a crucial relation which justifies the consideration of $\mathbb{P}_\beta$.

Theorem 5.23.

For $\beta \leqslant \beta_c$, if $\mu^{\mathrm{free}}_{G,\beta}[\sigma_x\sigma_y] \longrightarrow 0$, then $\mathbb{P}_\beta[0 \longleftrightarrow \infty] = 0$.

Proof. Let us start by proving that $\mathbb{P}_\beta[x \longleftrightarrow y]$ tends to 0 as x,y gets further and further away.

Let $L > 0$ and let $x, y \in \Lambda_L$. The switching lemma (Lemma 5.6) implies

$$\mathrm{P}^{\mathrm{free}}_{\Lambda_L,\beta} \otimes \mathrm{P}^{+}_{\Lambda_L,\beta}\left[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y \text{ in } \Lambda_L\right] := \frac{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}\left[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y \text{ in } \Lambda_L\right]}{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}$$

$$= \frac{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\{x,y\}\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}.$$

The representations of spin-spin correlations in terms of random currents then imply that

$$\mathrm{P}^{\mathrm{free}}_{\Lambda_L,\beta} \otimes \mathrm{P}^{+}_{\Lambda_L,\beta}\left[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} y \text{ in } \Lambda_L\right] = \mu^{\mathrm{free}}_{\Lambda_L,\beta}[\sigma_x\sigma_y]\mu^{+}_{\Lambda_L,\beta}[\sigma_x\sigma_y] \leqslant \mu^{\mathrm{free}}_{\Lambda_L,\beta}[\sigma_x\sigma_y].$$

The right-hand side converges to $\mu^{\mathrm{free}}_{\Lambda_L,\beta}[\sigma_x\sigma_y]$ as L tends to infinity. Since the event on the left-hand side can be expressed in terms of $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2$, the convergence of $\widehat{\mathrm{P}}^{+}_{\Lambda_L,\beta}$ and $\widehat{\mathrm{P}}^{\mathrm{free}}_{\Lambda_L,\beta}$ to $\widehat{\mathrm{P}}^{+}_{\beta}$ and $\widehat{\mathrm{P}}^{\mathrm{free}}_{\beta}$ provided by Proposition 5.19 implies that the left-hand side converges to $\mathbb{P}_\beta[x \longleftrightarrow y]$. (The percolation event does not depend on finitely many edges, but justifying passing to the limit is straightforward by first considering the events that x is connected to y in the box of size N.) Therefore,

$$\mathbb{P}_\beta[x \longleftrightarrow y] \leqslant \mu^{\mathrm{free}}_{\beta}[\sigma_x\sigma_y] \longrightarrow 0. \tag{5.15}$$

We now wish to prove that $\mathbb{P}_\beta[0 \leftrightarrow \infty] = 0$. Let $B \subset \mathbb{Z}^d$ be a finite subset. The Cauchy-Schwarz inequality applied to the random variable $X = \sum_{x\in B}\mathbf{I}[x \longleftrightarrow \infty]$ leads to

$$\big(|B|\mathbb{P}_\beta[0 \longleftrightarrow \infty]\big)^2 := \mathbb{E}_\beta[X]^2 \leqslant \mathbb{E}_\beta[X^2] =: \sum_{x,y\in B}\mathbb{P}_\beta[x, y \longleftrightarrow \infty].$$

The uniqueness of the infinite cluster (Theorem 5.21) thus implies

$$\big(|B|\mathbb{P}_\beta[0 \longleftrightarrow \infty]\big)^2 \leqslant \sum_{x,y\in B}\mathbb{P}_\beta[x, y \longleftrightarrow \infty] \leqslant \sum_{x,y\in B}\mathbb{P}_\beta[x \longleftrightarrow y]. \tag{5.16}$$

Combining this relation with (5.15) and optimizing over B we get:

$$\mathbb{P}_\beta[0 \longleftrightarrow \infty]^2 \leqslant \inf_{B\in\mathbb{Z}^d, |B|<\infty} \frac{1}{|B|^2}\sum_{x,y\in B}\mu^{\mathrm{free}}_{\beta}[\sigma_x\sigma_y] \longrightarrow 0$$

which proves the claim. □

Remark 5.24. In the last step of (5.16) one can see uniqueness of the infinite cluster used as a substitute for the classical percolation argument utilizing the FKG inequality, which we do not have for random currents.

Proof. We are now in a position to prove Theorem 5.16. Fix a pair of vertices x and y. Applying the switching lemma (Lemma 5.6) again, we find that for $L>0$:

$$\mu^+_{\Lambda_L,\beta}[\sigma_x\sigma_y]-\mu^{\rm free}_{\Lambda_L,\beta}[\sigma_x\sigma_y]=\frac{\displaystyle\sum_{\mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}w_\beta(\mathbf{n}_2)}{\displaystyle\sum_{\mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}w_\beta(\mathbf{n}_2)}-\frac{\displaystyle\sum_{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\{x,y\}}w_\beta(\mathbf{n}_1)}{\displaystyle\sum_{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing}w_\beta(\mathbf{n}_1)}$$

$$=\frac{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)-\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\{x,y\}\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}$$

$$=\frac{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\left(1-\mathbf{I}[x\overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow}y\text{ in }\Lambda_L]\right)}{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}.$$

Yet, any configuration $\mathbf{n}_2$ with sources at x and y such that x and y are not connected in Λ_L necessarily satisfies that x and y are connected to g. Therefore,

$$\mu^+_{\Lambda_L,\beta}[\sigma_x\sigma_y]-\mu^{\rm free}_{\Lambda_L,\beta}[\sigma_x\sigma_y]\leqslant\frac{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x\overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow}g]}{\displaystyle\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\varnothing}}w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)}. \tag{5.17}$$

We shall now estimate the sum in the numerator by comparing it to the corresponding sum in which the source condition of $(\mathbf{n}_1,\mathbf{n}_2)$ is changed to $\partial\mathbf{n}_1=\partial\mathbf{n}_2=\varnothing$.

Fix a sequence of vertices $x=x_0,\dots,x_m=y$ with $J_{x_ix_{i+1}}>0$ for any $0\leqslant i<m$. For any L large enough so that $x_i\in\Lambda_L$ for all $i\leqslant m$, consider the one-to-many mapping which assigns to each ω a modified current configuration $\mathbf{n}_2$ with the change limited to $\mathbf{n}_2$ along the set of edges $e_j=\{x_i,x_{i+1}\}$, $j=0,\dots,m-1$, at which the parity of all these variables is flipped, and the value of the new one is at least 1 at each edge. Under this mapping, the image of each pair $(\mathbf{n}_1,\mathbf{n}_2)$ that contributes in the numerator of (5.17) lies in the set for which the connection event $x\overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow}g$ remains satisfied, but the source set of $\mathbf{n}_2$ is reset to $\partial\mathbf{n}_2=\varnothing$.

Classifying the current pairs according to the values of all the unaffected variables of $\{(\mathbf{n}_1,\mathbf{n}_2)\}$, and the parity of $\mathbf{n}_2$ along the set of edges $e_0,\dots,e_{m-1}$, it is easy to see that under this one-to-many map the measure of each set is multiplied by a factor which is larger than or equal to

$$\Gamma_{xy}\;=\;\prod_{j=1}^m\min\left\{\frac{\sinh(\beta J_{e_j})}{\cosh(\beta J_{e_j})},\frac{\cosh(\beta J_{e_j})-1}{\sinh(\beta J_{e_j})}\right\}.$$

(*i.e.* the original measure multiplied by Γ_{xy} is dominated by the measure of the image set.) This allows us to conclude:

$$\sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2=\{x,y\}}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g] \leqslant \Gamma_{xy}^{-1} \sum_{\substack{\mathbf{n}_1\in\Omega_{\Lambda_L}:\partial\mathbf{n}_1=\varnothing\\ \mathbf{n}_2'\in\Omega_{\Lambda_L\cup\{g\}}:\partial\mathbf{n}_2'=\varnothing}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2')\mathbf{I}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2'}}{\longleftrightarrow} g].$$

Inserting this in (5.17), we find that

$$\mu^+_{\Lambda_L,\beta}[\sigma_x\sigma_y]-\mu^{\mathrm{free}}_{\Lambda_L,\beta}[\sigma_x\sigma_y] \leqslant \Gamma_{xy}^{-1}\, \mathrm{P}^{\mathrm{free}}_{\Lambda_L,\beta}\otimes\mathrm{P}^{+}_{\Lambda_L,\beta}[x \overset{\widehat{\mathbf{n}_1+\mathbf{n}_2}}{\longleftrightarrow} g].$$

Taking the limit $L\to\infty$ (which exists by Proposition 5.19) we obtain

$$0 \leqslant \mu^+_\beta[\sigma_x\sigma_y]-\mu^{\mathrm{free}}_\beta[\sigma_x\sigma_y] \leqslant \Gamma_{xy}^{-1}\, \mathbb{P}_\beta[x\leftrightarrow\infty].$$

(The percolation event on the right does not depend on finitely many edges, but justifying passing to the limit is straightforward by first considering the events that x is connected to distance N.)

We now consider $\beta\leqslant\beta_c$. Theorem 5.23 implies that $\mathbb{P}_\beta[x\longleftrightarrow\infty]=0$, and hence for any $x,y\in\mathbb{Z}^d$: $\mu^+_\beta[\sigma_x\sigma_y]=\mu^{\mathrm{free}}_\beta[\sigma_x\sigma_y]$. Thus, using the FKG inequality:

$$0\leqslant\mu^+_\beta[\sigma_0]\mu^+_\beta[\sigma_x]\leqslant\mu^+_\beta[\sigma_0\sigma_x]=\mu^{\mathrm{free}}_\beta[\sigma_0\sigma_x]$$

for any $x\in\mathbb{Z}^d$. The assumption that $\mu^{\mathrm{free}}_\beta[\sigma_0\sigma_x]$ averages to zero over translations leads to $\mu^+_{\beta_c}[\sigma_0]=0$. □

CHAPTER 6

EPILOGUE

This book presented a few graphical representations of lattice models. By lack of space and energy, we did not discuss a number of graphical representations. The zoo of these graphical representations is huge, and applications range from the lattice models described in this book to other models such as quantum chains or particle systems. In each case, correlations of the original model get rephrased into connectivity properties of a percolation-type model.

For completeness, we mention a few graphical representations which were omitted in this book: the low and high temperature expansions of the Ising model, loop $O(n)$ models, cluster expansions, random-walk representations, etc (see e.g. [DC 11] and references therein)... We refer to the extensive literature on these subjects for a comprehensive study of these models.

Going back to the graphical representations presented in this book, their study involves techniques coming from several fields of mathematics, including mathematical physics, probability, discrete complex analysis, combinatorics, exact solvability, etc. The theory lies at the interplay of all these domains, making it a rich and beautiful area of active research.

Bibliography

[ABF 87] **M. Aizenman, D. J. Barsky and R. Fernández**. The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.*, Vol. 47, N°3-4, pp. 343–374, 1987.

[ACCN 88] **M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman**. Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *J. Statist. Phys.*, Vol. 50, N°1-2, pp. 1–40, 1988.

[ADCS 15] **M. Aizenman, H. Duminil-Copin and V. Sidoravicius**. Random Currents and Continuity of Ising Model's Spontaneous Magnetization. *Communications in Mathematical Physics*, Vol. 334, N°2, pp. 719–742, 2015.

[AKN 87] **M. Aizenman, H. Kesten and C. M. Newman**. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.*, Vol. 111, N°4, pp. 505–531, 1987.

[Aiz 82] **M. Aizenman**. Geometric analysis of φ^4 fields and Ising models. I, II. *Comm. Math. Phys.*, Vol. 86, N°1, pp. 1–48, 1982.

[AB 87] **M. Aizenman and D. J. Barsky**. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, Vol. 108, N°3, pp. 489–526, 1987.

[BGN 91] **D. J. Barsky, G. R. Grimmett and C. M. Newman**. Percolation in half-spaces: equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields*, Vol. 90, N°1, pp. 111–148, 1991.

[Bax 71] **R. J. Baxter**. Generalized ferroelectric model on a square lattice. *Studies in Appl. Math.*, Vol. 50, pp. 51–69, 1971.

[Bax 78] **R. J. Baxter**. Solvable eight-vertex model on an arbitrary planar lattice. *Philos. Trans. Roy. Soc. London Ser. A*, Vol. 289, N°1359, pp. 315–346, 1978.

[Bax 73] **R. Baxter**. Potts model at the critical temperature. *Journal of Physics C: Solid State Physics*, Vol. 6, N°23, p. L445, 1973.

[Bax 89] **R. J. Baxter**. *Exactly solved models in statistical mechanics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989. Reprint of the 1982 original.

[BBMDG+ 14] **N. Beaton, M. Bousquet-Mélou, J. De Gier, H. Duminil-Copin and A. Guttmann**. The critical fugacity for surface adsorption of self-avoiding walks on the honeycomb lattice is $1+\sqrt{2}$. *Comm. Math. Phys.*, Vol. 326, N°3, pp. 727–754, 2014.

[BDC 12] **V. Beffara and H. Duminil-Copin**. Smirnov's fermionic observable away from criticality. *Ann. Probab.*, Vol. 40, N°6, pp. 2667–2689, 2012.

[BDC 13] **V. Beffara and H. Duminil-Copin**. Lectures on Planar percolation with a glimpse of Schramm Loewner Evolution. *Probability Surveys*, Vol. 10, pp. 1–50, 2013.

[BDCS 12] **V. Beffara, H. Duminil-Copin and S. Smirnov**. On the critical parameters of the $q \geqslant 4$ random-cluster model on isoradial graphs. *Journal of Physics A: Mathematical and Theoretical*, Vol. 48, N°48, p. 484003, 2012.

[BDC 12a] **V. Beffara and H. Duminil-Copin**. Critical point and duality in planar lattice models. 2012.

[BDC 12b] **V. Beffara and H. Duminil-Copin**. The self-dual point of the two-dimensional random-cluster model is critical for $q \geqslant 1$. *Probab. Theory Related Fields*, Vol. 153, N°3-4, pp. 511–542, 2012.

[BDCH 14] **S. Benoist, H. Duminil-Copin and C. Hongler**. Conformal invariance of crossing probabilities for the Ising model with free boundary conditions. 2014. arXiv:1410.3715.

[Ber 72] **V. Berezinskii**. Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum systems. *Soviet Journal of Experimental and Theoretical Physics*, Vol. 34, p. 610, 1972.

[Bis 09] **M. Biskup**. Reflection positivity and phase transitions in lattice spin models. Vol. 1970, pp. 1–86, 2009.

[BCC 06] **M. Biskup, L. Chayes and N. Crawford**. Mean-field driven first-order phase transitions in systems with long-range interactions. *J. Stat. Phys.*, Vol. 122, N°6, pp. 1139–1193, 2006.

[BR 06a] **B. Bollobás and O. Riordan**. The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields*, Vol. 136, N°3, pp. 417–468, 2006.

[BR 06b] **B. Bollobás and O. Riordan**. A short proof of the Harris-Kesten theorem. *Bull. London Math. Soc.*, Vol. 38, N°3, pp. 470–484, 2006.

[BR 10] **B. Bollobás and O. Riordan**. Percolation on self-dual polygon configurations. In: *An irregular mind*, pp. 131–217, János Bolyai Math. Soc., Budapest, 2010.

[BK 89] **R. M. Burton and M. Keane**. Density and uniqueness in percolation. *Comm. Math. Phys.*, Vol. 121, N°3, pp. 501–505, 1989.

[CDCH$^+$ 14] **D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen and S. Smirnov**. Convergence of Ising interfaces to Schramm's SLE curves. *C. R. Acad. Sci. Paris Math.*, Vol. 352, N°2, pp. 157–161, 2014.

[CDCH 16] **D. Chelkak, H. Duminil-Copin and C. Hongler**. Crossing probabilities in topological rectangles for the critical planar FK-Ising model. *Electronic Journal of Probability*, Vol. 21, N°1, 2016.

[CHI 15] **D. Chelkak, C. Hongler and K. Izyurov**. Conformal invariance of spin correlations in the planar Ising model. *Ann. of Math. (2)*, Vol. 181, N°3, pp. 1087–1138, 2015.

[CI 13] **D. Chelkak and K. Izyurov**. Holomorphic spinor observables in the critical Ising model. *Comm. Math. Phys.*, Vol. 322, N°2, pp. 303–332, 2013.

[CS 11] **D. Chelkak and S. Smirnov**. Discrete complex analysis on isoradial graphs. *Adv. Math.*, Vol. 228, N°3, pp. 1590–1630, 2011.

[CS 12] **D. Chelkak and S. Smirnov**. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, Vol. 189, N°3, pp. 515–580, 2012.

[CDC 13] **D. Cimasoni and H. Duminil-Copin**. The critical temperature for the Ising model on planar doubly periodic graphs. *Electron. J. Probab*, Vol. 18, N°44, pp. 1–18, 2013.

[DMNS 81] **E. Domany, D. Mukamel, B. Nienhuis and A. Schwimmer**. Duality relations and equivalences for models with O(N) and cubic symmetry. *Nuclear Physics B*, Vol. 190, N°2, pp. 279–287, 1981.

[DST 16] **H. Duminil, V. Sidoravicius and V. Tassion**. Absence of infinite cluster for critical bernoulli percolation on slabs. *Communications on Pure and Applied Mathematics*, Vol. 69, N°7, pp. 1397–1411, 2016.

[DCST 15] **H. Duminil-Copin, V. Sidoravicius and V. Tassion**. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leqslant q \leqslant 4$. 2015. arxiv:1505.04159.

[DC 11] **H. Duminil-Copin**. Phase transition in random-cluster and O (n)-models. *archive-ouverte.unige.ch/unige:18929*, p. 360 p, 2011.

[DC 12] **H. Duminil-Copin**. Divergence of the correlation length for critical planar FK percolation with $1 \leqslant q \leqslant 4$ via parafermionic observables. *Journal of Physics A: Mathematical and Theoretical*, Vol. 45, N°49, p. 494013, 2012.

[DC 13] **H. Duminil-Copin**. *Parafermionic observables and their applications to planar statistical physics models*. Vol. 25 in *Ensaios Matematicos*, Brazilian Mathematical Society, 2013.

[DCGP 14] **H. Duminil-Copin, C. Garban and G. Pete**. The near-critical planar FK-Ising model. *Comm. Math. Phys*, Vol. 326, N°1, pp. 1–35, 2014.

[DCHN 11] **H. Duminil-Copin, C. Hongler and P. Nolin**. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Appl. Math.*, Vol. 64, N°9, pp. 1165–1198, 2011.

[DCM 14] **H. Duminil-Copin and I. Manolescu**. The phase transitions of the planar random-cluster and Potts models with $q \geqslant 1$ are sharp. *Probability Theory and Related Fields*, pp. 1–28, 2014.

[DCPSS 14] **H. Duminil-Copin, R. Peled, W. Samotij and Y. Spinka**. Exponential decay of loop lengths in the loop $o(n)$ model with large n. 2014. arXiv:1412.8326.

[DCS 12a] **H. Duminil-Copin and S. Smirnov**. Conformal invariance of lattice models. In: *Probability and statistical physics in two and more dimensions*, pp. 213–276, Amer. Math. Soc., Providence, RI, 2012.

[DCS 12b] **H. Duminil-Copin and S. Smirnov**. The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. *Ann. of Math. (2)*, Vol. 175, N°3, pp. 1653–1665, 2012.

[DCT 15] **H. Duminil-Copin and V. Tassion**. A new proof of the sharpness of the phase A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. 2015. unpublished.

[FvdH 15] **R. Fitzner and R. van der Hofstad**. Generalized approach to the non-backtracking lace expansion. 2015. To appear in Probability Theory and Related Fields.

[FKG 71] **C. M. Fortuin, P. W. Kasteleyn and J. Ginibre**. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, Vol. 22, pp. 89–103, 1971.

[FSS 76] **J. Fröhlich, B. Simon and T. Spencer**. Infrared bounds, phase transitions and continuous symmetry breaking. *Comm. Math. Phys.*, Vol. 50, N°1, pp. 79–95, 1976.

[FS 81] **J. Fröhlich and T. Spencer**. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, Vol. 81, N°4, pp. 527–602, 1981.

[Geo 11] **H.-O. Georgii**. *Gibbs measures and phase transitions*. Vol. 9 in *de Gruyter Studies in Mathematics*, Walter de Gruyter & Co., Berlin, second Ed., 2011.

[Gla 13] **A. Glazman**. Connective constant for a weighted self-avoiding walk on $\mathbb{Z}^2$. 2013. Preprint.

[GG 11] **B. Graham and G. Grimmett**. Sharp thresholds for the random-cluster and Ising models. *Ann. Appl. Probab.*, Vol. 21, N°1, pp. 240–265, 2011.

[Gri 67] **R. Griffiths**. Correlation in Ising ferromagnets I, II. *J. Math. Phys.*, Vol. 8, pp. 478–489, 1967.

[GHS 70] **R. B. Griffiths, C. A. Hurst and S. Sherman**. Concavity of magnetization of an Ising ferromagnet in a positive external field. *J. Mathematical Phys.*, Vol. 11, pp. 790–795, 1970.

[Gri 99] **G. Grimmett**. *Percolation*. Vol. 321 in *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, second Ed., 1999.

[Gri 06] **G. Grimmett**. *The random-cluster model*. Vol. 333 in *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 2006.

[Gri 99] **G. R. Grimmett**. Inequalities and entanglements for percolation and random-cluster models. In: *Perplexing problems in probability*, pp. 91–105, Birkhäuser Boston, Boston, MA, 1999.

[HS 90] **T. Hara and G. Slade**. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, Vol. 128, N°2, pp. 333–391, 1990.

[Har 60] **T. E. Harris**. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, Vol. 56, pp. 13–20, 1960.

[HKW 78] **A. Hintermann, H. Kunz and F. Y. Wu**. Exact results for the Potts model in two dimensions. *J. Statist. Phys.*, Vol. 19, N°6, pp. 623–632, 1978.

[Hon 10] **C. Hongler**. *Conformal invariance of Ising model correlations*. PhD thesis, université de Genève, 2010.

[HK 13] **C. Hongler and K. Kytölä**. Ising interfaces and free boundary conditions. *J. Amer. Math. Soc.*, Vol. 26, N°4, pp. 1107–1189, 2013.

[HS 13] **C. Hongler and S. Smirnov**. The energy density in the planar Ising model. *Acta math.*, Vol. 211, N°2, pp. 191–225, 2013.

[ISV 02] **D. Ioffe, S. Shlosman and Y. Velenik**. 2D models of statistical physics with continuous symmetry: the case of singular interactions. *Commun. Math. Phys.*, Vol. 226, N°2, pp. 433–454, 2002.

[Isa 52] **R. Isaacs**. Monodiffric functions. Construction and applications of conformal maps. In: *Proceedings of a symposium*, pp. 257–266, U. S. Government Printing Office, Washington, D. C., 1952.

[Isa 41] **R. P. Isaacs**. A finite difference function theory. *Univ. Nac. Tucumán. Revista A.*, Vol. 2, pp. 177–201, 1941.

[Isi 25] **E. Ising**. Beitrag zur Theorie des Ferromagnetismus. *Z. Phys.*, Vol. 31, pp. 253–258, 1925.

[Izy 15] **K. Izyurov**. Smirnov's observable for free boundary conditions, interfaces and crossing probabilities. *Comm. Math. Phys.*, Vol. 337, N°1, pp. 225–252, 2015.

[KS 12] **A. Kemppainen and S. Smirnov**. Random curves, scaling limits and Loewner evolutions. 2012. arXiv:1212.6215.

[Kes 80] **H. Kesten**. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.*, Vol. 74, N°1, pp. 41–59, 1980.

[KT 73] **J. Kosterlitz and D. Thouless**. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C: Solid State Physics*, Vol. 6, N°7, pp. 1181–1203, 1973.

[KS 82] **R. Kotecký and S. B. Shlosman**. First-order phase transitions in large entropy lattice models. *Comm. Math. Phys.*, Vol. 83, N°4, pp. 493–515, 1982.

[Kup 80] **A. J. Kupiainen**. On the $1/n$ expansion. *Comm. Math. Phys.*, Vol. 73, N°3, pp. 273–294, 1980.

[LMR 86] **L. Laanait, A. Messager and J. Ruiz**. Phases coexistence and surface tensions for the Potts model. *Comm. Math. Phys.*, Vol. 105, N°4, pp. 527–545, 1986.

[LMMS$^+$ 91] **L. Laanait, A. Messager, S. Miracle-Solé, J. Ruiz and S. Shlosman**. Interfaces in the Potts model. I. Pirogov-Sinai theory of the Fortuin-Kasteleyn representation. *Comm. Math. Phys.*, Vol. 140, N°1, pp. 81–91, 1991.

[Len 20] **W. Lenz**. Beitrag zum Verständnis der magnetischen Eigenschaften in festen Körpern. *Phys. Zeitschr.*, Vol. 21, pp. 613–615, 1920.

[Li 12] **Z. Li**. Critical temperature of periodic Ising models. *Comm. Math. Phys.*, Vol. 315, N°2, pp. 337–381, 2012.

[LSS 97] **T. M. Liggett, R. H. Schonmann and A. M. Stacey**. Domination by product measures. *Ann. Probab.*, Vol. 25, N°1, pp. 71–95, 1997.

[MW 73] **B. McCoy and T. Wu**. *The two-dimensional Ising model*. Harvard University Press, Cambridge, MA, 1973.

[Men 86] **M. V. Menshikov**. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, Vol. 288, N°6, pp. 1308–1311, 1986.

[MW 66] **N. D. Mermin and H. Wagner**. Absence of ferromagnetism or antiferromagnetism in one-or two-dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, Vol. 17, N°22, p. 1133, 1966.

[Ons 44] **L. Onsager**. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, Vol. 65, pp. 117–149, 1944.

[Pol 75] **A. Polyakov**. Interaction of goldstone particles in two dimensions. Applications to ferromagnets and massive Yang-Mills fields. *Physics Letters B*, Vol. 59, N°1, pp. 79–81, 1975.

[Pot 52] **R. B. Potts**. Some generalized order-disorder transformations. In: *Proceedings of the Cambridge Philosophical Society*, pp. 106–109, Cambridge Univ Press, 1952.

[Rus 78] **L. Russo**. A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 43, N°1, pp. 39–48, 1978.

[Sch 00] **O. Schramm**. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, Vol. 118, pp. 221–288, 2000.

[SW 78] **P. D. Seymour and D. J. A. Welsh**. Percolation probabilities on the square lattice. *Ann. Discrete Math.*, Vol. 3, pp. 227–245, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[Sim 80] **B. Simon**. Correlation inequalities and the decay of correlations in ferromagnets. *Comm. Math. Phys.*, Vol. 77, N°2, pp. 111–126, 1980.

[Smi 01] **S. Smirnov**. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, Vol. 333, N°3, pp. 239–244, 2001.

[Smi 06] **S. Smirnov**. Towards conformal invariance of 2D lattice models. In: *International Congress of Mathematicians. Vol. II*, pp. 1421–1451, Eur. Math. Soc., Zürich, 2006.

[Smi 10] **S. Smirnov**. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math. (2)*, Vol. 172, N°2, pp. 1435–1467, 2010.

[Sta 68] **H. Stanley**. Dependence of critical properties on dimensionality of spins. *Physical Review Letters*, Vol. 20, N°12, pp. 589–592, 1968.

[Tas 14] **V. Tassion**. *Planarité et localité en percolation*. PhD thesis, ENS Lyon, 2014.

[Tas 16] **V. Tassion**. Crossing probabilities for Voronoi percolation. 2016. To appear in Annals of Probability.

[vdBK 85] **J. van den Berg and H. Kesten**. Inequalities with applications to percolation and reliability. *J. Appl. Probab.*, Vol. 22, N°3, pp. 556–569, 1985.

[Wer 09] **W. Werner**. Lectures on two-dimensional critical percolation. In: *Statistical mechanics*, pp. 297–360, Amer. Math. Soc., Providence, RI, 2009.

[Wu 82] **F. Y. Wu**. The Potts model. *Rev. Modern Phys.*, Vol. 54, N°1, pp. 235–268, 1982.

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