CHAPTER I

ARITHMETIC INTERSECTION

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1. Introduction

Intersection Theory in projective varieties is a topic in algebraic geometry which goes back to the eighteen century. An example is Bézout's theorem, which says that two projective plane curves C and D, of degrees c and d and which have no components in common, meet in at most cd points. This result can be extended to closed subvarieties Y and Z in the projective space $\mathbf{P}$ of dimension n over k, with $\dim(Y) + \dim(Z) = n$ and for which $Y \cap Z$ consists of a finite number of points: this number is at most $\deg(Y)\deg(Z)$. Here the degree $\deg(Y)$ of $Y \subset \mathbf{P}$ can be defined as follows. Let L be the canonical line bundle on $\mathbf{P}$. The integer $\deg(Y)$ is characterized by the following two properties:

i) When $Y = y$ is a closed point with residue field $\kappa(y)$

$$\deg(Y) = [\kappa(y) : k] .$$

ii) When s is a non-trivial rational section of L over Y, with divisor $\operatorname{div}(s) = \sum_\alpha n_\alpha Y_\alpha$,

$$\deg(Y) = \sum_\alpha n_\alpha \deg(Y_\alpha) .$$

Note that induction on the dimension of Y shows that i) and ii) define the degree $\deg(Y)$ uniquely. One proves that it does not depend on the choice of s made in ii). We refer the reader to [**7**] §2 for a brief introduction to classical intersection theory.

In 1974, Arakelov discovered an intersection theory on arithmetic surfaces. Namely, if C is a smooth projective curve over $\mathbf{Q}$, consider a regular projective scheme X over $\mathbf{Z}$ with generic fiber equal to C. Since the Krull dimension of X is two, one thinks of it as a surface. And since $\mathbf{Z}$ is affine, this surface X is not complete. To complete X, Arakelov adds to it the set of complex points $X(\mathbf{C})$, viewed as the fiber at ∞ of the map $X \to \mathrm{Spec}(\mathbf{Z})$. An *arithmetic divisor* is a formal sum $D+\lambda$ where D is a classical divisor on X and λ is a real number. Given two arithmetic divisors $D+\lambda$ and $D'+\lambda'$ (such that D and D' have no common components) Arakelov defines their intersection number, which is not an integer but a real number. He proves several properties of these numbers, e.g. an adjunction formula.

It appears that every notion or result in the classical algebraic geometry of varieties over fields has an *arithmetic analog* in the Arakelov geometry of schemes over $\mathbf{Z}$. In the eighties, the Arakelov intersection theory was extended to higher dimensions by Gillet and myself [**9**].

In this chapter, we shall discuss the arithmetic analog of the notion of degrees, namely heights of varieties. To be more precise, we fix a regular projective scheme X over $\mathbf{Z}$. As arithmetic analogs of algebraic line bundles we take *hermitian line bundles* , i.e. line bundles L on X equipped with a smooth hermitian metric h on the restriction of L to the set of complex points $X(\mathbf{C})$. If $\bar{L}=(L,h)$ is such an hermitian line bundle on X and if $Y \subset X$ is an integral closed subset of X, the height of Y is a real number $h_{\bar{L}}(Y)$ which can be defined in several ways. It was first introduced by Faltings using arithmetic intersection theory [**6**] in his work on diophantine approximation on abelian varieties. Alternatively, $h_{\bar{L}}(Y)$ can be defined axiomatically by axioms similar to i) and ii) above [**2**]. This is the point of view we shall take here. When the unicity of $h_{\bar{L}}(Y)$ is easy to deduce from the axioms by induction on the dimension of Y (see §2), it is more difficult to show that $h_{\bar{L}}(Y)$ is independent on choices. Actually, the existence of $h_{\bar{L}}(Y)$ is the main result of this chapter, as we try to make it self-contained (see §3). The section §4 is a survey, without proof, of arithmetic intersection theory [**9**]. We conclude with a third definition of $h_{\bar{L}}(Y)$, as the integral on Y of a suitable power of the first Chen class of $\bar{L}$ (Theorem 3.7).

The interested reader may want to read in [**2**] several properties of the height, including arithmetic Bézout's theorems.

2. Definition of the height

Let X be a regular projective flat scheme over $\mathbf{Z}$, and $\overline{L}$ an hermitian line bundle over X. For every integral closed subset $Y \subset X$ we shall define a real number $h_{\overline{L}}(Y)$, called the (Faltings) *height* of Y ([**6**]). For this we need a few preliminaries.

2.1. Algebraic preliminaries

2.1.1. *Length.* — Let A be a noetherian (commutative and unitary) ring, and M an A-module of finite type.

There exists a filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_r = 0 \tag{1}$$

such that $M_{i+1} \neq M_i$ and $M_i/M_{i+1} = A/\wp_i$, where $\wp_i$ is a prime ideal, $0 \leqslant i \leqslant r-1$ ([**3**], th.1, p.312).

Definition 2.1. — The module M has *finite length* when there exists a filtration as (1) above where, for all i, $\wp_i$ is a maximal ideal in A.

***Lemma 2.2* (Jordan-Hölder).** — *If M has finite length, r does not depend on the choice of the filtration (1) with $\wp_i$ maximal. We call this number the* length *of M and denote it $\ell(M) \in \mathbf{N}$.*

Lemma 2.3. — *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules of finite length. Then

$$\ell(M) = \ell(M') + \ell(M'') .$$

The proof of Lemma 2.2 (resp. Lemma 2.3) can be found in [**4**], th.6, p.41 (resp. [**5**], prop.16, p.21).

2.1.2. *Order.* — Let A be as above. The *dimension* of A is

$$\dim(A) = \max\{n | \exists \text{ a chain of prime ideals } \wp_0 \subset \wp_1 \subset \wp_2 \ldots \subset \wp_n \subset A, \text{ with } \wp_i \neq \wp_{i+1}\} .$$

Let A be an integral ring of dimension 1, and $a \in A$, $a \neq 0$.

Lemma 2.4. — *A/aA has finite length.*

Proof. — Let

$$\overline{\wp_0} \subset \ldots \subset \overline{\wp_n}$$

be a maximal chain in A/aA, with $\overline{\wp_i} \neq \overline{\wp_{i+1}}$, and $\varphi : A \to A/aA$ the projection. Let $\wp_i = \varphi^{-1}(\overline{\wp_i})$. We get a chain

$$\wp_0 \subset \ldots \subset \wp_n$$

with $\wp_i \neq \wp_{i+1}$. Since A is integral, (0) is a prime ideal. And $\wp_0 \neq (0)$ since $\wp_0$ contains a. We conclude that

$$\dim(A/aA) \leqslant \dim(A) - 1 .$$

Since $\dim(A) = 1$ this implies that every prime ideal of A/aA is maximal. Therefore A/aA has finite length. □

Let A be as in Lemma 2.4 and $K = \mathrm{frac}(A)$ be the field of fractions of A. If $x \in K - \{0\}$ we define, if $x = a/b$,

$$\mathrm{ord}_A(x) = \ell(A/aA) - \ell(A/bA) \in \mathbf{Z}\,.$$

Lemma 2.5. — *i)* $\mathrm{ord}_A(x)$ *does not depend on the choice of* a *and* b. *ii)* $\mathrm{ord}_A(xy) = \mathrm{ord}_A(x) + \mathrm{ord}_A(y)$.

The proof of Lemma 2.5 is left to the reader.

Example 2.6. — Assume A is *local* (i.e. A has only one maximal ideal $\mathscr{M}$) and *regular* (i.e. $\dim A = \dim(\mathscr{M}/\mathscr{M}^2)$). When $\dim(A) = 1$, K has a *discrete valuation*

$$v : K \to \mathbf{Z} \cup \{\infty\}\,,$$

$$A = \{x \in K \text{ such that } v(a) \geqslant 0\}$$

and $\mathrm{ord}_A(x) = n$ if and only if $x \in \mathscr{M}^n$ and $x \notin \mathscr{M}^{n+1}$. Therefore

$$\mathrm{ord}_A(x) = v(x)\,.$$

2.1.3. *Divisors.* — Let X be a nœtherian scheme and O_X be the sheaf of regular functions on X.

Definition 2.7. — A *line bundle* on X is a locally free O_X-module L of rank one.

In other words L is a sheaf of abelian groups on X with a morphism of sheaves

$$\mu : O_X \times L \to L$$

such that there exists an open cover

$$X = \bigcup_\alpha U_\alpha$$

such that

(i) $L(U_\alpha) \simeq O_X(U_\alpha)$;
(ii) μ on $L(U_\alpha)$ is the multiplication.

Assume now that X is integral (for every open subset $U \subset X$, $O(U)$ is integral). Let $\eta \in X$ be the generic point.

Definition 2.8. — A *rational section* of L is an element $s \in L_\eta$.

Let $Z^1(X)$ be the free abelian group spanned by the closed irreducible subsets $Y \subset X$ of codimension one. We call $Z^1(X)$ the group of *divisors of* X.

If $s \in L_\eta$ is a non-trivial rational section, its *divisor* is defined as

$$\mathrm{div}(s) = \sum_Y n_Y [Y] \in Z^1(X)\,,$$

where n_Y is computed as follows. If $Y \subset X$ has codimension 1 and $Y = \overline{\{y\}}$ is integral, the ring $A = O_{X,y}$ is local, integral, of dimension 1. Its fraction field is

$$K = O_{X,\eta}\,.$$

Choose an isomorphism $L_y \simeq A$, hence $L_\eta \simeq K$. If $s \in L_\eta - \{0\} = K^*$, we let

$$n_Y = \mathrm{ord}_A(s)$$

(we shall also write $n_Y = \mathrm{ord}_Y(s)$).

One can prove that n_Y does not depend on choices, and $n_Y = 0$ for almost all Y.

Example 2.9. — Let K be a number field and $X = \mathrm{Spec}\,(O_K)$. Giving L amounts to give

$$\Lambda = L(X)\,,$$

a projective O_K-module of rank one. If $s \in \Lambda$, $s \neq 0$, we have a decomposition

$$\Lambda / O_K s \simeq \prod_{\wp\,\mathrm{prime}} (O_K/\wp^{n_\wp})$$

where $n_\wp = \mathrm{ord}_{O_\wp}(s)$, hence

$$\mathrm{div}(s) = \sum_\wp n_\wp\,[\wp]\,.$$

2.2. Analytic preliminaries. — Let X be an analytic smooth manifold over $\mathbf{C}$, and $O_{X,\mathrm{an}}$ the sheaf of holomorphic functions on X.

Definitions 2.10. — a) An *holomorphic line bundle* on X is a locally free $O_{X,\mathrm{an}}$-module of rank one.
b) A *metric* $\|\cdot\|$ on L consists of maps

$$L(x) \xrightarrow{\|\cdot\|} \mathbf{R}_+$$

for any x, where $L(x) = L_x/\mathscr{M}_x$ is the fiber at x. We ask that

(i) $\|\lambda s\| = |\lambda|\,\|s\|$ if $\lambda \in \mathbf{C}$;
(ii) $\|s\| = 0$ iff $s = 0$;

(iii) Let $U \subset X$ be an open subset and s a section of L over U vanishing nowhere; then the map

$$x \longmapsto \|s(x)\|^2$$

is C^∞.

We write $\overline{L} = (L, \|\cdot\|)$.

Denote by $A^n(X)$ the $\mathbf{C}$-vector space of C^∞ complex differential forms of degree n on X. Recall that $A^n(X)$ decomposes as

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)\,,$$

where $A^{p,q}(X)$ consists of those differential forms which can be written locally as a sum of forms of type

$$u\, d\, z_{i_1} \wedge \ldots \wedge d\, z_{i_p} \wedge d\, \overline{z}_{j_1} \wedge \ldots \wedge d\, \overline{z}_{j_q}$$

where u is a C^∞ function, $d\, z_\alpha = d\, x_\alpha + i\, d\, y_\alpha$ and $d\, \overline{z}_\alpha = d\, x_\alpha - i\, d\, y_\alpha$.

The differential

$$d : A^n(X) \to A^{n+1}(X)$$

is a sum $d = \partial + \overline{\partial}$ where

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X)$$

and

$$\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)\,.$$

We have $\partial^2 = \overline{\partial}^2 = d^2 = 0$ and we let

$$d^c = \frac{\partial - \overline{\partial}}{4\,\pi\, i}\,,$$

so that

$$dd^c = \frac{\overline{\partial}\,\partial}{2\,\pi\, i}\,.$$

Lemma 2.11. — *Let $\overline{L} = (L, \|\cdot\|)$ be an analytic line bundle with metric. There exists a smooth form*

$$c_1(\overline{L}) \in A^{1,1}(X)$$

such that, if $U \subset X$ is an open subset and $s \in \Gamma(U, L)$ is such that $s(x) \neq 0$ for every $x \in U$,

$$c_1(\overline{L})_{|U} = -\, dd^c \log \|s\|^2\,.$$

Proof. — Let $s' \in \Gamma(U, L)$ be another section such that $s(x) \neq 0$ when $x \in U$. We need to show that

$$(2) \qquad - dd^c \log \|s'\|^2 = - dd^c \log \|s\|^2 \quad \text{in} \quad A^{1,1}(U)\,.$$

There exists $f \in \Gamma(U, O_{X_{\mathrm{an}}})$ such that

$$s' = f s\,.$$

We get

$$- dd^c \log \|s'\|^2 = - dd^c \log \|s\|^2 - dd^c \log |f|^2\,.$$

But

$$\partial\,\overline{\partial} \log |f|^2 = \partial \left[\frac{\overline{\partial} f}{f} + \frac{\overline{\partial}\,\overline{f}}{\overline{f}} \right] = - \overline{\partial}\, \frac{\partial\,(\overline{f})}{\overline{f}} = 0\,,$$

and (2) follows. □

The form $c_1(\overline{L})$ is called the *first Chern form* of $\overline{L}$.

2.3. Heights. — Let X be a regular, projective, flat scheme over $\mathbf{Z}$. We denote by $X(\mathbf{C})$ the set of complex points of X; it is an analytic manifold.

Definition 2.12. — An *hermitian line bundle* on X is a pair $\overline{L} = (L, \|\cdot\|)$, where L is a line bundle on X and $\|\cdot\|$ is a metric on the holomorphic line bundle

$$L_{\mathbf{C}} = L_{|X(\mathbf{C})}\,.$$

We also assume that $\|\cdot\|$ is invariant by the complex conjugation

$$F_\infty : X(\mathbf{C}) \to X(\mathbf{C})\,.$$

Let $\overline{L}$ be an hermitian line bundle on X. We let

$$c_1(\overline{L}) = c_1(\overline{L}_{\mathbf{C}}) \in A^{1,1}(X(\mathbf{C}))\,.$$

Theorem 2.13. — *There is a unique way to associate to every integral closed subset $Y \subset X$ a real number*

$$h_{\overline{L}}(Y) \in \mathbf{R}$$

in such a way that:

(i) *If $\dim(Y) = 0$, i.e. when $Y = \{y\}$ where $y \in X$ is a closed point, we let $\kappa(y) = O_{X,y}/\mathscr{M}_{X,y}$ be the residue field. Then $\kappa(y)$ is finite and*

$$h_{\overline{L}}(Y) = \log \#\,(\kappa(y))\,.$$

(ii) *If* $\dim(Y) > 0$, *let* s *be a non-trivial rational section of* L *over* Y. *If*

$$\mathrm{div}_Y(s) = \sum_\alpha n_\alpha \, Y_\alpha \, ,$$

then

$$h_{\overline{L}}(Y) = \sum_\alpha n_\alpha \, h_{\overline{L}}(Y_\alpha) - \int_{Y(\mathbf{C})} \log \|s\| \, c_1(\overline{L})^{\dim Y(\mathbf{C})}.$$

3. Existence of the height

3.1. Resolutions. — To prove Theorem 2.13, we first need to make sense of the integral in ii). For that we use Hironaka's resolution theorem.

Theorem 3.1 **(Hironaka).** — *Let* X *be a scheme of finite type over* $\mathbf{C}$, *and* $Z \subset X$ *a proper closed subset of* X *such that* $X - Z$ *is smooth. Then there exists a proper map*

$$\pi : \widetilde{X} \to X$$

such that:

(i) $\widetilde{X}$ *is smooth;*
(ii) $\widetilde{X} - \pi^{-1}(Z) \xrightarrow{\sim} X - Z$*;*
(iii) $\pi^{-1}(Z)$ *is a divisor with normal crossings.*

In the situation of ii) in Theorem 2.13, we apply Theorem 3.1 to $X = Y(\mathbf{C})$, and to the union $Z = |\mathrm{div}(s)| \cup Y(\mathbf{C})^{\mathrm{sing}}$ of the support of $\mathrm{div}(s)$ and the singular locus of $Y(\mathbf{C})$. Let $\pi : \widetilde{Y} \to Y(\mathbf{C})$ be a resolution of $Y(\mathbf{C})$, d the dimension of $Y(\mathbf{C})$ and $\omega \in A^{dd}(Y(\mathbf{C}) - Z)$ with compact support. Then we define

$$\int_{Y(\mathbf{C})} \log \|s\| \, \omega = \int_{\widetilde{Y}} \log \|\pi^*(s)\| \, \pi^*(\omega) \, .$$

To see that the integral converges choose local coordinates $z_1, \dots, z_d$ of $\widetilde{Y}$ such that

$$\pi^*(s) = z_1^n \, u \, ,$$

with u a non-trivial section. Therefore

$$\log \|\pi^*(s)\| = n \log |z_1| + \alpha \, ,$$

with α of class C^∞, and

$$\pi^*(\omega) = \beta \prod_{i=1}^{d} d\,z_i\, d\,\overline{z}_i\,,$$

with β of class $\mathbf{C}^\infty$. Since

$$\int_{|z|\leqslant\varepsilon} \log|z|\, d\,z\, d\,\overline{z} = 2\int_0^\varepsilon \int_0^{2\pi} \log(r)\, r\, dr\, d\theta < +\infty\,,$$

the integral converges.

3.2. — By induction on $\dim(Y)$, the unicity of $h_{\overline{L}}(Y)$ is clear.

Now we handle the case $X = \mathrm{Spec}\,(O_K)$ for a number field K . If Σ is the set of complex embeddings of K we have

$$X(\mathbf{C}) = \coprod_{\sigma\in\Sigma} \mathrm{Spec}\,(\mathbf{C})\,.$$

To give $\overline{L} = (L, \|\cdot\|)$ amounts to give a pair $\overline{\Lambda} = (\Lambda, \|\cdot\|_\sigma)$ where $\Lambda = L(X)$ is a projective O_K-module of rank one and, for any $\sigma\in\Sigma$, $\|\cdot\|_\sigma$ is a metric on $\Lambda \underset{\sigma}{\otimes} \mathbf{C} \simeq \mathbf{C}$ such that

$$\|F_\infty(x)\|_{F_\infty\circ\sigma} = \|x\|_\sigma\,.$$

If $s\in\Lambda$, $s\neq 0$, we have

$$\mathrm{div}(s) = \sum_{\wp} n_\wp\,[\wp]$$

and

$$h_{\overline{L}}(X) = \sum_{\wp} n_\wp \log(N\wp) - \sum_{\sigma\in\Sigma} \log\|\sigma(s)\|_\sigma\,,$$

where $N\wp = \#\,(O/\wp)$.

Since

$$\Lambda/O_K s = \prod_{\wp} (O_\wp/\wp^{n_\wp})$$

we get

$$\sum_{\wp} n_\wp \log(N\wp) = \log\#\,(\Lambda/O_K s)\,.$$

Lemma 3.2. — *$h_{\overline{L}}(X)$ does not depend on the choice of s.*

Proof. — Let

$$d(s) = \log\#\,(\Lambda/O_K s) - \sum_{\sigma\in\Sigma} \log\|\sigma(s)\|_\sigma\,.$$

If $s' \in \Lambda$, $s' \neq 0$, we have

$$s' = f\, s$$

with $f \in K^*$. Therefore

$$d(s') - d(s) = \sum_{\wp} v_{\wp}(f) \log(N\wp) - \sum_{\sigma \in \Sigma} \log \|\sigma(f)\| = 0$$

by the product formula. □

3.3. — Let us prove Theorem 2.13 when Y has dimension one and Y is horizontal, i.e. Y maps surjectively onto $\mathrm{Spec}\,(\mathbf{Z})$. We have then

$$Y = \overline{\{y\}}\,,$$

where y is a closed point in $X \underset{\mathbf{Z}}{\otimes} \mathbf{Q}$. The residue field $K = \kappa(y)$ is a number field and

$$Y = \mathrm{Spec}\,(R)$$

where R is an integral ring with fraction field K. Denote by $\widetilde{R}$ the integral closure of R in K (i.e. $\widetilde{R} = O_K$) and let

$$\pi : \widetilde{Y} = \mathrm{Spec}\,(\widetilde{R}) \to Y$$

be the projection. If

$$s \in \Gamma(Y, L) - \{0\}\,,$$

$$\pi^*(s) \in \Gamma(\widetilde{Y}, \pi^* L) - \{0\}\,.$$

We shall prove that

$$d(s) = d(\pi^*(s))\,. \tag{3}$$

By 2.2 this will imply that $d(s)$ is independent of the choice of s. To prove (3) we first notice that

$$Y(\mathbf{C}) = \widetilde{Y}(\mathbf{C}) = \coprod_{\sigma \in \Sigma} \mathrm{Spec}\,(\mathbf{C})\,,$$

hence

$$\sum_{\sigma \in \Sigma} \log \|s\|_{\sigma} = \sum_{\sigma \in \Sigma} \log \|\pi^*(s)\|_{\sigma}\,. \tag{4}$$

Next we consider the commutative diagram

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & 0 & & 0 & & K''' & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & R & \xrightarrow{s} & L(Y) & \longrightarrow & L/R\,s & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \widetilde{R} & \xrightarrow{\widetilde{s}} & \widetilde{L} & \longrightarrow & \widetilde{L}/\tilde{R}\,\widetilde{s} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & \widetilde{R}/R & \longrightarrow & \widetilde{L}/L & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\widetilde{s} = \pi^*(s) \in \widetilde{L} = \pi^*(L)$.

By diagram chase (snake lemma) we get

$$\# K''' = \# K' \,.$$

On the other hand, for any prime ideal $\wp$ in O_K, we have

$$\# \left(\frac{\widetilde{L}}{L} \right)_{\wp} = \# \left(\frac{\widetilde{R}}{R} \right)_{\wp}$$

since $\widetilde{L}_{\wp} = L_{\wp} \underset{R_{\wp}}{\otimes} \widetilde{R}_{\wp}$ and L and $\widetilde{L}$ are locally trivial. This implies

$$\# K' = \# K''$$

and $\# K''' = \# K''$. Therefore

$$\# (L/R\,s) = \# (\widetilde{L}/\widetilde{R}\,s) \,. \tag{5}$$

The assertion (3) follows from (4) and (5).

When $\dim(Y) = 1$ and Y is vertical i.e. its image in $\mathrm{Spec}\,(\mathbf{Z})$ is a closed point of finite residue field k, Theorem 2.13 is proved by considering the normalisation

$$\pi : \widetilde{Y} \to Y \,.$$

The proof is the same as in the case Y is horizontal, the product formula being replaced by the equality

$$\sum_{x \in \widetilde{Y}} v_x(f)\,[k(x):k] = 0$$

for any $f \in \kappa(y)^*$. Indeed,

$$\log \# k(x) = [k(x):k] \log \# k\,.$$

3.4. — Assume from now on that $\dim(Y) \geqslant 2$, with $Y \subset X$ a closed integral subscheme, $Y = \overline{\{y\}}$. If $s \in L_y$, $s \neq 0$,

$$\operatorname{div}(s) = \sum_\alpha n_\alpha\, Y_\alpha\,.$$

Lemma 3.3. — *There exists $t \in L_y$ such that, for every α, the restriction of t to Y_α is neither zero nor infinity.*

Proof. — Let $Y_\alpha = \overline{\{y_\alpha\}}$. The ring

$$R = \varinjlim_{U \text{s.t.}, \forall \alpha,\ y_\alpha \in U} O(U)$$

is *semi-local*, i.e. it has finitely many maximal ideals $\mathscr{M}_\alpha$, $\alpha \in A$. Let

$$I = \bigcap_{\alpha \in A} \mathscr{M}_\alpha$$

be the radical of R, and

$$\Lambda = \varinjlim_{U \text{s.t.}, \forall \alpha,\ y_\alpha \in U} L(U)\,.$$

Note that, for every α,

$$R_{\mathscr{M}_\alpha} = O_{y_\alpha}$$

and, for every pair $\alpha \neq \beta$

$$\mathscr{M}_\alpha + \mathscr{M}_\beta = R\,.$$

Since L is locally trivial

$$\Lambda \otimes R/I = \prod_\alpha (\Lambda \otimes O_{y_\alpha})/\mathscr{M}_\alpha \simeq \prod_\alpha O_{y_\alpha}/\mathscr{M}_\alpha = R/I\,.$$

Denote by $t \in \Lambda$ an element such that its class in $\Lambda \otimes R/I$ maps to $1 \in R/I$ by the above isomorphism. The module

$$M = \Lambda / R t$$

is such that $M = IM$. Therefore, by Nakayama's lemma, $M = 0$. Since

$$\Lambda = R t\,,$$

for every $\alpha \in A$ the restriction of t to Y_α does not vanish. □

3.5. — Given s and t as above we write

$$\operatorname{div}(s) = \sum_\alpha n_\alpha Y_\alpha$$

and

$$\operatorname{div}(t) = \sum_\beta m_\beta Z_\beta\,,$$

with $Z_\beta \neq Y_\alpha$ for all β and α. Consider

$$\operatorname{div}(s) \cdot \operatorname{div}(t) = \sum_\alpha n_\alpha \operatorname{div}(t \mid Y_\alpha)$$

and

$$\operatorname{div}(t) \cdot \operatorname{div}(s) = \sum_\beta m_\beta \operatorname{div}(s \mid Z_\beta)\,.$$

These are cycles of codimension two in Y.

Proposition 3.4. — *We have*

$$\operatorname{div}(s) \cdot \operatorname{div}(t) = \operatorname{div}(t) \cdot \operatorname{div}(s)\,.$$

The proof of Proposition 3.4 will be given later.
Assume $\dim Y(\mathbf{C}) = d$, and define

$$d(s) = h_{\overline{L}}(\operatorname{div}(s)) - \int_{Y(\mathbf{C})} \log \|s\|\, c_1(\overline{L})^d\,.$$

By induction hypothesis we have

$$\begin{aligned} d(s) &= \sum_\alpha n_\alpha\, h_{\overline{L}}(\operatorname{div}(t_{|Y_\alpha})) \\ &- \sum_\alpha n_\alpha \int_{Y_\alpha(\mathbf{C})} \log \|t\|\, c_1(\overline{L})^{d-1} - \int_{Y(\mathbf{C})} \log \|s\|\, c_1(\overline{L})^d \\ &= h_{\overline{L}}(\operatorname{div}(s) \cdot \operatorname{div}(t)) - I(s,t) \end{aligned}$$

where

$$I(s,t) = \sum_\alpha n_\alpha \int_{Y_\alpha(\mathbf{C})} \log \|t\|\, c_1(\overline{L})^{d-1} + \int_{Y(\mathbf{C})} \log \|s\|\, c_1(\overline{L})^d\,.$$

Proposition 3.5. — $I(s,t) = I(t,s)$.

From Proposition 3.4 and Proposition 3.5 we deduce that $d(s) = d(t)$ when $\operatorname{div}(s)$ and $\operatorname{div}(t)$ are transverse. When s and s' are two sections of L there exists a section t such that $\operatorname{div}(s)$ and $\operatorname{div}(t)$ (resp. $\operatorname{div}(s')$ and $\operatorname{div}(t)$) are transverse. Therefore

$$d(s) = d(t) = d(s')$$

and Theorem 2.13 follows.

3.6. — To prove Proposition 3.4 we write

$$\operatorname{div}(s) \cdot \operatorname{div}(t) = \sum_W n_W \, [W]$$

with $\operatorname{codim}_Y(W) = 2$. Let $W = \overline{\{w\}}$ and

$$R = O_{Y,w} \, .$$

Since $L_w \simeq O_{Y,w}$ one can assume that t (resp. s) corresponds to $a \in R$ (resp. $b \in R$). Since R is integral and $a \neq 0$, we know from the proof of Lemma 2.4 that, if $A = R/a\,R$,

$$\dim(A) \leqslant \dim(R) - 1 = 1 \, .$$

Let $\overline{b} \in A$ be the image of b and let $\overline{\wp} \subset A$ be a minimal prime ideal of A. The inverse image $\wp \subset R$ of $\overline{\wp}$ is a minimal nontrivial prime ideal. Since $a \in \wp$, the closed subset defined by $\wp$ in $X = \mathrm{Spec(R)}$ is contained in the image in X of the support of $\operatorname{div}(t)$. Since $\operatorname{div}(t)$ and $\operatorname{div}(s)$ are transverse, b does not belong to $\wp$, hence $\overline{b}$ does not belong to $\overline{\wp}$. According to Theorem 5.15, ii), in [**10**, chapter 2], it follows that

$$\dim(A/\overline{b}) = \dim(A) - 1 \, .$$

Since $\dim(A) \leqslant 1$ we get $\dim(A) = 1$ and $\dim(A/\overline{b}) = 0$. This implies that $A/\overline{b}$ has finite length. If $\langle a, b \rangle \subset R$ is the ideal spanned by a and b, $A/\overline{b} = R/\langle a, b \rangle$ and we shall prove that

$$n_W = \ell(R/\langle a, b \rangle) \, .$$

3.7. — Let A be as above and let M be an A-module of finite type. If $x \in A$ we have an exact sequence

$$0 \to M[x] \to M \xrightarrow{\times x} M \to M/x\,M \to 0 \, . \tag{6}$$

If $M[x]$ and $M/x\,M$ have finite length we define

$$e(x, M) = \ell(M/x\,M) - \ell(M[x]) \in \mathbf{Z} \, .$$

Lemma 3.6. — *i) $M[\bar{b}]$ and $M/\bar{b}\,M$ have finite length.*
ii)
$$e(\bar{b}, M) = \sum_{\substack{\wp \subset A \\ \wp \text{ minimal}}} \ell_{A_\wp}(M_\wp)\, e\,(\bar{b}, A/\wp)\,.$$
iii)
$$e(\bar{b}, A/\wp) = \ell(A/(\wp + b\,A))\,.$$

Proof of i) and ii). — Note that both sides in ii) are additive in M for exact sequences. Therefore we can assume that $M = A/q$ where q is a prime ideal. We distinguish two cases:
a) If q is maximal, for any minimal prime ideal $\wp$ we have $M_\wp = 0$. Therefore $\ell(M)$ is finite. From Lemma 2.3 and (6) we conclude that
$$e(\bar{b}, M) = 0\,.$$
b) Assume $q = \wp$ is minimal. If $\wp' \neq \wp$ is any prime ideal different from $\wp$ we have
$$M_{\wp'} = 0\,.$$
Therefore the right hand side reduces to one summand and i) holds. Furthermore
$$\ell_{A_\wp}(M_\wp) = 1$$
and
$$e(\bar{b}, M) = e(\bar{b}, A/\wp)$$
so ii) is true.

To prove iii) we let $M = A/\wp$. We saw that $b \notin \wp$ and $A/\wp$ is integral, therefore $M[\bar{b}] = 0$.

On the other hand
$$\dim(A/(\wp + bA)) \leqslant \dim(A/\wp) - 1 = 0\,.$$
Therefore
$$e(\bar{b}, A/\wp) = \ell(A/(\wp + bA))\,.$$
□

3.8. — We shall apply Lemma 3.6 to
$$M = A = R/a\,R\,.$$
Let $\wp$ be a minimal prime in A and $Y \subset |\mathrm{div}(s)|$ the corresponding component of the support of $\mathrm{div}(s)$. We have
$$\ell(A_\wp) = \mathrm{ord}_Y(t)$$
and
$$\ell(A/(\wp + bA)) = \mathrm{ord}_W(t_{|Y})\,.$$

Lemma 3.6 iii) says that

$$e(\overline{b}, A) = n_W\,.$$

But $\overline{b}$ does not divide zero, so

$$e(\overline{b}, A) = \ell(R/\langle a, b\rangle)\,.$$

Therefore $n_W = \ell(R/\langle a, b\rangle)$. Since $\langle a, b\rangle = \langle b, a\rangle$ we conclude that

$$\mathrm{div}(s)\cdot\mathrm{div}(t) = \sum_W n_W[W] = \mathrm{div}(t)\cdot\mathrm{div}(s)\,.$$

This ends the proof of Proposition 3.4.

3.9. — We shall now prove Proposition 3.5. For this we need some more analytic preliminaries. Let X be a smooth complex compact manifold of dimension d.

Definition 3.7. — A *current* $T \in D^{p,q}(X)$ is a $\mathbf{C}$-linear form

$$T : A^{d-p,d-q}(X) \to \mathbf{C}$$

which is continuous for the Schwartz' topology.

Examples 3.8. — i) If $\eta \in L^1(X) \underset{C^\infty(X)}{\otimes} A^{p,q}(X)$ is an integrable differential, η defines a current by the formula

$$\eta(\omega) = \int_X \eta \wedge \omega\,.$$

ii) If $Z = \sum\limits_\alpha n_\alpha\, Z_\alpha$ is a cycle of codimension p on X, it defines a Dirac current $\delta_Z \in D^{pp}(X)$ by the formula

$$\delta_Z(\omega) = \sum_\alpha n_\alpha \int_{Z_\alpha} \omega\,,$$

where the integrals converge by Hironaka's theorem.

We can derivate a current $T \in D^{p,q}(X)$ by the formula

$$\partial\, T(\omega) = (-1)^{p+q+1}\, T(\partial\, \omega)$$

and

$$\overline{\partial}\, T(\omega) = (-1)^{p+q+1}\, T(\overline{\partial}\, \omega)\,.$$

By the Stokes formula we get a commutative diagram

$$\begin{array}{ccc} D^{p,q}(X) & \xrightarrow{\partial} & D^{p+1,q}(X) \\ \cup & & \cup \\ A^{p,q}(X) & \xrightarrow{\partial} & A^{p+1,q}(X) \end{array}$$

and idem for $\overline{\partial}$ and $d = \partial + \overline{\partial}$.

***Proposition 3.9* (Poincaré-Lelong).** — *Let $\overline{L}$ be an hermitian line bundle on X and s a meromorphic section of L. Then we have the following formula in $D^{1,1}(X)$*

$$dd^c(-\log \|s\|^2) + \delta_{\mathrm{div}(s)} = c_1(\overline{L})\,. \tag{7}$$

3.10. — To prove Proposition 3.9 let $Z = |\mathrm{div}(s)|$ be the support of the divisor of s. By Theorem 3.1, there exists a birational resolution

$$\pi : \widetilde{X} \to X$$

where $\pi^{-1}(Z)$ has local equation $z_1 \dots z_k = 0$. Therefore

$$\pi^*(s) = z_1^{n_1} \dots z_k^{n_k}$$

locally. If Proposition 3.9 holds for $\pi^*(\overline{L})$ and $\pi^*(s)$, by applying π_* we get (7). So we can assume that $X = \widetilde{X}$. By additivity we can assume that

a) $\|s\| = |z_1|$

or

b) $\log \|s\| = \rho \in C^\infty(X)$.

In case b) $\mathrm{div}(s) = 0$ and (7) is true by definition of $c_1(\overline{L})$ (Lemma 2.11). In case a) we have to show that, for every differential form ω with compact support in U, and for $\varepsilon > 0$ small enough,

$$-\int_U \log |z_1|^2 \, dd^c(\omega) = \int_{|z_1|=\varepsilon} \omega\,.$$

But, by Stokes' theorem, we have

$$\begin{aligned} & -\lim_{\varepsilon\to 0} \int_{|z_1|\geqslant\varepsilon} \log |z_1|^2 \, dd^c(\omega) \\ = \; & \lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} \log |z_1|^2 \, d^c\,\omega + \lim_{\varepsilon\to 0} \int_{|z_1|\geqslant\varepsilon} d\log |z_1|^2 \, d^c\,\omega\,. \end{aligned}$$

The first summand vanishes and, applying Stokes' theorem again,

$$\begin{aligned} & \lim_{\varepsilon\to 0} \int_{|z_1|\geqslant\varepsilon} d\log |z_1|^2 \, d^c\,\omega = -\lim_{\varepsilon\to 0} \int_{|z_1|\geqslant\varepsilon} d^c\log |z_1|^2 \, d\,\omega \\ = \; & \lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} d^c\log |z_1|^2 \, \omega - \lim_{\varepsilon\to 0} \int_{|z_1|\geqslant\varepsilon} dd^c\log |z_1|^2 \, \omega\,. \end{aligned}$$

The second summand vanishes and, taking polar coordinates $z_1 = r_1\, e^{i\theta_1}$, we get

$$d^c \log |z_1|^2 = \frac{d\,\theta_1}{2\,\pi}$$

and

$$\lim_{\varepsilon\to 0}\int_{|z_1|=\varepsilon}\frac{d\,\theta_1}{2\,\pi}\,\omega=\int_{z_1=0}\omega\,.$$

□

3.11. — Coming back to Proposition 3.5 we consider the current

$$T_{s,t}=\delta_{\mathrm{div}(s)}\log\|t\|^2+\log\|s\|^2\,c_1(\overline{L})\,.$$

Then

$$I(s,t)=T_{s,t}(c_1(\overline{L})^{d-1})/2\,.$$

Proposition 3.9 implies

$$T_{s,t}=(c_1(\overline{L})+dd^c\log\|s\|^2)\log\|t\|^2+\log\|s\|^2\,c_1(\overline{L})$$

at least formally: we have to make sense of the product of currents $(dd^c\log\|s\|^2)\log\|t\|^2$. By Stokes' theorem we have (at least formally)

$$\begin{aligned}dd^c(T_1)\,T_2 &= d(d^c(T_1)\,T_2)+d^c(T_1)\,d(T_2)\\ &= d(d^c(T_1)\,T_2)+d^c(T_1\,d\,T_2)-T_1\,d^cd(T_2)\,.\end{aligned}$$

Since $d^cd=-dd^c$ and $d(c_1(\overline{L})^{d-1})=d^c(c_1(\overline{L})^{d-1})=0$ we get

$$2I(s,t)=T_{s,t}(c_1(\overline{L})^{d-1})=T_{t,s}(c_1(\overline{L})^{d-1})=2I(t,s)\,.$$

□

3.12. The height of the projective space. — Let $N\geqslant 1$ be an integer and $\mathbf{P}^N$ the N-dimensional projective space over $\mathbf{Z}$. The tautological line bundle $O(1)$ on $\mathbf{P}^N$ is a quotient of the trivial vector bundle of rank $N+1$

$$O_{\mathbf{P}^N}^{N+1}\to O(1)\to 0\,.$$

We equip $O_{\mathbf{P}^N}^{N+1}$ with the trivial metric and $O(1)$ with the quotient metric.

Proposition 3.10. — *The height of* $\mathbf{P}^N$ *is*

$$h_{\overline{O(1)}}(\mathbf{P}^N)=\frac{1}{2}\sum_{k=1}^{N}\sum_{m=1}^{k}\frac{1}{m}\,.$$

Proof. — Let s be the section of $O(1)$ defined by the homogeneous coordinate X_0. Then $\mathrm{div}(s)=\mathbf{P}^{N-1}$ and we get, from Theorem 2.13 ii),

$$h(\mathbf{P}^N)=h(\mathbf{P}^{N-1})-\int_{\mathbf{P}^N(\mathbf{C})}\log\|s\|\,d\mu$$

where $d\mu$ is the probability measure on $\mathbf{P}^N(\mathbf{C})$ invariant under rotation by $U(N+1)$. If dv is the probability measure on the sphere S^{2N+1} invariant under $U(N+1)$ we have

$$\int_{\mathbf{P}^N(\mathbf{C})} \log \|s\| \, d\mu = \int_{S^{2N+1}} \log |X_0| \, dv$$

and Proposition 3.10 follows from

Lemma 3.11. — *The integral on the sphere is given by*

$$\int_{S^{2N+1}} \log |X_0| \, dv = \frac{1}{2} \sum_{m=1}^{N} \frac{1}{m} \, .$$

4. Arithmetic Chow groups

4.1. Definition. — Let X be a regular projective flat scheme over $\mathbf{Z}$ and $p \geqslant 0$ an integer. Let $Z^p(X)$ be the group of codimension p cycles on X.

Definition 4.1. — A *Green current* for $Z \in Z^p(X)$ is a real current $g \in D^{p-1,p-1}(X(\mathbf{C}))$ such that $F_\infty^*(g) = (-1)^{p-1} g$ and

$$dd^c g + \delta_Z = \omega$$

for a smooth form $\omega \in A^{p,p}(X(\mathbf{C}))$.

We let $\widehat{Z}^p(X)$ be the group generated by pairs (Z, g), $Z \in Z^p(X)$, g Green current for Z, with $(Z_1, g_1) + (Z_2, g_2) = (Z_1 + Z_2, g_1 + g_2)$.

Examples 4.2. — i) Let $Y \subset X$ be a closed irreducible subset with $\mathrm{codim}_X(Y) = p - 1$, and $f \in \kappa(y)$ a rational function on Y. Define $\log |f|^2 \in D^{p-1,p-1}(X(\mathbf{C}))$ by the formula

$$(\log |f|^2)(\omega) = \int_{Y(\mathbf{C})} \log |f|^2 \, \omega$$

(which makes sense by Theorem 3.1). We may think of f as a rational section of the trivial line bundle on Y. Therefore Poincaré-Lelong formula (Proposition 3.9) reads

$$dd^c(-\log |f|^2) + \delta_{\mathrm{div}(f)} = 0 \, .$$

Hence the pair

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -\log |f|^2)$$

is an element of $\widehat{Z}^p(X)$.

ii) Given $u \in D^{p-2,p-1}(X(\mathbf{C}))$ and $v \in D^{p-1,p-2}(X(\mathbf{C}))$ we have

$$dd^c(\partial\, u + \overline{\partial}\, v) = 0\,,$$

so $(0, \partial\, u + \overline{\partial}\, v) \in \widehat{Z}^p(X)$.

We let $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ be the subgroup generated by all elements $\widehat{\operatorname{div}}(f)$ and $(0, \partial\, u + \overline{\partial}\, v)$.

Definition 4.3. — The *arithmetic Chow group* of codimension p of X is the quotient

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X) / \widehat{R}^p(X)\,.$$

4.2. Example. — Let $\widehat{\operatorname{Pic}}(X)$ be the group of isometric isomorphism classes of hermitian line bundles on X, equipped with the tensor product. If $\overline{L} = (L, \|\cdot\|) \in \widehat{\operatorname{Pic}}(X)$ and if $s \neq 0$ is a rational section of L we let

$$\widehat{\operatorname{div}}(s) = (\operatorname{div}(s), -\log \|s\|^2) \in \widehat{Z}^1(X)$$

(Proposition 3.9), and we define

$$\widehat{c}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(X)$$

to be the class of $\widehat{\operatorname{div}}(s)$. It does not depend on the choice of s: if s' is another section of L we have

$$s' = f\, s$$

with $f \in k(X)$. Therefore

$$\widehat{\operatorname{div}}(s') - \widehat{\operatorname{div}}(s) = \widehat{\operatorname{div}}(f) \in \widehat{R}^1(X)\,.$$

Proposition 4.4. — *The map $\widehat{c}_1$ induces a group isomorphism*

$$\widehat{c}_1 : \widehat{\operatorname{Pic}}(X) \to \widehat{\mathrm{CH}}^1(X)\,.$$

Proof. — To prove Proposition 4.4 we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\infty(X(\mathbf{C})) & \xrightarrow{a} & \widehat{\operatorname{Pic}}(X) & \xrightarrow{\zeta} & \operatorname{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \widehat{c}_1} & & \downarrow{\scriptstyle c_1} & & \\
0 & \longrightarrow & C^\infty(X(\mathbf{C})) & \xrightarrow{a'} & \widehat{\mathrm{CH}}^1(X) & \xrightarrow{\zeta'} & \mathrm{CH}^1(X) & \longrightarrow & 0
\end{array}$$

where $a(\varphi)$ is the trivial line bundle on X equipped with the norm such that $\|1\| = \exp(\varphi)$, $\zeta(\overline{L}) = L$, $a'(\varphi) = (0, -\log|\varphi|^2)$ and $\zeta(Z,g) = Z$. Since c_1 is an isomorphism the same is true for $\widehat{c}_1$. □

4.3. Products

4.3.1. — Denote by $\widehat{\mathrm{CH}}^p(X)_{\mathbf{Q}}$ the tensor product $\widehat{\mathrm{CH}}^p(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Q}$.

Theorem 4.5. — *When $p \geqslant 0$ and $q \geqslant 0$ there is an intersection pairing*

$$\begin{array}{ccc} \widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) & \longrightarrow & \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbf{Q}} \\ x \otimes y & \longmapsto & x \cdot y \end{array}$$

It turns $\underset{p \geqslant 0}{\oplus} \widehat{\mathrm{CH}}^p(X)_{\mathbf{Q}}$ into a commutative graded $\mathbf{Q}$-algebra.

Let $\zeta : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X)$ be the map sending the class of (Z, g) to the class of Z, and let $\omega : \widehat{\mathrm{CH}}^p(X) \to A^{pp}(X)$ be the map sending (Z, g) to $dd^c g + \delta_Z$. Then

$$\zeta(x \cdot y) = \zeta(x)\, \zeta(y)$$

and

$$\omega(x \cdot y) = \omega(x)\, \omega(y)\,.$$

4.3.2. — To sketch a proof of Theorem 4.5, let $y = (Y, g_Y) \in \widehat{Z}^p(X)$ and $z = (Z, g_Z) \in \widehat{Z}^q(X)$.

We first define a cycle $Y \cap Z$. For this we assume that the restrictions $Y_{\mathbf{Q}}$ and $Z_{\mathbf{Q}}$ of Y and Z to the generic fiber $X_{\mathbf{Q}}$ meet properly, i.e. the components of $|Y_{\mathbf{Q}}| \cap |Z_{\mathbf{Q}}|$ have codimension $p+q$ (the moving lemma allows one to make this hypothesis). It follows that there exists a well defined intersection cycle $Y_{\mathbf{Q}} \cdot Z_{\mathbf{Q}} \in Z^{p+q}(X_{\mathbf{Q}})$, supported on the closed set $|Y_{\mathbf{Q}}| \cap |Z_{\mathbf{Q}}|$. Let

$$\mathrm{CH}^p_Y(X) = \ker(\mathrm{CH}^p(X) \to \mathrm{CH}^p(X - Y))$$

be the Chow group with supports in Y, and $\mathrm{CH}^p_{\mathrm{fin}}(X)$ the union of the groups $\mathrm{CH}^p_Y(X)$ when $Y \subset X$ runs over all closed subsets with empty generic fiber. There is a canonical map

$$\mathrm{CH}^p_Y(X) \to \mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_{\mathbf{Q}})\,.$$

One can define an intersection paring

$$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y \cap Z}(X)_{\mathbf{Q}}\,.$$

One method to do so ([**8**], [**9**], [**11**]) is to interpret $\mathrm{CH}^p_Y(X)_{\mathbf{Q}}$ as the subspace of $K_0^Y(X)_{\mathbf{Q}}$ where the Adams operations ψ^k act by multiplication by k^p ($k \geqslant 1$), and to use the tensor product

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y \cap Z}(X)\,.$$

We let $Y \cap Z \in \mathrm{CH}^{p+q}_{\mathrm{fin}}(X)_{\mathbf{Q}} \oplus Z^{p+q}(X_{\mathbf{Q}})_{\mathbf{Q}}$ be the image of

$$[Y] \otimes [Z] \in \mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^p_Z(X)$$

by the maps

$$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y\cap Z}(X)_{\mathbf{Q}} \to \mathrm{CH}^{p+q}_{\mathrm{fin}}(X)_{\mathbf{Q}} \oplus Z^{p+q}(X)_{\mathbf{Q}}\,.$$

Next we define a Green current for $Y \cap Z$. For this we write

$$dd^c\, g_Y + \delta_Y = \omega_Y$$

and

$$dd^c\, g_Z + \delta_Z = \omega_Z\,,$$

and we let

$$g_Y * g_Z = \delta_Y\, g_Z + g_Y\, \omega_Z\,.$$

However $g_Y\, \delta_Z$, being a product of currents, is not well defined a priori. But g_Y is defined up to the addition of a term $\partial(u) + \overline{\partial}(v)$ and one shows that g_Y can be chosen to be an L^1-form on $X(\mathbf{C}) - Y(\mathbf{C})$, with restriction an L^1-form η on $Z(\mathbf{C}) - Z(\mathbf{C}) \cap Y(\mathbf{C})$. We let $g_Y\, \delta_Z$ be the current defined by η on $Z(\mathbf{C})$ (see above Example 3.8):

$$g_Y\, \delta_Z(\omega) = \int_{Z(\mathbf{C})-(Z(\mathbf{C})\cap Y(\mathbf{C}))} \eta\, \omega\,.$$

To see that $g_Y * g_Z$ is a Green current for $Y \cap Z$ we proceed formally:

$$\begin{aligned} dd^c(g_Y * g_Z) &= dd^c(\delta_Y\, g_Z) + dd^c(g_Y\, \omega_Z) \\ &= \delta_Y\, dd^c(g_Z) + dd^c(g_Y)\, \omega_Z \\ &= \delta_Y(\omega_Z - \delta_Z) + (\omega_Y - \delta_Y)\, \omega_Z \\ &= \omega_Y\, \omega_Z - \delta_Y\, \delta_Z \\ &= \omega_Y\, \omega_Z - \delta_{Y\cap Z}\,. \end{aligned}$$

We refer to [**9**] for the justification of this series of equalities.

4.4. Functoriality. — Let $f : X \to Y$ be a morphism.

Theorem 4.6. — *For every $p \geqslant 0$ there is a morphism*

$$f^* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)\,.$$

If the restriction of f to $X(\mathbf{C})$ is a smooth map of complex manifolds, there are morphisms

$$f_* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+\dim(Y)-\dim(X)}(Y)\,.$$

Both f^* and f_* are compatible to ζ and ω. Furthermore

$$f^*(x \cdot y) = f^*(x) \cdot f^*(y)$$

and

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y\,.$$

4.5. Heights and intersection numbers

4.5.1. — Let X be a projective regular flat scheme over $\mathbf{Z}$ and $Y \subset X$ a closed integral subscheme. We assume that X is equidimensional of dimension d and $\operatorname{codim}_X(Y) = p$. One can then define as follows a morphism

$$\int_Y : \widehat{\mathrm{CH}}^{d-p}(X) \to \mathbf{R}\,.$$

First, assume that $X = Y$ and that $x \in \widehat{\mathrm{CH}}^{d}(X)$ is the class of (Z, g_Z) where Z is a zero-cycle and $g_Z \in D^{d-1,d-1}(X(\mathbf{C}))$. The cycle Z is then a finite sum

$$Z = \sum_\alpha n_\alpha\, y_\alpha$$

where y_α is a closed point with finite residue field $k(y_\alpha)$, and there exist currents u and v such that $\eta_Z = g_Z + \partial(u) + \overline{\partial}(v)$ is smooth. By definition

$$\int_X x = \sum_\alpha n_\alpha \log \#\,(k(y_\alpha)) - \frac{1}{2}\int_{X(\mathbf{C})} \eta_Z\,.$$

In general we let g_Y be a Green current for Y in $X(\mathbf{C})$, and $y = (Y, g_Y)$. If $x \in \widehat{\mathrm{CH}}^{d-p}(Y)$ we have $x \cdot y \in \widehat{\mathrm{CH}}^{d}(X)$ and we define

$$\int_Y x = \int_X x \cdot y - \frac{1}{2}\int_{X(\mathbf{C})} \omega(x)\, g_Y\,.$$

One checks that this number is independent on the choice of g_Y.

Theorem 4.7. — *The height of* Y *is*

$$h_{\overline{L}}(Y) = \int_Y \widehat{c}_1(\overline{L})^{d-p}\,.$$

Proof. — To prove Theorem 4.7 we shall check that the two properties in Theorem 2.13 hold true for the number $\int_Y \widehat{c}_1(\overline{L})^{d-p}$.

When $p = d$, Y is a closed point y and, if x is the class of $(y, 0)$ in $\widehat{\mathrm{CH}}^{d}(X)$, we have

$$\int_X x = \log \#\,\kappa(y) = h_{\overline{L}}(Y)\,.$$

Assume $\dim(Y) > 0$. Let g_Y be a Green current for Y and $y = (Y, g_Y)$. Choose a rational section s of L on Y, and an extension $\widetilde{s}$ of s to X. Then

$$\widehat{c}_1(\overline{L}) = (\mathrm{div}(\widetilde{s}), -\log \|\widetilde{s}\|^2)\,.$$

If $x = \widehat{c}_1(\overline{L})^{d-p-1}$ we get, from the definition of $\int_Y$,

$$\int_Y x\,\widehat{c}_1(\overline{L}) = \int_X x \cdot \widehat{c}_1(\overline{L}) \cdot y - \frac{1}{2}\int_{X(\mathbf{C})} \omega(x\,\widehat{c}_1(\overline{L}))\, g_Y\,. \tag{8}$$

But

$$\begin{aligned} x \cdot \widehat{c}_1(\overline{L}) \cdot y &= x \cdot (\mathrm{div}(\widetilde{s} \mid Y), -\log \|\widetilde{s}\|^2 * g_Y) \\ &= x \cdot (\mathrm{div}(s), -\log \|\widetilde{s}\|^2\, \delta_Y + c_1(\overline{L})\, g_Y)\,. \end{aligned}$$

If $x =: \widehat{c}_1(\overline{L})^{d-p-1}$ is the class of (Z, g_Z), we get

$$x \cdot \widehat{c}_1(\overline{L}) \cdot y = (Z \cdot \mathrm{div}(s), \omega(x)(-\log \|\widetilde{s}\|^2\, \delta_Y + c_1(\overline{L})\, g_Y) + g_Z\, \delta_{\mathrm{div}(s)})\,. \tag{9}$$

Since

$$\int_X (Z \cdot \mathrm{div}(s), g_Z\, \delta_{\mathrm{div}(s)}) = \int_{\mathrm{div}(s)} x$$

we deduce from (9) that

$$\int_X x \cdot \widehat{c}_1(\overline{L}) \cdot y = \int_{\mathrm{div}(s)} x - \frac{1}{2}\int_{Y(\mathbf{C})} \omega(x) \log \|s\|^2 + \frac{1}{2}\int_{X(\mathbf{C})} \omega(x)\, c_1(\overline{L})\, g_Y\,. \tag{10}$$

Since $\omega(x\,\widehat{c}_1(\overline{L})) = \omega(x)\, c_1(\overline{L}_{\mathbf{C}})$, (8) and (10) imply that

$$\int_Y \widehat{c}_1(\overline{L})^{d-p} = \int_{\mathrm{div}(s)} \widehat{c}_1(\overline{L})^{d-p-1} - \frac{1}{2}\int_{Y(\mathbf{C})} c_1(\overline{L})^{d-p-1} \log \|s\|\,.$$

□

References

[1] Arakelov, S.Y.: Intersection theory of divisors on an arithmetic surface, Izv. Akad. Nauk SSSR Ser. Mat., **38** Issue 6 (1974), 1179–1192.

[2] Bost,J.-B.,Gillet, H.,Soulé, C.: Heights of projectivevarieties and positive Green forms.J. Amer. Math. Soc.**7** (1994), p. 903

[3] Bourbaki, N.: Algèbre commutative IV, Eléments de Mathématique, Hermann, Paris.

[4] Bourbaki, N.: Algèbre I, Eléments de Mathématique, Hermann, Paris.

[5] Bourbaki, N.: Algèbre II, Eléments de Mathématique, Hermann, Paris.

[6] Faltings, G.: Diophantine approximation on abelian varieties, Ann. of Math. (2) **133** (1991), no. 3, 549–576.

[7] Gillet, H.: K-Theory and Intersection Theory," in "Handbook of K-theory", E.M. Friedlander and D. R. Grayson, eds., Springer Verlag (2005), 235–294.

[8] Gillet, H., Soulé, C.: Intersection theory using Adams operations, *Inventiones Math.* **90** (1987),243-277.

[9] Gillet, H., Soulé, C.: Arithmetic intersection theory. *Inst. Hautes études Sci. Publ. Math.* **72** (1990), 93–174.

[10] Qing Liu: Algebraic Geometry and Arithmetic Curves, Oxford graduate texts in Mathematics (2002).

[11] Soulé, C., Abramovich, D., Burnol, J.-F. and Kramer, J.: Lectures on Arakelov Geometry. Cambridge University Press (1991).

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