# Linear projections and successive minima

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In arithmetic geometry, cohomology groups are not vector spaces as in classical algebraic geometry but rather euclidean lattices. As a consequence, to understand these groups we need to evaluate not only their rank, but also their successive minima, which are fundamental invariants in the geometry of numbers. The goal of this article is to perform this task for line bundles on projective curves.

Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers and $E$ a projective $\mathcal{O}_K$-module of finite rank $N$. We endow $E \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ with an hermitian metric $h$ and we let $\mu_1, \ldots, \mu_N$ be the logarithm of the successive minima of $(E, h)$. Assume $X_K \subset \mathbb{P}(E_K^\vee)$ is a smooth geometrically irreducible curve of genus $g > 0$. We shall find a lower bound for the numbers $\mu_i$, $g + 8 \leq i \leq N - 3$, in terms of a normalized height of $X_K$ and the average of the $\mu_i$'s (Theorem 2). This result is a complement to [12], Theorem 4, which gives a lower bound for $\mu_1$.

The method of proof is a variant of [12], *loc. cit.* It relies upon Morrison's proof of the fact that $X_K$ is Chow semi-stable [10]. We use a filtration $V_1 = E_K \supset V_2 \supset \ldots \supset V_N$ of the vector space $E_K$. But, contrary to [12], this filtration is chosen so that, for suitable values of $i$, the projection $\mathbb{P}(V_i^\vee) \cdots \to \mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of $X_K$. That such a choice is possible follows from a result of C. Voisin, namely an effective version of a theorem of Segre on linear projections of complex projective curves (Theorem 1). I thank her for proving this result and for helpful discussions. I am also greatful to the referee for useful corrections.

## 1 Linear projections of projective curves

Let $C \subset \mathbb{P}^n$ be an integral projective curve over $\mathbb{C}$ and $d$ its degree. Assume that $C$ is not contained in some hyperplane, $d \geq 3$ and $n \geq 3$.

**Theorem 1.** (C. Voisin) *There exists an integer $A(d)$ and a finite set $\Sigma$ of points in $(\mathbb{P}^n - C)(\mathbb{C})$, of order at most $A(d)$, such that, for every point $P \in \mathbb{P}^n(\mathbb{C}) - \Sigma \cup C(\mathbb{C})$, the linear projection $\mathbb{P}^n \cdots \to \mathbb{P}^{n-1}$ of center $P$ maps $C$ birationally onto its image.*

*Proof.* The existence of a finite set $\Sigma$ with the property above is a special case of a theorem of C. Segre [5]. The order of $\Sigma$ can be bounded as follows by a function of $d$.

If $n > 3$ a generic linear projection into $\mathbb{P}^3$ will map $C$ isomorphically onto its image [9] and the exceptional set $\Sigma \subset \mathbb{P}^n$ bijectively onto the exceptional set in $\mathbb{P}^3$. Therefore we can assume that $n = 3$.

When the projection with center $P \in \mathbb{P}^3(\mathbb{C})$ is not birational from the curve $C$ to its image $C' \subset \mathbb{P}^2$, we have $d' = \deg(C') \leq \frac{d}{2}$ hence $d' \leq d - 2$, and $P$ is the vertex of a cone $K$ with base $C'$ containing $C$. So we have to bound the number of such cones.

Let $N$ be the dimension of the kernel of the restriction map

$$\alpha : H^0(\mathbb{P}^3, \mathcal{O}(d')) \to H^0(C, \mathcal{O}(d')) .$$

Clearly $N$ is bounded as a function of $d$ and any $f \in \ker(\alpha)$ is an homogeneous polynomial of degree $d'$ which vanishes on $C$.

Let $Z \subset \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^{N-1}(\mathbb{C})$ be the set of pairs $(P, f)$ such that $f$ is the equation of a cone $K$ of vertex $P$. If $p_1 : \mathbb{P}^3 \times \mathbb{P}^{N-1} \to \mathbb{P}^3$ is the first projection, we have to bound the order of $p_1(Z)$. We note that this order is at most the number $c$ of connected components of $Z$.

Now $Z$ is defined by equations of bidegree $(\delta, 1)$, $\delta \leq d'$. Indeed $f$ is homogeneous of degree $d'$ and $(P, f) \in Z$ when all the derivatives of $f$, except those of order $d'$, vanish at $P$.

Let $L = \mathcal{O}(d', 1)$, $M = \dim H^0(\mathbb{P}^3 \times \mathbb{P}^N, L) - 1$, and

$$j : \mathbb{P}^3 \times \mathbb{P}^N \to \mathbb{P}^M$$

the Segre embedding. Since $j(Z)$ is the intersection of $j(\mathbb{P}^3 \times \mathbb{P}^N)$ with linear hyperplanes, Bézout theorem ([6], § 8.4) tells us that

$$c \leq \deg(j(\mathbb{P}^3 \times \mathbb{P}^N)) .$$

Hence $c$ is bounded by a function of $d$. □

**Corollary 1.** *Given any projective line $\Lambda \subset \mathbb{P}^n$, there exists a finite set $\Phi$ of order at most $A(d) + d$ in $\Lambda$ such that, if $P \in \Lambda - \Phi$, the linear projection of center $P$ maps $C$ birationally onto its image.*

*Proof.* Since $C$ is not equal to $\Lambda$, the cardinality of $C \cap \Lambda$ is at most $d$. So the corollary follows from Theorem 1.

*Remark.* The proof of Theorem 1 provides an upper bound for $A(d)$. Indeed

$$\deg(j) = \binom{3+N}{3} d'^3,$$

$d' \leq d/2$ and

$$N = \dim H^0(\mathbb{P}^3, \mathcal{O}(d')) = \frac{d'^3}{6} + d'^2 + \frac{11}{6}d' + 1 .$$

In particular, when $d \geq 12$ we get

$$\log(A(d) + d) \leq 12 \log(d) - 12 .$$

## 2 Successive minima

### 2.1

Let $K$ be a number field, $[K:\mathbb{Q}]$ its degree over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers, $S = \mathrm{Spec}(\mathcal{O}_K)$ the associated scheme and $\Sigma$ the set of complex embeddings of $K$. Consider an hermitian vector bundle $(E,h)$ over $S$, *i.e.* $E$ is a torsion free $\mathcal{O}_K$-module of finite rank $N$ and, for all $\sigma \in \Sigma$, the associated complex vector space $E_\sigma = E \underset{\mathcal{O}_K}{\otimes} \mathbb{C}$ is equipped with an hermitian scalar product $h_\sigma$. If $\bar{\sigma}$ is the conjugate of $\sigma$, we assume that the complex conjugation $E_\sigma \simeq E_{\bar{\sigma}}$ is an isometry. If $v \in E$ we define

$$\|v\| = \max_{\sigma \in \Sigma} \sqrt{h_\sigma(v,v)}\,.$$

If $i$ is a positive integer, $i \leq N$, we let $\mu_i$ be the infimum of the set of real numbers $r$ such that there exist $v_1, \ldots, v_i \in E$, linearly independent over $K$, such that $\log \|v_\alpha\| \leq r$ for all $\alpha \leq i$. The number $\mu_i$ is thus the logarithm of the $i$-th successive minimum of $(E,h)$. Let

$$\mu = \frac{\mu_1 + \cdots + \mu_N}{N}\,. \tag{1}$$

### 2.2

If $E^\vee = \mathrm{Hom}(E, \mathcal{O}_K)$ is the dual of $E$ we let $\mathbb{P}(E^\vee)$ be the associated projective space, representing lines in $E^\vee$. Let $E_K^\vee = E^\vee \underset{\mathcal{O}_K}{\otimes} K$ and $X_K \subset \mathbb{P}(E_K^\vee)$ a smooth geometrically irreducible curve of genus $g$ and degree $d$. We assume that the embedding of $X_K$ into $\mathbb{P}(E_K^\vee)$ is defined by a complete linear series on $X_K$???. We also assume that $d \geq 2g+1$. The rank of $E$ is thus $N = d+1-g$.

If $X$ is the Zariski closure of $X_K$ in $\mathbb{P}(E^\vee)$ and $\overline{\mathcal{O}(1)}$ the canonical hermitian line bundle on $\mathbb{P}(E^\vee)$, the Faltings height of $X_K$ is the real number

$$h(X_K) = \widehat{\deg}\,(\hat{c}_1(\overline{\mathcal{O}(1)})^2 \mid X)\,,$$

see [2] (3.1.1) and (3.1.5).

### 2.3

For any positive integer $i \leq N$ we define the integer $f_i$ by the formulae

$$f_i = i-1 \qquad \text{if} \qquad i-1 \leq d-2g$$

and

$$f_i = i-1+\alpha \qquad \text{if} \qquad i-1 = d-2g+\alpha\,, \qquad 0 \leq \alpha \leq g\,.$$

Assume $k$ and $i$ are two positive integers, $k \leq N$, $i \leq N$. We let

$$h_{i,k} = \begin{cases} f_i & \text{if} \quad i \leq k\,,\ i = N-1 \text{ or } i = N \\ f_k & \text{if} \quad k \leq i \leq N-2\,. \end{cases}$$

Finally, if $2 \leq k \leq N$, we let

$$B_k = \max_{i=2,\ldots,N} \frac{h_{i,k}^2}{(i-1)\, h_{i,k} - \sum\limits_{j=1}^{i-1} h_{j,k}}\,.$$

**Proposition 1.** *For every $k$ such that $2 \leq k \leq N-3$,* the following inequality holds:

$$B_k(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\,\mu \geq (2d - N\,B_k)(\mu - \mu_1) - 2d\log(A(d)+d)\,.$$

## 2.4

From Proposition 1 we shall deduce the following result:

**Theorem 2.** *If $g > 0$, for every integer $k$ such that $4 \leq k \leq d-2g-6$, the following inequality holds:*

$$\mu_{N+1-k} - \mu \geq -k\left(\frac{h(X_K)}{2d\,[K:\mathbb{Q}]} + \mu\right) - 12d\log(d) + 12d\,.$$

*Remarks.*

- It is proved in [12] that, when $d \geq 2g+1$,

$$\frac{h(X_K)}{2d\,[K:\mathbb{Q}]} + \mu \geq \frac{2g(d-2g)}{d^2+d-2g^2}(\mu - \mu_1).$$

In particular

$$\frac{h(X_K)}{2d\,[K:\mathbb{Q}]} + \mu \geq 0.$$

- From Bombieri-Vaaler's version of Minkowski's theorem on successive minima [4] we know that

$$-\mu \geq \frac{\widehat{\deg}\,(E,h)}{N\,[K:\mathbb{Q}]} - \frac{\log|D_K|}{2\,[K:\mathbb{Q}]} - K(N)\,,$$

where $D_K$ is the absolute discrimant of $K$ and the constant $K(N)$ depends only on $N$. If $h_{L^2}$ is the $L^2$-metric on $H^0(X,\mathcal{O}(1))$ ([3], (1.2.3)), the quantity

$$\frac{1}{[K:\mathbb{Q}]}\left(\frac{h(X_K)}{2d} - \frac{\widehat{\deg}\,(H^0(X,\mathcal{O}(1)),h_{L^2})}{N}\right)$$

is the normalized height of $X_K$ introduced by Bost in [3], (1.2.4).

## 2.5

To prove Proposition 1 fix a positive integer $k \leq N-3$ and choose elements $x_1, \ldots, x_N$ in $E$, linearly independent over $K$ and such that

$$\log \|x_i\| = \mu_{N-i+1}\,, \quad 1 \leq i \leq N\,.$$

Fix integers $n_\alpha$, $\alpha = k+1, \ldots, N-2$, to be specified later (in § 2.7). If $1 \leq i \leq N$ we define

$$v_i = \begin{cases} x_i + n_i\, x_{i-1} & \text{if } k+1 \leq i \leq N-2 \\ x_i & \text{else.} \end{cases} \tag{2}$$

We get a complete flag $E_K = V_1 \supset V_2 \supset \ldots \supset V_N$ by defining $V_i$ to be the linear span of $v_i, v_{i+1}, \ldots, v_N$.

When $m$ is large enough the cup-product map

$$\varphi : E_K^{\otimes m} \to H^0(X_K, \mathcal{O}(m))$$

is surjective, hence $H^0(X_K, \mathcal{O}(m))$ is generated by the monomials

$$v_1^{\alpha_1} \ldots v_N^{\alpha_N} = \varphi(v_1^{\otimes \alpha_1} \ldots v_N^{\otimes \alpha_N})\,,$$

$\alpha_1 + \cdots + \alpha_N = m$. A *special basis* of $H^0(X_K, \mathcal{O}(m))$ is a basis made of such monomials.

Let $r_1, r_2, \cdots, r_N$ be $N$ real numbers and $\boldsymbol{r} = (r_1, \ldots, r_N)$. We define the weight of $v_i$ to be $r_i$, the weight of a monomial in $E_K^{\otimes m}$ to be the sum of the weights of the $v_i$'s occuring in it, and the weight of a monomial $u \in H^0(X_K, \mathcal{O}(m))$ to be the minimum $wt_{\boldsymbol{r}}(u)$ of the weights of the monomials in the $v_i$'s mapping to $u$ by $\varphi$. The weight $wt_{\boldsymbol{r}}(\mathcal{B})$ of a special basis $\mathcal{B}$ is the sum of the weights of its elements, and $w_{\boldsymbol{r}}(m)$ is the minimum of the weights of special bases of $H^0(X_K, \mathcal{O}(m))$.

When $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ are natural integers there exists $e_{\boldsymbol{r}} \in \mathbb{N}$ such that, as $m$ goes to infinity,

$$w_{\boldsymbol{r}}(m) = e_{\boldsymbol{r}}\, \frac{m^2}{2} + O(m)$$

([11], [10] Corollary 3.3).

Our next goal is to find an upper bound for $e_{\boldsymbol{r}}$.

## 2.6

For every positive integer $i \leq N$ we let $e_i$ be the drop in degree of $X_K$ when projected from $\mathbb{P}(E_K^\vee)$ to $\mathbb{P}(V_i^\vee)$. A criterion of Gieseker ([7], [10] Corollary 3.8) tells us that $e_{\boldsymbol{r}} \leq S$ with

$$S = \min_{1=i_0<\ldots<i_\ell=N} \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}})\,.$$

Note that $S$ is an increasing function in each variable $e_i$. Furthermore, it follows from Clifford's theorem and Riemann-Roch that

$$e_i \leq f_i \tag{3}$$

for every positive $i \leq N$ – see [10] proof of Theorem 4.4 (N.B.: in [10] Theorem 4.4 the filtration of $V_0$ has length $n+1$, while $n = \dim V_0$. In our case, we start the filtration with $V_1$, hence the discrepancy between our definition of $f_i$ and [10] *loc. cit.*).

## 2.7

We now specify our choice of the integers $n_i$ in (2). This is where our argument will differ from [12], Theorem 4, which corresponds to the choice $n_i = 0$ for every $i$.

Let $w_1, \ldots, w_N \in E_K^\vee$ be the dual basis of $v_1, \ldots, v_N$. The linear projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ has center the image $\dot{w}_i$ of $w_i$.

If $y_1, \ldots, y_N \in E_K^\vee$ is the dual basis of $x_1, \ldots, x_N$, we get

$$w_i = \begin{cases} y_i + n_i\, z_i & \text{if } k \leq i \leq N-3 \\ y_i & \text{else,} \end{cases}$$

where $z_i + y_{i+1}$ is a linear combination of $y_{i+2}, y_{i+3}, \cdots$ with coefficients depending only on $n_{i+1}, n_{i+2}, \cdots$. When $n \neq m$ are two integers, the vectors $y_i + n\, z_i$ and $y_i + m\, z_i$ are linearly independent over $K$, therefore their images in $\mathbb{P}(V_i^\vee)$ are distinct. Since $e_{N-3} \leq f_{N-3}$ and $g > 0$ we get $e_{N-3} \leq d-3$, therefore the image of $X_K$ in $\mathbb{P}(V_i^\vee)$, $i \leq N-3$, has degree at least 3. Furthermore $\dim \mathbb{P}(V_i^\vee) \geq 3$. By Theorem 1 and Corollary 1, it follows that we can choose $n_i$ such that $0 \leq n_i < A(d) + d$ and the projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of $X_K$. We fix the integers $n_i$, $k \leq i \leq N-3$, with this property. Hence we have

$$e_i = e_k \quad \text{whenever} \quad k \leq i \leq N-2\,. \tag{4}$$

## 2.8

From (3) and (4) we conclude that

$$e_i \leq h_{i,k} \quad \text{if} \quad 1 \leq i \leq N$$

(see 2.3). Hence, by Morrison's main combinatorial theorem, [10] Corollary 4.3, for any decreasing sequence of real numbers $r_1 \geq r_2 \geq \cdots \geq r_N$ we have, if $k \geq 2$,

$$S \leq \psi(\boldsymbol{r})$$

with

$$\psi(\boldsymbol{r}) = B_k \cdot \sum_{i=1}^{N} (r_i - r_N)\,.$$

So, when $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ is a decreasing sequence of integers,

$$e_{\boldsymbol{r}} \leq \psi(\boldsymbol{r})\,. \tag{5}$$

Let

$$s_i = \begin{cases} \log \|x_{i-1}\| + \log(A+d) & \text{if } k+1 \leq i \leq N-2 \\ \log \|x_i\| + \log(A+d) & \text{else,} \end{cases} \tag{6}$$

and $r_i = s_i - s_N$. The sequence $r_1, \dots, r_N = 0$ is decreasing and (2) implies that

$$\log \|v_i\| \leq \log \|x_{i-1}\| + \log(1+n_i) \leq s_i \quad \text{if} \quad k+1 \leq i \leq N-2\,,$$

and

$$\log \|v_i\| = \log \|x_i\| \leq s_i \quad \text{else.}$$

We endow $\mathcal{O}(1)$ with the metric induced by the metric $h$ on $E$. We choose an hermitian metric, invariant by complex conjugation, on the complex points of $X$, and we endow $M = H^0(X, \mathcal{O}(m))$ with the associated $L^2$-metric. After multiplying the metric on $X$ by a fixed constant we know that, for every $m$, the morphism $\varphi$ is norm decreasing. Therefore, if $u = \varphi(v_1^{\otimes \alpha_1} \dots v_N^{\otimes \alpha_N})$ is a monomial in $M$ we have

$$\log \|u\| \leq \sum_{i=1}^N \alpha_i \log \|v_i\| \leq \sum_{i=1}^N \alpha_i\, s_i = \sum_{i=1}^N \alpha_i\, r_i + m\, s_N\,.$$

By definition of $wt_{\boldsymbol{r}}(u)$, we get

$$\log \|u\| \leq wt_{\boldsymbol{r}}(u) + m\, s_N\,.$$

Let $\widehat{deg}(\bar{M})$ be the arithmetic degree of the hermitian vector bundle $\bar{M} = (M, h_{L^2})$ over $S$. From the inequality above and the Hadamard inequality we deduce that, for any special basis $\mathcal{B}$ of $M$,

$$\widehat{deg}(\bar{M}) \geq -[K:\mathbb{Q}] \sum_{u \in \mathcal{B}} (wt_{\boldsymbol{r}}(u) + m\, s_N)\,.$$

This implies

$$\widehat{deg}(\bar{M}) \geq -[K:\mathbb{Q}](w_{\boldsymbol{r}}(m) + m\, h^0(X_K, \mathcal{O}(m))\, s_N)\,.$$

By (5) and the definition of $e_{\boldsymbol{r}}$, since both $w_{\boldsymbol{r}}(m)$ and $\psi(r)$ are linear functions of $r$, by approximating $r$ by a collection of rational numbers we get that, for every positive real number $\eta$,

$$w_{\boldsymbol{r}}(m) \leq (\psi(r) + \eta)\, \frac{m^2}{2} + O(m)$$

(cf. [12] § 2.2), so we get

$$\widehat{deg}(\bar{M}) \geq -[K:\mathbb{Q}](\psi(\boldsymbol{r}) + 2d\, s_N + \eta)\, \frac{m^2}{2} + O(m)\,.$$

On the other hand, by [8] Theorem 8, we have

$$\widehat{deg}(\bar{M}) = h(X_K)\, \frac{m^2}{2} + O(m \log(m))\,.$$

Therefore, for every $\eta > 0$,

$$h(X_K) \geq -[K:\mathbb{Q}](\psi(\boldsymbol{r}) + 2d\, s_N + \eta)\,. \tag{7}$$

By the definition of $\psi$, we deduce from (7) that

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\, s_N \geq -B_k \left( \sum_{i=1}^{N} r_i \right)\,,$$

and, using (6),

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\, \mu_1 \geq -B_k \left( \sum_{i=1}^{N} (\mu_i - \mu_1) + \mu_{N+1-k} - \mu_3 \right) - 2d \log(A(d) + d)\,. \tag{8}$$

Since $\mu_3 \geq \mu_1$, (8) implies the inequality in Proposition 1.

## 2.9

To make Proposition 1 more explicit, we need to evaluate $B_k$. For any $i = 2, \ldots, N$, we let

$$B_{i,k} = \frac{h_{i,k}^2}{(i-1)\, h_{i,k} - \sum\limits_{j=1}^{i-1} h_{j,k}}$$

so that

$$B_k = \sup_i \, B_{i,k}\,.$$

**Lemma 1.** *Assume that $k - 1 \leq d - 2g$. Then*

- *if $i \leq k$*

$$B_{i,k} = 2 - \frac{2}{i}\,,$$

- *if $k \leq i \leq N - 2$*

$$B_{i,k} = 2 - \frac{2}{k}\,.$$

## 2.10

To prove Lemma 1 we first assume that $i \le k$. Then, if $j \le i$ we have

$$h_{j,k} = f_j = j - 1\,.$$

Therefore

$$\sum_{j=1}^{i-1} h_{j,k} = \sum_{j=1}^{i-1} (j-1) = \frac{(i-2)(i-1)}{2}\,,$$

and

$$B_{i,k} = \frac{(i-1)^2}{(i-1)^2 - \frac{(i-2)(i-1)}{2}} = 2 - \frac{2}{i}\,.$$

Assume now that $k \le i \le N-2$. Then, if $1 \le j \le k-1$, we have

$$h_{j,k} = f_j = j - 1\,.$$

Furthermore, if $k \le j \le i-1$, we get

$$h_{j,k} = f_k = k - 1\,.$$

Therefore

$$B_{i,k} = \frac{(k-1)^2}{(i-1)(k-1) - \sum\limits_{j=1}^{k-1} (j-1) - \sum\limits_{j=k}^{i-1} (k-1)} = 2 - \frac{2}{k}\,.$$

q.e.d.

## 2.11

**Lemma 2.** *Assume that $g > 0$ and $4 \le k \le d - 2g - 6$. Then $B_k < 2$ and*

$$2d - NB_k \ge \frac{2d}{k}\,.$$

To prove Lemma 2 we first compute $B_{N-1,k}$ and $B_{N,k}$ under the assumption $k - 1 \le d - 2g$. When $d - 2g \le i - 1 < N$ we have

$$f_i = 2i - 2 - d + 2g\,.$$

Therefore

$$h_{N-1,k} = f_{N-1} = 2(N-2) - d + 2g = d - 2$$

and

$$h_{N,k} = f_N = 2(N-1) - d + 2g = d\,.$$

On the other hand

$$\sum_{j=1}^{N-2} h_{j,k} = \sum_{j=1}^{k} f_j + \sum_{j=k+1}^{N-2} f_k = \sum_{j=1}^{k} (j-1) + \sum_{j=k+1}^{N-2} (k-1)\,,$$

and

$$\sum_{j=1}^{N-1} h_{j,k} = \sum_{j=1}^{N-2} h_{j,k} + f_{N-1}\,.$$

Therefore

$$\begin{aligned} B_{N-1,k} &= \frac{(d-2)^2}{(N-2)(d-2) - \sum\limits_{j=1}^{k}(j-1) - \sum\limits_{j=k+1}^{N-2}(k-1)} \\ &= \frac{(d-2)^2}{(N-2)(d-1-k) + \frac{k(k-1)}{2}} \end{aligned} \tag{9}$$

and

$$\begin{aligned} B_{N,k} &= \frac{d^2}{(N-2)\,d + 2 - \sum\limits_{j=1}^{k}(j-1) - \sum\limits_{j=k+1}^{N-2}(k-1)} \\ &= \frac{d^2}{2 + (N-2)(d+1-k) + \frac{k(k-1)}{2}}\,. \end{aligned} \tag{10}$$

When $i \le N-2$ we know from Lemma 1 that

$$B_{i,k} \le 2 - \frac{2}{k} < 2\,,$$

hence

$$2d - N\,B_{i,k} \ge 2d - (d+1-g)\left(2 - \frac{2}{k}\right) \ge \frac{2d}{k}\,.$$

When $i = N-1$ we put $k = 4+p$, $p \ge 0$, and $d = 10+2g+p+t$, $t \ge 0$. Then (9) implies that $2 - B_{N-1,k}$ has the same sign as $2gt + 14g + p + t^2 + 12t + 38$, which is positive. On the other hand

$$2d - N\,B_{N-1,k} - \frac{2d}{k}$$

has the sign of the numerator of $k - 1 - NkB_{N-1,k}/(2d)$, namely

$$(2+p)t^3 + (p^2 + (5g+21)p + 10g + 36)t^2 + ((8+4g)p^2 + (8g^2+121+72g)p + 186 + 118g + 16g^2)t + (g-1)p^3$$
$$+ (5+26g+4g^2)p^2 + (172+4g^3+60g^2+240g)p + 244 + 328g + 92g^2 + 8g^3 \ > 0.$$

Similarly (10) implies that $2 - B_{N,k}$ has the same sign as $t^2 + (2g+12)t + 42 + 10g + p$, so it is positive, and

$$2d - N\,B_{N,k} - \frac{2d}{k}$$

has the sign of

$$(2+p)t^2 + ((9+3g)p + 12 + 6g)t + (g-1)p^2 + (11 + 18g + 2g^2)p - 14 + 22g + 4g^2 > 0.$$

This ends the proof of Lemma 2.

## 2.12

To prove Theorem 2 we first note that $B_k < 2$ and $2d - N\,B_k \geq 2d/k$ by Lemma 2. Since $\mu_1 \leq \mu_{N+1-k}$ and $\mu_1 \leq \mu$, Proposition 1 implies

$$2(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\,\mu \geq \frac{2d}{k}(\mu - \mu_1) - 2d\log(A(d) + d).$$

By dividing this inequality by $2d$ we obtain

$$\frac{1}{d}(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{2d\,[K:\mathbb{Q}]} + \mu \geq \frac{1}{k}(\mu - \mu_1) - \log(A(d) + d)\,. \tag{11}$$

Since $k < d$ we get

$$\mu_{N+1-k} - \mu_1 + k\left(\frac{h(X_K)}{2d\,[K:\mathbb{Q}]} + \mu\right) \geq \mu - \mu_1 - d\log(A(d) + d)\,,$$

and, since $d \geq 12$, the remark in §1 implies Theorem 2.

## References


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