HIGHER K-THEORY OF ALGEBRAIC INTEGERS AND THE COHOMOLOGY OF ARITHMETIC GROUPS

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Lecture one: Two theorems of Armand Borel. Let F be a number field, i.e, a finite field extension of $\mathbb{Q}$, and let $A = \mathcal{O}_F$ be its ring of integers, i.e., the integral closure of $\mathbb{Z}$ in F:

$$A = \mathcal{O}_F = \left\{ x \in F \mid x^n + a_1 x^{n-1} + \ldots + a_n = 0,\ a_i \in \mathbb{Z} \right\}.$$

Our goal in these lectures is to understand the algebraic K-theory of A.

First of all, observe that there is no negative K-theory because A is regular.

Proposition 1. *$K_0(A) \cong \mathbb{Z} \oplus \operatorname{Pic}(A)$.*

Here $\operatorname{Pic}(A)$ is the ideal class group of A, i.e., the set of isomorphism classes of invertible A-modules with addition given by the tensor product. Proposition 1 is true more generally for any Dedekind domain A, since every projective module is the sum of ideals, each of which is projective and satisfies $I \oplus J \cong IJ \oplus A$, see [Mil71].

For $A = \mathcal{O}_F$ Dirichlet proved that $\operatorname{Pic}(A)$ is finite.

Proposition 2. *$K_1(A) = A^\times$.*

In fact, Bass, Milnor, and Serre [BMS67] proved that $SK_1(A) = 0$, and for any commutative ring A one has $K_1(A) = A^\times \times SK_1(A)$.

For $A = \mathcal{O}_F$ Dirichlet proved that

$$\dim_{\mathbb{Q}}(A^\times \otimes \mathbb{Q}) = r_1 + r_2 - 1 = d_1$$

where

$$r_1 = \#\{\text{ real places of } F\,\} = \#\{\,\sigma\colon F \hookrightarrow \mathbb{R}\,\},$$

$$r_2 = \#\{\text{ complex places of } F\,\} = \frac{1}{2}\#\{\,\sigma\colon F \hookrightarrow \mathbb{C},\ \sigma \neq \overline{\sigma}\,\},$$

(the resulting decomposition of $F \otimes_{\mathbb{Q}} \mathbb{R}$ then shows that $[F : \mathbb{Q}] = r_1 + 2r_2$), and for any $n \geq 1$ we put

$$d_n = \begin{cases} r_1 + r_2 - 1 & \text{if } n = 1, \\ r_1 + r_2 & \text{if } n \text{ is odd and } \geq 3, \\ r_2 & \text{if } n \text{ is even.} \end{cases}$$

More precisely, Dirichlet proved that $A^\times$ is the product of the finite cyclic group $\mu(F)$ of roots of unity in F and a free abelian group of rank $r_1 + r_2 - 1 = d_1$.

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Theorem 3 (Quillen [Qui73]). *For all $m \geq 0$, $K_m(A)$ is finitely generated.*

Theorem 4 (Borel [Bor74]). *For all $m > 0$,*

- *if m is even then $K_m(A)$ is finite,*
- *if $m = 2n-1$ then $\dim_{\mathbb{Q}}(K_m(A) \otimes \mathbb{Q}) = d_n$.*

These results generalize the aforementioned theorems by Dirichlet.

Example 5. If $F = \mathbb{Q}$, $A = \mathbb{Z}$ then $r_1 = 1$ and $r_2 = 0$, and hence for $m > 0$

$$K_m(\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \text{finite} & m = 5, 9, 13, \dots, \\ \text{finite} & \text{otherwise.} \end{cases}$$

We will not discuss the proof of Quillen's theorem 3 here.

As we will see below, Borel's theorem 4 follows from the following theorem.

Theorem 6 (Borel). *Let $G = SL_N(\mathbb{R})^{r_1} \times SL_N(\mathbb{C})^{r_2} \supset \Gamma = SL_N(A)$.*

Assume $q + 1 \leq (N-1)/4$. Then the corestriction map $H^q_{\text{cont}}(G) \to H^q(\Gamma; \mathbb{R})$ is an isomorphism.

Here $H^q_{\text{cont}}(G)$ is the continuous cohomology of G with real coefficients. It can be defined as the cohomology of the complex

$$\cdots \longrightarrow C^q_{\text{cont}}(G)^G \xrightarrow{\partial} C^{q+1}_{\text{cont}}(G)^G \longrightarrow \cdots,$$

where $C^q_{\text{cont}}(G)$ is the real vector space of continuous maps from G^{q+1} to $\mathbb{R}$ and ∂ is given by the formula

$$\partial_\varphi(g_0, \dots, g_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(g_0, \dots, \widehat{g_i}, \dots, g_{q+1}).$$

Theorem 6 is actually a special case of the following more general result.

Theorem 7 (Borel [Bor74]). *Let $\underline{G}$ be a semi-simple algebraic group over $\mathbb{Q}$ such that $G = \underline{G}(\mathbb{R})$ is connected and let $\Gamma < \underline{G}(\mathbb{Q})$ be an arithmetic group.*

Assume $q + 1 \leq \operatorname{rank}_{\mathbb{Q}}(G)/4$. Then the corestriction map $H^q_{\text{cont}}(G) \to H^q(\Gamma; \mathbb{R})$ is an isomorphism.

Proof that theorem 6 implies theorem 4. Step 1: We first compute $H^*_{\text{cont}}(G)$ as follows. Consider the maximal compact subgroup K of G, and the symmetric space $X = K \backslash G$.

Example 8. If $G = GL_N(\mathbb{R})$ then $K = O(N)$ and X is the set of positive definite real quadratic forms. In fact, given $[g] \in X$ we can define $\varphi(x) = \|g(x)\|^2$ for $x \in \mathbb{R}^N$.

If $G = SL_N(\mathbb{R})$ then $K = SO(N)$ and X is the set of positive definite real quadratic forms modulo the action of $\mathbb{R}^\times_{>0}$.

The manifold X is contractible. Therefore the de Rham complex

$$0 \to \mathbb{R} \to \Omega^0(X) \to \Omega^1(X) \to \Omega^2(X) \to \dots$$

is exact. This yields a "strong" resolution of $\mathbb{R}$ by "relatively" injective G-modules (this means that the resolution is "good" from the point of view of continuous cohomology, see [Gui80]). Hence

$$H^q_{\text{cont}}(G) = H^q(\Omega^*(X)^G).$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. By restriction of differential forms at the origin we have

$$\Omega^q(X)^G = \hom_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), \mathbb{R}).$$

Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \supset i\mathfrak{p}$. Then the so-called unitarian trick yields that $\mathfrak{k} \oplus i\mathfrak{p} = \mathrm{Lie}(G_u)$, where G_u is a compact connected Lie group containing K. Then

$$\Omega^q(X)^G = \hom_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), \mathbb{R}) = \hom_{\mathfrak{k}}(\Lambda^q(\mathfrak{p}), \mathbb{R}) \cong \hom_{\mathfrak{k}}(\Lambda^q(i\mathfrak{p}), \mathbb{R}) = \Omega^q(K\backslash G_u)^{G_u}.$$

Since G_u is compact and connected, integration on G_u shows that the inclusion

$$\Omega^q(K\backslash G_u)^{G_u} \subseteq \Omega^q(K\backslash G_u)$$

is a homology equivalence [Gui80, rem. 7.1 and lemma E.2]. Therefore

$$H^q_{\mathrm{cont}}(G) = H^q(\Omega^*(X)^G) = H^q(\Omega^*(K\backslash G_u)^{G_u}) = H^q(\Omega^*(K\backslash G_u)) = H^q(K\backslash G_u; \mathbb{R}).$$

Example 9. If $G = SL_N(\mathbb{R})$, $K = SO(N)$ then

$$\mathfrak{g} = \{\, M \mid \operatorname{tr} M = 0 \,\}\ ,\quad \mathfrak{k} = \left\{\, M \mid M^t = -M \,\right\}\ ,\quad \mathfrak{p} = \left\{\, M \mid M^t = M \,\right\}$$

and therefore

$$\mathfrak{k} \oplus i\mathfrak{p} = \left\{ M \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \;\middle|\; \overline{M}^t = -M \right\} \cong \mathfrak{su}(N).$$

Hence $G_u = SU(N)$. Then we get

$$H^q_{\mathrm{cont}}(SL_N(\mathbb{R})) \cong H^*(SO(N)\backslash SU(N); \mathbb{R}).$$

The right-hand side is known (by previous work of Borel) and gives

$$H^*_{\mathrm{cont}}(SL_N(\mathbb{R})) \cong \Lambda^*(e_5, e_9, e_{13}, \dots, e_{4k+1})$$

with $e_q \in H^q(SO(N)\backslash SU(N); \mathbb{Z})$, $k = [\frac{N-1}{2}]$.

If $G = SL_N(\mathbb{C})$ then $K = SU(N)$ and $G_u = SU(N) \times SU(N)$. We get

$$H^*_{\mathrm{cont}}(SL_N(\mathbb{C})) \cong H^*(SU(N); \mathbb{R}) \cong \Lambda^*(\varepsilon_3, \varepsilon_5, \varepsilon_7, \dots, \varepsilon_{2N-1})$$

with $\varepsilon_q \in H^q(SU(N); \mathbb{Z})$.

For $G = SL_N(\mathbb{R})^{r_1} \times SL_N(\mathbb{C})^{r_2}$ this yields

$$H^*_{\mathrm{cont}}(G) \cong \Lambda^*(e_i)^{\otimes r_1} \otimes \Lambda^*(\varepsilon_j)^{\otimes r_2}.$$

Step 2: There is a homotopy equivalence

$$BSL(A)^+ \times B(A^\times) \xrightarrow{\simeq} BGL(A)^+$$

and hence for $m \geq 2$

$$K_m(A) \cong \pi_m BSL(A)^+.$$

For any CW-complex X consider the Hurewicz map

$$h_m\colon \pi_m(X) \otimes \mathbb{R} \to (IH^m(X; \mathbb{R}))^\vee$$

where $E^\vee = \hom_{\mathbb{R}}(E, \mathbb{R})$ and $IH^m(X; \mathbb{R}) = H^m(X; \mathbb{R})/\{\text{cup products}\}$.

Lemma 10. *If X is an H-space such that $\dim_{\mathbb{R}} H^m(X; \mathbb{R}) < \infty$ for all m, then h_m is an isomorphism.*

Proof. To prove this lemma, we define

$$PH_m(X;\mathbb{R}) = \{\, x \in H_m(X;\mathbb{R}) \mid \Delta_*(x) = x \otimes 1 + 1 \otimes x \,\}$$

where

$$\Delta_*\colon H_m(X;\mathbb{R}) \to H_m(X \times X;\mathbb{R}) \cong \bigoplus_{s+t=m} H_s(M;\mathbb{R}) \otimes H_t(M;\mathbb{R})$$

is induced by the diagonal map.

Then if X is an H-space there is an isomorphism

$$\pi_m(X) \otimes \mathbb{R} \xrightarrow{\cong} PH_m(X;\mathbb{R})$$

[MM65, Appendix], and under the finiteness assumption above $(IH^m(X;\mathbb{R}))^\vee \cong PH_m(X;\mathbb{R})$. □

Now $BSL(A)^+$ is an H-space satisfying the assumption of the previous lemma, because of (the proof of) Quillen's theorem 3, and therefore for $m \geq 2$ we get

$$\begin{aligned} K_m(A) \otimes \mathbb{R} \cong (IH^m(BSL(A)^+;\mathbb{R}))^\vee &= (IH^m(BSL(A);\mathbb{R}))^\vee \\ &= (IH^m(SL(A);\mathbb{R}))^\vee. \end{aligned}$$

Theorem 6 implies that for $N \gg m$

$$H^m(SL_N(A);\mathbb{R}) \cong H^m_{\text{cont}}(G) \cong \Lambda^*(e_i)^{\otimes r_1} \otimes \Lambda^*(\varepsilon_j)^{\otimes r_2}$$

and therefore

$$H^m(SL_{N+1}(A);\mathbb{R}) \cong H^m(SL_N(A);\mathbb{R}).$$

This yields

$$\begin{aligned} (IH^m(SL(A);\mathbb{R}))^\vee \overset{N \gg m}{\cong} IH^m_{\text{cont}}(G)^\vee \cong I(\Lambda^*(e_i)^{\otimes r_1} \otimes \Lambda^*(\varepsilon_j)^{\otimes r_2})^m \\ = \begin{cases} \mathbb{R}^{d_n} & \text{if } m = 2n-1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

completing the proof that Borel's theorem 4 follows from theorem 6. □

Example 11. If $F = \mathbb{Q}$ then $r_2 = 0$, $r_1 = 1$ and

$$I(\Lambda^*(e_5, e_9, e_{13}, \ldots))^m = \begin{cases} \mathbb{R} & \text{if } m = 5, 9, 13, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of proof of theorem 6. For simplicity we only consider

$$G = SL_N(\mathbb{R}) \;\supset\; \Gamma = SL_N(\mathbb{Z}).$$

Recall that $H^q_{\text{cont}}(G) = H^q(\Omega^*(X)^G)$ where X is the symmetric space $K\backslash G$.

Lemma 12 (Cartan)**.** *The differential* $d\colon \Omega^*(X)^G \to \Omega^{*+1}(X)^G$ *vanishes.*

Proof of lemma 12. Let $\theta\colon G \to G$ be the Cartan involution $\theta(g) = (g^{-1})^t$. It induces a map $\theta\colon X \to X$ and therefore a chain map $\theta^*\colon \Omega^*(X)^G \to \Omega^*(X)^G$.

Look at $\theta'\colon \mathfrak{g} \to \mathfrak{g}$, $\theta'(M) = -M^t$. Recall that $\Omega^q(X)^G = \hom_{\mathfrak{k}}(\Lambda^q\mathfrak{p}, \mathbb{R})$ and $\mathfrak{p} = \{\, M \mid M^t = M \,\}$. If $x \in \Lambda^q\mathfrak{p}$ then $\theta'(x) = (-1)^q x$. Hence if $\alpha \in \Omega^q(X)^G$ we compute

$$(-1)^q(d\alpha) = d\theta^*(\alpha) = \theta^* d(\alpha) = (-1)^{q+1}(d\alpha)$$

and therefore $d\alpha = 0$. □

Now assume first that $\Gamma = \{\, \gamma \in SL_N(\mathbb{Z}) \mid \gamma \equiv 1 \pmod 3 \,\}$.

Fact 13. Γ *is torsionfree.*

This fact implies that Γ is acting freely on $X = K\backslash G$, as we can see as follows. Let $\gamma \in \Gamma$ and $[g] \in K\backslash G$. If $[g]\gamma = [g]$, we get $g\gamma = kg$, i.e., $\gamma = g^{-1}kg \in g^{-1}Kg \cap \Gamma$. But $g^{-1}Kg \cap \Gamma$ is finite, being the intersection of a compact with a discrete group. Therefore γ has finite order, but, since Γ is torsionfree, this shows that $\gamma = 1$.

Since X is contractible, X/Γ is therefore a $K(\Gamma, 1)$-space. Then

$$H^q(\Gamma; \mathbb{R}) = H^q(X/\Gamma; \mathbb{R}) = H^q(\Omega^*(X/\Gamma)) = H^q(\Omega^*(X)^\Gamma)$$

and we have to study

$$\Omega^q(X)^G = H^q(\Omega^*(X)^G) \to H^q(\Omega^*(X)^\Gamma).$$

Fix a smooth G-invariant metric h on TX, and define

- the volume form $\mu = \sqrt{det(h^{i,j})}\, dx_1 \cdots dx_n \in \Omega^n(X)$, where $n = \dim(X)$,
- the star operator $\star\colon \Omega^q(X) \to \Omega^{n-q}(X)$ by $\omega \wedge \star\omega = h(\omega, \omega)\mu$,
- the Laplace operator $\Delta = dd^* + d^*d$, where

$$d^* = (-1)^{n(q+1)-1} \star d\star\colon \Omega^q(X) \to \Omega^{q-1}(X).$$

Cartan's lemma 12 above shows that $\Omega^*(X)^G \subset \ker \Delta$.

Main idea: Do Hodge theory on X/Γ.

Main difficulty: X/Γ is not compact, it has only finite volume.

First step:

$$H^q(\Omega^*(X)^\Gamma) = H^q(\Omega^*(X)^\Gamma_{\log})$$

where $\Omega^*(X)^\Gamma_{\log}$ is the complex of differential forms ω such that both ω and $d\omega$ have "logarithmic growth at infinity". For instance, when $G = SL_2(\mathbb{R})$, in which case X is the Poincaré upper half-plane

$$X = G/K = \{\, z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \,\},$$

a form ω is said to have logarithmic growth at infinity when its restriction to a Siegel set

$$\mathfrak{G} = \Big\{\, z \in X \;\Big|\; |\operatorname{Re}(z)| \le b,\ \operatorname{Im}(z) > t \,\Big\}$$

can be written

$$\omega_{|\mathfrak{G}} = \sum_{I,J} a_{I,J}(z)(dx)^I \left(\frac{dy}{y}\right)^J$$

with

$$|a_{I,J}(z)| \le C|\log(y)|^k$$

for some integer k.

The proof of this step relies upon a Poincaré lemma with logarithmic growth.

Next, assume $\omega \in \Omega^q(X)^\Gamma_{\log}$ and q is small. Then ω is L^2, i.e.,

$$\|\omega\|^2_{L^2} = \int_{X/\Gamma} h(\omega, \omega)\mu < \infty.$$

In other words, $\Omega^*(X)^\Gamma_{\log} \subset \Omega^*(X)^\Gamma_{L^2}$ for q small.

Now we can do L^2-Hodge theory:

(a) If ω is L^2 and $d\omega = 0$ then $\omega = h + d\eta$ with h harmonic and L^2.
(b) If h is harmonic and L^2, and $h = d\eta$ where η is L^2, then $h = 0$.

(E.g., in order to prove (b) compute

$$(h,h)_{L^2} = (h, d\eta)_{L^2} = (d^*h, \eta)_{L^2} = 0$$

and therefore $h = 0$.)

The next step is to show that $\Omega^*(X)^G \subset \Omega^*(X)^\Gamma_{L^2}$.

And then the crucial step, due essentially to Garland and Matsushima, is to prove that if q is small and $h \in \Omega^q(X)^\Gamma_{L^2}$ with $\Delta(h) = 0$, then $h \in \Omega^q(X)^G$.

Putting all together we get

$$H^q(\Omega^*(X)^\Gamma) \cong \ker(\Delta) \cap \Omega^q(X)^\Gamma_{L^2} \cong \Omega^q(X)^G.$$

Finally, for $\Gamma_0 = SL_N(\mathbb{Z})$ we have

$$H^q(\Gamma_0; \mathbb{R}) \cong H^q(\Gamma; \mathbb{R})^{\Gamma_0/\Gamma}$$

which is then equal to $\Omega^q(X)^G$ since the action of Γ_0 is trivial. □

Lecture two: Regulators. Let F be a number field and A its ring of integers. Let $\mathfrak{a} \subset A$ be a non-zero ideal. The norm of $\mathfrak{a}$ is $N\mathfrak{a} = \#(A/\mathfrak{a}) < \infty$.

Definition 14. $\zeta_F(s) = \displaystyle\sum_{\mathfrak{a}\neq 0} \frac{1}{(N\mathfrak{a})^s}$.

Example 15. If $F = \mathbb{Q}$ then

$$\zeta_{\mathbb{Q}}(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

is the classical zeta function ζ.

Fact 16.
- *$\zeta_F(s)$ is absolutely convergent where $\Re(s) > 1$;*
- *$\zeta_F(s)$ has a meromorphic continuation to $\mathbb{C}$;*
- *$\zeta_F(s)$ has a pole of order 1 at $s = 1$;*
- *Let $\xi(s) = A^s\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_F(s)$, where $\Gamma(s)$ is the classical gamma function, $A = 2^{-r_2}\sqrt{|D|}\pi^{r_1+2r_2}$, and D is the discriminant of F. Then ξ satisfies the functional equation $\xi(1-s) = \xi(s)$.*

Corollary 17. *If $n \geq 1$ then $\zeta_F(s)$ has a zero of order d_n at $s = 1-n$.*

Proof. The gamma function $\Gamma(s)$ has poles of order 1 at $s = 0, -1, -2, -3, \ldots$, hence $\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$ has a pole of order r_2 (respectively $r_1 + r_2$) at $s = 1-n$ when $n \geq 0$ is even (respectively, odd). Note that $\zeta_F(n) \neq 0$ hence $\xi(n) \neq 0$ when $n > 1$.

Now let $s \to n$ and consider $A^{1-s}\Gamma((1-s)/2)^{r_1}\Gamma(1-s)^{r_2}\zeta_F(1-s) = \xi(s)$, by the functional equation. Therefore $\zeta_F(s)$ has a zero of order d_n at $s = 1-n$. □

Definition 18. $\zeta_F^*(1-n) = \displaystyle\lim_{s\to 1-n} \frac{\zeta_F(s)}{(s+n-1)^{d_n}} \in \mathbb{R}^\times$.

Theorem 19 (Dirichlet's class number formula)**.**

$$\zeta_F^*(0) = -\frac{hR}{w}$$

where

$$h = \#\operatorname{Pic}(A) = \#K_0(A)_{\text{tors}}$$
$$w = \#\mu(F) = \#K_1(A)_{\text{tors}}$$

and R is the regulator defined below.

Definition 20 (of the regulator)**.** If $u \in A^\times$ and v is an archimedean place of F, put

$$\|u\|_v = \begin{cases} |\sigma(u)| & \text{if } v = \sigma\colon F \hookrightarrow \mathbb{R}, \\ |\sigma(u)|^2 & \text{if } v = \{\sigma, \overline{\sigma}\},\ \sigma\colon F \hookrightarrow \mathbb{C}. \end{cases}$$

There is the so-called product formula: $\prod_v \|u\|_v = 1$, where v ranges over all archimedean places. This yields a map

$$\rho\colon A^\times \to \mathbb{R}^{d_1} = \mathbb{R}^{r_1+r_2-1} = \ker(\Sigma\colon \mathbb{R}^{r_1+r_2} \to \mathbb{R}) , \quad u \mapsto (\log \|u\|_v)_v .$$

Fact 21 (Dirichlet)**.** $\operatorname{im}(\rho)$ *is a lattice.*

Now endow $\mathbb{R}^{d_1}$ with the restriction of the Lebesgue measure on $\mathbb{R}^{r_1+r_2}$, and define

$$R = \operatorname{vol}\left(\frac{\mathbb{R}^{d_1}}{\rho(A^\times)}\right).$$

We now want to see how Dirichlet's class number formula 19 generalizes to higher K-theory.

Recall that $H^*_{\text{cont}}(SL_N(\mathbb{R})) \cong \Lambda^*(e_5, e_9, e_{13}, \ldots)$. Given $\sigma\colon F \hookrightarrow \mathbb{R}$, then for $m = 4k+1$ we get

$$\sigma^*(e_m) \in H^m(SL_N(A); \mathbb{R}) \overset{N \gg 0}{\cong} H^m(SL(A); \mathbb{R})$$

and hence a map

$$K_m(A) \to H^m(SL(A); \mathbb{R})^\vee \to \mathbb{R}$$

given by the composition of the Hurewicz homomorphism and the map sending $\varphi \in H^m(SL(A); \mathbb{R})^\vee$ to $2\pi\varphi(\sigma^*(e_m))$. Similarly for $\sigma\colon F \hookrightarrow \mathbb{C}$ and $m = 3, 5, 7, \ldots$.

In this way we get for $n > 1$ a map

$$\rho_n\colon K_{2n-1}(A) \to \mathbb{R}^{d_n}$$

called the *higher (Borel) regulator map.*

The proof of theorem 4 actually shows that $\operatorname{im}(\rho_n)$ is a lattice. Define

$$R_n = \operatorname{vol}\left(\frac{\mathbb{R}^{d_n}}{\rho_n(K_{2n-1}(A))}\right).$$

Theorem 22 (Siegel; Borel [Bor77])**.** *For any $n > 1$ there is a $q_n \in \mathbb{Q}^\times$ such that*

$$\zeta_F^*(1-n) = q_n R_n.$$

Example 23. Assume $r_2 = 0$ and $n > 1$ is even. Then $d_n = 0$, and theorem 22 (in this case due to Siegel) gives

$$\zeta_F(1-n) \in \mathbb{Q}^\times.$$

For instance, if $F = \mathbb{Q}$ and $n > 1$ is even then

$$\zeta(1-n) = -\frac{b_n}{n}$$

where b_n are the Bernoulli numbers defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$

E.g.,

$$\zeta(1-n) = \begin{cases} -1/2 \\ 1/12 \\ 0 \\ 1/120 \\ 0 \\ \dots \\ 691/32760 \end{cases} \quad \text{if} \quad 1-n = \begin{cases} 0 \\ -1 \\ -2 \\ -3 \\ -4 \\ \dots \\ -11 \end{cases} .$$

Lecture three: Étale cohomology. Let E be a finite extension of the number field F and let B be the ring of integers of E. Let p be a prime.

Definition 24. E is said to be *unramified outside* p when, for any prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \nmid pA$, one has in B that $\mathfrak{p}B = q_1 \cdots q_t$ with $q_1, \dots, q_t$ distinct prime ideals.

Example 25. For any integer $k \geq 1$ then $F(\mu_{p^k})|F$ is unramified outside p, where μ_{p^k} are the p^k-th roots of unity.

Define

$$\Phi = \bigcup_{\substack{E|F \\ \text{unramified} \\ \text{outside p}}} E$$

and notice that $\mu_{p^k} \subset \Phi^\times$ for all $k \geq 1$. Define a character

$$\epsilon\colon \operatorname{Gal}(\Phi|F) \to \mathbb{Z}_p^\times$$

by the equation $g(\xi) = \xi^{\epsilon(g)}$ for any $g \in \operatorname{Gal}(\Phi|F)$ and for any $\xi \in \mu_{p^k}$.

The abelian group $\mathbb{Z}_p$ carries then for any n a new $\operatorname{Gal}(\Phi|F)$-module structure, denoted $\mathbb{Z}_p(n)$ and defined as

$$g \cdot \alpha = \epsilon(g)^n \alpha$$

for $g \in \operatorname{Gal}(\Phi|F)$ and $\alpha \in \mathbb{Z}_p(n)$.

Define *étale cohomology* as

$$H^q_{\text{ét}}\big(\operatorname{Spec} A\big[\tfrac{1}{p}\big]; \mathbb{Z}_p(n)\big) = H^q_{\text{cont}}(\operatorname{Gal}(\Phi|F); \mathbb{Z}_p(n))$$

and abbreviate these groups to $H^q(A; \mathbb{Z}_p(n))$.

The following theorem is in quotation marks because it depends on the proof of the so-called Bloch-Kato conjecture, announced by Voevodsky and Rost but not yet fully written-up (cf. [Wei05]). Notice however that the corresponding surjectivity statements were proved by Soulé [Sou79] and Dwyer-Friedlander [DF85].

"Theorem" 26. *If p is odd and $n \geq 2$ there are canonical isomorphisms*

$$K_{2n-1}(A) \otimes \mathbb{Z}_p \xrightarrow{\cong} H^1(A; \mathbb{Z}_p(n)),$$
$$K_{2n-2}(A) \otimes \mathbb{Z}_p \xrightarrow{\cong} H^2(A; \mathbb{Z}_p(n)).$$

The natural maps in the theorem above were first constructed by Dwyer and Friedlander [DF85] using étale K-theory $K_m^{\text{ét}}(A;\mathbb{Z}_p)$, which can be thought of as topological K-theory of the étale homotopy type of $\operatorname{Spec} A[\frac{1}{p}]$. (There is also a more modern description using motivic cohomology instead of étale cohomology.) There is a natural map

$$K_m(A) \to K_m^{\text{ét}}(A;\mathbb{Z}_p)$$

and there is also a spectral sequence converging to $K_{2n-q}^{\text{ét}}(A;\mathbb{Z}_p)$ with

$$E_2^{q,-2n} = H^q(A;\mathbb{Z}_p(n)).$$

But, assuming that p is odd, $H^q(A;\mathbb{Z}_p(n)) = 0$ if $n > 0$ and $q \neq 1$ or 2. So the spectral sequence degenerates, i.e., $E_2 = E_\infty$ and in any diagonal there is only one non-zero E_2-term, therefore

$$K_{2n-1}^{\text{ét}}(A;\mathbb{Z}_p) = H^1(A;\mathbb{Z}_p(n)),$$
$$K_{2n-2}^{\text{ét}}(A;\mathbb{Z}_p) = H^2(A;\mathbb{Z}_p(n)).$$

Corollary 27. *The group $H^2(A;\mathbb{Z}_p(n))$ is always finite, and it vanishes for almost all primes p; moreover $\dim_{\mathbb{Q}_p} H^1(A;\mathbb{Z}_p(n)) \otimes \mathbb{Q}_p = d_n$.*

This corollary is not in quotation marks because the surjectivity statements in theorem 26, combined with theorem 4, are enough for it.

Theorem 28 (Wiles [Wil90])**.** *If $r_2 = 0$ (i.e., if F is totally real) and n is even then*

$$|\zeta_F(1-n)| = 2^? \frac{\prod_{p>2} \# H^2(A;\mathbb{Z}_p(n))}{\prod_{p>2} \# H^1(A;\mathbb{Z}_p(n))}.$$

"Corollary" 29. *If $r_2 = 0$ and n is even then*

$$|\zeta_F(1-n)| = 2^? \frac{\# K_{2n-2}(A)}{\# K_{2n-1}(A)}.$$

"Example" 30. If $F = \mathbb{Q}$ and n is even then

$$|\zeta(1-n)| = 2\frac{\# K_{2n-2}(\mathbb{Z})}{\# K_{2n-1}(\mathbb{Z})}.$$

Combining "theorem" 26 with work of Fleckinger, Kolster, and Nguyen Quang Do [KNQDF96] we get:

"Theorem" 31. *If F is abelian and $n \geq 1$ then*

$$|\zeta_F^*(1-n)| = 2^? \frac{\# K_{2n-2}(A)}{\# K_{2n-1}(A)_{\text{tors}}} R_n.$$

Conjecture 32 (Vandiver)**.** *For an odd prime p define*

$$C = \operatorname{Pic}(\mathbb{Q}(\mu_p)) \otimes \mathbb{Z}/p\mathbb{Z}\ , \quad C^+ = \{\, x \in C \mid \overline{x} = x \,\}\,.$$

Then $C^+ = 0$.

Recall that p is called *regular* if $C = 0$ (and that Kummer proved Fermat for regular primes—Vandiver hoped that $C^+ = 0$ would also imply Fermat).

Using computers one can show that Vandiver's conjecture is true if $p < 10^7$.

"Theorem" 33 ([Kur92])**.** *The Vandiver conjecture is true for all primes if and only if $K_{4k}(\mathbb{Z}) = 0$ for all $k \geq 1$.*

Example 34. $K_4(\mathbb{Z}) = 0$ by a theorem of Rognes [Rog00], but $K_8(\mathbb{Z})$ is still unknown.

Proof. For $i \in \mathbb{Z}$ denote

$$C^{(i)} = \left\{ x \in C \;\middle|\; g(x) = \epsilon(g)^i x \quad \forall g \in \mathrm{Gal}(\mathbb{Q}(\mu_p)|\mathbb{Q}) \overset{\epsilon}{\cong} (\mathbb{Z}/p\mathbb{Z})^\times \right\}.$$

Then $C^+ = \bigoplus_{i \text{ even}} C^{(i)}$. The last needed ingredient is the computation

$$H^2(\mathbb{Z}; \mathbb{Z}_p(n)) \otimes \mathbb{Z}/p\mathbb{Z} \cong C^{(p-n)},$$

which combined with "theorem" 26 finishes the proof. □

Example 35. Obviously $C^{(p-1)} = 0$; there is a surjection $K_4(\mathbb{Z}) \twoheadrightarrow C^{(p-3)}$, and therefore, as noticed by Kurihara [Kur92], $C^{(p-3)} = 0$.

Assuming the Bloch-Kato and Vandiver conjectures we get the following computation of $K_*(\mathbb{Z})$ (cf. [Wei05]; note that the 2-torsion is known [RW00]): setting

$$\begin{aligned} w_n &= \text{denominator of } \frac{1}{2}\zeta(1-n), \\ c_n &= \text{numerator} \quad \text{of } \frac{1}{2}\zeta(1-n), \\ k_m &= \left[1 + \frac{m}{4}\right], \end{aligned}$$

then for all $m > 1$ we have

$$K_m(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } m \equiv 1 \\ \mathbb{Z}/2c_{2k_m} & \text{if } m \equiv 2 \\ \mathbb{Z}/2w_{2k_m} & \text{if } m \equiv 3 \\ 0 & \text{if } m \equiv 4 \\ \mathbb{Z} & \text{if } m \equiv 5 \\ \mathbb{Z}/c_{2k_m} & \text{if } m \equiv 6 \\ \mathbb{Z}/w_{2k_m} & \text{if } m \equiv 7 \\ 0 & \text{if } m \equiv 8 \end{cases} \qquad \pmod 8.$$

Lecture four: Perfect forms. Let $N \geq 2$ and let $\phi(x) = \sum_{1 \leq i,j \leq N} a_{ij} x_i x_j$ be a positive definite real quadratic form in N-variables, i.e., $a_{ij} = a_{ji} \in \mathbb{R}$, $\phi(x) \geq 0$, with equality if and only if $x = 0$.

Define $M(\phi) = \left\{ x \in \mathbb{Z}^N - 0 \mid \phi(x) \text{ is minimal} \right\}$. This is a finite set.

Definition 36. We say that ϕ is *perfect* when $M(\phi)$ determines ϕ up to scalar multiplication.

Example 37. $M(x^2 + y^2) = \{(1,0), (-1,0), (0,1), (0,-1)\} = M(x^2 + \frac{1}{2}xy + y^2)$, hence these forms are not perfect. On the other hand $x^2 + xy + y^2$ is perfect, and

$$M(x^2 + xy + y^2) = \{(1,0), (-1,0), (0,1), (0,-1), (1,-1), (-1,1)\}.$$

The group $\Gamma = SL_N(\mathbb{Z})$ acts on forms by $(\phi\gamma)(x) = \phi(\gamma(x))$.

Theorem 38 (Voronoï [Vor08])**.** *Modulo the action of* Γ *and scalar multiplication there are only finitely many perfect forms of a given rank* N*.*

For small values of N perfect forms have been classified and

$$\#\{\text{perfect forms in } N\text{-variables}\} = \begin{cases} 1 \\ 1 \\ 2 \\ 3 \\ 7 \\ 33 \\ 10916 \end{cases} \quad \text{if} \quad N = \begin{cases} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{cases}$$

(the last two numbers were obtained by computers).

Define

$$C_N^* = \left\{ \phi(x) = \sum_{1 \le i,j \le N} a_{ij} x_i x_j \;\middle|\; \begin{array}{l} a_{ij} = a_{ji} \in \mathbb{R} \ , \quad \phi(x) \ge 0 \ , \\ \exists\, V \subsetneqq \mathbb{Q}^N : \ker(\phi) = V \otimes \mathbb{R} \end{array} \right\}$$

and $X_N^* = C_N^* / \mathbb{R}_{>0}^{\times}$, together with a projection $\pi \colon C_N^* \to X_N^* \supset X_N$.

For $v \in \mathbb{Z}^N - 0$ define $\widehat{v} \in C_N^*$ by $\widehat{v}(x) = (v|x)^2$. If ϕ is perfect define

$$\sigma(\phi) = \pi\left(\{ \textstyle\sum_i \lambda_i \widehat{v}_i \mid \forall i \ \lambda_i \ge 0 \text{ and } v_i \in M(\phi) \}\right)$$

Theorem 39 (Voronoï [Vor08]). *The family of cells $\sigma(\phi)$ for ϕ perfect and their intersections give a Γ-invariant cell decomposition of X_N^*.*

This can be used to compute $H^*(\Gamma; \mathbb{Z})$. Endow X_N^* with the CW-topology coming from this cell decomposition (warning: this is different from the usual topology on matrices). For $\partial X_N^* = X_N^* - X_N$, consider the equivariant homology of $(X_N^*, \partial X_N^*; \mathbb{Z})$.

There is a first spectral sequences $E_{pq}^2 = H_p(\Gamma, H_q(X_N^*, \partial X_N^*; \mathbb{Z}))$ converging to $H_{p+q}^{\Gamma}(X_N^*, \partial X_N^*; \mathbb{Z})$.

Proposition 40. *The space X_N^* is contractible and ∂X_N^* is homotopy equivalent to the spherical Tits building of $SL_N(\mathbb{Q})$, i.e., has the homotopy type of a bouquet of infinitely many spheres of dimension $N-2$.*

Therefore

$$H_q(X_N^*, \partial X_N^*; \mathbb{Z}) = \widetilde{H}_{q-1}(\partial X_N^*; \mathbb{Z}) = \begin{cases} St_N & \text{if } q = N-1, \\ 0 & \text{otherwise,} \end{cases}$$

where St_N is the Steinberg module, and so

$$H_m^{\Gamma}(X_N^*, \partial X_N^*; \mathbb{Z}) = H_{m-N+1}(\Gamma; St_N).$$

There is a second spectral sequence with $E_{pq}^1 = \bigoplus_{\dim(\sigma)=p} H_q(\Gamma_\sigma; \mathbb{Z}_\sigma)$ also converging to $H_{p+q}^{\Gamma}(X_N^*, \partial X_N^*; \mathbb{Z})$ (here Z_σ is the orientation module of the cell σ).

Lemma 41. *If a prime p divides $\#\Gamma_\sigma$ then $p \le N+1$.*

Proof. If $\gamma^p = 1$ then $\gamma^{p-1} + \gamma^{p-2} + \ldots + 1 = 0$, but since the minimal polynomial divides the characteristic polynomial we get that $p - 1 \le N$. □

Denote by $\mathcal{S}_{N+1}$ the Serre subcategory of finite abelian groups A such that if $p | \# A$ then $p \le N+1$. We will now compute modulo $\mathcal{S}_{N+1}$.

If $q > 0$ then $\#\Gamma_\sigma$ annihilates $H_q(\Gamma_\sigma; \mathbb{Z}_\sigma)$, hence $E_{pq}^1 \equiv 0 \pmod{\mathcal{S}_{N+1}}$.

If Γ_σ acts non-trivially on $\mathbb{Z}_\sigma$ then 2 annihilates $H_0(\Gamma_\sigma;\mathbb{Z}_\sigma)$.

Let V_n be the free abelian group spanned by all Γ-orbits of cells σ of dimension n such that σ meets X_N and Γ_σ preserves the orientation of σ, and $V=(V_*,d^1)$. We get

$$H_n(V)\equiv H_{n-N+1}(\Gamma,St_N)\pmod{\mathcal{S}_{N+1}}.$$

According to Borel-Serre duality and an additional argument of Farrell

$$H_m(\Gamma;St_N)\equiv H^{d-m}(\Gamma;\mathbb{Z})\pmod{\mathcal{S}_{N+1}}$$

where $d=N(N-1)/2$. One gets:

Theorem 42.

(a) $$H^m(SL_2(\mathbb{Z});\mathbb{Z})\equiv H^m(SL_3(\mathbb{Z});\mathbb{Z})\equiv\begin{cases}\mathbb{Z} & \text{if } m=0\\ 0 & \text{otherwise}\end{cases}\pmod{\mathcal{S}_3};$$

(b) $$H^m(SL_4(\mathbb{Z});\mathbb{Z})\equiv\begin{cases}\mathbb{Z} & \text{if } n=0,3\\ 0 & \text{otherwise}\end{cases}\pmod{\mathcal{S}_5};$$

(c) $$H^m(SL_5(\mathbb{Z});\mathbb{Z})\equiv\begin{cases}\mathbb{Z} & \text{if } n=0,5\\ 0 & \text{otherwise}\end{cases}\pmod{\mathcal{S}_5};$$

(d) $$H^m(SL_6(\mathbb{Z});\mathbb{Z})\equiv\begin{cases}\mathbb{Z} & \text{if } m=0,8,9\\ \mathbb{Z}^2 & \text{if } n=5\\ 0 & \text{otherwise}\end{cases}\pmod{\mathcal{S}_7};$$

(e) $$H^m(SL_7(\mathbb{Z});\mathbb{Q})\cong\begin{cases}\mathbb{Q} & \text{if } m=0,5,11,14,15\\ 0 & \text{otherwise}\end{cases}.$$

Here part (b) is due to Lee and Szczarba, and (c)-(d)-(e) to Elbaz-Vincent, Gangl, and Soulé [EVGS02], involving computer calculations.

This result can be used to compute $K_4(\mathbb{Z})=0$. The classification of perfect forms for $N=8$ is also known, but the computation of the cohomology of $SL_8(\mathbb{Z})$ seems too complicated for today's computers.

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