

MOTIVIC WEIGHT COMPLEXES FOR ARITHMETIC VARIETIES

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Date: May 28, 2009.

This research partially supported by NSF grant DMS-0500762.

INTRODUCTION

In this paper we extend the results in the paper [14] to arithmetic varieties. While in *op. cit.*, our main results were framed in terms of Chow motives with integral coefficients, in this paper we shall use K_0 -motives with rational coefficients. This is because we do not have resolution of singularities, but rather must appeal to De Jong's results in [6].

In addition, De Jong's results in [6] lead us to extend the theory of weight complexes to Deligne-Mumford stacks. However this is not a substantial generalization, since the weight complexes of a stack and of its "coarse space" are homotopy equivalent.

In this paper we shall prove the following theorem, where S is a base scheme satisfying the condition (C) below.

Theorem 0.1. *There is a covariant functor $h : \text{Stack}_S \rightarrow \text{Ho}(C_*(K(S)))$ from the category of (separated) Deligne-Mumford stacks of finite type over S , to the category of homotopy classes of maps of bounded complexes of (homological) K_0 -motives over S with rational coefficients, having the following properties:*

- *If X is a regular scheme, projective over S , then $h(X)$ is the usual motive of X .*
- *If X is a regular scheme, projective over S , and G is a finite group acting on X , then $h([X/G]) = h(X)^G$. Here $[X/G]$ is the quotient stack associated to the action.*
- *If $\mathfrak{Y} \subset \mathfrak{X}$ is a closed substack with complement \mathfrak{U} , then we have a triangle*

$$h(\mathfrak{Y}) \rightarrow h(\mathfrak{X}) \rightarrow h(\mathfrak{U}) \rightarrow h(\mathfrak{Y})[+1].$$

An immediate consequence of the theorem is:

Corollary 0.2. *One can associate to any reduced separated Deligne-Mumford stack \mathfrak{X} of finite type over S an element $\chi_c(\mathfrak{X})$ in the Grothendieck group $K_0(\mathbf{KM}_S)$ of the category \mathbf{KM}_S of K_0 -motives over S , with the following properties:*

- (i) *If X is a regular projective scheme over S equipped with an action by a finite group G then $\chi_c([X/G])$ is the class of*

$$\left(X, \frac{1}{\#(G)} \sum_{g \in G} [\mathcal{O}_{\Gamma(g_*)}]\right);$$

here $[X/G]$ is the quotient stack associated to the action of G on X and $\Gamma(g_)$ is the graph of the action $g_* : X \rightarrow X$ of an element $g \in G$.*

- (ii) *If $\mathfrak{Y} \subset \mathfrak{X}$ is a closed substack, with complement $\mathfrak{X} \setminus \mathfrak{Y}$,*

$$\chi_c(\mathfrak{X}) = \chi_c(\mathfrak{Y}) + \chi_c(\mathfrak{X} \setminus \mathfrak{Y}).$$

We work over a fixed base scheme S which is regular, excellent, and finite dimensional. Such an S is sufficient to define the category of K_0 -motives. However, in order to apply the results of de Jong to construct weight complexes, we must additionally assume:

- (C) for every finite morphism $\pi : T \rightarrow S$, and finite group G acting on T over S , the pair (T, G) satisfies 5.12.1 of [6].

It is straightforward to check that examples of such an S are:

- i) $S = \text{Spec}(k)$ with k a field
- ii) $S = \text{Spec}(\Lambda)$ with Λ an excellent Dedekind domain. In particular $S = \text{Spec}(\mathcal{O}_K)$ with \mathcal{O}_K the ring of integers in a number field.

This article is organized as follows.

In the first section, after deriving some basic properties of algebraic stacks, we deduce from [6] two results about resolution of singularities, one for arbitrary Deligne-Mumford stacks over a base scheme S satisfying the condition (C) above, and one about quotient stacks.

In the second paragraph, we discuss simplicial schemes and hypercovers. We introduce the notion of proper hypercover of a stack \mathfrak{X} by a simplicial scheme and derive some of its properties, *e.g.* the fact that is stable by base change by any morphism from a simplicial scheme to \mathfrak{X} (Lemma 2.15). We show that any simplicial variety admits a split proper hypercover (Prop. 2.16) and we also show that these coverings can be compactified (Prop. 2.18).

The next section discusses homological descent, following the method of SGA4 [16], for arbitrary covariant functors from the category of proper morphisms between schemes to the category of connective spectra which satisfy appropriate axioms. This result applies in particular to rational G -theory. The main statement, Theorem 3.4, says that any proper hypercover between simplicial varieties with proper face maps induces a weak equivalence of the associated spectra with rational coefficients. We also show, in Theorem 3.9, that any hyperenvelope between simplicial varieties with proper face maps induces a weak equivalence of the associated spectra with integral coefficients.

The fourth section is a review of the G -theory of stacks and simplicial schemes. In particular we show that G -theory may be extended to a covariant functor on the category of *all* proper, not necessarily representable, morphisms between Deligne-Mumford stacks. The next section contains the main results about weight complexes. After defining correspondences and homological motives over S , we associate a weight complex of motives $\Gamma_*(\alpha.)$ to an arbitrary arrow $\alpha.$ of simplicial projective varieties (Theorem 5.8). We use this result to define weight complexes for arbitrary simplicial varieties over S (Theorem 5.9) and then to prove Theorem 0.1 and Corollary 0.2. We then show how this implies the existence of a virtual Chow motive (with rational coefficients) for every variety over a base field of arbitrary characteristic. Finally using a result of O. Gabber, we show that given a prime ℓ

invertible in S one can define weight complexes for Deligne-Mumford stacks over S with values in the category of homotopy classes of complexes of K_0 -motives over S with $\mathbb{Z}_{(\ell)}$ -coefficients. However, we do not know if this latter construction leads to bounded complexes or not.

The last section is devoted to the proof of contravariance properties of weight complexes. We assume that S is the spectrum of a field. Given two varieties X and Y over S we define a Waldhausen category $\mathbf{C}(X, Y)$ of complexes of sheaves on $X \times Y$ and we prove that it is contravariant in X and covariant in Y (Lemma 6.4). When X is regular and projective the K -theory $\mathbf{KC}(X, Y)$ of $\mathbf{C}(X, Y)$ coincides with the G -theory of $X \times Y$ (Proposition 6.9). We then extend the construction to an arbitrary pair of maps (α, β) of simplicial varieties, getting a spectrum $\mathbf{KC}(\alpha, \beta)$. When the simplicial varieties involved are regular and projective, we define a morphism γ from $\pi_0 \mathbf{KC}(\alpha, \beta)$ to the group of homotopy classes of maps from $\Gamma_*(\alpha)$ to $\Gamma_*(\beta)$. Using this, given two projective varieties X and Y we define a map from $\pi_0 \mathbf{KC}(X, Y)$ to $\text{Hom}(h(X), h(Y))$ which extends earlier constructions, and in particular, we attach to any morphism $f : X \rightarrow Y$ of finite tor-dimension between varieties, a morphism of weight complexes $f^* : h(Y) \rightarrow h(X)$. When f is an open immersion, we show that f^* coincide with the map already defined.

We would like to thank the referee for useful comments on the paper.

Conventions.

- All schemes and stacks will be assumed to be separated.
- By a variety we will simply mean a scheme which is of finite type over our base S . The category of such varieties and proper morphisms between them will be denoted $\mathcal{V}ar_S$.
- A scheme is said to be integral if it is reduced and irreducible.
- A simplicial scheme X . over S will be said to be *proper* over S , if each X_n is proper over S .
- A subsimplicial scheme $U. \subset X.$ will be said to be *strongly open* if its complement $X. \setminus U.$ is a closed sub-simplicial scheme. This is equivalent to requiring that $U. = \text{cosk}^{X.}(U_0)$.

1. STACKS

1.1. Quotient stacks. For definitions and terminology relating to stacks see [23].

All our stacks will be Deligne-Mumford stacks of finite type over S .

If \mathfrak{X} is a stack, we denote its set of points by $|\mathfrak{X}|$, which is a finite dimensional noetherian topological space ([23], Chapter 5). Given $x \in |\mathfrak{X}|$, it has a residue field $\mathbf{k}(x)$ which is an S -field (*op. cit.* 11.2). We write $\mathfrak{X}_{(i)}$ for the set of points $x \in |\mathfrak{X}|$ which are the generic points of i -dimensional subsets.

Following [22], [5] we know that $|\mathfrak{X}|$ is the set of points of an algebraic space, which we shall also denote $|\mathfrak{X}|$ and call the coarse space of \mathfrak{X} .

Proposition 1.1. ([23] thm. 6.1.) *If \mathfrak{X} has a finite étale cover by a variety, then there exists a Galois cover by a variety $\pi : V \rightarrow \mathfrak{X}$. I.e., there is a finite group G acting on V , and $\mathfrak{X} = [V/G]$ is the quotient stack for the action.*

Definition 1.2. We shall refer to stacks of the form $[V/G]$ as *quotient stacks*.

Putting together 1.1 and [23], corollaire 6.6.1, we get:

Proposition 1.3. *Let \mathfrak{X} be a Deligne-Mumford stack. Then there is a non empty open substack $\mathfrak{U} \subset \mathfrak{X}$ which is a quotient stack, and we can choose \mathfrak{U} so that $\mathfrak{U} \simeq [U/G]$ with U quasi-projective over S .*

Lemma 1.4. *Let $\mathfrak{Z} = [Z/G]$ for G a finite group acting on an S -variety V . Then there is a G -equivariant compactification $Z \subset W$ of Z over S , and hence an open immersion $\mathfrak{Z} \rightarrow \mathfrak{W} = [W/G]$ with \mathfrak{W} proper over S .*

Proof. It will be sufficient to find a G -equivariant compactification of Z . Recall the following standard argument. Let $i : Z \rightarrow \bar{Z}$ be any compactification of Z . (This exists by Nagata's theorem.) Consider the G -fold fiber product \bar{Z}_S^G , which is proper over S . There is a natural morphism of schemes $\eta : Z \rightarrow \bar{Z}_S^G$, the “ g -th” component, for $g \in G$, of which is equal to $i \cdot \rho_g : Z \rightarrow \bar{Z}$, where $\rho_g : Z \rightarrow Z$ is the action of g . Observe that G acts on \bar{Z}_S^G by permuting the factors in the product, and, by construction, η is G -equivariant. Hence the Zariski closure W of Z in \bar{Z}_S^G is a G -equivariant compactification of Z . \square

Lemma 1.5. *A map $f : X \rightarrow Y$ of quotient stacks is proper if and only if there are finite étale covers $U \rightarrow X$ and $V \rightarrow Y$ by varieties, and a map $\tilde{f} : U \rightarrow V$ which is proper, such that the diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes.

Proof. Let $V \rightarrow Y$ be any finite étale cover of Y by a variety. Since f being proper is local in the étale topology of Y , f is proper if and only if

$f_V : V \times_Y X \rightarrow V$ is proper. Now let $W \rightarrow X$ be a finite étale cover of X by a variety. Then the induced map $g : U := V \times_Y W \rightarrow V \times_Y X$ is finite, étale, and surjective, and so f_V is proper if and only if the composition $\tilde{f} = f_V \circ g : U \rightarrow V$ is proper. \square

Definition 1.6. Recall that a map $f : X \rightarrow Y$ of schemes is *radicial* if equivalently:

- For every field F , the induced map $X(F) \rightarrow Y(F)$ is injective.
- f is injective as a map of schemes, and for every point $x \in X$, the field extension $\mathbf{k}(f(x)) \subset \mathbf{k}(x)$ is purely inseparable.

Following [23] 3.10, we say that a representable morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ between Deligne-Mumford stacks is radicial if there is an étale cover $p : U \rightarrow \mathfrak{Y}$ with U a scheme, such that $U \times_{\mathfrak{Y}} \mathfrak{X}$ is a scheme, and $U \times_{\mathfrak{Y}} \mathfrak{X} \rightarrow U$ is radicial.

1.2. Resolution of singularities.

Theorem 1.7. *If \mathfrak{X} is a Deligne-Mumford stack of finite type over S , there is a proper surjective morphism $p : X \rightarrow \mathfrak{X}$ with X a regular variety over S .*

Proof. By Chow's lemma ([23] Corollaire 16.6.1) there is a proper surjective morphism $p_0 : X_0 \rightarrow \mathfrak{X}$ with X_0 a projective variety over S . Hence it suffices to show that there is a proper surjective map $\pi : X \rightarrow X_0$ such that X is regular. Let $f : X_0 \rightarrow S$ be the structural map, and $f = q \cdot g$ its Stein factorization, with $g : X_0 \rightarrow T$ and $q : T \rightarrow S$. Then q is finite, and so by the hypothesis (C) on S , T satisfies 5.12.1 of [6]. Furthermore, by Remark 4.3.4 of EGA III [15], g has geometrically connected fibres. Let T' be the disjoint union of the irreducible components of T , and X'_0 the pull back of X_0 over T' . Then $g' : X'_0 \rightarrow T'$ still has geometrically connected fibres, and so the inverse image by g' of each component of T' is an irreducible component of X'_0 . By the standing assumption on S , each component of T' satisfies [6] 5.12.1, and hence by [6] Theorem 5.13 each component of X'_0 admits a non-singular alteration, and we set X equal to the disjoint union of these. \square

We also need resolution of singularities for quotient stacks:

Theorem 1.8. *Let $\mathfrak{X} = [X/G]$ be a reduced quotient stack of finite type over S . Then there is a family of proper morphisms of quotient stacks*

$$p_i : [Y_i/H_i] \rightarrow \mathfrak{X},$$

where each Y_i is regular and integral, and a dense open substack $\mathfrak{U} \subset \mathfrak{X}$ such that, if

$$p : \coprod_i [Y_i/H_i] \rightarrow \mathfrak{X}$$

is the disjoint union of the p_i 's, the induced map

$$p^{-1}(\mathfrak{U}) \rightarrow \mathfrak{U}$$

is representable and radicial.

Proof. If $\mathfrak{X} = \bigcup_i \mathfrak{X}_i$ is the decomposition of \mathfrak{X} into irreducible components, since the map $\coprod_i \mathfrak{X}_i \rightarrow \mathfrak{X}$ is an isomorphism on a dense open subset of both source and target, it suffices to consider the case when \mathfrak{X} is irreducible. Since \mathfrak{X} is irreducible, G acts transitively on the irreducible components of X .

Let X_0 be one such component, and $G_0 \subset G$ its stabilizer. Then since the inclusion $i : X_0 \rightarrow X$ is G_0 -equivariant, there is an induced map

$$f : [X_0/G_0] \rightarrow [X/G]$$

which is clearly surjective (since the pull back of the étale cover $X \rightarrow [X/G]$ by f is isomorphic to the disjoint union of the irreducible components of X). Furthermore, since the map from the disjoint union of the irreducible components of X to X induces an isomorphism between dense open subsets of source and target, the same is true for f .

It suffices, therefore, to show that there is a map $\mathfrak{Y} = [Y/H] \rightarrow [X_0/G_0]$ with the desired properties.

Consider the map $g_0 : X_0 \rightarrow S$. This is G_0 -equivariant, with G_0 acting trivially on S . Let

$$X_0 \xrightarrow{q} T \xrightarrow{\pi} S$$

be its Stein factorization, which is again G_0 -equivariant. Then the morphism q has geometrically irreducible fibres, and (T, G_0) , by the standing assumption on S , satisfies 5.12.1 of [6], hence (*op. cit.*, 5.13) there is a Galois alteration $(Y, H) \rightarrow (X_0, G_0)$ with Y regular. Set $\mathfrak{Y} = [Y/H]$. The field extension $k(\mathfrak{Y})/k(\mathfrak{X})$ is purely inseparable by (*op. cit.*, 5.13). Since Y is integral, we may assume that the kernel of $H \rightarrow G_0$ is the Galois group of this extension. Therefore there is a dense open substack $\mathfrak{U} \subset \mathfrak{X}$ such that the map $p^{-1}(\mathfrak{U}) \rightarrow \mathfrak{U}$ is representable and radicial. This proves our result. \square

2. HYPERCOVERS

The material in this section is based on exposé V^{bis} of [16], as well as [7]. Throughout this section \mathcal{C} will be a category with finite limits and colimits. In particular \mathcal{C} has a final object.

2.1. Simplicial objects and coskeleta. Recall that Δ is the category of finite non-empty totally ordered sets and order preserving maps. For $n \in \mathbb{N}$, we write $[n] := \{0 < \dots < n\}$; these objects (and the morphisms between them) form a skeletal subcategory of Δ . Let $\Delta_{\leq n} \subset \Delta$ be the full subcategory consisting of objects with cardinality at most n .

A simplicial object in \mathcal{C} is a *contravariant* functor $X : \Delta \rightarrow \mathcal{C}$. We write X_n for $X([n])$. The category of simplicial objects in \mathcal{C} will be denoted by $\Delta^{\text{op}}(\mathcal{C})$.

An n -truncated simplicial object is a contravariant functor $X : \Delta_{\leq n} \rightarrow \mathcal{C}$. The corresponding category will be denoted $\Delta_{\leq n}^{\text{op}}(\mathcal{C})$.

There is an obvious restriction functor

$$\text{sk}_n : \Delta^{\text{op}}(\mathcal{C}) \rightarrow \Delta_{\leq n}^{\text{op}}(\mathcal{C}) .$$

This functor has two adjoints. Since \mathcal{C} has finite inverse limits, sk_n has a right adjoint

$$\text{cosk}_n : \Delta_{\leq n}^{\text{op}}(\mathcal{C}) \rightarrow \Delta^{\text{op}}(\mathcal{C}).$$

Note that $\text{sk}_n \circ \text{cosk}_n = \text{Id}$. In addition, sk_n has a left adjoint which we denote ι_n , and which is a fully faithful functor.

We shall write Cosk_n for the composition $\text{cosk}_n \circ \text{sk}_n$, and Sk_n for the composition $\iota_n \circ \text{sk}_n$. Note that $(\text{Sk}_n, \text{Cosk}_n)$ are an adjoint pair.

If Y is a fixed object in $\Delta^{\text{op}}(\mathcal{C})$, we can consider the category $\Delta^{\text{op}}(\mathcal{C})_Y$ of simplicial objects over Y , as well as the category of n -truncated objects $\Delta_{\leq n}^{\text{op}}(\mathcal{C})_Y$ over $\text{sk}_n(Y)$.

There is an obvious restriction functor

$$\text{sk}_n^Y : \Delta^{\text{op}}(\mathcal{C})_Y \rightarrow \Delta_{\leq n}^{\text{op}}(\mathcal{C})_Y$$

which has a right adjoint cosk_n^Y , given by

$$\text{cosk}_n^Y(X) = \text{cosk}_n(X) \times_{\text{Cosk}_n(Y)} Y.$$

We write Cosk_n^Y for the composition $\text{cosk}_n^Y \circ \text{sk}_n^Y$. We also set $\text{Cosk}_{-1}^Y := Y$.

The following is straightforward, so we omit the proof.

Lemma 2.1. *Let $f : X \rightarrow Y$ be a map of simplicial objects in \mathcal{C} .*

1) *If $m \geq n \geq 0$, then the natural map*

$$\text{Cosk}_n^Y(X) \rightarrow \text{Cosk}_m^Y(\text{Cosk}_n^Y(X))$$

is an isomorphism.

2) *If $n \geq p \geq 0$, the natural map*

$$X_p \rightarrow \text{Cosk}_n^Y(X)_p$$

is an isomorphism.

3) *If $n > p \geq 0$, then*

$$\text{Cosk}_n^Y(X)_p \rightarrow \text{Cos}_{n-1}^Y(X)_p$$

is an isomorphism.

If A is a finite set, and $X \in \mathcal{C}$ is an object, we write $X \times A$ for $\sqcup_{a \in A} X$. If A is the simplicial set associated to a finite simplicial complex, and $X \in \Delta^{\text{op}}\mathcal{C}$ we define $X \times A$ by $(X \times A)_k := X_k \times A_k$.

Two maps $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ are said to be homotopic if there is a map

$$h : X \times \Delta[1] \rightarrow Y$$

such that $h|_{X \times \{i\}} = f_i$.

Lemma 2.2. *Let X and Y be two objects of \mathcal{C} . If $f_i : X \rightarrow Y$ for $i = 0, 1$ are two arbitrary maps, then the maps $\text{cosk}(f_0) : \text{cosk}_0(X) \rightarrow \text{cosk}_0(Y)$ and $\text{cosk}(f_1) : \text{cosk}_0(X) \rightarrow \text{cosk}_0(Y)$ are homotopic.*

Proof. We have to construct maps, for $p \geq 0$:

$$(\mathrm{cosk}_0(X)_p = X^{p+1}) \times \mathrm{Hom}_\Delta([p], [1]) \rightarrow \mathrm{cosk}_0(Y)_p = Y^{p+1}$$

which are compatible with the morphisms in Δ . Given $\phi : [p] \rightarrow [1]$, define

$$h_p(\phi) := (h_p(\phi)_0, \dots, h_p(\phi)_p) : X^{p+1} \rightarrow Y^{p+1}$$

by, for $i = 0, \dots, p$,

$$h_p(\phi)_i = f_{\phi(i)} : X \rightarrow Y.$$

It is straightforward to check that this works. \square

More generally, we have the following result, which extends Lemme 3.0.2.4 of exposé V^{bis} of [16]:

Lemma 2.3. *Let S . be a fixed simplicial object in \mathcal{C} . Suppose that $n \geq 0$ and that $f_i : X \rightarrow Y$. for $i = 0, 1$ are two maps in the category $\Delta_S^{\mathrm{op}}(\mathcal{C})$ of simplicial objects over S ., such that $(f_0)_p = (f_1)_p$ for $p < n$. Note that if $n = 0$ this last condition is vacuous. Then $\mathrm{Cosk}_n^{S\cdot}(f_0)$ and $\mathrm{Cosk}_n^{S\cdot}(f_1)$ are homotopic.*

Proof. The two maps $\mathrm{Cosk}_n^{S\cdot}(f_0)$. and $\mathrm{Cosk}_n^{S\cdot}(f_1)$. are homotopic if there is a map

$$h. : \mathrm{Cosk}_n^{S\cdot}(X) \times \Delta[1] \rightarrow \mathrm{Cosk}_n^{S\cdot}(Y).$$

such that $h.|_{\mathrm{Cosk}_n^{S\cdot}(X) \times \{i\}} = f_i$. Since

$$\mathrm{Cosk}_n^{S\cdot}(X) = \mathrm{Cosk}_n(X) \times_{\mathrm{Cosk}_n(S)} S.$$

it is enough to find a homotopy

$$h. : \mathrm{Cosk}_n(X) \times \Delta[1] \rightarrow \mathrm{Cosk}_n(Y).$$

which is a map of simplicial objects over $\mathrm{Cosk}_n(S)$.

To give the map h . is equivalent to giving a map

$$\mathrm{sk}_n(h.) : \mathrm{sk}_n(X) \times \mathrm{sk}_n(\Delta[1]) \rightarrow \mathrm{sk}_n(Y),$$

i.e. to giving maps, for $k \leq n$,

$$h_{k,\phi} : X_k \rightarrow Y_k$$

for $\phi \in \Delta[1]_k = \mathrm{Hom}_\Delta([k], [1])$ compatible with faces and degeneracies, such that $h_{k,\partial_i} = (f_i)_k$ for $\partial_i : [k] \rightarrow [1] = \{0, 1\}$ the constant map with value i .

Since $(f_0)_k = (f_1)_k$ for $k < n$, we can set $h_{k,\phi} = (f_0)_k = (f_1)_k$ for $k < n$. For $k = n$, and $\phi \neq \partial_0, \partial_1$, we can choose $h_{k,\phi} = f_0$ or f_1 arbitrarily. It is now straightforward to check that such a choice defines a homotopy, and that since f_0 and f_1 are maps of simplicial objects over S ., h . is a map of objects over $\mathrm{Cosk}_n(S)$. \square

If \mathcal{C} is a category with finite products, if A is a finite set, and X is an object of \mathcal{C} we define $\mathrm{Hom}(A, X) := X^A$. Notice that this is a functor

$$\mathrm{Finite\ Sets}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathcal{C}.$$

Let $n \geq 0$. Given an n -truncated simplicial finite set A . and n -truncated simplicial object X . in \mathcal{C} , $\text{Hom}(A., X.)$ defines a functor $(\Delta_{\leq n})^{\text{op}} \times (\Delta_{\leq n}^{\text{op}})^{\text{op}} \rightarrow \mathcal{C}$ and we define

$$\text{Hom}_{\Delta_{\leq n}}(A., X.) := \int_{[p] \in \Delta_{\leq n}} \text{Hom}(A_p, X_p) \in \mathcal{C}.$$

Here $\int_{[p] \in \Delta_{\leq n}}$ denotes the end as in [24]. We say that a simplicial set A . is *finite* if all the A_i are finite, and there is an $n \geq 0$ such that $A. = \text{Sk}_n(A.)$. Notice that if K is a finite simplicial complex, the associated simplicial set is finite.

If $A.$ is a finite simplicial set with $A. = \text{Sk}_n A.$, and $X.$ is a simplicial object in \mathcal{C} , we define:

$$\text{Hom}_{\Delta}(A., X.) := \text{Hom}_{\Delta_{\leq n}}(\text{Sk}_n A., \text{Sk}_n X.).$$

Note that if \mathcal{C} had infinite products so that we could define $\text{Hom}_{\Delta}(A., X.)$ directly, then we would have, since $A. = \text{Sk}_n A. = \iota_n \circ \text{sk}_n A.$:

$$\begin{aligned} \text{Hom}_{\Delta}(A., X.) &= \text{Hom}_{\Delta}(\iota_n \cdot \text{sk}_n A., X.) \\ &= \text{Hom}_{\Delta_{\leq n}}(\text{sk}_n A., \text{sk}_n X.). \end{aligned}$$

It is straightforward to check that this definition does not depend on the choice of n , so long as $A. = \text{Sk}_n A.$

Proposition 2.4. *The functor $\text{Hom}_{\Delta} : [(\Delta^{\text{op}} \text{Sets})_{\text{finite}}]^{\text{op}} \times \Delta^{\text{op}} \mathcal{C} \rightarrow \mathcal{C}$ is left exact with respect to the first variable, i.e., suppose that $A : I \rightarrow (\Delta^{\text{op}} \text{Sets})_{\text{finite}}$ is a diagram with I finite, then*

$$\text{Hom}_{\Delta}(\varinjlim_{i \in I} A(i)., X.) \simeq \varinjlim_{i \in I} \text{Hom}_{\Delta}(A(i)., X.).$$

Proof. First observe that

$$A \mapsto X^A$$

is left exact with respect to A , and that direct limits in $(\Delta^{\text{op}} \text{Sets})_{\text{finite}}$ are computed degreewise. Since I is finite, there is an $n \geq 0$ such that

$$\varinjlim_{i \in I} \text{Sk}_n(A(i).) = \text{Sk}_n(\varinjlim_{i \in I} A(i).).$$

Therefore

$$\begin{aligned}
 \mathrm{Hom}_\Delta(\varinjlim_{i \in I} A(i), X) &:= \int_{p \in \Delta_{\leq n}} \mathrm{Hom}(\varinjlim_{i \in I} A(i)_p, X_p) \\
 &= \int_{p \in \Delta_{\leq n}} \mathrm{Hom}_{\Delta_{\leq n}}(\mathrm{sk}_n(\varinjlim_{i \in I} A(i)_p), \mathrm{sk}_n(X_p)) \\
 &= \int_{p \in \Delta_{\leq n}} \varprojlim_{i \in I} (\mathrm{Hom}(A(i)_p, X_p)) \\
 &\simeq \varprojlim_{i \in I} \int_{p \in \Delta_{\leq n}} \mathrm{Hom}(A(i)_p, X_p) \\
 &\quad (\text{by the Fubini theorem, [24], IX.8}) \\
 &= \varprojlim_{i \in I} \mathrm{Hom}_\Delta(A(i), X).
 \end{aligned}$$

□

2.2. Hypercovers and right lifting. Recall that a map $f : X \rightarrow Y$ between simplicial sets is said to be a trivial fibration (II.2.2 of [27]), if it has the right lifting property for all monomorphisms $i : A \rightarrow B$ of simplicial sets. That is, for all commutative squares:

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & X \\
 i \downarrow & & \downarrow f \\
 B & \xrightarrow{\beta} & Y
 \end{array}$$

there is a lifting $\gamma : B \rightarrow X$ such that the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & X \\
 i \downarrow & \nearrow \gamma & \downarrow f \\
 B & \xrightarrow{\beta} & Y
 \end{array}$$

commutes. We can remove the explicit mention of the maps α and β , by defining $\mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A, Y)} \mathrm{Hom}(B, Y)$ to be the pull back in the square

$$\begin{array}{ccc}
 & \mathrm{Hom}(B, A) & \\
 & \downarrow i^* & \\
 \mathrm{Hom}(A, X) & \xrightarrow{f_*} & \mathrm{Hom}(A, Y)
 \end{array}$$

Then the lifting property becomes the assertion that the natural map

$$\mathrm{Hom}(B, X) \xrightarrow{(i^*, f_*)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A, Y)} \mathrm{Hom}(B, Y)$$

is surjective.

Suppose now that $f : X. \rightarrow Y.$ is a morphism in $\Delta^{\text{op}}\mathcal{C}$. If $i : A. \rightarrow B.$ is a monomorphism between finite simplicial sets, we define

$$\text{Hom}_{\Delta}(A., X.) \times_{\text{Hom}_{\Delta}(A., Y.)} \text{Hom}_{\Delta}(B., Y.)$$

to be the pull back in the square

$$\begin{array}{ccc} & & \text{Hom}_{\Delta}(B., A.) \\ & & \downarrow i^* \\ \text{Hom}_{\Delta}(A., X.) & \xrightarrow{f_*} & \text{Hom}_{\Delta}(A., Y.) \end{array}$$

Let \mathcal{P} be a class of morphisms in \mathcal{C} which contains isomorphisms, is closed under composition, and is closed under base change. It is straightforward to check that these two conditions imply that \mathcal{P} is closed under products.

Definition 2.5. We say that a morphism $f : X. \rightarrow Y.$ of simplicial objects in \mathcal{C} has the right \mathcal{P} -lifting property with respect to an inclusion $i : A. \hookrightarrow B.$ between finite simplicial sets, if the map

$$\text{Hom}_{\Delta}(B., X.) \rightarrow \text{Hom}(A., X.) \times_{\text{Hom}(A., Y.)} \text{Hom}(B., Y.)$$

is in \mathcal{P} .

Recall that the morphism $f : X \rightarrow Y$ in $\Delta^{\text{op}}\mathcal{C}$ is called a \mathcal{P} -hypercover, if for all $n \geq 0$, the morphism

$$X_n \rightarrow \text{Cosk}_{n-1}^Y(X.)_n$$

is in \mathcal{P} . By definition of $\text{Cosk}_{n-1}^Y(X.)_n$, this is equivalent to f having, for each $n \geq 1$, the right lifting property with respect to the inclusion

$$\text{sk}_{n-1}(\Delta_n) \subset \Delta_n,$$

and for $n = 0$, this means that $X_0 \rightarrow Y_0$ is in \mathcal{P} , *i.e.* the lifting property with respect to $\emptyset \hookrightarrow \Delta_0$. Applying II.3.8. of [9] and Proposition 2.4, we obtain:

Lemma 2.6. *A map $f : X \rightarrow Y$ in $\Delta^{\text{op}}\mathcal{C}$ is a \mathcal{P} -hypercover if and only if, for all monomorphisms $i : A. \rightarrow B.$ of finite simplicial sets, the natural map*

$$\text{Hom}_{\Delta}(B., X.) \xrightarrow{(i^*, f_*)} \text{Hom}_{\Delta}(A., X.) \times_{\text{Hom}_{\Delta}(A., Y.)} \text{Hom}_{\Delta}(B., Y.)$$

*is in \mathcal{P} , *i.e.* f has the right lifting property with respect to all injective maps between finite simplicial sets.*

From which it immediately follows that:

Corollary 2.7. *If $f : X \rightarrow Y$ in $\Delta^{\text{op}}\mathcal{C}$ is a \mathcal{P} -hypercover, then for all n , $f_n \in \mathcal{P}$.*

Proof. For any given n , this is simply the right lifting property with respect to the inclusion $\phi \hookrightarrow \Delta_n$. \square

Lemma 2.8. *If $f : X \rightarrow Y$ is a \mathcal{P} -hypercover, then the tower of maps:*

$$\begin{aligned} \dots \rightarrow \mathrm{Cosk}_{n+1}^Y(X.) \xrightarrow{f_{n+1}} \mathrm{Cosk}_n^Y(X.) \rightarrow \\ \dots \mathrm{Cosk}_0^Y(X.) \xrightarrow{f_0} \mathrm{Cosk}_{-1}^Y(X.) = Y. \end{aligned}$$

has the following properties for all $n \geq 0$:

1) the natural map

$$X_p \rightarrow \mathrm{Cosk}_n^Y(X.)_p$$

is an isomorphism for $p \leq n$.

2) $(f_n)_p$ is an isomorphism for $p < n$.

3) $(f_n)_n \in \mathcal{P}$.

Proof. 1) and 2) are already in Lemma 2.1 while 3) follows from 1) and the definition of a hypercover. \square

For schemes and stacks of finite type over S , we shall be interested in hypercovers with respect to two possible choices of \mathcal{P} :

- i) \mathcal{P} consists of those maps $f : X \rightarrow Y$ which are *envelopes* i.e. are proper and surjective on F -valued points for all fields F . We shall refer to such hypercovers as *hyperenvelopes*.
- ii) \mathcal{P} consists of all maps $f : X \rightarrow Y$ which are proper and surjective, i.e., surjective on F -valued points for all *algebraically closed* fields. We shall refer to such hypercovers as *proper hypercovers*.

The proof of the following proposition is a straightforward consequence of the definition of right \mathcal{P} -lifting, and the valuative criterion of properness:

Proposition 2.9. *Let $f : X \rightarrow Y$ be a map of simplicial varieties. Then f is a proper hypercover if, for all injections $i : A \hookrightarrow B$ between finite simplicial sets, the following two conditions hold:*

i) *For all algebraically closed fields F , and each commutative diagram:*

$$\begin{array}{ccc} A \times \mathrm{Spec}(F) & \xrightarrow{\alpha} & X \\ \downarrow i & \nearrow \rho & \downarrow f \\ B \times \mathrm{Spec}(F) & \xrightarrow{\beta} & Y \end{array}$$

there is a lifting $\rho : B \times \mathrm{Spec}(F) \rightarrow X$.

ii) *For every valuation ring Λ with fraction field F , and each commutative diagram:*

$$\begin{array}{ccc} A \times \mathrm{Spec}(\Lambda) \cup B \times \mathrm{Spec}(F) & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow \rho & \downarrow \\ B \times \mathrm{Spec}(\Lambda) & \xrightarrow{\beta} & Y \end{array}$$

there is a lifting $\rho : B \times \mathrm{Spec}(\Lambda) \rightarrow X$.

f is a hyperenvelope if in (i), the field F is any field.

2.3. Hypercovers of schemes and stacks. Recall that a stack \mathfrak{X} over S is in particular a category over the category of S -varieties:

$$\alpha : \mathfrak{X} \rightarrow \mathcal{V}ar_S$$

with $\alpha^{-1}(X)$ the groupoid $\text{Hom}(X, \mathfrak{X})$. If $f : X \rightarrow \mathfrak{X}$ and $g : Y \rightarrow \mathfrak{X}$ are two morphisms, then $f \times_{\mathfrak{X}} g$ is the product of f and g in the category \mathfrak{X} . Then $\alpha(f \times_{\mathfrak{X}} g)$ is the scheme parametrizing isomorphisms:

$$\theta : (f \circ p_X = p_X^*(f)) \simeq (g \circ p_Y = p_Y^*(g))$$

where $p_X : X \times_S Y \rightarrow X$ and $p_Y : X \times_S Y \rightarrow Y$ are the projections.

Definition 2.10. A morphism from a simplicial scheme to a stack \mathfrak{X} , $f : X \rightarrow \mathfrak{X}$, consists of a simplicial object f in \mathfrak{X} , such that $\alpha(f \cdot) = X$.

Lemma 2.11. *Given f as above, let $\text{cosk}_0^{\mathfrak{X}}(X_0)$ be the simplicial scheme $n \mapsto X_0 \times_{\mathfrak{X}} \dots \times_{\mathfrak{X}} X_0 = \alpha(f_0 \times_{\mathfrak{X}} \dots \times_{\mathfrak{X}} f_0)$, in which $\text{cosk}_0^{\mathfrak{X}}(X_0)_n$ parametrizes n -tuples*

$$(\theta_{0,1}, \dots, \theta_{n-1,n})$$

with $\theta_{i,i+1} : f_0 \rightarrow f_0$ an isomorphism in the groupoid $\alpha^{-1}(X) = \text{Hom}(X, \mathfrak{X})$. Then giving the full simplicial object f in $\mathfrak{X}^{\Delta^{\text{op}}}$ is equivalent to giving a map of simplicial schemes

$$X \cdot \rightarrow \text{cosk}_0^{\mathfrak{X}}(X_0).$$

Proof. A map

$$X_n \rightarrow \text{cosk}_0^{\mathfrak{X}}(X_0)_n$$

is equivalent to giving a map

$$\varphi_n : X_n \rightarrow X_0 \times_S \dots \times_S X_0 = \text{cosk}_0^S(X_0)_n$$

and isomorphisms, for $i = 1 \dots n$

$$\theta_{i-1,i} : (p_{i-1} \cdot \varphi_n)^*(f_0) \rightarrow (p_i \cdot \varphi_n)^*(f_0).$$

This is equivalent, up to canonical isomorphism, to giving an object $f_n \in \mathfrak{X}$ with $\alpha(f_n) = X_n$ and isomorphisms, for $i = 0, \dots, n$:

$$\theta_i : f_n \rightarrow (p_i \varphi_n)^*(f_0).$$

Given f_n , and the θ_i , set $\theta_{i-1,i} = \theta_i \cdot \theta_{i-1}^{-1}$; notice that there is then a canonical isomorphism $f_n \simeq (p_0 \varphi_n)^*(f_0)$ while given the $\theta_{i-1,i}$, we can set f_n equal to $(p_0 \varphi_n)^*(f_0)$. \square

Definition 2.12. We say that $f : X \cdot \rightarrow \mathfrak{X}$ is a proper hypercover if

- 0) $f_0 : X_0 \rightarrow \mathfrak{X}$ is proper and surjective.
- 1) For all $n \geq 1$, the map

$$\alpha(f_n) = X_n \rightarrow \alpha(\text{cosk}_{n-1}(f \cdot)_n)$$

is proper and surjective (i.e. the natural map of simplicial schemes $X \cdot \rightarrow \text{cosk}_0^{\mathfrak{X}}(X_0)$ is a proper hypercover).

Given any stack \mathfrak{X} there exists a proper hypercover $X. \rightarrow \mathfrak{X}$ as above. Indeed, Chow's lemma implies the existence of $f_0 : X_0 \rightarrow \mathfrak{X}$ proper and surjective. Then choose $X. = \text{cosk}_0^{\mathfrak{X}}(X_0)$.

Lemma 2.13. *If \mathfrak{X} is a stack and $f. : X. \rightarrow \mathfrak{X}$ is a morphism from a simplicial scheme to \mathfrak{X} , then for all morphisms $g : T \rightarrow \mathfrak{X}$ with T a scheme, $f.$ induces a map $f_T. : X. \times_{\mathfrak{X}} T \rightarrow T$ which is a proper hypercover if $f.$ is. Furthermore, if $g : T \rightarrow \mathfrak{X}$ is étale and surjective, then f is a proper hypercover if and only if $f_T.$ is.*

Proof. First we need a technical lemma about fibre products.

Lemma 2.14. *Let $f : X \rightarrow \mathfrak{X}$, $g : Y \rightarrow \mathfrak{X}$, $h : Z \rightarrow \mathfrak{X}$ be three morphisms from schemes to a stack \mathfrak{X} . Suppose we are given a map of schemes $p : X \rightarrow Y$, and a map $\tilde{p} : f \rightarrow g$ in \mathfrak{X} covering p . Then the commutative square*

$$\begin{array}{ccc} \alpha(f \times h) = X \times_{\mathfrak{X}} Z & \longrightarrow & Y \times_{\mathfrak{X}} Z = \alpha(g \times h) \\ \downarrow & & \downarrow \\ \alpha(f = \tilde{p} \cdot g) = X & \xrightarrow{p} & Y = \alpha(g) \end{array}$$

is cartesian. I.e. there is a canonical isomorphism

$$X \times_{\mathfrak{X}} Z \xrightarrow{\sim} X \times_Y Y \times_{\mathfrak{X}} Z.$$

Proof. Straightforward using the explicit description of fiber products in [23], 2.2.2. \square

Suppose now that $f.$ is a proper hypercover. Then, for $n = 0$, $f_0 : X_0 \rightarrow \mathfrak{X}$ is proper and surjective. Therefore $f_0 \times_{\mathfrak{X}} 1_T : X_0 \times_{\mathfrak{X}} T \rightarrow T$ is proper and surjective, by the definition of properness ([23], Definition (7.11)).

For $n \geq 1$, we know that

$$\alpha(f_n) = X_n \rightarrow \alpha(\text{Cosk}_{n-1}(f._)_n)$$

is proper and surjective. Now since Cosk_{n-1} commutes with products (since it is a right adjoint),

$$\text{cosk}_{n-1}(f. \times g)_n = \text{cosk}_{n-1}(f._)_n \times g$$

and so

$$\alpha(f_n \times g) \rightarrow \alpha(\text{cosk}_{n-1}(f. \times g)_n)$$

is equal to

$$X_n \times_{\mathfrak{X}} T \rightarrow \alpha(\text{Cosk}_{n-1}(f._)_n \times g) \simeq \alpha(\text{Cosk}_{n-1}(f._)) \times_{\mathfrak{X}} T$$

which is again proper and surjective.

Suppose now that $g : T \rightarrow \mathfrak{X}$ is étale and surjective, and that $X. \times_{\mathfrak{X}} T \rightarrow T$ is a proper hypercover. Then

$$X_0 \times_{\mathfrak{X}} T \rightarrow T$$

is proper and surjective, and hence by [23] Remark (7.11.1) $X_0 \rightarrow \mathfrak{X}$ is proper and surjective. Similarly, for all $n \geq 1$,

$$X_n \rightarrow \alpha(\mathrm{Cosk}_{n-1}(f)_n)$$

is proper and surjective. This proves Lemma 2.13. \square

Lemma 2.15. *If \mathfrak{X} is a stack, and $f : X \rightarrow \mathfrak{X}$ is a proper hypercover of \mathfrak{X} by a simplicial variety, then for all morphisms $g : Y \rightarrow \mathfrak{X}$ with Y a simplicial variety, the induced map $f_Y : Y \times_{\mathfrak{X}} X \rightarrow Y$ is a proper hypercover.*

Proof. We must show that for all $n \geq 0$ the natural map

$$(*) \quad Y_n \times_{\mathfrak{X}} X_n \rightarrow \mathrm{Cosk}_{n-1}^Y(Y \times_{\mathfrak{X}} X)_n$$

is proper and surjective. It is straightforward to verify, knowing the adjunction between Sk_{n-1} and Cosk_{n-1} , that there is a canonical isomorphism of schemes over Y_n :

$$\mathrm{Cosk}_{n-1}^Y(Y \times_{\mathfrak{X}} X)_n \rightarrow Y_n \times_{\mathfrak{X}} \mathrm{Cosk}_{n-1}^{\mathfrak{X}}(X)_n$$

and that the composition of these two maps is the base change, by $Y_n \rightarrow \mathfrak{X}$, of the proper surjective map $X_n \rightarrow \mathrm{Cosk}_{n-1}^{\mathfrak{X}}(X)_n$. But by Lemma 2.14,

$$Y_n \times_{\mathfrak{X}} X_n \simeq (Y_n \times_{\mathfrak{X}} \mathrm{Cosk}_{n-1}^{\mathfrak{X}}(X)_n) \times_{\mathrm{Cosk}_{n-1}^{\mathfrak{X}}(X)_n} X_n$$

and so the map (*) is proper and surjective by base change with respect to the map

$$Y_n \times_{\mathfrak{X}} \mathrm{cosk}_{n-1}^{\mathfrak{X}}(X)_n \rightarrow \mathrm{cosk}_{n-1}^{\mathfrak{X}}(X)_n.$$

\square

2.4. Compactification of simplicial varieties. Recall that for $n \geq 0$, an n -truncated simplicial variety X is said to be *split* if for all $k \leq n$, the complement in X_k of the image of the degeneracies:

$$NX_k = X_k \setminus \bigcup_{s_i : X_{n-1} \rightarrow X_n} s_i(X_{n-1})$$

is both open and closed in X_n . It then follows that for $0 \leq k \leq n$:

$$X_k = \prod_{\alpha : [k] \rightarrow [\ell]} X(\alpha)(NX_\ell).$$

If X is a simplicial variety, we say that it is *n-split* if $\mathrm{sk}_n(X)$ is split, and *split* if it is n -split for all $n \geq 0$.

Proposition 2.16. *Given a simplicial variety X with proper face maps, there is a proper hypercover $f : Y \rightarrow X$ with Y split.*

Proof. We shall construct, by induction on $n \geq 0$, simplicial varieties $Y(n)$ (with proper face maps), together with maps

$$f_{n+1} : Y(n+1) \rightarrow Y(n).$$

with the following properties:

- 1) $Y(n)$. is n -split for all n
- 2) $Y(0)$. = X . (note that *any* simplicial variety is 0-split)
- 3) For all $n \geq 1$, f_n is a proper hypercover
- 4) For all $n \geq 1$, $\text{sk}_{n-1}(f_n)$ is an isomorphism
- 5) The natural map

$$Y(n) \rightarrow \text{Cosk}_n^{X.}(Y(n)).$$

is an isomorphism. (Notice that this is trivially true for $n = 0$.)

Having constructed such a tower of maps, set (for $n \geq 1$) $\varphi_n : Y(n) \rightarrow Y(0) = X$. equal to $f_1 \circ \dots \circ f_n$, and then define $Y = \varinjlim_n Y(n)$, and $\varphi : Y \rightarrow X$. equal to $\varinjlim_n \varphi_n$. Notice that the natural map $Y \rightarrow Y(n)$ induces an isomorphism $\text{sk}_n(Y) \rightarrow \text{sk}_n(Y(n))$, and hence $\varphi : Y \rightarrow X$. is a proper hypercover with Y . split. The induction has already been started by setting $Y(0)$. equal to X .

Now suppose that $n \geq 1$ and that $Y(k)$. and f_k have been defined for $k < n$ (if $n = 1$, then we only need the existence of $Y(0)$. = X .), satisfying conditions 1)–5) above. We start by setting

$$\text{sk}_{n-1} Y(n) = \text{sk}_{n-1} Y(n-1).$$

Following [16] 5.1.3 we know that, given a variety N and a map

$$\beta : N \rightarrow \text{Cosk}_{n-1}(\text{Sk}_{n-1}(Y(n)))_n$$

there is, up to isomorphism, a unique split n -truncated variety V . with $NV_n = N$, $\text{sk}_{n-1}(V) = \text{sk}_{n-1}(Y(n)) = \text{sk}_{n-1}(Y(n-1))$, and β the restriction of the natural map $V_n \rightarrow \text{cosk}_{n-1}(\text{sk}_{n-1}(V))$ to NV_n .

Thus constructing the split n -truncated simplicial object $\text{sk}_n(Y(n))$ is equivalent to giving a variety V_n , and a map $V_n \rightarrow \text{Cosk}_{n-1}(\text{Sk}_{n-1} Y(n-1))_n$. By [7] Proposition 6.2.4, to give a map

$$\text{sk}_n(f_n) : \text{sk}_n(Y(n)) \rightarrow \text{sk}_n(Y(n-1))$$

is the same as giving morphisms:

$$\text{sk}_{n-1}(f_n) : \text{sk}_{n-1}(Y(n)) \rightarrow \text{sk}_{n-1}(Y(n-1))$$

and

$$\tilde{f} : N \rightarrow Y(n-1)_n = \text{Cosk}_{n-1}^{X.}(Y(n-1))_n$$

such that the diagram:

$$\begin{array}{ccc} N & \xrightarrow{\beta} & \text{Cosk}_{n-1}(Y(n))_n \\ \downarrow \tilde{f} & & \parallel \\ \text{Cosk}_{n-1}^{X.}(Y(n-1))_n & \xrightarrow{\pi} & \text{Cosk}_{n-1}(Y(n-1))_n \\ \parallel & & \\ \text{Cosk}_{n-1}(Y(n-1))_n \times_{\text{Cosk}_{n-1}(X)_n} X_n & & \end{array}$$

commutes. Here π is the projection onto the first factor in the fibre product. Hence the map \tilde{f} determines β , the split object $\mathrm{sk}_n(Y(n))$, and the map $\mathrm{sk}_n(f_n)$.

Let us choose N and \tilde{f} so that \tilde{f} is proper and surjective; for example, take \tilde{f} to be the identity. Having defined $\mathrm{sk}_n(Y(n))$ we now set $Y(n) = \mathrm{cosk}_n^X(\mathrm{sk}_n(Y(n)))$.

To define f_n , observe that, by 2.1,

$$Y(n-1) = \mathrm{Cosk}_{n-1}^X(Y(n-1)) = \mathrm{Cosk}_n^X(Y(n-1)).$$

and so we set $f_n = \mathrm{cosk}_n^X(\mathrm{sk}_n(f_n))$.

It remains to show that f_n is a proper hypercover. Since $(f_n)_p$ is an isomorphism for $p < n$, we know that

$$Y(n)_p \rightarrow \mathrm{Cosk}_{p-1}^{Y(n-1)}(Y(n))_p$$

is an isomorphism, hence proper and surjective for $p < n$. Now suppose $p \geq n$.

First observe that

$$\begin{aligned} \mathrm{Cosk}_{n-1}^{Y(n-1)}(Y(n))_n &= \\ \mathrm{Cosk}_{n-1}(Y(n))_n \times_{\mathrm{Cosk}_{n-1}(Y(n-1))_n} Y(n-1)_n &= Y(n-1)_n \end{aligned}$$

since $\mathrm{sk}_{n-1}Y(n) \xrightarrow{\sim} \mathrm{sk}_{n-1}Y(n-1)$, and so

$$\mathrm{Cosk}_{n-1}(Y(n)) \simeq \mathrm{Cosk}_{n-1}(Y(n-1)).$$

However, $f(n)_n : Y(n)_n \rightarrow Y(n-1)_n$ is proper and surjective, since β is proper and surjective, and the face (and hence also the degeneracy) maps of $Y(n-1)$ are proper by the induction hypothesis. Hence $\mathrm{sk}_n Y(n) \rightarrow \mathrm{sk}_n(Y(n-1))$ is a proper hypercover of n -truncated varieties (2.14). We then conclude with the following lemma:

Lemma 2.17. *Let Y and Z be n -truncated simplicial varieties over the simplicial variety X , and suppose we are given a proper hypercover $f : Y \rightarrow Z$ (over X). Then*

$$\mathrm{cosk}_n^X(Y) \rightarrow \mathrm{cosk}_n^X(Z)$$

is a proper hypercover.

Proof. We have to prove the lifting property for all inclusions of finite simplicial sets $i : A \hookrightarrow B$. Suppose then that we are given a commutative diagram, with $S = \mathrm{Spec}(F)$ for F an algebraically closed field:

$$(*) \quad \begin{array}{ccc} A \times S & \longrightarrow & \mathrm{Cosk}_n^X(Y) \\ \downarrow & \nearrow \tilde{\theta} & \downarrow \\ B \times S & \longrightarrow & \mathrm{Cosk}_n^X(Z) \end{array}$$

This is then equivalent to giving a commutative diagram

$$\begin{array}{ccc}
 \mathrm{sk}_n(A) \times S & \longrightarrow & Y. \\
 \downarrow & \nearrow \theta & \downarrow \\
 \mathrm{sk}_n(B) \times S & \longrightarrow & Z. \\
 \downarrow & & \downarrow \\
 B \times S & \longrightarrow & X.
 \end{array}$$

and hence a lifting θ exists, since $Y. \rightarrow Z.$ is a proper hypercover, and θ defines a lifting $\tilde{\theta}$ in diagram (*). To see that, for every valuation ring Λ , with fraction field F , a lifting ψ exists in every diagram, with $i : A \hookrightarrow B$ as above,

$$\begin{array}{ccc}
 A \times \mathrm{Spec}(\Lambda) \cup B \times \mathrm{Spec}(F) & \longrightarrow & \mathrm{Cosk}_n^X(Y.) \\
 \downarrow & \nearrow \tilde{\psi} & \downarrow \\
 B \times \mathrm{Spec}(\Lambda) & \longrightarrow & \mathrm{Cosk}_n^X(Z.)
 \end{array}$$

it is enough to produce a lifting ψ in the diagram:

$$\begin{array}{ccc}
 \mathrm{sk}_n(A) \times \mathrm{Spec}(\Lambda) \cup \mathrm{sk}_n B \times \mathrm{Spec}(F) & \longrightarrow & Y. \\
 \downarrow & \nearrow \psi & \downarrow \\
 \mathrm{sk}_n(B) \times \mathrm{Spec}(\Lambda) & \longrightarrow & Z. \\
 \downarrow & & \downarrow \\
 B \times \mathrm{Spec}(\Lambda) & \longrightarrow & X.
 \end{array}$$

which again exists since $Y. \rightarrow Z.$ is a proper hypercover. \square

Proposition 2.18. *Let $X.$ be a split simplicial variety with proper face maps. Then there is a map of simplicial varieties $j. : X. \rightarrow \bar{X}.$ such that:*

- 1) *For all $k \geq 0$, $j_k : X_k \hookrightarrow \bar{X}_k$ is an open immersion with \bar{X}_k proper over S*
- 2) *$\bar{X}.$ is split.*

Proof. We construct $\mathrm{sk}_n(\bar{X}.)$ by induction on $n \geq 0$. For $n = 0$, we set \bar{X}_0 equal to a ‘‘compactification’’ of X_0 over S , which exists by Nagata’s theorem.

Suppose now that $n \geq 1$, and that we have constructed $\mathrm{sk}_{n-1}(\bar{X}.)$, together with a map

$$\mathrm{sk}_{n-1}(j.) : \mathrm{sk}_{n-1}(X.) \rightarrow \mathrm{sk}_{n-1}(\bar{X}.)$$

satisfying 1) above.

Giving $\mathrm{sk}_n(\bar{X}.)$ is the same as giving $N\bar{X}_n$ and a map $\bar{\beta}_n : N\bar{X}_n \rightarrow \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(\bar{X}.)_n)$. Since $X.$ is split, we have maps

$$\beta_n : NX_n \rightarrow \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(X.)_n)$$

and

$$\mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(j.)) : \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(X.)) \rightarrow \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(\bar{X}.)).$$

By Nagata's theorem, there exists a factorization of $\mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(j.)) \circ \beta_n$:

$$\begin{array}{ccc} NX_n & \xrightarrow{i_n} & \overline{NX_n} \\ \downarrow \beta_n & & \downarrow \bar{\beta}_n \\ \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(X.)) & \xrightarrow{\mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(j.))} & \mathrm{cosk}_{n-1}(\mathrm{sk}_{n-1}(\bar{X}.) \end{array}$$

with $\bar{\beta}_n$ proper, and i_n an open immersion.

By [16] there is then a unique split n -truncated simplicial variety $\mathrm{sk}_n(\bar{X}.)$ with $\mathrm{sk}_{n-1}(\mathrm{sk}_n(\bar{X}.)_n) = \mathrm{sk}_{n-1}(\bar{X}.)$ and $N\bar{X}_n$ equal to $\overline{NX_n}$. Then, following [7], the composition of i_n with the inclusion $N\bar{X}_n \hookrightarrow \bar{X}_n$, induces a map

$$\mathrm{sk}_n(j) : \mathrm{sk}_n(X.) \rightarrow \mathrm{sk}_n(\bar{X}.)$$

with j_n an open immersion, since it is a disjoint union of open immersions. This completes the induction step. \square

3. HOMOLOGICAL DESCENT

In the following, E will be a covariant functor from the category of proper morphisms between schemes to the category of connective spectra. This includes the case of functors to the category of chain complexes of abelian groups concentrated in degrees ≥ 0 .

We assume in addition that E is contravariant with respect to open immersions. We suppose that the two types of functoriality are related as follows:

1. If $j : U \subset X$ is an open subset with complement $Y = X \setminus U$, then:

$$E(Y) \rightarrow E(X) \rightarrow E(U)$$

is a (co-)fibration sequence.

2. If $j : U \subset X$ is an open subset, and $f : Z \rightarrow X$ is proper, the diagram:

$$\begin{array}{ccc} E(Z) & \xrightarrow{(j|_Z)^*} & E(f^{-1}(U)) \\ f_* \downarrow & & \downarrow (f|_U)_* \\ E(X) & \xrightarrow{j^*} & E(U) \end{array}$$

commutes.

Properties 1. and 2. imply that if X and Y are schemes then the inclusion maps of X and Y into the disjoint union $X \sqcup Y$ induce a weak equivalence

$$E(X) \vee E(Y) \rightarrow E(X \cup Y).$$

We also assume that E satisfies devissage:

3. For all X , the natural map $j : X^{\text{red}} \rightarrow X$ from the reduced structure on X to X induces a weak equivalence on E .

In this section we will prove that a functor E having properties (1.) to (3.) above satisfies descent with respect to hyperenvelopes.

If in addition E is contravariant with respect to finite flat morphisms, and satisfies properties (4.) and (5.) below, we shall show that E , with rational coefficients, satisfies descent with respect to all proper hypercovers.

4. The diagram analogous to that in item (2.) above, in which $U \rightarrow X$ is finite and flat, is also strictly commutative.
5. If $p : X \rightarrow Y$ is finite and flat, with $p_*(\mathcal{O}_X)$ a free \mathcal{O}_Y -module of rank n , then

$$p_* p^* : E_*(Y) \rightarrow E_*(Y)$$

is multiplication by n .

The K -theory of coherent sheaves (G -theory), or the homology theory corresponding to a cycle complex in the sense of Rost, [29], and in particular the homology theory associated to the Gersten complexes [11], all satisfy properties (1.) to (5.).

In the paper [12], descent for K' -theory with respect to hyperenvelopes was proved first for the homology of the Gersten complexes and then using a spectral sequence argument, for K' -theory. Here we shall prove descent for the map on homology with rational coefficients induced by an arbitrary proper hypercover of simplicial schemes, following the method of SGA 4 [16].

We start by extending E from varieties to simplicial varieties:

Definition 3.1. If $X.$ is a simplicial variety with proper face maps, and E is any covariant functor from the category of proper morphisms between schemes to the category of connective spectra, then we define:

$$E(X.) := \text{hocolim}_{i \in \Delta^{\text{op}}} E(X_i).$$

This is clearly covariant functorial with respect to proper morphisms between simplicial schemes with proper face maps.

Lemma 3.2. *Let $X.$ be a simplicial scheme with proper face maps, and suppose that E is any covariant functor from the category of proper morphisms between schemes to the category of connective spectra. Then there is a (convergent) first quadrant homological spectral sequence:*

$$E_{p,q}^1 = E_p(X_q) \Rightarrow E_{p+q}(X.).$$

Proof. This is the standard spectral sequence for hocolim [4]. □

Lemma 3.3. *Suppose that E is a covariant functor from the category of proper morphisms between schemes to the category of connective spectra which takes disjoint unions of schemes to direct sums of spectra (in particular if E satisfies properties (1.) and (2.)). Then if $f, g : X. \rightarrow Y.$ are homotopic maps in the category of proper morphisms between simplicial schemes with proper face maps, $E(f), E(g) : E(X.) \rightarrow E(Y.)$ are equal in the homotopy category of spectra.*

Proof. There is a map $h : X. \times \Delta^1 \rightarrow Y.$ such that $h|_{X. \times 0} = f$ and $h|_{X. \times 1} = g$. It suffices to observe that the inclusions $X. \times \{i\} \rightarrow X. \times \Delta^1$, for $i = 0, 1$, induce weak equivalences of E -theory spectra inverse to the weak equivalence induced by the projection $X. \times \Delta^1 \rightarrow X.$. By Lemma 3.2, it is enough to check that these maps induce isomorphisms on the chain complexes obtained by applying the functor E_q for all q . But E_q takes disjoint unions of schemes to direct sums of abelian groups, and for any functor H from schemes to abelian groups with this property, the maps of simplicial abelian groups (for $i = 0, 1$):

$$H(X. \times \{i\}) \rightarrow H(X. \times \Delta^1) \simeq H(X.) \otimes \mathbf{Z}(\Delta^1) \rightarrow H(X.)$$

are weak equivalences. \square

Theorem 3.4. *Suppose that E is a covariant functor from the category of proper morphisms between schemes to the category of connective spectra satisfying properties (1.) to (5.). Let $f. : X. \rightarrow Y.$ be a proper hypercover between simplicial S -varieties, each having proper face maps. Then $E(f.)_{\mathbb{Q}} : E(X.)_{\mathbb{Q}} \rightarrow E(Y.)_{\mathbb{Q}}$ is a weak equivalence.*

There are several steps to the proof.

Lemma 3.5. *Let $p : X \rightarrow Y$ be a proper morphism that admits a section $s : Y \rightarrow X$. Then for any covariant functor E from the category of proper morphisms between schemes to the category of connective spectra which takes a disjoint union of schemes to a direct sum of spectra, in particular for any E which satisfies properties (1.) and (2.) above, we have that:*

$$\text{cosk}_0^Y(p) : E(\text{cosk}_0^Y(X).) \rightarrow E(Y)$$

is a weak equivalence.

Proof. Let $Y.$ be the constant simplicial object $[n] \mapsto Y$. Then s induces a section $\tilde{s} : Y. \rightarrow \text{cosk}_0^Y(X).$ of the projection $\text{cosk}_0^Y(p)$. On the other hand, by Lemma 2.2, $\tilde{s} \circ \text{cosk}_0^Y(p) = \text{cosk}_0^Y(s \circ p)$ is homotopic to the identity and so we are done. (Note that here we do not need to assume that E has rational coefficients.) \square

Lemma 3.6. *Let $p : X \rightarrow Y$ be a finite flat morphism, such that $p_*(\mathcal{O}_X) \simeq \mathcal{O}_Y^d$ for some $d \geq 0$. (Note that if p is finite and flat this will be true on a Zariski dense open subset of Y .) Then*

$$E(\text{cosk}_0^Y(X).)_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$$

is a weak equivalence.

Proof. Following Lemma 3.2, it is enough to prove that, for all $q \geq 0$,

$$H_p(i \mapsto E_q(\mathrm{cosk}_i^Y(X))_{\mathbb{Q}}) \simeq \begin{cases} 0, & i > 0; \\ E_q(Y)_{\mathbb{Q}}, & i = 0. \end{cases}$$

Consider, therefore, the corresponding augmented chain complex C_* concentrated in degrees ≥ -1 , with $C_n = E_q(\mathrm{cosk}^Y(X)_n)_{\mathbb{Q}}$ for $n \geq 0$, $C_{-1} = E_q(Y)_{\mathbb{Q}}$, and differentials $\delta_n = \sum_{i=0}^n (-1)^i (d_i)_*$ for $n \geq 0$, where we set $d_0 = p : X = \mathrm{cosk}^Y(X)_0 \rightarrow \mathrm{cosk}^Y(X)_{-1} = Y$ when $n = 0$.

For all $n \geq 0$, and all $i \leq n$, one may easily check that the following square is cartesian, with all maps finite and flat :

$$\begin{array}{ccc} \mathrm{cosk}^Y(X)_{n+1} & \xrightarrow{d_{n+1}} & \mathrm{cosk}^Y(X)_n \\ d_i \downarrow & & \downarrow d_i \\ \mathrm{cosk}^Y(X)_n & \xrightarrow{d_n} & \mathrm{cosk}^Y(X)_{n-1} \end{array}$$

and hence that

$$(d_n)^*(d_i)_* = (d_i)_*(d_{n+1})^* : E_q(\mathrm{cosk}^Y(X)_n) \rightarrow E_q(\mathrm{cosk}^Y(X)_n).$$

It is then straightforward to check that the sequence of maps, for $n \geq -1$:

$$h_n = \frac{(-1)^{(n+1)}}{d} (d_{n+1})^* : E_q(\mathrm{cosk}^Y(X)_n)_{\mathbb{Q}} \rightarrow E_q(\mathrm{cosk}^Y(X)_{n+1})_{\mathbb{Q}}$$

is a contracting homotopy for the complex C_* , given that for all $n \geq -1$, $(d_{n+1})_*(d_{n+1})^* : E_q(\mathrm{cosk}^Y(X)_n) \rightarrow E_q(\mathrm{cosk}^Y(X)_n)$ is (by the projection formula) multiplication by d . \square

Proposition 3.7. *Let $p : X \rightarrow Y$ be a proper surjective morphism. Then*

$$E(\mathrm{cosk}^Y(X)_{\cdot})_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$$

is a weak equivalence.

Proof. By devissage we may assume that Y is reduced. We proceed by noetherian induction on the closed subsets of Y . The assertion is trivially true if Y is empty. Now suppose that Y is non-empty, and let $\eta \in Y$ be a generic point of a component of Y . Then there exists a point $\xi \in X_{\eta}$ (the fiber over the point η) such that the residue field $\mathbf{k}(\xi)$ is a finite extension of $\mathbf{k}(\eta)$. Taking the Zariski closure $\bar{\xi}$ of ξ in X , we obtain a subscheme of X which is finite over an open neighborhood of η . By generic flatness, there is then an open neighborhood U of η , the inverse image V in $\bar{\xi}$ of which is finite and flat over U . Additionally, we may assume that $p_*(\mathcal{O}_V) \simeq \mathcal{O}_U^d$.

By the induction hypothesis the assertion of the proposition is true for $p_Z : Z \times_Y X \rightarrow Z$, where $Z = Y \setminus U$.

Now consider the diagram:

$$\begin{array}{ccccc} E(\mathrm{cosk}^Z(p^{-1}(Z)).)_{\mathbb{Q}} & \longrightarrow & E(\mathrm{cosk}^Y(X).)_{\mathbb{Q}} & \longrightarrow & E(\mathrm{Cosk}^{p^{-1}(U)}(U).)_{\mathbb{Q}} \\ f \downarrow & & g \downarrow & & h \downarrow \\ E(Z)_{\mathbb{Q}} & \longrightarrow & E(X)_{\mathbb{Q}} & \longrightarrow & E(U)_{\mathbb{Q}}. \end{array}$$

The map f is a weak equivalence by the induction hypothesis. Next, observe that:

$$E(Z)_{\mathbb{Q}} \rightarrow E(X)_{\mathbb{Q}} \rightarrow E(U)_{\mathbb{Q}}$$

is a fibration sequence by localization, and similarly, for all $n \geq 0$,

$$\mathrm{cosk}^Z(p^{-1}(Z))_n = \mathrm{cosk}^Y(X)_n \setminus \mathrm{cosk}^{p^{-1}(U)}(U)_n$$

and so, again by localization, we have a fibration sequence of connective spectra

$$E(\mathrm{cosk}^Z(p^{-1}(Z))_n) \rightarrow E(\mathrm{cosk}^Y(X)_n) \rightarrow E(\mathrm{cosk}^{p^{-1}(U)}(U)_n).$$

Hence, since by [4], hocolim preserves fibration sequences of connective spectra, we have a fibration sequence:

$$E(\mathrm{cosk}^{p^{-1}(Z)}(Z).)_{\mathbb{Q}} \rightarrow E(\mathrm{cosk}^Y(X).)_{\mathbb{Q}} \rightarrow E(\mathrm{cosk}^{p^{-1}(U)}(U).)_{\mathbb{Q}}.$$

Hence, in order to prove that $E(\mathrm{cosk}^Y(X).)_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$ is a weak equivalence, it remains to show (writing Y for U and X for $p^{-1}(U)$) that if $p : X \rightarrow Y$ is a proper surjective morphism such that there is a finite flat map $\pi : V \rightarrow Y$, and a section $s : V \rightarrow X$, *i.e.* such that $p \circ s = \pi$, then $p_* : E(\mathrm{cosk}^Y \cdot (X))_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$ is a weak equivalence.

Consider the bisimplicial scheme $\mathrm{cosk}^Y(V) \times_Y \mathrm{cosk}^Y(X)$. For each $n \geq 0$, $E(\mathrm{cosk}^Y(V) \times_Y \mathrm{cosk}^Y(X)_n)_{\mathbb{Q}} \rightarrow E(\mathrm{cosk}^Y(X)_n)_{\mathbb{Q}}$ is a weak equivalence by Lemma 3.5. Hence (by the homotopy colimit theorem [4]) $E(\mathrm{cosk}^Y(V) \times_Y \mathrm{cosk}^Y(X).)_{\mathbb{Q}} \rightarrow E(\mathrm{cosk}^Y(X).)_{\mathbb{Q}}$ is too.

On the other hand, for each $m \geq 0$,

$$\mathrm{cosk}_m^Y(V) \times_Y \mathrm{cosk}^Y(X) \simeq \mathrm{cosk}^{\mathrm{cosk}_m^Y(V)}(X \times_Y \mathrm{cosk}^Y(V)_m).$$

and since there is a section $V \rightarrow X$ over Y , there is a section $\mathrm{cosk}_m^Y(V) = \prod_Y^m(V) \rightarrow X \times_Y \mathrm{cosk}_m^Y(V)$. Therefore, the augmentation

$$E(\mathrm{cosk}_m^Y(V) \times_Y \mathrm{cosk}^Y(X).) \rightarrow E(\mathrm{cosk}_m^Y(V)_m)$$

is a weak equivalence (even without tensoring with \mathbb{Q}), by 3.5, and so again by the homotopy colimit theorem [4], we have a weak equivalence: $E(\mathrm{cosk}_m^Y(V) \times_Y \mathrm{cosk}^Y(X).) \rightarrow E(\mathrm{cosk}_m^Y(V).)$

Finally, by Lemma 3.6, we know that $E(\mathrm{cosk}_m^Y(V).)_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$ is a weak equivalence, and it follows therefore that $E(\mathrm{cosk}_m^Y(X).)_{\mathbb{Q}} \rightarrow E(Y)_{\mathbb{Q}}$ is too. \square

Proof of Theorem 3.4. Suppose that $f : X. \rightarrow Y.$ is a proper hypercover of simplicial varieties with proper face maps. By 2.1, we know that the sequence of simplicial varieties:

$$\dots \rightarrow \text{cosk}_n^Y(X.) \rightarrow \text{cosk}_{n-1}^Y(X.) \rightarrow \dots \rightarrow Y.$$

satisfies properties 1)–3) of Lemma 2.8. \square

Since $E_q = 0$ for $q < 0$, it follows from the descent spectral sequence 3.2, that if $p : V. \rightarrow W.$ is a proper map between simplicial varieties with proper face maps, and there is an $n \geq 1$ such that p_i is an isomorphism for $i \leq n$, then $E_q(V.) \rightarrow E_q(W.)$ is an isomorphism for $q < n$. It follows that for a given $q \geq 0$, the map

$$E_q(X.) \rightarrow E_q(\text{cosk}_n^Y(X.))$$

is an isomorphism once $n > q$. It will therefore be sufficient to show, for all $n \geq 0$, that

$$E(\text{cosk}_n^Y(X.))_{\mathbb{Q}} \rightarrow E(\text{cosk}_{n-1}^Y(X.))_{\mathbb{Q}}$$

is weak equivalence. First we need the following lemma:

Lemma 3.8. *Let $f : X. \rightarrow Y.$ be a proper hypercover. Then for all $n \geq 0$ and all $p \geq 0$,*

$$(f_n)_p : \text{Cosk}_n^Y(X.)_p \rightarrow \text{Cosk}_{n-1}^Y(X.)_p$$

is proper and surjective.

Proof. We have to show that $(f_n)_p$ is surjective on F -valued points for all algebraically closed fields, and that the valuative criterion of properness holds. We shall only prove the valuative criterion, since the proof of surjectivity is similar, but easier.

Suppose then that we are given a commutative square, for Λ a valuation ring, with fraction field F :

$$\begin{array}{ccc} \text{Spec}(F) & \xrightarrow{i} & \text{Cosk}_n^Y(X.)_p \\ \downarrow & \nearrow \tilde{j} & \downarrow (f_n)_p \\ \text{Spec}(\Lambda) & \longrightarrow & \text{Cosk}_{n-1}^Y(X.)_p \end{array}$$

Since

$$\text{Cosk}_n^Y(X.)_p \cong \text{Cosk}_n(X.)_p \times_{\text{Cosk}_n(Y.)_p} Y_p$$

the commutative square above is equivalent to a commutative diagram

$$\begin{array}{ccc} (\text{sk}_n(\Delta^p) \times \text{Spec}(F)) \cup (\text{sk}_{n-1}(\Delta^p) \times \text{Spec}(\Lambda)) & \longrightarrow & X. \\ \downarrow & \nearrow \theta & \downarrow f \\ \text{sk}_n(\Delta^p) \times \text{Spec}(\Lambda) & \longrightarrow & Y. \\ \downarrow & & \parallel \\ \Delta^p \times \text{Spec}(\Lambda) & \longrightarrow & Y. \end{array}$$

and giving the lifting \tilde{j} is equivalent to giving the map θ , which exists since f is a proper hypercover. \square

We now have that the map

$$f_n : \text{cosk}_n^Y(X.) \rightarrow \text{cosk}_{n-1}^Y(X.).$$

has the following properties (see 2.1)

- (1) For all p , $(f_n)_p$ is proper and surjective.
- (2) For $p < n$, $(f_n)_p$ is an isomorphism.
- (3) For $p \geq n$, the natural map

$$\text{Cosk}_n^Y(X) \rightarrow \text{Cosk}_p^Y(\text{Cosk}_n^Y(X))$$

is an isomorphism.

Suppose now that $n \geq 0$ and that $f : V. \rightarrow W.$ is a map of simplicial varieties over $Y.$ with proper face maps satisfying (1)–(3) above. We claim that $E(V.)_{\mathbb{Q}} \rightarrow E(W.)_{\mathbb{Q}}$ is a weak equivalence. To see this, consider the bisimplicial scheme:

$$([k], [\ell]) \mapsto \text{cosk}_0^{W_\ell}(V_\ell)_k = V_\ell \times_{W_\ell} \times \dots \times_{W_\ell} V_\ell \quad (k \text{ factors}).$$

By [4], we have:

$$\begin{aligned} & \text{hocolim}_{\Delta^{\text{op}} \times \Delta^p}([k], [\ell]) \mapsto E(\text{cosk}_0^{W_\ell}(V_\ell)_k)_{\mathbb{Q}} \\ \simeq & \quad \text{hocolim}_{\Delta^{\text{op}}}([\ell] \mapsto \text{hocolim}_{\Delta^{\text{op}}}([k] \mapsto E(\text{cosk}_0^{W_\ell}(V_\ell)_k)_{\mathbb{Q}})) \\ \simeq & \quad \text{hocolim}_{\Delta^{\text{op}}}([k] \mapsto \text{hocolim}_{\Delta^{\text{op}}}([\ell] \mapsto E(\text{cosk}_0^{W_\ell}(V_\ell)_k)_{\mathbb{Q}})). \end{aligned}$$

Since f_ℓ is proper and surjective for all $\ell \geq 0$, we have that, for all $\ell \geq 0$:

$$E(\text{cosk}_0^{W_\ell}(V_\ell).)_{\mathbb{Q}} = \text{hocolim}_{\Delta^{\text{op}}}([k] \mapsto E(\text{cosk}_0^{W_\ell}(V_\ell))_k)_{\mathbb{Q}} \rightarrow E(W_\ell)_{\mathbb{Q}}$$

is a weak equivalence (by Proposition 3.7). Hence the natural map from the first iterated hocolim to $E(W.)_{\mathbb{Q}}$ is a weak equivalence.

On the other hand, consider, for a given $k \geq 1$, the face maps

$$d_i^W : \text{cosk}_0^W(V.)_k = W. \times_V \times \dots \times_V W. \rightarrow \text{cosk}_0^W(V.)_{k-1} = W. \times_V \times \dots \times_V W..$$

Each map d_i^W has a section, either s_{i-1}^W or s_i^W . Also, since $V_p \rightarrow W_p$ is an isomorphism for $p < n$, the same is true for $d_i^{W_p}$. Furthermore, since $V. \rightarrow \text{Cosk}_p^W(V.)$ is an isomorphism for all $p \geq n$, it is straightforward to check that the same is true for $\text{Cosk}_0^W(V.)_k$ instead of $V.$ for all k . Thus, for $i < n$, the two maps $s_i^W \cdot d_i^W : \text{Cosk}_0^W(V.)_k \rightarrow \text{Cosk}_0^W(V.)_k$ and the identity are homotopic, by Lemma 2.3. Similarly, if $i > 0$, the maps $s_{i-1}^W \cdot d_i^W$ and the identity are homotopic.

Applying the functor E , we see that $E(d_i^W)_{\mathbb{Q}}$ is an equivalence, independent of i , with inverse $E(s_i^W)_{\mathbb{Q}}$ and/or $E(s_{i-1}^W)_{\mathbb{Q}}$. It follows that the natural map

$$V. \rightarrow \text{Cosk}_0^W(V.)..$$

induces an equivalence on $E_{\mathbb{Q}}$, since for each $q \geq 0$,

$$k \mapsto E_q(\underbrace{V. \times_W. \times \dots \times_W. V.}_{k\text{-times}})$$

is a constant simplicial group. Hence the map from $E(V.)_{\mathbb{Q}}$ to the second iterated hocolim above is also a weak equivalence, and the map

$$E(V.)_{\mathbb{Q}} \rightarrow E(W.)_{\mathbb{Q}}$$

is a weak equivalence. \square

Theorem 3.9. *Suppose that E is a covariant functor from proper morphisms between schemes to connective spectra which satisfies properties (1)-(3) but not necessarily properties (4) and (5). Then, if $f. : X. \rightarrow Y.$ is a proper hyperenvelope between simplicial S -varieties, each having proper face maps, $E(f.) : E(X.) \rightarrow E(Y.)$ is a weak equivalence.*

Proof. The proof of this result is essentially already in [12] and [14], but for completeness we indicate how the proof differs from that of Theorem 3.4. The only difference is in the proof of the analog of Proposition 3.7, which states that if $p : X \rightarrow Y$ is an envelope, then

$$E(\text{cosk}^Y(X).) \rightarrow E(Y)$$

is a weak equivalence. As in the proof of Proposition 3.7, using localization and noetherian induction on the closed subsets of Y it is enough to know that for any Y there is a non-empty Zariski open subset $U \subset Y$ such that

$$E(\text{cosk}^U(p^{-1}(U)).) \rightarrow E(U)$$

is a weak equivalence. However, since p is an envelope, if η is a generic point of a component of Y there is an open neighbourhood U of η such that the map $p_U : p^{-1}(U) \rightarrow U$ has a section, and hence the map above is a weak equivalence by Lemma 3.5. \square

4. REVIEW OF K -THEORY

4.1. The homology theory associated to G -theory. As in [14], we view the K -theory of coherent sheaves as a covariant functor from the category of proper morphisms between schemes to the category of Waldhausen categories (here we follow the terminology of [32], 1.2.3, rather than use the original terminology “category with cofibrations and weak equivalences”), by sending X to the category of bounded below complexes of flasque sheaves of \mathcal{O}_X -modules which have bounded coherent cohomology; see *op. cit.* 3.16. Indeed this is a strict functor, and composing with the K -theory spectrum functor, we get, as in [10], a covariant functor from the category of proper morphisms between S -schemes to the category **Spectra** of symmetric spectra of [19]. We then compose with the \mathbb{Q} -localization functor (smash product with the Eilenberg-MacLane spectrum $H\mathbb{Q}$) **Spectra** \rightarrow **Spectra** $_{\mathbb{Q}}$, to obtain the K -theory functor that we will use: $X. \mapsto \mathbf{G}(X.) \in \mathbf{Spectra}_{\mathbb{Q}}$. Note

that we are following Thomason's notation (G -theory), rather than Quillen's (K' -theory).

Since we also want G -theory to be contravariant with respect to flat (and in particular étale) morphisms, it will be important to consider our complexes of sheaves to be sheaves of \mathcal{O}_X -modules in the $fppf$ topology. Since being flasque is compatible with flat base change, the resulting Waldhausen category is equivalent, and hence has the same G -theory, as the category of complexes of sheaves in the Zariski topology.

We extend the G -theory functor first of all to the category of (degreewise) proper morphisms between simplicial schemes with proper face maps by taking the homotopy colimit of the associated simplicial spectra. We may then extend one step further to the category $Ar(\mathbf{sP})$ with objects proper morphisms $f : Y \rightarrow Z$ between simplicial schemes with proper face maps, and morphisms $(g_Y, g_Z) : (f_1 : Y_1 \rightarrow Z_1) \rightarrow (f_2 : Y_2 \rightarrow Z_2)$ pairs of proper maps such that $g_Z \circ f_1 = f_2 \circ g_Y$, by setting $\mathbf{G}(f : Y \rightarrow Z) := \text{Cone}(\mathbf{G}(f))$.

From theorem 3.4, we have:

Theorem 4.1. *Let $f : X \rightarrow Y$ be a morphism between simplicial schemes with proper face maps which is a proper hypercover. Then $G(f)$ is a weak equivalence.*

4.2. G -theory of stacks.

Definition 4.2. If \mathfrak{X} is a stack, we define the G -theory spectrum $\mathbf{G}(\mathfrak{X})$ to be $\mathbf{H}_{\text{ét}}(\mathfrak{X}, \mathbf{G})$, where \mathbf{G} is the presheaf of (\mathbf{Q} -localized) spectra on the étale site induced by the functor \mathbf{G} . See [13], though here we are using the hypercohomology of (pre-)sheaves of spectra in the sense of [20].

It follows from étale descent for rational G -theory (due to Thomason, (Theorem 2.15 of [31])) that this definition is consistent with the definition for schemes. In particular, we have:

Lemma 4.3. *If $\pi : V \rightarrow \mathfrak{X}$ is an étale hypercover of a Deligne-Mumford stack, the natural map:*

$$G(\mathfrak{X}) \rightarrow \text{holim}_i(G(V_i))$$

is a weak equivalence.

Lemma 4.4. *Let \mathfrak{Y} be a closed substack of the stack \mathfrak{X} , with complement \mathfrak{U} . Then there is a fibration sequence*

$$\mathbf{G}(\mathfrak{Y}) \xrightarrow{i_*} \mathbf{G}(\mathfrak{X}) \xrightarrow{j^*} \mathbf{G}(\mathfrak{U})$$

in which $i : \mathfrak{Y} \rightarrow \mathfrak{X}$ and $j : \mathfrak{U} \rightarrow \mathfrak{X}$ are the inclusions.

Proof. Since $\mathbf{G}(\mathfrak{X})$ is the hypercohomology $\mathbf{R}\Gamma(\mathfrak{X}, \mathbf{G}_{\mathfrak{X}})$ of the presheaf $\mathbf{G}_{\mathfrak{X}} : V \mapsto G(V)$ on the étale site of \mathfrak{X} , it suffices to observe that for each étale

morphism $p : V \rightarrow \mathfrak{X}$ from a scheme V to \mathfrak{X} , we have a localization fibration sequence

$$\mathbf{G}(V \times_{\mathfrak{X}} \mathfrak{Y}) \rightarrow \mathbf{G}(V) \rightarrow \mathbf{G}(V \times_{\mathfrak{X}} \mathfrak{U})$$

and hence an isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{Y}}(\mathbf{G}) \approx i_* \mathbf{G}_{\mathfrak{Y}}.$$

Note that this also follows easily from the previous lemma and the fact that $\mathbf{H}\text{olim}$ preserves fibration sequences. \square

The functoriality of \mathbf{G} for morphisms of stacks is a more complicated question. In [13] pushforward maps were only constructed for proper *representable* morphisms of stacks. However we may replace stacks by simplicial varieties, as follows:

Let $p : X. \rightarrow \mathfrak{X}$ be a proper morphism to a stack from a simplicial variety with proper face maps. Let $\pi : V. \rightarrow \mathfrak{X}$ be an étale hypercover of \mathfrak{X} , such as the nerve $\text{Cosk}_0^{\mathfrak{X}}(V)$ of an étale presentation $V \rightarrow \mathfrak{X}$. Then we have a commutative square:

$$\begin{array}{ccc} X. & \xleftarrow{\tilde{\pi}} & X. \times_{\mathfrak{X}} V. \\ p \downarrow & & \tilde{p} \downarrow \\ \mathfrak{X} & \xleftarrow{\pi} & V. \end{array}$$

For each i , since $X. \times_{\mathfrak{X}} V_i \rightarrow V_i$ is proper, we have a map

$$\mathbf{G}(X. \times_{\mathfrak{X}} V_i) = \text{hocolim}_j(\mathbf{G}(X_j \times_{\mathfrak{X}} V_i)) \rightarrow \mathbf{G}(V_i)$$

which is contravariant with respect to i (since we have constructed G to be strictly compatible with flat base change). Therefore, we get a diagram:

$$\begin{array}{ccc} \mathbf{G}(X.) & \xrightarrow{\tilde{\pi}^*} & \text{holim}_i \mathbf{G}(X. \times_{\mathfrak{X}} V_i) \\ & & \tilde{p}_* \downarrow \\ \mathbf{G}(\mathfrak{X}) & \xrightarrow{\pi^*} & \text{holim}_i(\mathbf{G}(V.)) \end{array}$$

in which the bottom horizontal arrow is a weak equivalence. Hence we get a map (up to homotopy) $p_* : \mathbf{G}(X.) \rightarrow \mathbf{G}(\mathfrak{X})$, which it is straightforward to check (again using the fact that G is strictly compatible with flat base change) does not depend on the choice of étale hypercover $\pi : V. \rightarrow \mathfrak{X}$. Furthermore, if $f. : Y. \rightarrow X.$ is a map of simplicial varieties, using the fact that push forward commutes with flat base-change, we have that

$$(p \cdot f)_* = p_* \cdot f_* : \mathbf{G}(Y.) \rightarrow \mathbf{G}(\mathfrak{X})$$

in the rational stable homotopy category.

In order to show that this construction gives an extension of \mathbf{G} -theory from simplicial varieties to stacks, we need:

Lemma 4.5. *Suppose that $p : X. \rightarrow \mathfrak{X}$ is a proper hypercover. Then $p_* : \mathbf{G}(X.) \rightarrow \mathbf{G}(\mathfrak{X})$ is a weak equivalence.*

Proof. Since both the G -theory of $X.$ and of \mathfrak{X} are compatible with localization, we may proceed by noetherian induction on \mathfrak{X} . By Proposition 1.3, every Deligne-Mumford stack has a non-empty dense open set which is a quotient stack. It therefore suffices to show that if $\mathfrak{X} = [V/\Gamma]$ for Γ a finite group, then $\mathbf{G}(X.) \rightarrow \mathbf{G}(X. \times_{\mathfrak{X}} \mathrm{Cosk}_0^{\mathfrak{X}}(V))$ is a weak equivalence. However $\mathrm{Cosk}_0^{\mathfrak{X}}(V)$ is just the bar construction $(E.\Gamma \times V)/\Gamma$ for the action of Γ on V , hence

$$X. \times_{\mathfrak{X}} \mathrm{Cosk}_0^{\mathfrak{X}}(V) \simeq (E.\Gamma \times (X. \times_{\mathfrak{X}} V))/\Gamma .$$

It is then straightforward to check, since we are taking G -theory with rational coefficients, and since for each j , $X_j \times_{\mathfrak{X}} V \rightarrow X_j$ is a *finite étale* Galois cover with group Γ , that

$$\mathrm{holim}_j(j \mapsto \mathbf{G}(E_j.\Gamma \times (X_j \times_{\mathfrak{X}} V))/\Gamma)$$

is simply the Γ -invariants of $\mathbf{G}(X. \times_{\mathfrak{X}} V)$ and the map

$$\mathbf{G}(X.) \rightarrow \mathrm{holim}_j(j \mapsto \mathbf{G}(E_j.\Gamma \times (X_j \times_{\mathfrak{X}} V))/\Gamma)$$

is a weak equivalence with inverse induced by the transfer $\mathbf{G}(X. \times_{\mathfrak{X}} V) \rightarrow \mathbf{G}(X.)$ (divided by the order of Γ). \square

Theorem 4.6. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a proper, not necessarily representable, morphism of stacks. Then there exists a canonical map (in the homotopy category)*

$$\mathbf{G}(\mathfrak{X}) \rightarrow \mathbf{G}(\mathfrak{Y})$$

with the property that for any commutative square:

$$\begin{array}{ccc} X. & \xrightarrow{\tilde{f}} & Y. \\ p. \downarrow & & \downarrow q. \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

in which $p.$ and $q.$ are proper morphisms with domains simplicial schemes with proper face maps, we have a commutative square in the stable homotopy category:

$$\begin{array}{ccc} \mathbf{G}(X.) & \xrightarrow{\tilde{f}_*} & \mathbf{G}(Y.) \\ \mathbf{G}(p.) \downarrow & & \downarrow \mathbf{G}(q.) \\ \mathbf{G}(\mathfrak{X}) & \xrightarrow{\mathbf{G}(f)} & \mathbf{G}(\mathfrak{Y}) \end{array}$$

Proof. To define the map $f_* : \mathbf{G}(\mathfrak{X}) \rightarrow \mathbf{G}(\mathfrak{Y})$, pick any proper hypercover $p : X. \rightarrow \mathfrak{X}$, and set $f_* = (f \cdot p)_* \cdot (p_*)^{-1}$. Since p_* is functorial with respect to $p : X. \rightarrow \mathfrak{X}$, the map f_* does not depend on the choice of p . Given a commutative square as above, the fact that associated square of G -theory spectra is also commutative is a consequence of the same functoriality for simplicial schemes over \mathfrak{Y} . \square

Definition 4.7. We shall call a morphism of stacks $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a \mathbf{G} -equivalence if $\mathbf{G}(f)$ is an equivalence.

Theorem 4.8. *If \mathfrak{X} is a stack, the natural map $\mathfrak{X} \rightarrow |\mathfrak{X}|$ from the stack to its coarse moduli space is a \mathbf{G} -equivalence.*

Proof. Assume G is a finite group acting on some affine scheme U , and $H \subset G$ is a normal subgroup acting trivially on U and such that G/H acts freely on U . A standard transfer argument shows that the map $\mathbf{G}([U/G]) \rightarrow \mathbf{G}(U/G)$ is an equivalence. Since every stack has a dense open subset which is of this type one can use localization and noetherian induction to conclude the proof. \square

5. WEIGHT COMPLEXES FOR VARIETIES AND STACKS

5.1. Introduction. In this section we prove the extension of the main theorem of [14] for varieties and stacks over S , where S is a base scheme satisfying the conditions of the introduction. In particular, this includes the cases $S = \text{Spec}(\mathcal{O}_K)$ for \mathcal{O}_K the ring of integers in a number field, and $S = \text{Spec}(k)$ for k a field of characteristic different from zero.

As in section 5 of *op. cit.* we shall use K_0 -motives (but with rational coefficients) rather than Chow motives. (Manin, in [25], seems to have been the first to mention using K_0 -motives.) The proofs in this section are often variations on those in [14], but we shall give proofs again where the current situation merits it.

5.2. K_0 -correspondences.

Definition 5.1. If X and Y are regular, projective, S -varieties, we will write $KC_S(X, Y)$ for $G_0(X \times_S Y)_{\mathbb{Q}}$, and call it the group of “ K_0 -correspondences from X to Y ”.

Note that $X \times_S Y$ is not in general regular; however since Y is regular, \mathcal{O}_Y is of finite global tor-dimension, and hence there is a bilinear product, given X, Y and Z regular, projective, S -varieties :

$$\begin{aligned} * : G_0(X \times_S Y) \times G_0(Y \times_S Z) &\rightarrow G_0(X \times_S Y \times_S Z) \\ ([\mathfrak{F}], [\mathfrak{G}]) &\mapsto \sum_{i \geq 0} (-1)^i [\text{Tor}_i^{\mathcal{O}_Y}(\mathfrak{F}, \mathfrak{G})]. \end{aligned}$$

Composing with the direct image map ($p_{XZ} : X \times_S Y \times_S Z \rightarrow X \times_S Z$ being the natural projection):

$$(p_{XZ})_* : G_0(X \times_S Y \times_S Z) \rightarrow G_0(X \times_S Z)$$

we get a bilinear pairing:

$$G_0(X \times_S Y) \times G_0(Y \times_S Z) \rightarrow G_0(X \times_S Z)$$

and hence:

$$KC_S(X, Y) \times KC_S(Y, Z) \rightarrow KC_S(X, Z).$$

The proofs of the following lemmas are straightforward so we omit them.

Lemma 5.2. *Given regular varieties X, Y, Z and W projective over S , and elements $\alpha \in KC_S(X, Y)$, $\beta \in KC_S(Y, Z)$, $\gamma \in KC_S(Z, W)$ we have*

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

Given a morphism $f : X \rightarrow Y$ of regular varieties, projective over S , with graph $\Gamma_f \subset X \times_S Y$, we write $\Gamma(f)$ for the class $[\mathcal{O}_{\Gamma(f)}] \in KC_S(X, Y)$.

Lemma 5.3. *If $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are morphisms of regular varieties, projective over S , we have*

$$\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f).$$

Furthermore, if X and Y are regular projective varieties over S , and $\alpha \in KC_S(X, Y)$, then $\Gamma(1_Y) \circ \alpha = \alpha \circ \Gamma(1_X) = \alpha$.

Definition 5.4. We write KC_S for the category with objects regular projective varieties over S , and homsets the $KC_S(X, Y)$ for X and Y objects in KC_S , and identity $\Gamma(1_X) \in KC_S(X, X)$ for each X .

Clearly Γ is a covariant functor from the category of regular projective varieties over S to the category KC_S . Note that KC_S is a \mathbb{Q} -linear category.

Observe that Γ extends to a functor from the category of simplicial varieties which, in each degree, are regular and projective over S to the category of chain complexes in KC_S by associating (in the usual fashion) to X . the complex

$$n \mapsto \Gamma(X_n)$$

with differential $\sum_{i=0}^n (-1)^i (d_i)_* : \Gamma(X_n) \rightarrow \Gamma(X_{n-1})$.

Lemma 5.5. *The functors G_i for $i \geq 0$, where $G_i(X)$ is K_i of the category of coherent sheaves of \mathcal{O}_X modules, factor through Γ .*

Proof. Give $\alpha \in KC_S(X, Y)$, we need to define $G_i(\alpha) : G_i(X) \rightarrow G_i(Y)$. We start by observing that for X regular, $K_i(X) \simeq G_i(X)$, and so if $p_X : X \times_S Y \rightarrow X$ and $p_Y : X \times_S Y \rightarrow Y$ are the projective, we can define

$$G_i(\alpha) : x \mapsto p_{Y*}(p_X^*(x) \cap \alpha)$$

where

$$\cap : K_i(X \times_S Y) \otimes G_0(X \times_S Y) \rightarrow G_i(X \times_S Y)$$

is the natural cap product.

Let Z be a regular projective variety and $\beta \in KC_S(Y, Z)$. We want to show that

$$(B). \quad G_i(\beta) \circ G_i(\alpha) = G_i(\beta \circ \alpha).$$

Consider the diagram of maps

$$\begin{array}{ccccc}
 & & X \times_S Y \times_S Z & & \\
 & & \downarrow q & & \\
 & p \swarrow & & \searrow r & \\
 X \times_S Y & & Y \times_S Z & & X \times_S Z \\
 \swarrow p_X & & \swarrow q_Y & & \swarrow r_Z \\
 X & & Y & & Z \\
 & & & & \searrow r_X \\
 & & & & X
 \end{array}$$

and call $u : X \times_S Y \times_S Z \rightarrow X$ the obvious projection. If $x \in K_i(X)$ we have

$$G_i(\beta) \circ G_i(\alpha)(x) = q_{Z*}(q_Y^*(p_{Y*}(p_X^*(x) \cap \alpha)) \cap \beta)$$

and

$$\begin{aligned}
 G_i(\beta \circ \alpha)(x) &= r_{Z*}(r_X^*(x) \cap (r_*(\alpha * \beta))) \\
 &= r_{Z*}(r_*(u^*(x) \cap (\alpha * \beta))) \quad (\text{projection formula for } r) \\
 &= q_{Z*}q_*(u^*(x) \cap (\alpha * \beta)).
 \end{aligned}$$

Letting $\xi = p_X^*(x)$, it is enough to show that

$$(A) \quad q_Y^*(p_{Y*}(\xi \cap \alpha)) \cap \beta = q_*(p^*(\xi) \cap (\alpha * \beta)).$$

Let \mathcal{F} be a complex of coherent sheaves on $X \times_S Y$ which is acyclic with respect to p_Y and represents $\alpha \in G_0(X \times_S Y)$. For any locally free sheaf \mathcal{E} on $X \times_S Y$, the complex $\mathcal{E} \otimes \mathcal{F}$ is still acyclic with respect to p_Y and its derived direct image by p_Y is $p_{Y*}(\mathcal{E} \otimes \mathcal{F})$. On the other hand, let $Z \subset \mathbb{P}_S^N$ be a projective embedding of Z and \mathcal{G} a complex of coherent sheaves on $Y \times_S \mathbb{P}_S^N$ which is flat on \mathcal{O}_Y and acyclic outside $Y \times_S Z$, and which represents β in the K -theory with supports

$$K_0^{Y \times_S Z}(Y \times_S \mathbb{P}_S^N) = G_0(Y \times_S Z).$$

Let $\tilde{q}_Y : Y \times_S \mathbb{P}_S^N \rightarrow Y$, $\tilde{q} : X \times_S Y \times_S \mathbb{P}_S^N \rightarrow Y \times_S \mathbb{P}_S^N$ and $\tilde{p} : X \times_S Y \times_S \mathbb{P}_S^N \rightarrow X \times_S Y$ be the obvious projections. By flat base change we know that

$$\tilde{q}_Y^*(p_{Y*}(\mathcal{E} \otimes \mathcal{F})) = \tilde{q}_* \tilde{p}^*(\mathcal{E} \otimes \mathcal{F}).$$

Therefore, by the projection formula for \tilde{q} , and since \mathcal{G} is flat over \mathcal{O}_Y ,

$$\begin{aligned}
 \tilde{q}_Y^*(p_{Y*}(\mathcal{E} \otimes \mathcal{F})) \otimes_{\mathcal{O}_{Y \times_S \mathbb{P}_S^N}} \mathcal{G} &= \tilde{q}_*(\tilde{p}^*(\mathcal{E} \otimes \mathcal{F})) \otimes_{\mathcal{O}_{X \times_S Y \times_S \mathbb{P}_S^N}} \tilde{q}^*(\mathcal{G}) \\
 &= \tilde{q}_*(\tilde{p}^*(\mathcal{E}) \otimes \tilde{p}^*(\mathcal{F}) \otimes \tilde{q}^*(\mathcal{G})).
 \end{aligned}$$

The functor $\mathcal{E} \mapsto \tilde{q}_Y^*(p_{Y*}(\mathcal{E} \otimes \mathcal{F})) \otimes \mathcal{G}$ induces the map $\xi \mapsto q_Y^*(p_{Y*}(\xi \cap \alpha)) \cap \beta$ on higher K -theory, and the functor

$$\mathcal{E} \mapsto \tilde{q}_*(\tilde{p}^*(\mathcal{E}) \otimes \tilde{p}^*(\mathcal{F}) \otimes \tilde{q}^*(\mathcal{G}))$$

induces the map $\xi \mapsto q_*(p^*(\xi) \cap (\alpha * \beta))$. Therefore (A) and (B) follow. \square

Definition 5.6. We set \mathbf{KM}_S equal to the idempotent completion of KC_S . Note that these are homological motives.

The proof of the following theorem, given Lemma 5.5, is the same as that of Theorem 6 of [14] (which itself is a variation of Theorem 1 of *op. cit.*):

Theorem 5.7. *Let*

$$\begin{array}{ccc} X. & \xrightarrow{g} & Y. \\ p \downarrow & & \downarrow q \\ Z. & \xrightarrow{f} & W. \end{array}$$

be a commutative square of maps between simplicial objects in the category of regular projective varieties over S . Suppose that for all regular projective varieties V over S , the associated square of spectra:

$$\begin{array}{ccc} \mathbf{G}(V \times_S X.) & \longrightarrow & \mathbf{G}(V \times_S Y.) \\ \downarrow & & \downarrow \\ \mathbf{G}(V \times_S Z.) & \longrightarrow & \mathbf{G}(V \times_S W.) \end{array}$$

is homotopy cartesian. Then the associated square of complexes in KC_S is homotopy cartesian, i.e., the associated total complex is contractible, or equivalently the natural map $\text{Cone}(\Gamma_(g.)) \rightarrow \text{Cone}(\Gamma_*(f.))$ is a homotopy equivalence.*

5.3. Weight complexes for simplicial varieties. Following section 2.2 of [14], we write $\text{Ar}(P_S)$ for the category of morphisms in P_S , where P_S is the category of proper, not necessarily regular S -varieties, with objects $f : Y \rightarrow X$, and morphisms $g : f' \rightarrow f$ commutative squares :

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g_Y \downarrow & & \downarrow g_X \\ Y & \xrightarrow{f} & X \end{array}$$

We can also consider the category $\text{Ar}(P_S^{\Delta\text{op}})$ of arrows between simplicial objects on P_S . A morphism

$$g. : (Y' \xrightarrow{f'} X') \rightarrow (Y. \xrightarrow{f} X.)$$

in $\text{Ar}(P_S^{\Delta\text{op}})$ will be called a proper hypercover if both g_Y and g_X are proper hypercovers. We have a functor

$$\Gamma_* : \text{Ar}(RP_S^{\Delta\text{op}}) \rightarrow C_*(KC_S)$$

from the category of arrows between simplicial *regular* projective varieties to the category of chain complexes in KC_S ,

$$\Gamma_* : (f. : X. \rightarrow Y.) \mapsto \text{Cone}(\Gamma_*(f.)).$$

We shall now show, following *op. cit.*, that this functor induces a functor from $\text{Ar}(P_S^{\Delta\text{op}})$ to the homotopy category of $C_*(KC_S)$.

Theorem 5.8. *The functor*

$$\begin{array}{ccc} \mathrm{Ar}(RP_S^{\Delta^{\mathrm{op}}}) & \rightarrow & H_0(C_*(KC_S)) \\ f. \mapsto & & \Gamma_*(f.) \end{array}$$

has a unique extension to $\mathrm{Ar}(P_S^{\Delta^{\mathrm{op}}})$ with the property that proper hypercovers map to homotopy equivalences.

Proof. The proof is the same as in [14], § 2.2, using Theorem 1.7 instead of Hironaka's resolution of singularities. The theorem is also Theorem 5.3.c) of [21], applied to D^{op} the category $P_S^{\Delta^{\mathrm{op}}}$, to C^{op} the full subcategory $RP_S^{\Delta^{\mathrm{op}}}$, to S the category of proper hypercovers, and to $E = \Delta^1$. (Note that Theorem 5.3 in [21] was partly inspired by [14].) \square

We shall call a morphism $g.$ in $\mathrm{Ar}(P_S^{\Delta^{\mathrm{op}}})$ a *universal G -equivalence* if after multiplying all schemes involved by any regular projective variety V , the corresponding square of spectra obtained by applying $G_{\mathbb{Q}}$ is homotopy cartesian. Theorem 5.7 tells us that if $g.$ is a universal G -equivalence and all schemes involved are regular then $\Gamma_*(g.)$ is a homotopy equivalence.

Using Theorem 1 and the same proof as in [14], end of § 2.2, we get from this that if $g.$ is a universal G -equivalence among arbitrary projective schemes, then $\Gamma_*(g.)$ is a homotopy equivalence.

Theorem 5.9. *There is a covariant functor*

$$h : \mathrm{Var}_S^{\Delta^{\mathrm{op}}} \rightarrow \mathrm{Ho}(C_*(KC_S))$$

from the category of simplicial varieties over S with proper face maps and proper morphisms to the category of homotopy classes of maps of complexes of K_0 -motives over S with rational coefficients, satisfying the following properties:

- i) *If, for all $n \geq 0$, X_n is a regular projective scheme over S and $h(X_n)$ is the usual motive of X_n , $h(X.)$ is the complex of motives*

$$\dots \rightarrow h(X_n) \xrightarrow{\sum_{i=0}^n (-1)^i d_{i*}} h(X_{n-1}) \rightarrow \dots$$

- ii) *If $U. \subset X.$ is a strongly open simplicial subvariety of $X.$ with proper face maps and with complement $T.$, then we have a triangle*

$$h(T.) \rightarrow h(X.) \rightarrow h(U.) \rightarrow h(T.)[1].$$

- iii) *If $X. \rightarrow Y.$ is a proper hypercover, the induced map $h(X.) \rightarrow h(Y.)$ is a homotopy equivalence.*

Proof. Let $X.$ be a simplicial variety over S with proper face maps. According to Proposition 2.16 there exists a proper hypercover $\tilde{X}. \rightarrow X.$ with $\tilde{X}.$ a split simplicial variety. By Proposition 2.18 $\tilde{X}.$ admits a compactification $\bar{X}.$ with complement $\bar{X}.\setminus\tilde{X}.$. Let $h(X.)$ be the image in $\mathrm{Ho}(C_*(KC_S))$ of the arrow $\bar{X}.\setminus\tilde{X}. \rightarrow \bar{X}.$ (Theorem 5.8). We claim that $h(X.)$ is functorial in $X.$ and does not depend on the choices made to define it.

Indeed, let $f : X. \rightarrow Y.$ be a map in $\mathcal{V}ar_S^{\Delta, \text{op}}$ and $\tilde{X}. \rightarrow X.$ and $\tilde{Y}. \rightarrow Y.$ proper hypercovers admitting compactifications $\bar{X}.$ and $\bar{Y}.$. The fiber product $\tilde{X}. \times_Y \tilde{Y}.$ has a compactification $\overline{\bar{X}. \times \bar{Y}.}$ which is the Zariski closure of $\tilde{X}. \times_Y \tilde{Y}.$ in $\bar{X}. \times_S \bar{Y}.$. Then we get two commutative squares

$$\begin{array}{ccc} \overline{\bar{X}. \times \bar{Y}.} \setminus \tilde{X}. \times \tilde{Y}. & \longrightarrow & \overline{\bar{X}. \times \bar{Y}.} \\ \downarrow & & \downarrow \\ \bar{X}. \setminus \tilde{X}. & \longrightarrow & \bar{X}. \end{array}$$

and

$$\begin{array}{ccc} \overline{\bar{X}. \times \bar{Y}.} \setminus \tilde{X}. \times \tilde{Y}. & \longrightarrow & \overline{\bar{X}. \times \bar{Y}.} \\ \downarrow & & \downarrow \\ \bar{Y}. \setminus \tilde{Y}. & \longrightarrow & \bar{Y}. \end{array}$$

the first of which is a universal G -equivalence. Hence we get a map

$$h(f) : h(X.) = \Gamma_*(\bar{X}. \setminus \tilde{X}. \rightarrow \bar{X}.) \rightarrow \Gamma_*(\bar{Y}. \setminus \tilde{Y}. \rightarrow \bar{Y}.) = h(Y.).$$

When $X. = Y.$ and f is the identity, both squares above are universal G -equivalences and it follows that $h(X.)$ does not depend on choices made to define it.

Assume now that $f : X. \rightarrow Y.$ and $g : Y. \rightarrow Z.$ are proper maps of simplicial varieties. Then the same argument as in [14], §2.3 (except that we replace ‘‘Gersten acyclic’’ by ‘‘universal G -equivalence’’) shows that $h(g) \circ h(f) = h(g \circ f)$, *i.e.* $h(X.)$ is functorial in $X.$.

When every X_n is regular and projective over S , $n \geq 0$, we can take $\tilde{X}. = \bar{X}. = X.$, so Property i) is clear.

To check ii), let $\tilde{X}.$ be a proper hypercover of $X.$, with compactification $\bar{X}.$. Let $\tilde{U}.$ (resp. $\tilde{T}.$) be the inverse image of $U.$ (resp. $T.$) in $\tilde{X}.$ and define $Y. = \bar{X}. - \tilde{X}.$, $Z. = \bar{X}. - \tilde{U}.$. We have inclusions $Y. \rightarrow Z. \rightarrow \bar{X}.$. Consider a proper hypercover of this diagram

$$\begin{array}{ccccc} Y'. & \longrightarrow & Z'. & \longrightarrow & X'. \\ \downarrow & & \downarrow & & \downarrow \\ Y. & \longrightarrow & Z. & \longrightarrow & \bar{X}. \end{array}$$

with Y' , Z' and X' in RP_S . We get a triangle

$$\Gamma_*(Y' \rightarrow Z') \rightarrow \Gamma_*(Y' \rightarrow X') \rightarrow \Gamma_*(Z' \rightarrow X') \rightarrow \Gamma_*(Y' \rightarrow Z')[+1].$$

Since $Z. \setminus Y. = \tilde{T}.$ and $\bar{X}.$ is a compactification of both $\tilde{X}.$ and $\tilde{U}.$, using Theorem 5.8, we can write this triangle as

$$h(T.) \rightarrow h(X.) \rightarrow h(U.) \rightarrow h(T.)[1].$$

Finally, to check iii), if $f : X. \rightarrow Y.$ is a proper hypercover, $\tilde{X}. \rightarrow X.$ a proper hypercover and $\bar{X}.$ a compactification of $\tilde{X}.$, we notice that the

composite map $\tilde{X} \rightarrow Y$ is a proper hypercover so that $h(Y) = \Gamma_*(\bar{X} \setminus \tilde{X} \rightarrow \bar{X}) = h(X)$.

5.4. Weight complexes of stacks. We now prove Theorem 0.1 of the introduction. Given a stack \mathfrak{X} and $X \rightarrow \mathfrak{X}$ a proper hypercover by a simplicial variety we define $h(\mathfrak{X}) = h(X)$.

Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a proper map of stacks and $X \rightarrow \mathfrak{X}$ and $Y \rightarrow \mathfrak{Y}$ proper hypercovers of these. According to Lemma 2.15 the induced map $X \times_{\mathfrak{Y}} Y \rightarrow X$ is a proper hypercover. Therefore, by Theorem 5.9 iii), we get a map

$$h(f) : h(\mathfrak{X}) = h(X \times_{\mathfrak{Y}} Y) \rightarrow h(Y) = h(\mathfrak{Y}).$$

If $g : \mathfrak{Y} \rightarrow \mathfrak{Z}$ is a proper map of stacks one easily checks that $h(g \circ f) = h(g) \circ h(f)$. When $f = \text{id}_{\mathfrak{X}}$ we get $h(f) = \text{id}_{h(\mathfrak{X})}$, *i.e.* $h(\mathfrak{X})$ does not depend on the choice of X .

Remark 5.10. If \mathfrak{X} is a stack, then for any variety (and hence for any simplicial variety) X , $\text{Hom}_S(X, \mathfrak{X})$ is a groupoid. If $f : X' \rightarrow \mathfrak{X}$ and $g : X'' \rightarrow \mathfrak{X}$ are two proper hypercovers, then an isomorphism $\theta : f \rightarrow g$ induces a section $\tilde{\theta} : X' \rightarrow X' \times_{\mathfrak{X}} X''$ of the projection $p' : X' \times_{\mathfrak{X}} X'' \rightarrow X'$ and hence the map $h(X') \rightarrow h(X' \times_{\mathfrak{X}} X'')$ induced by θ coincides with the homotopy equivalence $h(p'') h(p')^{-1}$, where $p'' : X' \times_{\mathfrak{X}} X'' \rightarrow X''$ is the second projection. Therefore the functor $\mathfrak{X} \rightarrow h(\mathfrak{X})$ from the 2-category of stacks to the category of homotopy classes of maps between complexes of motives maps each 2-morphism to the identity.

If $\mathfrak{Y} \subset \mathfrak{X}$ is a closed substack with complement \mathfrak{U} and if $X \xrightarrow{f} \mathfrak{X}$ is a proper hypercover, $U = f^{-1}(\mathfrak{U})$ is strongly open in X with complement a proper hypercover of \mathfrak{Y} . Therefore Theorem 5.9 ii) gives a triangle

$$h(\mathfrak{Y}) \rightarrow h(\mathfrak{X}) \rightarrow h(\mathfrak{U}) \rightarrow h(\mathfrak{Y})[1].$$

If G is a finite group acting on a regular projective scheme X , and $[X/G]$ the quotient stack, a proper hypercover of $[X/G]$ is the simplicial scheme $\text{Cosk}_0^{[X/G]}(X)$, which is isomorphic to $(X \times EG)/G$, where EG is the standard contractible simplicial set with free action of G . In degree $k \geq 1$ we have

$$h(X \times G^k) = \bigoplus_{(g_1, \dots, g_k) \in G^k} h(X)$$

and so $h((X \times EG)/G)$ is the chain complex computing the homology groups $H_*(G, h(X))$ in the Karoubian category of motives. Since the order of G is invertible, this homology vanishes except for $H_0(G, h(X)) = h(X)^G$, the image of the projector $\frac{1}{\#G} \sum_{g \in G} g_*$.

To check that $h(\mathfrak{X})$ is homotopy equivalent to a bounded complex we first need a lemma.

Lemma 5.11. *If $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a finite, representable radicial map of stacks, the induced map $\mathbf{G}(\mathfrak{X}) \rightarrow \mathbf{G}(\mathfrak{Y})$ is a weak equivalence.*

Proof of Lemma. Let $Y. \rightarrow \mathfrak{Y}$ be a proper hypercover of \mathfrak{Y} . Since f is representable, $X. : \mathfrak{X} \times_{\mathfrak{Y}} Y.$ is a simplicial scheme and the natural map $X. \rightarrow \mathfrak{X}$ is a proper hypercover. Let $g : X. \rightarrow Y.$ be the induced map. Since f is finite and radicial each $g_i : X_i \rightarrow Y_i$ is finite and radicial and so ([28]) $g_{i*} : \mathbf{G}(X_i) \rightarrow \mathbf{G}(Y_i)$ is a weak equivalence. Taking homotopy colimits we get that $g_* : \mathbf{G}(X.) \rightarrow \mathbf{G}(Y.)$ is a weak equivalence, and by 4.6 $\mathbf{G}(\mathfrak{X}) \rightarrow \mathbf{G}(\mathfrak{Y})$ is a weak equivalence. \square

To prove that $h(\mathfrak{X})$ is bounded we can assume, by noetherian induction, that \mathfrak{X} is irreducible. By Proposition 1.3, there is a dense open $\mathfrak{U} \subset \mathfrak{X}$ which is a quotient stack and, by noetherian induction, we just need to show that $h(\mathfrak{U})$ is bounded. The quotient stack \mathfrak{U} admits some equivariant compactification (Lemma 1.4) $[V/G] = \mathcal{V}$ proper over S . By induction on dimension it is enough to show that $R(\mathcal{V})$ is bounded. By resolution of singularities (Theorem 1.8) there exists a proper morphism of quotient stacks

$$p : [Y/H] \rightarrow \mathcal{V}$$

where Y is regular and integral and a dense open substack $\mathcal{W} \subset \mathcal{V}$ such that the induced map $p^{-1}(\mathcal{W}) \rightarrow \mathcal{W}$ is representable and radicial. We need to show that $h(\mathcal{V})$ is bounded *i.e.*, again by induction, that $h(\mathcal{W})$ is bounded. From the previous lemma, $p^{-1}(\mathcal{W}) \rightarrow \mathcal{W}$ is a universal G -equivalence, therefore $h(\mathcal{W}) = h(p^{-1}(\mathcal{W}))$. But, since Y is proper and regular $h([Y/H])$ is bounded hence, by induction on dimensions, $h(p^{-1}(\mathcal{W}))$ is bounded.

This ends the proof of Theorem 0.1. \square

Corollary 0.2 follows from Theorem 0.1 by letting $\chi_c(\mathfrak{X})$ be the class in $K_0(\mathbf{KM}_S)$ of the complex $h(\mathfrak{X})$ (compare [14], Lemma 3 and Theorem 4).

5.5. Chow motives. If $S = \mathrm{Spec}(k)$ with k a perfect field (of arbitrary characteristic), there are three possible categories of motives that we could consider:

- (1) The category \mathbf{KM}_k of K_0 -motives over $\mathrm{Spec}(k)$.
- (2) The category \mathbf{CHM}_k^0 of effective Chow motives over $\mathrm{Spec}(k)$, which is the pseudo-abelian completion of the additive category in which morphisms are *degree zero* correspondences, modulo linear equivalence, between smooth projective varieties over k , as in section 5 of [25].
- (3) The category \mathbf{CHM}_k which is the pseudo-abelian completion of the additive category in which morphisms are *all* correspondences, modulo linear equivalence, between smooth projective varieties over k .

We can also take any or all of these categories with rational coefficients, getting categories $\mathbf{KM}_{k\mathbb{Q}}$, $\mathbf{CHM}_{k\mathbb{Q}}^0$ and $\mathbf{CHM}_{k\mathbb{Q}}$.

Remark 5.12. If we do not assume that k is perfect then we should take the category of all regular projective varieties over k . We can still define

\mathbf{CHM}_k^0 and \mathbf{CHM}_k as above. Since the product of two regular varieties will no longer necessarily be regular the composition of correspondences is harder to define.

For any smooth variety V over k the Chern character $ch : K_0(V) \rightarrow \mathbf{CH}^*(V)_{\mathbb{Q}}$ induces an isomorphism $ch : K_0(V)_{\mathbb{Q}} \rightarrow \mathbf{CH}^*(V)_{\mathbb{Q}}$. Therefore, given varieties X and Y , the map:

$$\begin{aligned} \tau : K_0(X \times Y)_{\mathbb{Q}} &\rightarrow \mathbf{CH}^*(X \times Y)_{\mathbb{Q}} \\ \alpha &\mapsto ch(\alpha)p_Y^*(\mathrm{Td}(Y)) \end{aligned}$$

where $p_Y : X \times Y \rightarrow Y$ is the projection onto the second factor, is an isomorphism of \mathbb{Q} -vector spaces, since the Todd genus is a unit in the Chow ring.

Suppose that $\alpha \in K_0(X \times Y)$ and $\beta \in K_0(Y \times Z)$ are K_0 -correspondences from X to Y and from Y to Z respectively. Then, by the Grothendieck Riemann-Roch theorem, [3], we have:

$$\begin{aligned} \tau(\beta \cdot \alpha) &= ch(p_{XZ*}(p_{XY}^*(\alpha)p_{YZ}^*(\beta)))p_Z^*(\mathrm{Td}(Z)) \\ &= p_{XZ*}(ch(p_{XY}^*(\alpha)p_{YZ}^*(\beta))p_Y^*(\mathrm{Td}(Y)))p_Z^*(\mathrm{Td}(Z)) \\ &= p_{XZ*}(p_{XY}^*ch(\alpha)p_{YZ}^*ch(\beta)p_Y^*(\mathrm{Td}(Y)))p_Z^*(\mathrm{Td}(Z)) \\ &= p_{XZ*}(p_{XY}^*\tau(\alpha)p_{YZ}^*\tau(\beta)) \\ &= \tau(\beta) \cdot \tau(\alpha). \end{aligned}$$

Therefore τ is compatible with composition of correspondences, and hence gives an isomorphism of categories:

$$\mathbf{KM}_{k\mathbb{Q}} \rightarrow \mathbf{CHM}_{k\mathbb{Q}}.$$

If $f : X \rightarrow Y$ is a morphism, and $\gamma_f : X \rightarrow X \times Y$ is the induced map, which is an isomorphism onto the graph $\Gamma(f)$ of f , since the normal bundle of $\Gamma(f) \subset X \times Y$ is isomorphic to $f^*(T_Y)$ (the pull back of the tangent bundle of Y), the Grothendieck Riemann-Roch theorem for γ_f gives $\tau([\mathcal{O}_{\Gamma(f)}]) = [\Gamma(f)]$, and so the functor τ respects the natural functors from the category of smooth projective varieties to the two categories of motives.

Since the functor τ is an isomorphism, we have an isomorphism:

$$K_0(\mathbf{KM}_{k\mathbb{Q}}) \rightarrow K_0(\mathbf{CHM}_{k\mathbb{Q}}).$$

Since the degree zero correspondences from X to Y are a subgroup of the group of all correspondences from X to Y , and the inclusion is compatible with composition, we get a functor $\mathbf{CHM}_k^0 \rightarrow \mathbf{CHM}_k$. This functor preserves idempotents and is therefore exact, and so induces a homomorphism on the associated Grothendieck groups:

$$K_0(\mathbf{CHM}_k^0) \rightarrow K_0(\mathbf{CHM}_k).$$

From Corollary 0.2 and the isomorphism $K_0(\mathbf{KM}_{k\mathbb{Q}}) \rightarrow K_0(\mathbf{CHM}_{k\mathbb{Q}})$, we get:

Corollary 5.13. *Given any reduced variety X over k one can define an element $\chi_c(X) \in K_0(\mathbf{CHM}_{k\mathbb{Q}})$ in such a way that*

- i) *If X is smooth and projective, $\chi_c(X)$ is the class of (X, Δ_X) , where Δ_X is the diagonal in $X \times X$.*
- ii) *If $Y \subset X$ is a closed subset, the equality*

$$\chi_c(X) = \chi_c(Y) + \chi_c(X \setminus Y)$$

holds in $K_0(\mathbf{CHM}_{k\mathbb{Q}})$.

One can also prove the existence in $K_0(\mathbf{CHM}_{k\mathbb{Q}}^0)$ of an Euler characteristic satisfying i) and ii) above (and mapping to $\chi_c(X)$ in $K_0(\mathbf{CHM}_{k\mathbb{Q}})$) by using the method of [14]. Indeed Theorem 3.4 above shows that if $f : X \rightarrow Y$ is a proper hypercover between simplicial varieties having proper face maps, the induced map of Gersten complexes tensored with \mathbb{Q} is a quasi-isomorphism. If we use this fact instead of Proposition 1 in [14], the whole argument in *op. cit.* remains valid (after tensoring with \mathbb{Q}) and the existence of an Euler characteristic in $K_0(\mathbf{CHM}_{k\mathbb{Q}}^0)$ follows as in [14], Theorem 4. When k has characteristic zero, this Euler characteristic is the image of the Euler characteristic in $K_0(\mathbf{CHM}^0 k)$ constructed in [14], [17], [2].

5.6. A variant. Let ℓ be a prime integer.

Definition 5.14. If $f : X \rightarrow Y$ is proper and surjective morphism, we say that f is an ℓ' -envelope if, for every point $y \in Y$, there is an $x \in X$ s.t. $f(x) = y$ and $k(y) \subset k(x)$ is a finite algebraic extension of degree prime to ℓ .

It follows from a result of Gabber [8] that if X is an irreducible reduced separated scheme flat and of finite type over $\mathrm{Spec}(R)$, for R an excellent Dedekind ring with ℓ invertible in R , there exists an ℓ' -envelope $X' \rightarrow X$ with X' regular.

If \mathcal{P} is the class of ℓ' -envelopes; then \mathcal{P} satisfies the conditions of §2.2, and so we can talk of a map $f : X \rightarrow Y$ of simplicial schemes being a “hyper ℓ' -envelope”. One may then show, by the methods of section 3.4, that any homology theory E on the category of varieties taking values in the category of spectra, localized away from ℓ , satisfies descent with respect to hyper ℓ' -envelopes, and that one has a functor, for $S = \mathrm{Spec}(R)$

$$\mathrm{Stack}_S \rightarrow \mathrm{Ho}(C_*(KC_S; \mathbb{Z}_{(\ell)}))$$

from the category of Deligne-Mumford stacks of finite type over S to the category of homotopy classes of complexes of K_0 -motives over S with $\mathbb{Z}_{(\ell)}$ -coefficients with the properties of Theorem 0.1, *except* that we do not know if it takes values in the subcategory of complexes homotopy equivalent to bounded complexes.

6. CONTRAVARIANCE OF WEIGHT COMPLEXES

6.1. A Category of complexes of sheaves. In this section \mathcal{V} will be the category of varieties (= reduced separated schemes of finite type) over a field k .

Definition 6.1. If X is a variety, we say that a quasi-coherent sheaf \mathcal{F} on X is *strongly acyclic* if for every morphism of varieties $f : X \rightarrow Y$ and every quasi-coherent sheaf \mathcal{G} of \mathcal{O}_X -modules,

$$R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for $i > 0$. We say that \mathcal{F} is *universally strongly acyclic*, if for every morphism of varieties $f : X \rightarrow Y$, all morphisms $Z \rightarrow Y$, and every quasi-coherent sheaf \mathcal{G} of $\mathcal{O}_{Z \times_Y X}$ -modules,

$$R^i f_{Z*}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

where f_Z is the base change of f by the morphism $Z \rightarrow Y$.

The example that we have in mind is:

Lemma 6.2. *If $j : U = \text{Spec}(A) \hookrightarrow X$ is an affine open subset, and if $\mathcal{F} = j_* \widetilde{M}$, for \widetilde{M} the quasi-coherent sheaf of \mathcal{O}_U -modules associated to the A -module M , then \mathcal{F} is universally strongly acyclic.*

Proof. First of all observe that since varieties are separated, any morphism $g : U \rightarrow Y$ of varieties with affine domain is an affine morphism, and so $R^i g_*(\widetilde{M}) = 0$ for $i > 0$. It then follows by a standard spectral sequence argument that $j_* \widetilde{M}$ is acyclic with respect to any morphism $f : X \rightarrow Y$.

Furthermore, if \mathcal{G} is a quasi-coherent sheaf on X , then $j^* \mathcal{G}$ is quasi-coherent (= $\widetilde{\mathcal{G}(U)}$), and

$$j_* \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq j_*(\widetilde{M} \otimes_{\mathcal{O}_U} j^* \mathcal{G}).$$

We see, therefore, that \widetilde{M} is strongly acyclic. Finally, let $f : X \rightarrow Y$ and $Z \rightarrow Y$ be morphisms of varieties; since higher direct images with respect to

$$f_Z : Z \times_Y X \rightarrow Z$$

can be computed locally on Z , we may suppose that Z is affine. But then (separation again!) $Z \times_Y U$ is an affine open subset of $Z \times_Y X$ and writing $j_Z : Z \times_Y U \rightarrow Z \times_Y X$ for the inclusion, we have:

$$j_* \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z \times_Y X} \simeq (j_Z)_*(\widetilde{M} \otimes_{\mathcal{O}_U} \mathcal{O}_{Z \times_Y U}).$$

But $\widetilde{M} \otimes_{\mathcal{O}_U} \mathcal{O}_{Z \times_Y U}$ is again a quasi-coherent sheaf on an affine open, and so $(j_Z)_*(\widetilde{M} \otimes_{\mathcal{O}_U} \mathcal{O}_{Z \times_Y U})$ is strongly acyclic. Thus $j_* \widetilde{M}$ is universally strongly acyclic. □

Note also, that by a standard spectral sequence argument:

Lemma 6.3. *If $f : X \rightarrow Y$ is a morphism, and \mathcal{F} is a strongly acyclic (resp. universally strongly acyclic) quasi-coherent sheaf of \mathcal{O}_X -modules, then $f_*\mathcal{F}$ is strongly acyclic (resp. universally strongly acyclic).*

Given varieties X and Y , we let $\mathbf{C}(X, Y)$ be the following Waldhausen category (also referred to as a category with cofibrations and weak equivalences, but here we follow the terminology of [32]).

The objects of $\mathbf{C}(X, Y)$ are bounded complexes of \mathfrak{F}^* of quasi-coherent $\mathcal{O}_{X \times Y}$ -modules, such that

- each \mathfrak{F}^i is flat over X ;
- the cohomology sheaves of \mathfrak{F}^\cdot are coherent with support proper over Y
- each \mathfrak{F}^i is universally strongly acyclic.

The cofibrations in $\mathbf{C}(X, Y)$ are the monomorphisms $\mathfrak{F}^\cdot \rightarrow \mathcal{G}^\cdot$ with cokernel in $\mathbf{C}(X, Y)$, and the weak equivalencies are the quasi-isomorphisms.

Lemma 6.4.

- 1) *Let $f : X' \rightarrow X$ be a morphism of varieties and \mathcal{F}^\cdot an object in $\mathbf{C}(X, Y)$. Then $(f \times 1_Y)^*(\mathcal{F}^\cdot)$ lies in $\mathbf{C}(X', Y)$.*
- 2) *Let $p : Y \rightarrow Y'$ be a proper morphism of varieties. Then $(1_X \times p)_*(\mathcal{F}^\cdot)$ lies in $\mathbf{C}(X, Y')$.*

Proof. The first assertion is a basic property of sheaves of flat modules. Turning to the second assertion, given the previous lemma, we need only show that $(1_X \times p)_*(\mathcal{F}^\cdot)$ is \mathcal{O}_X -flat. If $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ is an exact sequence of \mathcal{O}_X -modules, then $0 \rightarrow \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}^\cdot \rightarrow \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{F}^\cdot \rightarrow \mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{F}^\cdot \rightarrow 0$ is exact. By universal acyclicity, the push forward of this sequence is still exact, and hence $(1_X \times p)_*(\mathcal{F}^\cdot)$ is still \mathcal{O}_X -flat. \square

By the hypothesis on the objects in $\mathbf{C}(X, Y)$, both p_* and f^* are exact functors preserving weak equivalences. Furthermore, we can rigidify these categories by choosing inverse images with respect to all $f : X' \rightarrow X$. Direct images are already “rigid”, and so we get a functor

$$\begin{array}{ccc} \mathcal{V}^{\text{op}} \times \mathcal{V} & \rightarrow & \text{Waldhausen Categories} \\ (X, Y) & \mapsto & \mathbf{C}(X, Y) \end{array}$$

If X is a variety, let us write $\mathbf{C}(X)$ for the Waldhausen category of complexes of quasi-coherent sheaves of \mathcal{O}_X -modules with bounded coherent cohomology, and weak equivalences given by quasi-isomorphisms. Recall [32] that the K -theory of this category is naturally isomorphic to the G -theory of X .

If X and Y are varieties, and $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$ are the projections, observe that if \mathcal{F}^\cdot is an object of $\mathbf{C}(X, Y)$, then

$$\mathcal{G}^\cdot \mapsto q_*(p^*(\mathcal{G}^\cdot) \otimes_{\mathcal{O}_{X \times Y}} \mathcal{F}^\cdot)$$

defines an exact functor

$$\mathbf{C}(X) \rightarrow \mathbf{C}(Y)$$

and furthermore it is straightforward to check:

Lemma 6.5.

$$\mathbf{C}(X) \times \mathbf{C}(X, Y) \rightarrow \mathbf{C}(Y)$$

is a bi-exact functor.

Equivalently, we have an exact functor from $\mathbf{C}(X, Y)$ to the exact category $\mathbf{Exact}(\mathbf{C}(X), \mathbf{C}(Y))$ of exact functors from $\mathbf{C}(X)$ to $\mathbf{C}(Y)$. Not surprisingly, we have a composition law on the categories $\mathbf{C}(X, Y)$, which it is easy to see is compatible with composition:

$$\mathbf{Exact}(\mathbf{C}(X), \mathbf{C}(Y)) \times \mathbf{Exact}(\mathbf{C}(Y), \mathbf{C}(Z)) \rightarrow \mathbf{Exact}(\mathbf{C}(X), \mathbf{C}(Z)) .$$

Lemma 6.6. *Given a triple of proper varieties X, Y, Z , then we have a bi-exact functor:*

$$\begin{aligned} \gamma_{X,Y,Z} : \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) &\rightarrow \mathbf{C}(X, Z) \\ (\mathcal{G}, \mathcal{F}) &\mapsto r_*(p^*(\mathcal{G}) \otimes_{\mathcal{O}_Y} q^*(\mathcal{F})) , \end{aligned}$$

where $p : X \times Y \times Z \rightarrow X \times Y$, $q : X \times Y \times Z \rightarrow Y \times Z$, and $r : X \times Y \times Z \rightarrow X \times Z$ are the projections. Furthermore, given four varieties we an obvious isomorphism,

$$\gamma_{X,Z,W} \cdot (\gamma_{X,Y,Z} \times I_{\mathbf{C}(Z,W)}) \simeq \gamma_{X,Y,W} \cdot (I_{\mathbf{C}(X,Y)} \times \gamma_{Y,Z,W})$$

Proof. It is a straightforward exercise using the definition of $\mathbf{C}(X, Y)$ to check both that $r_*(p^*(\mathcal{G}) \otimes_{\mathcal{O}_Y} q^*(\mathcal{F}))$ is indeed an object in $\mathbf{C}(X, Z)$ and that $\gamma_{X,Y,Z}$ is biexact. \square

6.2. Enriching the category of varieties over the category of chain complexes. It will be convenient to work not with K -theory spectra but with *chain complexes* which compute rational K -theory. Recall that if \mathbf{A} is a spectrum, then $\pi_*(\mathbf{A}) \otimes \mathbb{Q} \simeq H_*(\mathbf{A}, \mathbb{Q})$. If \mathcal{E} is an exact category then, following [26], we have an explicit chain complex $M_*(\mathcal{E})$ which computes $H_*(\mathbf{K}(\mathcal{E}), \mathbb{Q})$. It is the complex associated to the cubical object which in degree n is the \mathbb{Q} -vector space spanned by the set of n -cubes of exact sequences. (See [30] for a nice exposition of this construction.) This construction is functorial in the sense that an exact functor $\gamma : \mathcal{E} \rightarrow \mathcal{F}$ induces a map of chain complexes $M_*(\gamma) : M_*(\mathcal{E}) \rightarrow M_*(\mathcal{F})$, and that an isomorphism between functors $\gamma \rightarrow \gamma'$ induces a homotopy between $M_*(\gamma)$ and $M_*(\gamma')$.

One important feature of this construction is that if \mathcal{E}, \mathcal{F} and \mathcal{G} are exact categories, and $\mu : \mathcal{E} \times \mathcal{F} \rightarrow \mathcal{G}$ is a biexact functor, then given an m -cube α of exact sequences in \mathcal{E} and an n -cube β of exact sequences in \mathcal{F} , $\mu(\alpha, \beta)$ is an $(m+n)$ -cube of exact sequences in \mathcal{G} . Clearly if either α or β is degenerate, their product is too, and thus we have a pairing:

$$M_*(\mu) : M_*(\mathcal{E}) \otimes_{\mathbb{Q}} M_*(\mathcal{F}) \rightarrow M_*(\mathcal{G}) .$$

Suppose that \mathcal{E} is an exact category and \mathbf{w} is a subcategory of \mathcal{E} so that $(\mathcal{E}, \mathbf{w})$ is a category with cofibrations (= to the admissible monomorphisms

in \mathcal{E}) and weak equivalences. Then we have a fibration sequence of K -theory spectra

$$\mathbf{K}(\mathcal{E}^{\mathbf{w}}) \rightarrow \mathbf{K}(\mathcal{E}) \rightarrow \mathbf{K}(\mathbf{w}\mathcal{E})$$

[33], 1.6.4 and [32], 1.8.2. Observe that $M_*(\mathcal{E}^{\mathbf{w}})$ is a subcomplex of $M_*(\mathcal{E})$, since a degenerate cube of exact sequences in \mathcal{E} which lies in $\mathcal{E}^{\mathbf{w}}$ is degenerate as a cube in $\mathcal{E}^{\mathbf{w}}$. Hence we have:

Lemma 6.7. *The rational K -theory of the Waldhausen category $(\mathcal{E}, \mathbf{w})$ is computed by the chain complex*

$$M_*(\mathbf{w}\mathcal{E}) = M_*(\mathcal{E})/M_*(\mathcal{E}^{\mathbf{w}}) .$$

Furthermore, $(\mathcal{E}, \mathbf{w}) \mapsto M_*(\mathbf{w}\mathcal{E})$ is a covariant functor from exact categories to chain complexes.

Definition 6.8. If X and Y are varieties, we shall write $M_*(X, Y)$ for $M_*(\mathbf{C}(X, Y))$. We then have a functor:

$$\begin{aligned} \mathcal{V}^{\text{op}} \times \mathcal{V} &\rightarrow \text{Chain}(\mathbb{Q}) \\ (X, Y) &\mapsto M_*(X, Y) \end{aligned}$$

where $\text{Chain}(\mathbb{Q})$ is the category of chain complexes of \mathbb{Q} -vector spaces.

Proposition 6.9. *If X is regular and projective, the natural functor from $\mathbf{C}(X, Y)$ to the Waldhausen category of all bounded complexes of quasi-coherent sheaves with coherent cohomology on $X \times Y$ induces an isomorphism on K -theory, and hence a weak equivalence of K -theory spectra:*

$$\mathbf{K}\mathbf{C}(X, Y) \simeq \mathbf{G}(X \times Y) .$$

Proof. By [32] 1.9.7 and 1.9.8 it is enough to show:

- i) Any bounded complex \mathcal{G} of quasi-coherent sheaves on $X \times Y$ is isomorphic in the derived category to a bounded complex \mathfrak{F} of universally strongly acyclic quasi-coherent sheaves with each \mathfrak{F}^i flat over X .
- ii) Any bounded complex \mathcal{G} of quasi-coherent sheaves on $X \times Y$ with coherent cohomology is quasi-isomorphic to a bounded complex of coherent sheaves.

Assertion ii) is a standard fact, since any quasi-coherent sheaf is a direct limit of coherent sheaves.

To prove i), we have to deal with two issues, flatness and universal strong acyclicity. The approach we take addresses both issues simultaneously. If V is a quasi-compact separated scheme, and $\mathfrak{U} = \{U_i\}_{i=1}^n$ is a finite affine cover of V , \mathfrak{U} determines a diagram in the category of affine schemes, with vertices the non-empty intersections U_I ($\emptyset \neq I \subset \{1, \dots, n\}$) of opens in the cover and arrows the inclusions. Following [1] we define a quasi-coherent \mathfrak{U} -module \mathfrak{M} to be a family M_I of $\mathcal{O}_V(U_I)$ -modules, together with maps $M_I \rightarrow M_J$ if $I \subset J$ which are $\mathcal{O}_V(U_J)$ linear and which satisfy the obvious presheaf condition. A sequence of \mathfrak{U} -modules is exact if the induced sequences of

$\mathcal{O}_V(U_I)$ modules are exact for all I , and a \mathfrak{U} -module \mathfrak{M} is flat if all the M_I are flat.

There is an obvious exact functor from quasi-coherent sheaves on V to quasi-coherent \mathfrak{U} -modules, sending a quasi-coherent sheaf \mathcal{F} to the \mathfrak{U} -module $U_I \mapsto \mathcal{F}(U_I)$. On the other hand, given a quasi-coherent \mathfrak{U} -module \mathfrak{M} , there is a Čech complex $\check{C}^*(\mathfrak{M})$ which is a bounded complex of quasi-coherent sheaves, and which is functorial with respect to \mathfrak{M} . Since the sheaves in the complex are all direct sums of direct images of quasi-coherent sheaves on affine opens they are universally strongly acyclic. If \mathfrak{F} is the \mathfrak{U} -module associated to a quasi-coherent sheaf \mathcal{F} on V , then $\check{C}^*(\mathfrak{F})$ is just the standard Čech resolution $\check{C}^*(\mathfrak{U}, \mathcal{F})$ which is quasi-isomorphic to \mathfrak{F} . It is shown in section (1.2) of *op. cit* that the functor $\mathfrak{M} \mapsto \check{C}^*(\mathfrak{M})$ is exact, and that if \mathfrak{M} is flat as a \mathfrak{U} -module, then $\check{C}^*(\mathfrak{M})$ is a complex of flat quasi-coherent sheaves on V . Furthermore any quasi-coherent \mathfrak{U} -module is a quotient of a flat quasi-coherent \mathfrak{U} -module. Suppose now that \mathcal{G}^* is a bounded complex of quasi-coherent sheaves on V . Let \mathfrak{G}^* be the associated complex of quasi-coherent \mathfrak{U} -modules, and choose a resolution \mathfrak{F}^* of \mathfrak{G}^* by flat quasi-coherent \mathfrak{U} -modules. If we then apply the functor \check{C}^* we obtain a complex $\check{C}^*(\mathfrak{F}^*)$ of universally strongly acyclic flat quasi-coherent \mathcal{O}_V -modules quasi-isomorphic to \mathcal{G}^* .

Returning now to the situation of part ii) of the theorem, suppose that \mathcal{G}^* is a bounded complex of quasi-coherent sheaves of $X \times Y$ -modules. Since we assume that X is regular, \mathcal{O}_X has finite global tor-dimension. Hence if \mathfrak{F}^* is a flat resolution of \mathfrak{G}^* as above, for sufficiently large k , the kernel of $\mathfrak{F}^{-k} \rightarrow \mathfrak{F}^{-k+1}$ while not flat as an $\mathcal{O}_{X \times Y}$ module, will be flat as an \mathcal{O}_X -module. Hence taking the “good” truncation of the complex \mathfrak{F}^* at that point we obtain a finite resolution $\overline{\mathfrak{F}}^*$ of \mathfrak{G}^* by \mathcal{O}_X -flat \mathfrak{U} -modules, and so on applying the functor \check{C}^* we obtain a bounded complex, isomorphic in the derived category to \mathcal{G}^* , of universally strongly acyclic quasi-coherent sheaves which are flat over \mathcal{O}_X . \square

Corollary 6.10. *Suppose that X is regular and projective, Y is an arbitrary variety and $Y' \subset Y$ is a closed subscheme with open complement U . There is map $M_*(X, Y) \rightarrow M_*(X, U)$ (since the restriction of an object \mathfrak{F} in $\mathbf{C}(X, Y)$ to $X \times U$ is clearly in $\mathbf{C}(X, U)$). Then the associated sequence of maps of complexes:*

$$M_*(X, Y') \rightarrow M_*(X, Y) \rightarrow M_*(X, U).$$

induces a quasi-isomorphism from the cone of the map

$$M_*(X, Y') \rightarrow M_*(X, Y)$$

to $M_(X, U)$.*

Proof. This follows from Quillen’s localization theorem, which implies that

$$\mathbf{G}(X \times Y') \rightarrow \mathbf{G}(X \times Y) \rightarrow \mathbf{G}(X \times U)$$

is a fibration sequence of spectra. \square

Definition 6.11. Let X be a variety, and Y a simplicial variety with proper face maps. Then we define

$$M_*(X, Y) := \text{Tot}_*(j \mapsto M_*(X, Y_j)).$$

If $\beta : Y \rightarrow Z$ is a morphism between simplicial varieties with proper maps which is proper in each degree, we define

$$M_*(X, \beta) := \text{cone}(M_*(X, Y) \rightarrow M_*(X, Z)).$$

If X is a simplicial variety, we define (with notation as above)

$$M_*(X., Y) := \text{Tot}_*^{\prod}(i \mapsto M_*(X_i, Y)),$$

i.e., $M_k(X., Y) = \prod_i M_{i+k}(X_i, Y)$, and

$$M_*(X., \beta) := \text{cone}(M_*(X., Y) \rightarrow M_*(X., Z)).$$

Note that since infinite products are exact, $M_*(X., \beta)$ is isomorphic to

$$\text{Tot}_*^{\prod}(i \mapsto M_*(X_i, \beta)).$$

If $\alpha : W \rightarrow X$ is an arrow between simplicial varieties, we set

$$M_*(\alpha., \beta) := \text{cone } M_*(X., \beta) \rightarrow M_*(W., \beta)[-1].$$

These constructions are all contravariant in the first variable, and covariant in the second variable (e.g. with respect to morphisms between arrows of simplicial objects in the category of proper maps between varieties). There are also natural equivalences when two of these constructions should agree ; for example if $X = X$ is a constant simplicial variety, the natural map

$$M_*(X, Y) \rightarrow M_*(X., Y)$$

is a quasi-isomorphism.

Suppose that X, Y, Z are all simplicial proper varieties. Then we have a composition:

$$M_*(X., Y) \otimes M_*(X., Z) \rightarrow M_*(X., Z)$$

which is associative up to homotopy, defined by

$$M_i(X., Y) \otimes M_m(Y., Z) \rightarrow M_{i+m}(X., Z)$$

$$\begin{aligned} \prod_j \left(\bigoplus_{k+l=j+i} M_l(X_j, Y_k) \right) \otimes \prod_n \left(\bigoplus_{p+q=m+n} M_p(Y_n, Z_q) \right) \\ \rightarrow \prod_r \left(\bigoplus_{t+s=i+m+r} M_t(X_r, Z_s) \right) \end{aligned}$$

which for $l + p = t$, k and $s = q$, is induced by the pairings

$$M_l(X_j, Y_k) \otimes M_p(Y_n, Z_q) \rightarrow M_t(X_r, Z_s)$$

and is zero otherwise.

It is straightforward to check that this defines a category enriched over the category of chain complexes of vector spaces (with respect to the monoidal structure given by tensor product), with objects simplicial varieties.

Lemma 6.12. *Let $\pi_* : Y' \rightarrow Y$ be a proper hypercover of simplicial varieties with proper faces maps. Then, if X_\bullet is a simplicial regular projective variety, the map*

$$M_*(X_\bullet, Y') \rightarrow M_*(X_\bullet, Y)$$

is a quasi-isomorphism. Similarly, if $\pi : \beta' \rightarrow \beta$ is a map of arrows

$$\begin{array}{ccc} Y' & \xrightarrow{\beta'} & Z' \\ \pi_Y \downarrow & & \downarrow \pi_Z \\ Y & \xrightarrow{\beta} & Z \end{array}$$

with π_Y and π_Z proper hypercovers, we have a weak equivalence, for all arrows α_\bullet between simplicial regular projective varieties

$$M_*(\alpha_\bullet, \beta') \rightarrow M_*(\alpha_\bullet, \beta).$$

Proof. For a fixed X_i we know that $M_*(X_i, Y') \rightarrow M_*(X_i, Y)$ is a quasi-isomorphism, by theorem 4.1. Since $M_*(X_\bullet, Y)$ is the inverse limit, over $n \geq 0$, of $\text{Tot}_*^{\prod}(i \mapsto M_*(X_i, Y), i \leq n)$ a standard \lim^1 argument shows that $M_*(X_\bullet, Y') \rightarrow M_*(X_\bullet, Y)$ is a quasi-isomorphism. A similar argument gives the second assertion of the lemma. \square

Lemma 6.13. *Suppose that Z_\bullet is a simplicial variety with proper face maps, and $Y_\bullet \subset Z_\bullet$ is a closed simplicial subvariety such that the complement $U_\bullet = Z_\bullet \setminus Y_\bullet$ is an open simplicial subvariety. Then, for all simplicial regular projective varieties X_\bullet , the localization sequences*

$$M_*(X_i, Y_j) \rightarrow M_*(X_i, Z_j) \rightarrow M_*(X_i, U_j)$$

induce a fibration sequence

$$M_*(X_\bullet, Y_\bullet) \rightarrow M_*(X_\bullet, Z_\bullet) \rightarrow M_*(X_\bullet, U_\bullet)$$

and similarly, if $\alpha_\bullet : W_\bullet \rightarrow X_\bullet$ is an arrow in the category of simplicial regular projective varieties, we have a (co)fibration sequence:

$$M_*(\alpha_\bullet, Y_\bullet) \rightarrow M_*(\alpha_\bullet, Z_\bullet) \rightarrow M_*(\alpha_\bullet, U_\bullet).$$

Proof. Again a standard \lim^1 argument. \square

Lemma 6.14. *Let $g : X \rightarrow Y$ be a proper morphism in \mathcal{V} , and $\Gamma_g : X \rightarrow X \times Y$ its graph. Let \mathcal{O}_{Γ_g} be the object $(\Gamma_g)_*(\mathcal{O}_X)$ in $\mathbf{C}(X, Y)$. Then if $p : Y' \rightarrow Y$ is proper, $p_*(\mathcal{O}_{\Gamma_g}) = \mathcal{O}_{\Gamma_{p \circ g}}$. (Note that this is an equality rather than an isomorphism!). While if $f : X' \rightarrow X$ is a morphism, we have a canonical isomorphism:*

$$f^*(\mathcal{O}_{\Gamma_g}) \simeq \mathcal{O}_{\Gamma_{f \circ g}}.$$

Hence we have a well defined object in the big Zariski site over X , $(f : X' \rightarrow X) \mapsto (\Gamma_{(g \circ f)})_(\mathcal{O}_{X'})$.*

Proof. Exercise. □

On the category $\text{Chain}(\mathbb{Q})$ of chain complexes concentrated in positive degrees, H_0 gives a functor to the category of \mathbb{Q} -vector spaces, viewed as chain complexes concentrated in degree zero, together with a natural transformation

$$\eta : \text{Id} \rightarrow H_0 .$$

This is the homological equivalent of taking the 0-th stage of the Postnikov tower of a spectrum.

Hence, given a pair of objects $X_., Y_.$ in $RP^{\Delta^{\text{op}}}$, we get a map of cosimplicial simplicial chain complexes:

$$(i, j) \mapsto (M_*(X_i, Y_j) \rightarrow KC(X_i, Y_j))$$

since $H_0(M_*(X_i, Y_j)) \simeq KC(X_i, Y_j)$, and therefore we have a homomorphism

$$H_0(M_*(X_., Y_)) \rightarrow H_0(\text{Tot}_*(KC(X_i, Y_j)) .$$

But the target of this homomorphism is equal to the homotopy classes of maps $\Gamma_*(X_.) \rightarrow \Gamma_*(Y_.)$ (cf. §5.3). More generally, given arrows $\alpha_.$ and $\beta_.$ in $RP^{\Delta^{\text{op}}}$, we get in a similar fashion a morphism

$$\gamma : H_0(M_*(\alpha_., \beta_)) \rightarrow \{\text{Homotopy classes of maps } \Gamma_*(\alpha_.) \rightarrow \Gamma_*(\beta_.)\} .$$

Proposition 6.15. *Let $f : \alpha_. = (W_. \xrightarrow{a} X_.) \rightarrow \beta_. = (Y_. \xrightarrow{b} Z_.)$ be a map of arrows in $RP^{\Delta^{\text{op}}}$, so that we have a commutative diagram :*

$$\begin{array}{ccc} W_ . & \xrightarrow{a} & X_ . \\ f \downarrow & & \downarrow g \\ Y_ . & \xrightarrow{b} & Z_ . \end{array}$$

of maps of simplicial regular projective varieties. Then there is a 0-cycle $[f]$ in $M_(\alpha_., \beta_.)$ such that $\gamma[f]$ is the homotopy class of the map $\Gamma_*(f)$.*

Proof. It follows from Lemma 6.14, that the sheaves $\mathcal{O}_{\Gamma_{f_i}}$ and $\mathcal{O}_{\Gamma_{g_j}}$ determine elements $[\mathcal{O}_{\Gamma_{f_i}}]$ and $[\mathcal{O}_{\Gamma_{g_j}}]$ in $M_0(W_i, Y_i)$ and $M_0(X_j, Z_j)$, respectively, such that for all i , $b_*[\mathcal{O}_{\Gamma_{f_i}}] = a^*[\mathcal{O}_{\Gamma_{g_i}}]$, and which are compatible with all face and degeneracy maps, and therefore define a 0-cycle $[f]$ in $M_*(\alpha_., \beta_.)$. On the other hand, under the map γ , for each i , the class of $[\mathcal{O}_{\Gamma_{f_i}}]$ in $K_0 C(W_i, Y_i)$ is the map $\Gamma_*(f_i)$, and similarly for Γ_{g_j} , and so $\gamma[f] = \Gamma_*(f)$. □

Let $\mathbf{C}'(X, Y)$ be the Waldhausen category of bounded complexes of quasi-coherent sheaves of $\mathcal{O}_{X \times Y}$ modules which are flat over X , and which have coherent cohomology having support proper over Y .

Lemma 6.16. 1) *The inclusion of Waldhausen categories*

$$\mathbf{C}(X, Y) \subset \mathbf{C}'(X, Y)$$

induces an isomorphism on K-theory.

- 2) $\mathbf{C}'(X, Y)$ is contravariant with respect to X : given an object \mathfrak{F} in $\mathbf{C}'(X, Y)$ and a morphism $f : X' \rightarrow X$, $f^*(\mathfrak{F}) \in \mathbf{C}'(X', Y)$.

Proof. Straightforward. \square

We write $M'_*(X, Y)$ for the complex $M_*(\mathbf{C}'(X, Y))$ computing the rational K -theory of $\mathbf{C}'(X, Y)$. Suppose that $f : X \rightarrow Y$ is a flat morphism of varieties. Pick a compactification $i : Y \hookrightarrow \bar{Y}$ with complement Z . Note that $i \cdot f : X \rightarrow \bar{Y}$ is also flat. The structure sheaf of the graph of f , \mathcal{O}_{Γ_f} , is an object in $\mathbf{C}'(X, Y)$ while $\mathcal{O}_{\Gamma_{i \circ f}}$ is an object in $\mathbf{C}'(\bar{Y}, X)$. Notice that the pull back of $\mathcal{O}_{\Gamma_{i \circ f}}$ to $\mathbf{C}'(Z, X)$ is identically zero. Hence $[\mathcal{O}_{\Gamma_{i \circ f}}]$ gives a 0-cycle in the homotopy fibre of $M'_*(\bar{Y}, X) \rightarrow M'_*(Z, X)$, which we denote $M'_*((Z \rightarrow \bar{Y}), X)$. Now pick a non-singular hypercover $\pi : (\tilde{Z} \rightarrow \tilde{Y}) \rightarrow (Z \rightarrow \bar{Y})$. Again by contravariance of $\mathbf{C}'(\cdot, \cdot)$ in the first variable, we get a class (which we denote $\pi^*([\mathcal{O}_{\Gamma_{i \circ f}}])$) in $H_0 M'_*((\tilde{Z} \rightarrow \tilde{Y}), X) \simeq H_0 M_*((\tilde{Z} \rightarrow \tilde{Y}), X)$

Now pick a compactification \bar{X} of X with complement W . By localization, we have a fibration sequence, writing $\alpha = (\tilde{Z} \rightarrow \tilde{Y})$,

$$M_*(\alpha, W) \rightarrow M_*(\alpha, \bar{X}) \rightarrow M_*(\alpha, X)$$

and hence a class in

$$\zeta \in H_0(M_*(\alpha, (W \rightarrow \bar{X})))$$

which maps to $\pi^*([\mathcal{O}_{\Gamma_{i \circ f}}])$.

If $p : \beta = (\tilde{W} \rightarrow \tilde{X}) \rightarrow (W \rightarrow \bar{X})$ is a regular proper hypercover of the arrow $(W \rightarrow \bar{X})$, by descent (3.4), we have that

$$p_* : M_*(\alpha, \beta) \rightarrow M_*(\alpha, (W \rightarrow \bar{X}))$$

is a quasi-isomorphism, and hence we get a class $p_*(\zeta)^{-1} \in H_0(M_*(\alpha, \beta))$. Now applying γ , we get a homotopy class of maps $\Gamma_*(\alpha) \rightarrow \Gamma_*(\beta)$, *i.e.* a map $f^* : h(Y) \rightarrow h(X)$.

We can compare this construction with the one situation in which we have already defined a pull back map, *i.e.* when $f : U \rightarrow X$ is an open immersion:

Proposition 6.17. *Let $i : U \rightarrow X$ is an open immersion between varieties. Then the associated map of weight complexes $h(X) \rightarrow h(U)$ which comes from the definition of weight complexes, agrees with the map induced by viewing \mathcal{O}_{Γ_i} as being quasi-isomorphic to an object of $\mathbf{C}(U, X)$.*

Proof. Recall that the map of weight complexes is constructed as follows. Pick a compactification \bar{X} of X , and set $Y = \bar{X} \setminus X$ and $Z = \bar{X} \setminus U$, so that $Y \subset Z \subset X$, with all three proper varieties. We can find a diagram:

$$\begin{array}{ccccc} \tilde{Y} & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & \longrightarrow & X \end{array}$$

with $\tilde{Y} \rightarrow Y$, $\tilde{Z} \rightarrow Z$ and $\tilde{X} \rightarrow X$ all non-singular proper hypercovers. Consider the induced commutative square:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\alpha} & \tilde{X} \\ j \downarrow & & \downarrow 1_{\tilde{X}} \\ \tilde{Z} & \xrightarrow{\beta} & \tilde{X} \end{array}$$

Following Proposition 6.15, we know that there is a corresponding 0-cycle $M_*(\alpha, \beta)$ which induces the corresponding map of weight complexes. By localization and descent, $M_*(\alpha, \beta) \simeq M_*(\alpha, U)$ and by the compatibility of the objects \mathcal{O}_{Γ_g} of lemma 6.14 with push-forward and pull-back on the source and target of f , we see that the image of this zero cycle is exactly the 0-cycle used to define pull back with respect to the flat morphism $i : U \rightarrow X$. \square

Theorem 6.18. *Let X, Y be projective varieties. Then there is a map*

$$\gamma : K_0(\mathbf{C}(X, Y))_{\mathbb{Q}} \simeq H_0(M_*(X, Y)) \rightarrow \text{Hom}(h(X), h(Y))$$

which is covariant with respect to Y and contravariant with respect to X , and which is the identity if X and Y are regular and projective (note that then $K_0(\mathbf{C}(X, Y)) = G_0(X \times Y)$).

Proof. Let $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ be non-singular proper hypercovers. Given $\varphi \in K_0(\mathbf{C}(X, Y))$, we have $p^*(\varphi) \in H_0(M_*(\tilde{X}, Y))$ and $q_* : M_*(\tilde{X}, \tilde{Y}) \rightarrow M_*(\tilde{X}, Y)$ is a weak equivalence. Hence there is a unique $\tilde{\varphi} \in H_0(M_*(\tilde{X}, \tilde{Y}))$ such that $q_*(\tilde{\varphi}) = p^*(\varphi)$.

Applying the natural transformation γ defined before 6.15, we get an element

$$\gamma(\tilde{\varphi}) \in H_0(\text{Hom}(\Gamma_*(\tilde{X}), \Gamma_*(\tilde{Y}))).$$

That the homotopy class of this map does not depend on the choice of $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ follows the same pattern as 5.9. Functoriality with respect to X and Y is an immediate consequence of the fact that the complex $M_*(X, Y)$ is contravariant with respect to X and covariant with respect to Y . If X and Y are already regular, we take p and q to be the identity and the assertion is clear. \square

Notice that it follows from this result that if $f : Y \rightarrow X$ is a morphism of finite tor-dimension between projective varieties, then there is a well defined map

$$f^* : h(X) \rightarrow h(Y),$$

since \mathcal{O}_{Γ_f} is quasi-isomorphic to an object of $\mathbf{C}(X, Y)$.

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