From Isometric Embeddings to Turbulence

László S´ Jr. (Universität Bonn)

The following dichotomy concerning isometric embeddings of the shere is well-known: whereas the only C^2 isometric embedding of S^2 into $\mathbb{R}^3$ is the standard embedding modulo rigid motion, there exist many C^1 isometric embeddings which can "wrinkle" S^2 into arbitrarily small regions. The latter "flexibility", known as the Nash-Kuiper theorem [8, 7], involves an iteration scheme called convex integration which turned out to have surprisingly wide applicability.

More generally, this type of flexibility appears in a variety of different geometric contexts and is known as the "h-principle" [6]. But one has to distinguish two contrasting cases: in problems which are formally highly undetermined, such as isometric embeddings into Euclidean space with high codimension, one might expect to find flexibility among smooth solutions. On the other hand in problems which are formally determined, like embedding a surface into $\mathbb{R}^3$, the flexibility can only be expected at very low regularity. In these lectures I will focus on this latter case and in particular show how the same ideas can be applied to the Euler equations in fluid mechanics.

After a discussion of the proof of the Nash-Kuiper theorem, we show that - at least if we relax C^1 to Lipschitz -, the ideas can be applied in a general framework originally due to L. Tartar [12], which consists of a wave-plane analysis in the phase space. We then show that with this framework at hand, the celebrated results of Scheffer and Shnirelman [10, 11] concerning the existence of weak solutions to the Euler equations with compact support in space-time, can be recovered [4, 5].

Finally, we take another look at the Nash-Kuiper theorem and analyse whether the construction can be extended to produce more regular solutions [1, 2, 3]. The motivation for this comes from Onsager's theory of turbulence [9], which predicts the existence of certain weak solutions of the Euler equations.

Prerequisites Familiarity with basic PDE theory, conservation laws and differential geometry is assumed.

References

[1] J.F. B, $C^{1,\alpha}$ *Isometric Immersions of Riemannian Spaces*, Doklady **163** (1965), 869-871.

[2] J.F. B, *Irregular $C^{1,\beta}$ Surfaces with Analytic Metric*, Sib. Mat. Zh. **45** (2004), 25-61.

[3] S. C , C. D L , L. S ´ , *h-Principle and Rigidity for $C^{1,\alpha}$ Isometric Embeddings*, arxiv.org/abs/0905.0370v1

[4] C. D L , L. S ´ , *The Euler Equations as a Differential Inclusion*, Ann. Math. **170** (2009), 1417-1436.

[5] C. D L , L. S ´ , *Admissibility Criteria for Weak Solutions of the Euler Equations*, Arch. Rat. Mech. Anal. **172** (2008).

[6] M. G , *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete **9** (3), Springer-Verlag, Berlin, 1986.

[7] N. K , *On C^1 Isometric Embeddings I. II.*, Proc. Kon. Acad. Wet. Amsterdam **A 58** (1955), 545-556, 683-689.

[8] J. N , *C^1 Isometric Embeddings*, Ann. Math. **60** (1954), 383–396.

[9] L. O , *Statistical Hydrodynamics*, Nuovo Cimento (9) 6, Supplemento, 2 (Convegno Internazionale di Meccanica Statistica) (1949), 279–287.

[10] V. S , *An Inviscid Flow with Compact Support in Space-Time*, J. Geom. Anal. **3** (1993), 343–401.

[11] A. S , *On the Nonuniqueness of Weak Solutions of the Euler Equation*, Commun. Pure Appl. Math. **50** (1997), 1261–1286.

[12] L. T , *The Compensated Compactness Method Applied to Systems of Conservation Laws*, In Systems of nonlinear partial differential equations (Oxford, 1982), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **111**, Reidel, Dordrecht, 1983, 263–285.