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Limit of the Wulff Crystal when approaching criticality for site percolation on the triangular lattice

Hugo Duminil-Copin*

Abstract

The understanding of site percolation on the triangular lattice progressed greatly in the last decade. Smirnov proved conformal invariance of critical percolation, thus paving the way for the construction of its scaling limit. Recently, the scaling limit of near-critical percolation was also constructed by Garban, Pete and Schramm. The aim of this article is to explain how these results imply the convergence, as p tends to p_c, of the Wulff crystal to a Euclidean disk. The main ingredient of the proof is the rotational invariance of the scaling limit of near-critical percolation proved by these three mathematicians.

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1 Introduction

Definition of the model Percolation as a physical model was introduced by Broadbent and Hammersley in the fifties [5]. For general background on percolation, we refer the reader to [21, 24, 7].

Let $\mathbb{T}$ be the regular triangular lattice given by the vertices $m + e^{i\pi/3}n$ where $m, n \in \mathbb{Z}$, and edges linking nearest neighbors together. In this article, the vertex set will be identified with the lattice itself. For $p \in (0, 1)$, *site percolation* on $\mathbb{T}$ is defined as follows. The set of configurations is given by $\{\text{open}, \text{closed}\}^{\mathbb{T}}$. Each vertex, also called *site*, is *open* with probability p and *closed* otherwise, independently of the state of other vertices. The probability measure thus obtained is denoted by $\mathbb{P}_p$.

A *path* between a and b is a sequence of sites $v_0, \dots, v_k$ such that $v_0 = a$ and $v_k = b$, and such that (v_i, v_{i+1}) is an edge of $\mathbb{T}$ for any $0 \le i < k$. A path is said to be *open* if all its sites are open. Two sites a and b of the triangular lattice are *connected* (this is denoted by $a \longleftrightarrow b$) if there exists an open path between them. A *cluster* is a maximal connected graph for the relation $\longleftrightarrow$ on sites of $\mathbb{T}$.

The different phases Bernoulli percolation undergoes a phase transition at $p_c = 1/2$: in the *sub-critical phase* $p < p_c$, there is almost surely no infinite cluster, while in the *super-critical phase* $p > p_c$, there is almost surely a unique one.

The understanding of the *critical phase* $p = p_c$ has progressed greatly these last few years. In [29], Smirnov proved Cardy's formula, thus providing the first rigorous proof of the conformal invariance of the model (see also [33, 4] for details and references).

*Université de Genève, Switzerland. E-mail: hugo.duminil@unige.ch

This result led to many applications describing the critical phase. Among others, the convergence of interfaces was proved in [12, 13], and critical exponents were computed in [32].

Another phase of interest is given by the so-called *near-critical phase*. It is obtained by letting p go to p_c as a well-chosen function of the size of the system (see below for more details). This phase was first studied in the context of percolation by Kesten [25], who used it to relate fractal properties of the critical phase to the behavior of the correlation length and the density of the infinite cluster (as p tends to p_c). Recently, the scaling limit of near-critical percolation was proved to exist in [20]. This result will be instrumental in the proof of our main theorem.

Main statement Mathematicians and physicists are particularly interested in the following quantity, called the *inverse correlation length*. For $p < p_c$ and for any u on the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, define

$$\tau_p(u) \ := \ \lim_{n\to\infty} -\tfrac{1}{n}\log \mathbb{P}_p(0 \longleftrightarrow \widehat{nu}),$$

where $\widehat{nu}$ is the site of $\mathbb{T}$ closest to nu. In [32], the inverse correlation length $\tau_p(u)$ was proved to behave like $(p_c-p)^{4/3+o(1)}$ as $p \nearrow p_c$.

Interestingly, conformal invariance at criticality is a strong indication that $\tau_p(u)$ becomes isotropic, meaning that it does not depend on $u \in \mathbb{U}$. The aim of this note is to show that this is indeed the case.

Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^2$.

Theorem 1.1. *For percolation on the triangular lattice, $\tau_p(u)/\tau_p(|u|) \longrightarrow 1$ uniformly in the direction $u \in \mathbb{U}$ as $p \nearrow p_c$.*

While this result is very intuitive once conformal invariance has been proved, it does not follow directly from it. More precisely, it requires some understanding of the near-critical phase mentioned above. The main input used in the proof is the spectacular and highly non-trivial result of [20]. In that paper, the scaling limit for near-critical percolation is proved to exist and to be invariant under rotations. This result constitutes the heart of the proof of Theorem 1.1, which then consists in connecting the inverse correlation length to properties of this near-critical scaling limit (in some sense, the proof can be understood as an exchange of two limits).

Wulff crystal Theorem 1.1 has an interesting corollary. Consider the cluster C_0 of the origin (its cardinal is denoted by card(C_0)). When $p < p_c$, there exists a deterministic shape W_p such that for any $\varepsilon > 0$,

$$\mathbb{P}_p\left(\mathbf{d}_{\mathrm{Hausdorff}}\Big(\frac{\mathsf{C}_0}{\sqrt{n}}, \frac{W_p}{\sqrt{\mathrm{Vol}(W_p)|}}\Big) > \varepsilon \ \Big| \ \mathrm{card}(\mathsf{C}_0) \geq n\right) \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $\mathrm{Vol}(E)$ denotes the volume of the set E, and $\mathbf{d}_{\mathrm{Hausdorff}}$ is the Hausdorff distance. In the previous formula, W_p is the *Wulff crystal* defined by

$$W_p \ := \ \{x \in \mathbb{C} : \langle x|u\rangle \leq \tau_p(u), u \in \mathbb{U}\},$$

where $\langle\cdot|\cdot\rangle$ is the standard scalar product on $\mathbb{C}$.

The Wulff crystal appears naturally when studying phase coexistence. Originally, the Wulff crystal was constructed rigorously in the context of the planar Ising model by Dobrushin, Kotecký and Shlosman [18] for very low temperature (see [28, 22] for extensions of this result). In the case of planar percolation, the first result is due to [1]. Let us mention that the Wulff construction was extended to higher dimensional percolation by Cerf [8] (see also [6, 14] for the Ising case). We refer to [9] for a comprehensive exposition of the subject.

The geometry of the Wulff crystal has been studied extensively since then. Let us mention that for any $p < p_c$, it is a strictly convex body with analytic boundary [1, 2, 10].

The expression of W_p in terms of the inverse correlation length, together with Theorem 1.1, implies the following result.

Corollary 1.2. *When $p \nearrow p_c$, $W_p/\sqrt{\mathrm{Vol}(W_p)}$ tends to the disk $\{u \in \mathbb{C} : |u| \le 1\}$.*

This corollary has a strong geometric interpretation. As $p \nearrow p_c$, the typical shape of a cluster conditioned to be large becomes round.

Super-critical phase For the super-critical phase, the previous results can be translated in the following way. For $p > p_c$, define

$$\tau_p^{\mathrm{f}}(u) \;:=\; \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_p(0 \longleftrightarrow \widehat{nu}, \mathrm{card}(\mathsf{C}_0) < \infty).$$

One can prove that $\tau_p^{\mathrm{f}}(u) = 2\tau_{1-p}(u)$; see [11, Theorem A] for a much more precise (and much harder) result. This fact, together with Theorem 1.1, immediately implies that $\tau_p^{\mathrm{f}}(u)/\tau_p^{\mathrm{f}}(|u|) \to 1$, uniformly in the direction $u \in \mathbb{U}$, as $p \searrow p_c$. The Wulff construction can also be extended to the super-critical phase. When $p > p_c$, we find that for any $\varepsilon > 0$,

$$\mathbb{P}_0\left(\mathbf{d}_{\mathrm{Hausdorff}}\Big(\frac{\mathsf{C}_0}{\sqrt{n}}, \frac{W_{1-p}}{\sqrt{\mathrm{Vol}(W_{1-p})}}\Big) > \varepsilon \;\Big|\; n \le \mathrm{card}(\mathsf{C}_0) < \infty\right) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Other models Let us mention that conformal invariance has been proved for a number of models, including the dimer model [23] and the Ising model [30, 15]; see [17] for lecture notes on the subject. In both cases, exact computations (see [26] for the Ising model) allow one to show that the inverse correlation length becomes isotropic, hence providing an extension of Theorem 1.1. For the Ising model, we refer to [26] for the original computation, and to [3] for a recent computation. For percolation, no exact computation is available and the passage via the near-critical regime seems required. For the Ising model, the near-critical phase was also studied in [16].

An open question To conclude, let us mention the following question, which was asked by I. Benjamini: let $p > p_c$ and condition 0 to be connected to infinity. Consider the sequence of balls of center 0 and radius n for the graph distance on the infinite cluster. Show that these balls possess a limiting shape U_p which becomes round as $p \searrow p_c$.

2 Proof of Theorem 1.1

We will use standard tools of percolation theory such as correlation inequalities (for instance the FKG and BK inequalities). The reader is referred to [21] for precise definitions.

Points will be considered as elements of the plane and we use complex numbers to position them. For $r > 0$ and $u \in \mathbb{R}^2$, let $B_r(u) = \{z \in \mathbb{R}^2 : |z - u| \le r\}$ be the Euclidean ball of radius r around u. For two sets A and B in $\mathbb{R}^2$, we say that $A \longleftrightarrow B$ if there exists $a \in A \cap \mathbb{T}$ and $b \in B \cap \mathbb{T}$ such that $a \longleftrightarrow b$.

2.1 An important input

Let $\mathcal{A}_4(1,n)$ be the event that there exist four disjoint paths from neighbors of the origin to distance n, indexed in the clockwise order by $\gamma_1, \gamma_2, \gamma_3$ and γ_4, with the property that γ_1 and γ_3 are open, while γ_2 and γ_4 are closed (meaning that they contain closed sites only). For any $\lambda > 0$ and $p < p_c$, we set

$$L_p^\lambda := \inf\left\{ n \geq 0 : n^2 \mathbb{P}_{p_c}[\mathcal{A}_4(1,n)] \geq \tfrac{\lambda}{p_c - p} \right\}.$$

Let us mention that this quantity can be proved to be within bounded multiplicative constants from the *correlation length* $1/\tau_p(u)$.

Proposition 2.1. *There exists $f : \mathbb{R} \times \mathbb{R}_+^* \to [0,1]$ such that for any $v \in \mathbb{R}^2$ and $\lambda > 0$,*

$$\lim_{p \nearrow p_c} \mathbb{P}_p\big[B_{L_p^\lambda}(0) \longleftrightarrow B_{L_p^\lambda}(L_p^\lambda v)\big] \quad = \quad f(\lambda, |v|). \tag{2.1}$$

The previous proposition has two interesting features. First, the quantity on the left possesses a limit as p tends to p_c. Second, this limit is invariant under rotations. This result is very difficult. Let us briefly explain how it can be obtained.

The scaling limit described in [20, Corollary 1.7] is a limit, in the sense of the Quad-topology introduced in [31], of percolation configurations $\mathbb{P}_p$ on $\frac{1}{L_p^\lambda}\mathbb{T}$. This topology is sufficiently strong to control events considered in Proposition 2.1. The existence of the scaling limit is justified by a careful study of macroscopic “pivotal points”. Scales L_p^λ correspond to the scales for which a variation of $p_c - p$ will alter the pivotal points, and therefore the scaling limit, but not too drastically. This fact enabled Garban, Pete and Schramm to construct the scaling limit of near-critical percolation from the scaling limit of critical percolation. The invariance under rotation of the near-critical scaling limit is then a consequence of the invariance under rotation of the critical one. The existence of the near-critical scaling limit and its invariance under rotations imply (2.1). We refer to [20] and also to [19] for more details.

In the proof, the near-critical phase will be used at its full strength. On the one hand, the scaling limit is still invariant under rotations. On the other hand, as $\lambda \to \infty$, the “crossing probabilities” tend to 0. The existence of such a phase is crucial here.

2.2 Proof of the theorem

The proof consists in estimating the inverse correlation length $\tau_p(u)$ using $f(\lambda, |u|)$. It is known since [25] that the inverse correlation length is related to crossing probabilities. Yet, previous studies were interested in relations which are only valid up to bounded multiplicative constants. Here, we will need a slightly better control (roughly speaking that these constants tend to 1 as p goes to p_c).

In order to relate $\tau_p(u)$ and $f(\lambda, |u|)$, we use the existence of different parameters R, λ, p_0 with some specific properties presented in the next proposition.

Let $\mathcal{C}_{\mathrm{circuit}}(x,n)$ be the event that there exists an open circuit (meaning a path starting and ending at the same site) in $B_{2n}(x) \setminus B_n(x)$ surrounding x.

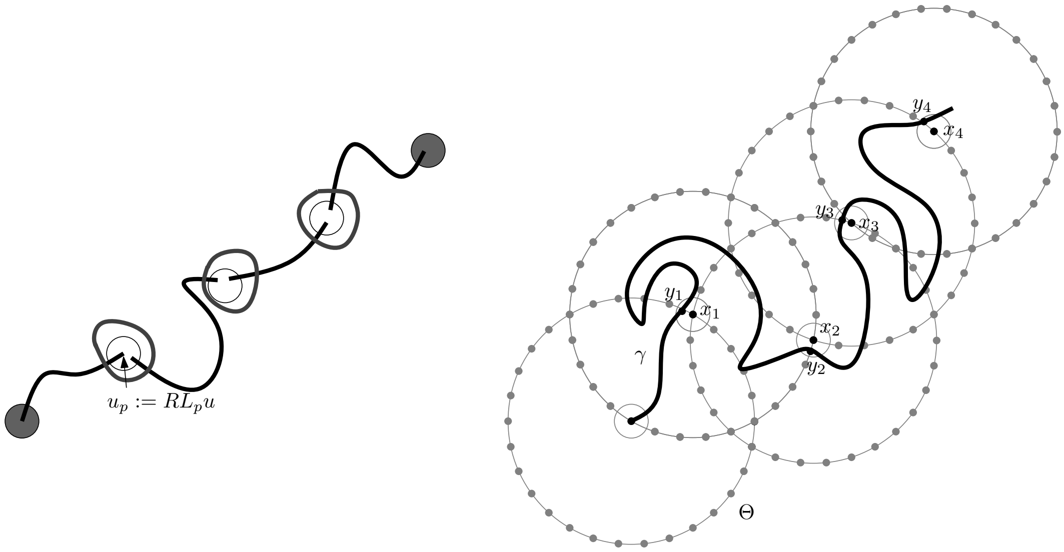

Figure 1: On the left, the events $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$. On the right, the construction corresponding to the upper bound.

Proposition 2.2. *Let $\varepsilon > 0$. There exist $\lambda, R > 0$ and $p_0 < p_c$ such that for any $p \in [p_0, p_c)$,*

P1 *for any $u \in \mathbb{U}$,* $f(\lambda, R)^{1+\varepsilon} \;\leq\; \mathbb{P}_p\big[B_{L_p^\lambda}(0) \longleftrightarrow B_{L_p^\lambda}(L_p^\lambda\, Ru)\big] \;\leq\; f(\lambda, R)^{1-\varepsilon}$,

P2 $\mathbb{P}_p\big[\mathcal{C}_{\mathrm{circuit}}(0, L_p^\lambda)\big] \;\geq\; f(\lambda, R)^{\varepsilon}$,

P3 $1 \;\geq\; 4R \cdot f(\lambda, R)^{\varepsilon}$.

Proposition 2.2 follows from very classical arguments using the Russo-Seymour-Welsh theory and the study of the near-critical window. We therefore choose to present the proof of Theorem 1.1 before sketching the proof of the proposition.

Proof of Theorem 1.1. Fix $\varepsilon > 0$ and $u \in \mathbb{U}$. Define $\lambda, R > 0$ and $p_0 < p_c$ such that Proposition 2.2 holds true. Let $p_0 < p < p_c$. We drop the dependency in λ by setting $L_p := L_p^\lambda$ and we introduce $u_p := L_pRu$.

For $K \geq 1$, consider the following three events:

$\mathcal{E}_1$= "$B_{L_p}(0)$ and $B_{L_p}(Ku_p)$ are full ",

$\mathcal{E}_2$= " $B_{L_p}(ku_p) \longleftrightarrow B_{L_p}((k+1)u_p)$ for every $0 \leq k < K$ ",

$\mathcal{E}_3$= " $\mathcal{C}_{\mathrm{circuit}}(ku_p, L_p)$ for every $0 \leq k \leq K$ ".

As shown on Fig. 1, if all these events occur, then 0 and the site of $\mathbb{T}$ closest to Ku_p, denoted by $\widehat{Ku_p}$, are connected by an open path. The FKG inequality (see [21, Theorem 2.4]) implies that

$$\mathbb{P}_p\big[0 \longleftrightarrow \widehat{Ku_p}\big] \;\geq\; \mathbb{P}_p\left[\mathcal{E}_1\right] \mathbb{P}_p\left[\mathcal{E}_2\right] \mathbb{P}_p\left[\mathcal{E}_3\right] \;\geq\; p^{8(L_p)^2} \cdot f(\lambda, R)^{(1+\varepsilon)K} \cdot f(\lambda, R)^{\varepsilon K}.$$

We have used P1 and P2 to bound the probabilities of $\mathcal{E}_2$ and $\mathcal{E}_3$ in the second inequality. The bound on $\mathbb{P}_p[\mathcal{E}_1]$ comes from the fact that there are less than $8(L_p)^2$ sites in $B_{L_p}(0) \cup B_{L_p}(Ku_p)$.

By taking the logarithm and letting K tend to infinity, we obtain that for $p_0 < p < p_c$,

$$\tau_p(u) \le -(1+2\varepsilon)\frac{\log f(\lambda, R)}{RL_p}. \tag{2.2}$$

This provides us with an upper bound that we will match with the lower bound below. Assume that 0 and $\widehat{Ku_p}$ are connected. Define

$$\Theta \ := \ \big\{\, u_p\, ,\, e^{\frac{\pi i}{2R}} u_p\, ,\, e^{2\frac{\pi i}{2R}} u_p\, ,\, \dots\, ,\, e^{(2R-1)\frac{\pi i}{2R}} u_p \,\big\}.$$

We claim that if 0 and $\widehat{Ku_p}$ are connected, then there must exist a sequence of sites $0 = x_0, x_1, \dots, x_K$ such that $x_{i+1} - x_i \in \Theta$ and $B_{L_p}(x_i) \longleftrightarrow B_{L_p}(x_{i+1})$ occurs for every $0 \le i < K$. Furthermore, the events $B_{L_p}(x_i) \longleftrightarrow B_{L_p}(x_{i+1})$ occur disjointly in the sense of [21, Section 2.3].

In order to prove this claim, consider a self-avoiding open path $\gamma = (\gamma_i)_{0\le i\le r}$ from 0 to $\widehat{Ku_p}$. Let y_1 be the first point of this path which is outside the Euclidean ball of radius L_p around 0. Define $x_1 \in \Theta$ such that $|y_1 - x_1| \le L_p$ (the point x_1 exists since $2R\sin(\frac{\pi}{2R}) \ge 1$ and therefore the balls of radius L_p and center in Θ cover well the boundary of the Euclidean ball of radius L_p). Let y_2 be the first point of $\gamma[y_1, r]$ outside the Euclidean ball of radius L_p around x_1. We pick x_2 such that $x_2 - x_1 \in \Theta$ and $|y_2 - x_2| \le L_p$. We construct $(x_i)_{0\le i\le K}$ iteratively. See Fig. 1 for an illustration. By construction, the events occur *disjointly* since the path γ is self-avoiding. The disjoint occurrence of events A and B will be denoted by $A \circ B$ (see [21, Theorem 2.12] for a formal definition of disjoint occurrence). The union bound and the BK inequality give

$$\begin{aligned}
\mathbb{P}_p[0 \longleftrightarrow \widehat{Ku_p})] &\le \sum_{(x_i)_{i\le K}} \mathbb{P}_p\Big[\big\{B_{L_p}(x_0) \longleftrightarrow B_{L_p}(x_1)\big\} \circ \dots \circ \big\{B_{L_p}(x_{K-1}) \longleftrightarrow B_{L_p}(x_K)\big\}\Big] \\
&\le (4R)^K \max\Big\{\mathbb{P}_p[B_{L_p}(0) \longleftrightarrow B_{L_p}(RL_p u)] : u \in \mathbb{U}\Big\}^K \\
&\le \big(4R\, f(\lambda, R)^{1-\varepsilon}\big)^K \le f(\lambda, R)^{(1-2\varepsilon)K}.
\end{aligned}$$

In the second inequality, we used the fact that the cardinality of Θ is bounded by $4R$. In the last line, we used P1 and then P3. By taking the logarithm and letting K go to infinity, we obtain that for $p_0 < p < p_c$

$$\tau_p(u) \ge -(1-2\varepsilon)\frac{\log f(\lambda, R)}{RL_p}. \tag{2.3}$$

Therefore, for every $p_0 < p < p_c$ and $u \in \mathbb{U}$, (2.2) and (2.3) imply that

$$\frac{1-2\varepsilon}{1+2\varepsilon} \ \le\ \frac{\tau_p(u)}{\tau_p(1)} \ \le\ \frac{1+2\varepsilon}{1-2\varepsilon}.$$

□

Proof of Proposition 2.2 (sketch). Property P1 follows directly from the definition of L_p^λ and $f(\lambda, R)$. Thus, we simply need to prove that R and λ can be chosen in such a way that properties P2 and P3 are satisfied.

First, recall the definition of the *characteristic length* from [27, Equation (7.1)]: for $p < p_c$ and $\alpha > 0$,

$$L_\alpha(p) := \inf\Big\{n \ge 0 : \mathbb{P}_p\big(\mathcal{C}_{\mathrm{H}}([0,n]^2)\big) \le \alpha\Big\},$$

where $\mathcal{C}_{\mathrm{H}}([0,n]^2)$ is the event that the box $[0,n]^2 = \{k + \mathrm{e}^{\mathrm{i}\pi/3}\ell : 0 \le k, \ell \le n\}$ is crossed from left to right by an open path. (This definition is yet again related to the correlation length $1/\tau_p(u)$.) With this definition, [27, Proposition 34] yields that for any $\alpha > 0$ there exist $c_\alpha, C_\alpha > 0$ such that

$$c_\alpha \leq (p_c - p) L_\alpha(p)^2 \mathbb{P}_{p_c}[\mathcal{A}_4(1, L_\alpha(p))] \leq C_\alpha \tag{2.4}$$

for $p < p_c$ close enough to p_c. Therefore, there exists $\lambda = \lambda(\alpha) > 0$ large enough so that

$$\lim_{p \nearrow p_c} \mathbb{P}_p\big(\mathcal{C}_{\mathrm{H}}([0, L_p^\lambda]^2)\big) \leq \alpha \tag{2.5}$$

(the fact that the limit on the left-hand side exists comes from the convergence to the near-critical regime proved in [20, Corollary 1.7]).

Define $\mathcal{C}_{\mathrm{in/out}}(x, n)$ to be the event that there exists an open path from $B_n(x)$ to the boundary of $B_{2n}(x)$. For $\beta > 0$, the RSW theorem [27, Theorem 2] and (2.5) easily imply the existence of $\lambda = \lambda(\beta)$ and $c(\beta) > 0$ so that for $p < p_c$ close enough to p_c,

$$\mathbb{P}_p\big[\mathcal{C}_{\mathrm{in/out}}(0, L_p^\lambda)\big] \leq \beta \quad \text{and} \quad \mathbb{P}_p\big[\mathcal{C}_{\mathrm{circuit}}(0, L_p^\lambda)\big] \geq c(\beta). \tag{2.6}$$

Now, define $\Theta := \big\{ 2L_p^\lambda\,,\, e^{\frac{\pi i}{4}}\, 2L_p^\lambda\,,\, e^{2\frac{\pi i}{4}}\, 2L_p^\lambda\,,\, \ldots\,,\, e^{7\frac{\pi i}{4}}\, 2L_p^\lambda \big\}$ and let $u \in \mathbb{U}$. Following the proof of the upper bound in the previous theorem, we may show that if $B_{L_p^\lambda}(0)$ and $B_{L_p^\lambda}(L_p^\lambda\, Ru)$ are connected, then there must exist a sequence of sites $0 = x_0, x_1, \ldots, x_K$ such that $x_{i+1} - x_i \in \Theta$ and $\mathcal{C}_{\mathrm{in/out}}(x_i, L_p^\lambda)$ occurs disjointly for every $0 \leq i \leq K$ with $K = \lfloor R/2 \rfloor - 1$. As before, the union bound and the BK inequality give

$$\mathbb{P}_p[B_{L_p^\lambda}(0) \longleftrightarrow B_{L_p^\lambda}(L_p^\lambda\, Ru)] \leq \sum_{(x_i)_{i \leq K}} \mathbb{P}_p\Big[\mathcal{C}_{\mathrm{in/out}}(x_0, L_p^\lambda) \circ \cdots \circ \mathcal{C}_{\mathrm{in/out}}(x_K, L_p^\lambda)\Big] \leq (8\beta)^{K+1}$$

uniformly in the choice of $u \in \mathbb{U}$ and p close enough to p_c. In the second inequality, we used the fact that the cardinality of Θ is bounded by 8.

We are now ready to conclude. First, we choose $\beta := \frac{1}{9}$ and $\lambda := \lambda(\beta)$. Then, we choose $R > 0$ so that $c(\beta)$ and $1/(4R)$ are larger than $f(\lambda, R)^\varepsilon$, which is possible since

$$f(\lambda, R)^\varepsilon \leq (\tfrac{8}{9})^{\varepsilon(K+1)} \leq (\tfrac{8}{9})^{\varepsilon \lfloor R/2 \rfloor}$$

decays exponentially fast in R. The claim follows by setting $p_0 = p_0(\lambda, R)$ close enough to p_c. □

Remark 2.3. *In fact, [20, Theorem 11.1] shows that for any $\lambda, u > 0$, $f(\lambda, u) = f(1, \lambda^{4/3} u)$ and therefore the exponential decay holds for every $\lambda > 0$.*

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