A proof of first order phase transition for the planar random-cluster and Potts models with $q \gg 1$

Hugo Duminil-Copin

April 28, 2016

Abstract

We provide a proof that the random-cluster model on the square lattice undergoes a discontinuous phase transition for large values of the cluster-weight q. This implies discontinuity of the phase transition for Potts model on the square lattice provided that the number of colors q is large enough. Let us remind the reader that this result is classical and that we simply provide an alternative approach based on the loop representation.

1 Introduction

The *random-cluster model*, also called *Fortuin-Kasteleyn percolation*, is a percolation model on $\mathbb{Z}^d$ introduced in [FK72]. This model has been proved to be useful both as a geometric representation for the Potts model, and as an archetypical example of a dependent percolation model.

Let G be a graph with vertex and edge sets V_G and E_G respectively. A *percolation configuration* ω on a graph G is a subgraph of G with the same vertex set as G. Let $o(\omega)$ and $k(\omega)$ be the number of edges and connected components in ω. The probability measure $\phi^{\text{free}}_{G,p,q}$ of the random-cluster model on a finite graph G with $p \in [0,1]$, $q > 0$ and free boundary conditions is defined by

$$\phi^{\text{free}}_{G,p,q}[\{\omega\}] := \frac{p^{o(\omega)}(1-p)^{|E_G|-o(\omega)}\ q^{k(\omega)}}{Z(G,p,q)} \tag{1.1}$$

for every percolation configuration ω in G. The constant $Z(G,p,q)$ is the *partition function* defined in such a way that the sum over all configurations equals 1. For $q \geq 1$, a measure $\phi^{\text{free}}_{\mathbb{Z}^2,p,q}$ on percolation configurations on $\mathbb{Z}^2$ can be defined by taking sub-sequential limits along a sequence of graphs G_n increasing to $\mathbb{Z}^2$.

Let $0 \longleftrightarrow x$ denote the event that 0 and x are in the same connected component of ω. Also let $0 \longleftrightarrow \infty$ denote the event that 0 is in an infinite connected component of ω.

The random-cluster model undergoes a phase transition when $q \geq 1$: there exists $p_c = p_c(q) \in (0,1)$ such that

$$\phi^{\mathrm{free}}_{\mathbb{Z}^2,p,q}[0 \longleftrightarrow \infty] = \begin{cases} 0 & \text{if } p < p_c, \\ \theta(p,q) > 0 & \text{if } p > p_c. \end{cases}$$

The model admits a dual model (see below for a definition) defined on the dual lattice $(\mathbb{Z}^2)^* = (\frac{1}{2},\frac{1}{2}) + \mathbb{Z}^2$. It is known that the dual measure of the random-cluster measure with free boundary conditions $\phi^{\mathrm{free}}_{\mathbb{Z}^2,p,q}$ is the random-cluster measure with wired boundary conditions $\phi^{\mathrm{wired}}_{(\mathbb{Z}^2)^*,p^*,q^*}$, where $p^* = p^*(p,q)$ and $q^* = q^*(p,q)$ satisfy

$$\frac{pp^*}{(1-p)(1-p^*)} \quad \text{and } q^* = q.$$

We refer to [Gri06] for a formal definition of the wired boundary conditions. Let us simply mention that $\phi^{\mathrm{free}}_{\mathbb{Z}^2,p,q} = \phi^{\mathrm{wired}}_{\mathbb{Z}^2,p,q}$ for any $p \neq p_c$. The value $p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ satisfies that $p_{\mathrm{sd}}(q)^* = p_{\mathrm{sd}}(q)$ and is equal to $p_c(q)$ in [BD12, DRT].

The main result of this note is the following.

Theorem 1 *For $q > 256$, there exists $c = c(q) > 0$ such that for every $x \in \mathbb{Z}^2$,*

$$\phi^{\mathrm{free}}_{\mathbb{Z}^2,p_c(q),q}[0 \longleftrightarrow x] \leq \exp(-c|x|). \tag{1.2}$$

Furthermore,

$$\phi^{\mathrm{wired}}_{\mathbb{Z}^2,p_c(q),q}[0 \longleftrightarrow \infty] \; > \; 0. \tag{1.3}$$

A similar result was proved for $q \geq 25.72$ in [LMR86, LMMS+91, KS82]. Here, we provide an alternative proof of this last statement on $\mathbb{Z}^2$.

Discussion Let us discuss the context in which this theorem takes place. Once $p_c(q)$ is determined, mathematicians and physicists are interested in the behavior at criticality. In [DST15], the authors proved that for any $q \geq 1$, the following properties are equivalent:

(1) $\lim_{p \searrow p_c} \theta(p,q) = 0$,

(2) $\phi^{\mathrm{wired}}_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow \infty] = 0$,

(3) $\sum_{x \in \mathbb{Z}^2} \phi^{\mathrm{free}}_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow x] = \infty$,

(4) $\phi^{\mathrm{free}}_{\mathbb{Z}^2,p_c,q} = \phi^{\mathrm{wired}}_{\mathbb{Z}^2,p_c,q}$.

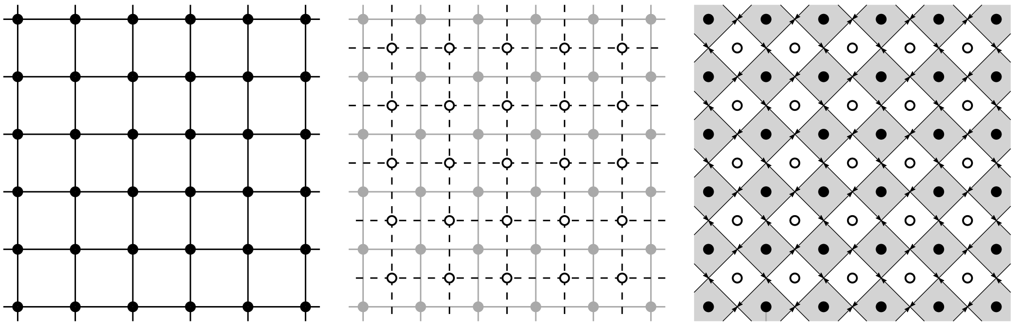

Figure 1: The square lattice (left), its dual lattice (center), its medial lattice with the orientation of its edges (right).

If one of (and therefore all) these properties are satisfied, we say that the phase transition is *continuous*. Otherwise, the phase transition is *discontinuous*. Notice that each one of these properties corresponds to a classical facet of a phase transition: continuity of the infinite-component density, absence of infinite-component at criticality, infinite susceptibility, uniqueness of the infinite-volume measure at criticality.

With this vocabulary, the main theorem states that the phase transition is discontinuous for $q > 256$ (or $q > 25.72$ if one refers to [LMR86, LMMS+91, KS82]). In [DST15], the phase transition was proved to be continuous when $q \in [1,4]$. Let us mention the following conjecture, due to Baxter [Bax71, Bax73, Bax78, Bax89], claiming that the phase transition of the planar random-cluster model is continuous if $q \in [0,4]$, and discontinuous if $q > 4$.

Let us conclude this introduction by reminding the reader that this result implies the discontinuity of the phase transition for the Potts model on $\mathbb{Z}^2$ for $q > 256$. We refer to [Dum15] for details on this fact.

2 The loop representation of the random-cluster model

Let us start by defining the dual configuration ω^* and the loop configuration $\overline{\omega}$ associated to a percolation configuration ω.

Consider the dual lattice $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ of $\mathbb{Z}^2$. Let e^* be the unique edge of $(\mathbb{Z}^2)^*$ crossing e in its middle, and G^* be the subgraph whose edge set is $E_{G^*} = \{e^* : e \in E_G\}$, and vertex set is given by the endpoints of these edges.

Definition 2 *The* dual configuration ω^* *of* ω *is a percolation configuration on* G^* *defined as follows: an edge* e^* *is in* ω^* *if and only if* e *is not in* ω.

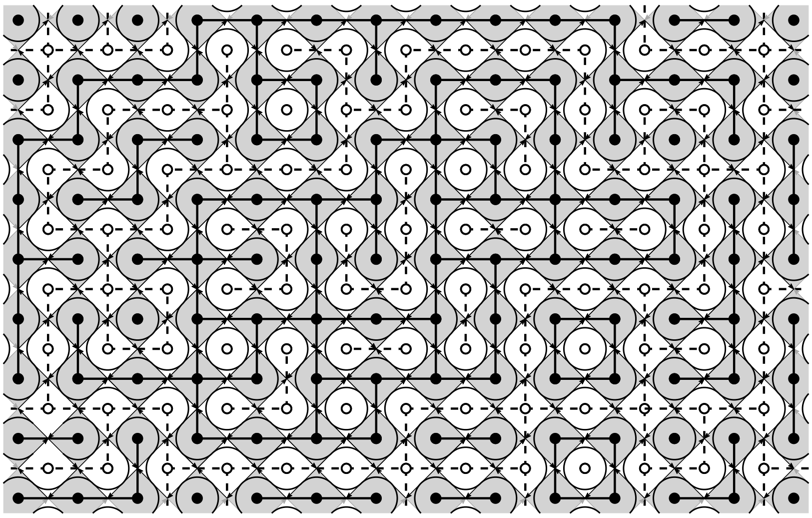

Figure 2: The configurations ω (in bold lines), ω^* (in dashed lines) and $\overline{\omega}$ (in plain lines).

Consider the medial lattice $(\mathbb{Z}^2)^\diamond$ whose vertices are the midpoints of edges of the lattice $\mathbb{Z}^2$, and edges between nearest neighbors. It is a rotated and rescaled version of the square lattice, see Fig. 1. Faces of $(\mathbb{Z}^2)^\diamond$ correspond to vertices of $\mathbb{Z}^2$ or $(\mathbb{Z}^2)^*$. We call them *black* and *white* faces respectively. Furthermore, edges are oriented counter-clockwise around black faces. Let $G^\diamond$ be the subgraph of $(\mathbb{Z}^2)^\diamond$ induced by the set of vertices bordering a face of $(\mathbb{Z}^2)^\diamond$ corresponding to a vertex of G.

Definition 3 *The* loop configuration $\overline{\omega}$ *of* ω *is the collection of (oriented) loops drawn on* $G^\diamond$ *as follows: a loop arriving at a vertex of* $G^\diamond$ *always makes a* $\pm\pi/2$ *turn so as not to cross any edge of* ω *or* ω^**, see Fig. 2.*

The loop configuration is defined in an unequivocal way since:

- there is either an edge of ω or an edge of ω^* crossing non-boundary vertices in $G^\diamond$, and therefore there is exactly one coherent way for the loop to turn at non-boundary vertices.

- there is only one possible $\pm\pi/2$ turn at boundary vertices keeping the loops in $G^\diamond$.

Identify $\phi^{\text{free}}_{G,p,q}$ with the push forward of $\phi^{\text{free}}_{G,p,q}$ by the map $\omega \mapsto \overline{\omega}$. In other words, $\phi^{\text{free}}_{G,p,q}$ now denotes a measure on loop configurations in $G^\diamond$, i.e. configurations of loops that can be obtained from a percolation configuration on G following the procedure above.

Remark 4 *The loop representation of the random-cluster model is an intermediate step in a sequence of mappings going from the Potts model to a* *solid-on-solid ice-type model named six vertex model. The six vertex model is exactly solvable, in the sense that its free energy can be computed via transfer matrices and the so-called Bethe Ansatz (see [Bax89] and references therein).*

The law of the loop configuration can be easily expressed in terms of the number of loops as follows. Set

$$x = x(p,q) := \frac{p}{\sqrt{q}(1-p)}.$$

Proposition 5 *Let G be a finite subgraph of $\mathbb{Z}^2$. Let $p \in (0,1)$ and $q > 0$. For any configuration ω,*

$$\phi^{\text{free}}_{G,p,q}[\overline{\omega}] \;=\; \frac{x^{o(\omega)}\sqrt{q}^{\ell(\overline{\omega})}}{\widehat{Z}(G,p,q)},$$

where $\ell(\overline{\omega})$ is the number of loops in $\overline{\omega}$ and $\widehat{Z}(G,p,q)$ is a normalizing constant.

Note that $x(p,q) = 1$ when $p = p_c(q)$. As a consequence, the law of $\overline{\omega}$ depends only on the number of loops in $\overline{\omega}$ in this case.

Proof An induction on the number of edges of ω shows that

$$\ell(\overline{\omega}) = 2k(\omega) + o(\omega) - |V_G|. \tag{2.1}$$

Indeed, if there is no edge, then $\ell(\overline{\omega}) = k(\omega) = |V_G|$. Now, adding an edge can either:

- join two connected components of ω, thus decreasing the number of loops and connected components by 1,
- close a cycle in ω, thus increasing the number of loops by 1 and not changing the number of connected components.

Equation (2.1) implies that

$$\begin{aligned} p^{o(\omega)}(1-p)^{|E_G|-o(\omega)}q^{k(\omega)} &= (1-p)^{|E_G|}\sqrt{q}^{|V_G|}\left(\frac{p}{(1-p)\sqrt{q}}\right)^{o(\omega)}\sqrt{q}^{2k(\omega)+o(\omega)-|V_G|} \\ &= (1-p)^{|E_G|}\sqrt{q}^{|V_G|}x^{o(\omega)}\sqrt{q}^{\ell(\overline{\omega})}. \end{aligned}$$

The proof follows by setting

$$\widehat{Z}(G,p,q) := \frac{Z(G,p,q)}{(1-p)^{|E_G|}\sqrt{q}^{|V_G|}}.$$

□

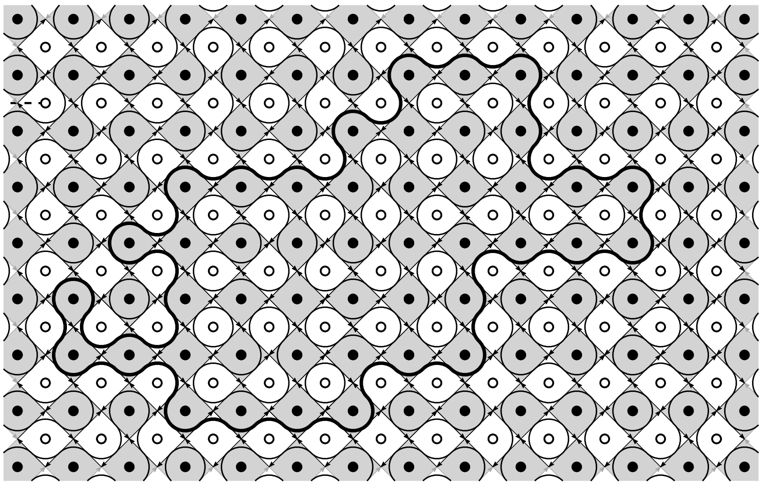

Figure 3: Consider a loop configuration $\overline{\omega}$ containing the loop L (in bold).

3 Proof of the theorem

Let G be a finite graph and L be a loop on $G^\diamond$ which is oriented counter-clockwise. Let n be the number of edges of $(\mathbb{Z}^2)^\diamond$ on L.

Let A_L be the event that the loop L is a loop of the configuration $\overline{\omega}$. Our goal is to bound $\phi^{\text{free}}_{G,p_c,q}[A_L]$. In order to do so, we construct a one-to-one "repair map" f_L from A_L to the set of loop configurations on $G^\diamond$ such that the image $f_L(\overline{\omega})$ has much larger probability than the probability of $\overline{\omega}$. This will imply a bound on the probability of A_L (see below).

Let $\overline{\omega}$ be a loop configuration in A_L. A loop of $\overline{\omega}$ is said to be *inside* (resp. *outside*) L if it is included in the bounded (resp. unbounded) connected component of $\mathbb{R}^2 \setminus L$. Perform the following three successive modifications on $\overline{\omega}$:

Step 1 Remove the loop L from $\overline{\omega}$.

Step 2 Translate the loops of $\overline{\omega}$ which are inside L by the vector $\frac{1-i}{2}$.

Step 3 Complete the configuration thus obtained by putting loops of length four around black faces of $G^\diamond$ bordered by an edge which is not covered by any loop after Step 2.

(See Figures 3–6 for an illustration.) The configuration thus obtained is denoted by $f_L(\overline{\omega})$.

Claim 1. *The configuration $f_L(\overline{\omega})$ is a loop configuration on $G^\diamond$.*

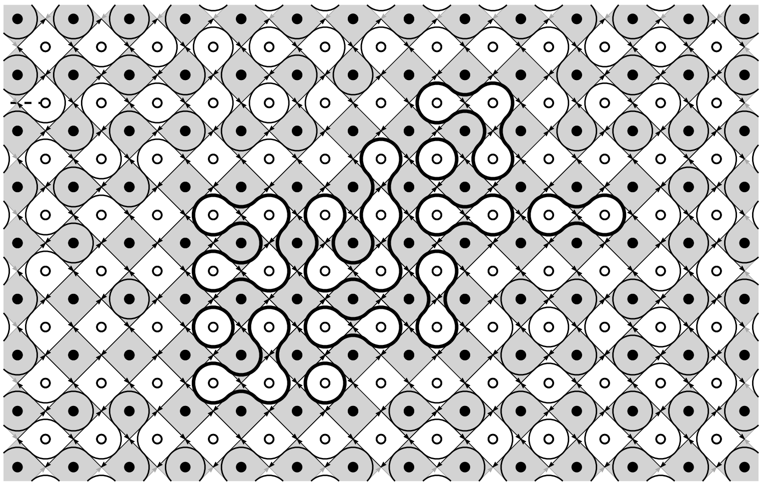

Figure 4: (Step 1) Remove the loop L from $\overline{\omega}$. The loops inside L are depicted in bold.

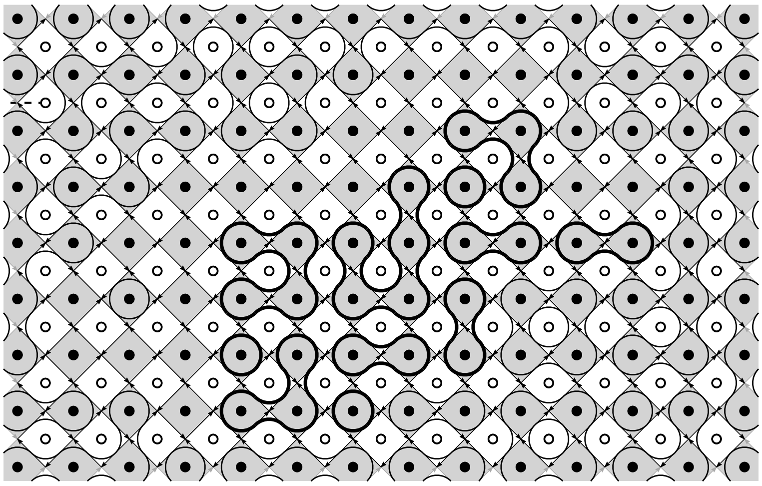

Figure 5: (Step 2) Translate the loops inside L in the south-east direction.

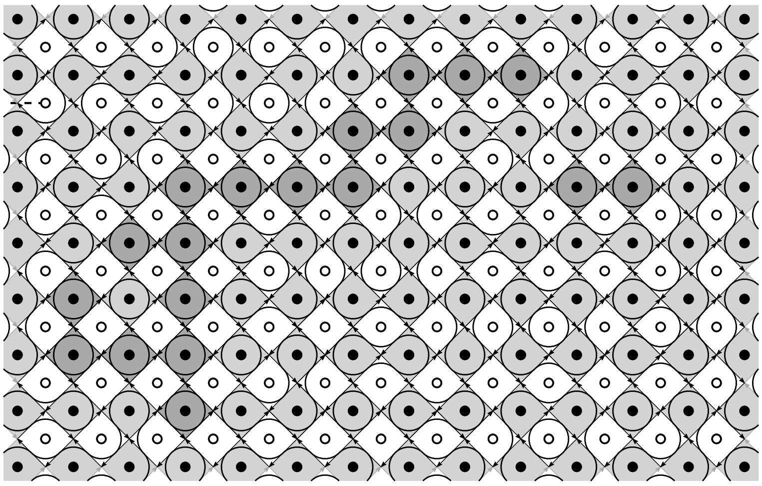

Figure 6: (Step 3) Fill the "holes" (depicted in darker gray) with loops of length four.

Proof We need to prove that each edge is covered by exactly one loop after Step 3. Step 3 rules out the fact that an edge is not covered by any loop. Furthermore, there is no edge inside L whose translate by $\frac{1-i}{2}$ is outside L. Therefore, there is no edge covered by two loops after Step 2. Hence, we only need to prove that Step 3 cannot add a loop on an edge already covered by a loop after Step 2. Equivalently, it is sufficient to prove that if a black face F is bordered by an edge which is not covered by a loop after Step 2, then none of the edges bordering F is covered by a loop after Step 2. Indeed, in such case the loop of length four which will be added in Step 3 will not cover an already covered edge. We prove this property now. Let e be the edge pointing south-west which is bordering F:

- If e is outside L, then the counter-clockwise orientation of L implies that all the edges bordering F also are. In particular they are all covered by a loop after Step 2. This contradicts the assumption on F.

- If e is inside L, then the counter-clockwise orientation of L prevents the existence of an edge in L bordering the translate F' of F by the vector $-\frac{1-i}{2}$. But this means that none of the edges bordering F' were removed in Step 1, and therefore their translates by $\frac{1-i}{2}$ are all covered in Step 2. This again contradicts the assumption on F.

- If $e \in L$, then F is bordering L from the inside while F' (defined above) is bordering L from the outside. In this case, a loop of $\overline{\omega}$ covering an edge bordering F is either removed in Step 1 (if it is L) or translated (or both, for instance for e). In both cases it does not cover any edge bordering F. On the contrary, every edge bordering F' is either on L or outside of L. Therefore, no loop covering such an edge is translated onto an edge of F during Step 2.

□

Claim 2. *For any counter-clockwise loop L included in G, $\phi^{\text{free}}_{G,p_c,q}[A_L] \le q^{1/2-n/8}$.*

Proof Let G be a finite graph containing L. Step 1 of the construction removes a loop from $\overline{\omega}$, but Step 3 adds one loop per edge of L pointing south-west. Since the number of edges added in the last step is four times this number, and that the final configuration has as many edges as the first one, we deduce that this number is equal to $n/4$. Thus, we have

$$\phi^{\text{free}}_{G,p_c,q}[\overline{\omega}] = \sqrt{q}^{\ell(\overline{\omega})-\ell(f_L(\overline{\omega}))}\phi^{\text{free}}_{G,p_c,q}[f_L(\overline{\omega})] = \sqrt{q}^{1-n/4}\phi^{\text{free}}_{G,p_c,q}[f_L(\overline{\omega})].$$

Using the previous equality in the second line and the fact that f_L is one-to-one in the third, we deduce that

$$\begin{aligned}\phi^{\text{free}}_{G,p_c,q}[A_L] &= \sum_{\overline{\omega}\in A_L}\phi^{\text{free}}_{G,p_c,q}[\overline{\omega}]\\ &= q^{1/2-n/8}\sum_{\overline{\omega}\in A_L}\phi^{\text{free}}_{G,p_c,q}[f_L(\overline{\omega})]\\ &= q^{1/2-n/8}\,\phi^{\text{free}}_{G,p_c,q}[f_L(A_L)] \le q^{1/2-n/8}.\end{aligned}$$

□

Let us now prove (1.2). Let G be a finite graph containing 0 and x. If 0 and x are connected to each others, then there must exist a loop in $\overline{\omega}$ surrounding 0 and x which is oriented counter-clockwise (simply take the exterior-most such loop). Since any such loop contains at least $|x|$ edges, we deduce that

$$\begin{aligned}\phi^{\text{free}}_{G,p_c,q}[0\longleftrightarrow x] &\le \sum_{\substack{L\text{ surrounding}\\ 0\text{ and }x}}\phi^{\text{free}}_{G,p_c,q}[A_L]\\ &\le \sum_{n\ge|x|}\sum_{\substack{L\text{ of length }n\\ \text{surrounding }0}} q^{1/2-n/8}\\ &\le \sum_{n\ge|x|} n2^n\cdot q^{1/2-n/8}.\end{aligned}$$

In the last inequality, we used the following easy claim:

Claim 3. *The number of loops with n edges on $G^\diamond$ is smaller than $n2^n$.*

Proof The number of loops with n edges passing through a vertex of $(\mathbb{Z}^2)^\diamond$ is bounded by 2^n (simply notice that there are at most two choices for every new step of the loop, except for the first/last ones for which there are respectively four/one choices). Therefore, the number of loops with n edges surrounding the origin is smaller than $n \cdot 2^n$ since the loop must go through one of the vertices of $(\mathbb{Z}^2)^\diamond$ of the form $\frac{1}{2} + k$ with $k \in \{0, \dots, n-1\}$. □

Letting G tend to the full lattice $\mathbb{Z}^2$, the weak convergence of $\phi^{\text{free}}_{G,p_c,q}$ to $\phi^{\text{free}}_{\mathbb{Z}^2,p_c,q}$ implies that

$$\phi^{\text{free}}_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow x] \le \sum_{n \ge |x|} n 2^n \cdot q^{1/2 - n/8}.$$

The existence of $c = c(q) > 0$ such that (1.2) holds true follows from the assumption $2q^{-1/8} < 1$.

Let us now turn to (1.3). First, the previous computation applied to $x = 0$ gives that

$$\sum_{L \text{ surrounding } 0} \phi^{\text{free}}_{\mathbb{Z}^2,p_c,q}[A_L] < \infty.$$

The Borel-Cantelli lemma thus implies that there is almost surely finitely many loops oriented counter-clockwise surrounding the origin in $\overline{\omega}$. Since nested loops have alternating orientations, this implies that there is almost surely finitely many loops (oriented clockwise or counter-clockwise) surrounding the origin in $\overline{\omega}$.

This translates into the following property of ω and ω^*: there is almost surely an infinite connected component either in ω or in ω^*. The bound (1.3) implies immediately that there is no infinite cluster in ω almost surely, which implies that there exists almost surely an infinite cluster in ω^*. Duality [Gri06] implies that

$$\phi^{\text{wired}}_{\mathbb{Z}^2,p_c,q}[\text{there exists an infinite connected component in } \omega] = 1.$$

Remark 6 *The value 256 is not optimal. Taking into account that the connective constant of the Manhattan lattice (which is counting the number of possible loops) is smaller than* 1.733535 *[Jen15], we deduce that the proof works for $q \geq 82$. We could also improve this constant by observing that sometimes, one may put a loop with eight edges instead of two loops with four edges, thus offering us more possible images by the repair map. We did not try to estimate from which value of q does the reasoning work if we try this approach. Anyway, it most probably does not beat* 25.72*.*

Acknowledgments This work was supported by a grant from the Swiss FNS and the NCCR SwissMap also funded by the Swiss FNS. We thank Dmitry Krachun for a useful suggestion.

References

[Bax71] R. J. Baxter, *Generalized ferroelectric model on a square lattice*, Studies in Appl. Math. **50** (1971), 51–69. MR 0307646 (46 #6766)

[Bax73] ______, *Potts model at the critical temperature*, Journal of Physics C: Solid State Physics **6** (1973), no. 23, L445.

[Bax78] ______, *Solvable eight-vertex model on an arbitrary planar lattice*, Philos. Trans. Roy. Soc. London Ser. A **289** (1978), no. 1359, 315–346. MR 0479213 (57 #18660)

[Bax89] ______, *Exactly solved models in statistical mechanics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989, Reprint of the 1982 original. MR 998375 (90b:82001)

[BD12] V. Beffara and H. Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$*, Probab. Theory Related Fields **153** (2012), no. 3-4, 511–542. MR 2948685

[DRT] H. Duminil-Copin, A. Raoufi, and V. Tassion, *A new computation of the critical point for the planar random-cluster model with $q \geq 1$*, arXiv:1604.03702, 2016.

[DST15] H. Duminil-Copin, V. Sidoravicius, and V. Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$*, arXiv:1505.04159, 2015.

[Dum15] H. Duminil-Copin, *Geometric representations of lattice spin models*, book, Edition Spartacus, 2015.

[FK72] C. M. Fortuin and P. W. Kasteleyn, *On the random-cluster model. I. Introduction and relation to other models*, Physica **57** (1972), 536–564. MR 0359655 (50 #12107)

[Gri06] G. Grimmett, *The random-cluster model*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 333, Springer-Verlag, Berlin, 2006. MR 2243761 (2007m:60295)

[Jen15] I. Jensen, *private communications (t. guttmann)*, 2015.

[KS82] R. Kotecký and S. B. Shlosman, *First-order phase transitions in large entropy lattice models*, Comm. Math. Phys. **83** (1982), no. 4, 493–515. MR 649814 (83i:82033)

[LMMS+91] L. Laanait, A. Messager, S. Miracle-Solé, J. Ruiz, and S. Shlosman, *Interfaces in the Potts model. I. Pirogov-Sinai theory of the Fortuin-Kasteleyn representation*, Comm. Math. Phys. **140** (1991), no. 1, 81–91. MR 1124260 (93d:82021a)

[LMR86] L. Laanait, A. Messager, and J. Ruiz, *Phases coexistence and surface tensions for the Potts model*, Comm. Math. Phys. **105** (1986), no. 4, 527–545. MR 852089 (88i:82030)

Département de Mathématiques Université de Genève Genève, Switzerland E-mail: hugo.duminil@unige.ch