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Internal diffusion-limited aggregation with uniform starting points

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Abstract. We study internal diffusion-limited aggregation with uniform starting points on $\mathbb{Z}^d$. In this model, each new particle starts from a vertex chosen uniformly at random on the existing aggregate. We prove that the limiting shape of the aggregate is a Euclidean ball.

Résumé. Nous étudions le modèle d'agrégation limitée par diffusion interne avec points de départ uniformes sur $\mathbb{Z}^d$. Dans ce modèle, chaque nouvelle particule est ajoutée à un point choisi uniformément au hasard parmi ceux de l'agrégat existant. Nous prouvons que l'agrégat normalisé admet comme forme limite la boule euclidienne.

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1. Introduction

1.1. *Historical introduction and motivation*

Internal diffusion-limited aggregation (IDLA) was introduced by Diaconis and Fulton in [6], and gives a protocol for recursively building a random aggregate of particles. At each step, the first vertex visited outside the current aggregate by a random walk started at the origin is added to the aggregate. In a number of settings, this model is known to have a deterministic limit-shape, meaning that a random aggregate with a large number of particles has a typical shape. On $\mathbb{Z}^d$, Lawler, Bramson and Griffeath [17] were the first to identify this limit-shape, in the case of simple random walks, as the Euclidean ball. Their result was later sharpened by Lawler [16], and was recently drastically improved with the simultaneous works of Asselah and Gaudillère [1,2] and Jerison, Levine and Sheffield [12–14], where logarithmic bounds are proved for fluctuations of the boundary.

The IDLA model has been extended in several contexts including drifted random walks [20], Cayley graphs of finitely generated groups [3,4,7,11] and random environments [8,22].

Another interesting growth model is provided by Richardson's model [21], which is defined as follows. At time 0, only the origin is occupied. A vacant site becomes occupied at an exponential time with a rate proportional to the number of occupied neighbours, and once occupied a site remains occupied. The set of vertices occupied by time t is the ball of radius t centered at the origin in first passage percolation with exponential clocks (see [15]). Eden [9] first asked about the shape of this process on Euclidean lattices and Richardson proved that a limiting shape exists. It is believed that the convex centrally symmetric limiting shape is not a Euclidean ball. This was established by Kesten in high dimensions (unpublished, but see [5]) together with the fact that the boundary has $t^{1/3}$ fluctuations, a long standing conjecture.

Internal diffusion-limited aggregation with uniform starting points (from here on shortened to uIDLA) is a growth model interpolating between standard internal diffusion-limited aggregation and Richardson's model. In uIDLA, particles are born uniformly on the shape and relocate to the outer boundary according to harmonic measure seen from the site they appeared at. While usual IDLA approaches rely on estimating the number of visits to a given point by particles starting from the origin, either directly or as the solution to a discrete partial differential equation, the study of uIDLA is more difficult because of the self-dependence involved in the construction.

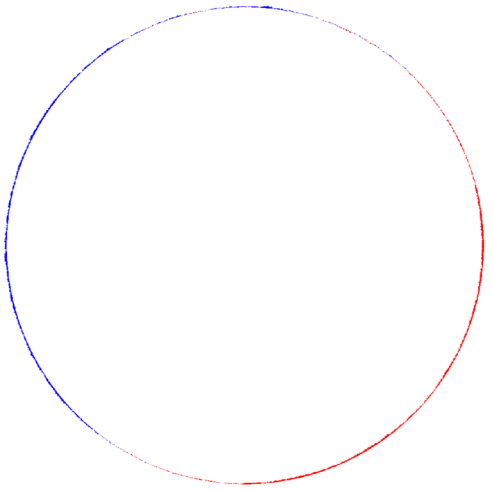

Fig. 1. Symmetric difference between the two-dimensional uIDLA aggregate and the Euclidean ball, with 10^6 particles. Blue points are present in the aggregate but not the ball, whereas it is the other way around for red points.

Another related model is *excited to the center*. In this model a single particle walks around the lattice $\mathbb{Z}^d$ doing simple random walk, except when it arrives at a vertex it never visited before ("a new vertex"), in which case it gets a drift towards the point 0. To compare excited to the center to the models described so far, think about standard IDLA as a single particle which, upon reaching a new vertex, is teleported to 0; and about uIDLA as a single particle which, upon reaching a new vertex, is teleported to a random location in the visited area. Very little is known about random walk excited to the center – there is an unpublished result showing that it is recurrent in all dimensions, but the shape of visited vertices is very far from being understood. Simulations and some heuristics indicate that at time t the set of visited vertices should be a ball with radius approximately $t^{1/(d+1)}$.

For uIDLA, we show that the limiting shape is a Euclidean ball, hence showing a behaviour close to the standard IDLA behaviour. Yet, the boundary fluctuations are expected to be slightly stronger than that of standard IDLA. This is not surprising, since part of the growth is due to particles emerging near the boundary thus behaving very roughly like the Richardson model. This suggests that the local regularity will be determined by some competition between particles born locally *à la* Richardson and particles arriving from far away as in standard IDLA. Furthermore, simulations like the one we present below seem to indicate a mesoscopic shift in the center of mass of the cluster, which occurs in a random direction. This paper deals with the limiting shape and not the fluctuations.

1.2. *Definition of the model and statement of the main theorem*

We consider the lattice $\mathbb{Z}^d$ with $d \geq 1$. Let $S \subseteq \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$.

In order to define both standard and uniform starting point IDLA, first define the action of adding a particle to an existing aggregate S. Let $\xi = (\xi(0), \xi(1), \ldots)$ be a random walk on $\mathbb{Z}^d$ and let t_S be the first time this walk is not in S. By random walk we mean the simple random walk choosing one of its $2d$ neighbours uniformly and independently at random at each step. Define

$$\mathrm{Add}[\xi, S] := S \cup \{\xi(t_S)\}.$$

Standard IDLA. Fix an integer $n \geq 0$. Let DA_n be the aggregate with n particles started at 0, constructed inductively as follows: $DA_0 = \varnothing$ and

$$DA_{n+1} := \mathrm{Add}\big[\xi_n^0, DA_n\big]$$

where ξ_n^0 is a random walk starting at 0 which is independent from $\xi_0^0, \ldots, \xi_{n-1}^0$. This process is referred to as IDLA.

Note that the equivalent initialisation $DA_1 = \{0\}$ is sometimes used.

IDLA with uniform starting point. Fix an integer $n \geq 0$. Let A_n be the uniform starting point aggregate with n particles constructed inductively as follows: $A_1 = \{0\}$ and

$$A_{n+1} := \mathrm{Add}\big[\xi_n^{X_n}, A_n\big]$$

where X_n is a point chosen uniformly on A_n, and $\xi_n^{X_n}$ is a random walk starting at X_n and independent of $\xi_0^{X_0}, \dots, \xi_{n-1}^{X_{n-1}}$. This process is referred to as uIDLA. Let $|\cdot|$ be the Euclidean distance in $\mathbb{R}^d$. For $n > 0$, let $B[n] := \{y \in \mathbb{Z}^d : |y| \leq n\}$ and $b_n := |B[n]|$.

Theorem 1.1. *Let $d \geq 2$. There exists positive constants c_1, c_2, C_1 and C_2 depending only on the dimension, such that almost surely,*

$$B\big[n\big(1 - C_1 n^{-c_1}\big)\big] \subseteq A_{b_n} \subseteq B\big[n\big(1 + C_2 n^{-c_2}\big)\big]$$

for n large enough.

Remarks. In dimension 1, the uIDLA aggregate with n points is a set of consecutive integers of length n, therefore it is entirely determined by the position of its middle point (called M_n). It is clear, either from a quick computation using the gambler's ruin or from a symmetry argument, that the probability for the cluster to grow on either of the two sides is exactly $1/2$. Therefore the process M_n is exactly a simple random walk on integers and half-integers, and the behaviour of the cluster is obvious, with a law of large numbers and CLT fluctuations.

In dimensions bigger than 2 we expect much smaller fluctuations, and our theorem is not satisfactory in this regard. We have chosen not to optimize the n^{-c}, mainly in order to help alleviate notations, but also because we do not hope to capture the true order of the error term with our method.

1.3. *Structure of the paper*

The first section contains five lemmas. They provide useful information on comparing uIDLA to IDLA. As they are of interest on their own, we isolate them from the proof of the theorem.

The second section of the article deals with the stability properties of the Euclidean ball under the uIDLA process. We first investigate the claim that the process started from a configuration that includes a ball will contain a growing ball with high probability. Then we take the converse and prove that the process started from any configuration inside a ball will stay contained in a slightly bigger growing ball. To prove this statement we examine the cluster together with the genealogical tree describing the starting points of the random walks. Our proof involves a comparison with a First Passage Percolation process on random trees.

In our third section, we bring these elements together for a proof of our theorem. The inner bound is proved first, using a refinement method that relies heavily on our coupling properties. The outer bound is then proved using the genealogical construction from the previous section.

Further notation. For every $y \in \mathbb{Z}^d$, let $\mathbb{P}_y$ denote the law of a simple random walk on $\mathbb{Z}^d$ starting from y. For a set $S \subset \mathbb{Z}^d$ we will denote by ∂S the set of vertices in $\mathbb{Z}^d \setminus S$ with a neighbour (or more than one) in S.

2. Comparison lemmas

We start with the following notations which will enable us to state our lemmas more easily. Given vertices $x_1, \dots, x_k$ in $\mathbb{Z}^d$, define $DA_{x_1,\dots,x_k}(S)$ to be the IDLA aggregate formed by launching additional particles from points $x_1, \dots, x_k$. Note that x_i need not be in the set S. Naturally, for $x \notin S$, $DA_x(S) = S \cup \{x\}$ deterministically. Recall that, classically, the law of the aggregate does not depend on the order in which these particles are added. Therefore, we also define $DA_X(S) := DA_{x_1,\dots,x_k}(S)$, where X is the multi-set $X = \{x_1, \dots, x_k\}$. If the multi-set X is just k repetitions of the origin, we denote for conciseness $DA_k(S) := DA_X(S)$. Remark that this notation is consistent with our initial definition, in that $DA_n = DA_n(\varnothing)$.

Similarly, for the uIDLA process, we denote $A_1(S)$ the result of adding a particle started uniformly on S to the set S. We also denote $A_k(S)$ for $k \in \mathbb{N}$ the result of the recursive process of adding k particles to the aggregate S, where the first one starts uniformly on S, and the jth particle starts uniformly on $A_{j-1}(S)$.

We start with the following lemma. It states that the aggregate obtained by launching k particles from arbitrary points in $B[n/2]$ is bigger than the aggregate obtained by launching a smaller yet comparable number of particles from the origin.

Lemma 2.1. *There exists $\eta > 0$ (depending only on the dimension) such that for any multi-set X of cardinality k in $B[n/2]$, $DA_X(B[n])$ stochastically dominates $DA_\kappa(B[n])$, where κ follows a binomial distribution $\mathcal{B}(k, \eta)$.*

Proof. Let x be a point in $B[n/2]$ and A a set containing $B[n]$. We consider the function evaluating the probability that the random walk starting at x exits A through a point y. We consider the stopping time $\tau_A = \inf\{t : \xi^x(t) \notin A\}$ and the function

$$h_y(x) = \mathbb{P}_x\big(\xi^x(\tau_A) = y\big).$$

This function is harmonic in x on $B[n-1]$, hence the Harnack inequality [18, Theorem 6.3.9] implies that there exists $\eta > 0$ such that for all $A \supseteq B[n]$, $x \in B[n/2]$ and $y \notin A$,

$$h_y(x) \geq \eta h_y(0).$$

This inequality allows to construct a coupling between DA_X and DA_κ as follows. Let E_k and F_k be constructed recursively. Set $E_0 = F_0 = B[n]$. Index sites of X by $\{x_1, \ldots, x_{|X|}\}$. Assume that $F_k \subseteq E_k$ have been constructed. Construct $E_{k+1} = \mathrm{Add}[\xi_{k+1}^{x_{k+1}}, E_k]$. Consider a killed random walk ξ_{k+1}^0 coupled with $\xi_{k+1}^{x_{k+1}}$ in such a way that:

- ξ_{k+1}^0 is killed at 0 with probability $1-\eta$,
- if ξ_{k+1}^0 exit E_k through y, so does $\xi_{k+1}^{x_{k+1}}$.

The existence of this coupling is guaranteed by $h_y(x) \geq \eta h_y(0)$. After exiting $B[n]$ we couple the walks in the usual way: they walk together until exiting their respective aggregates (since $F_k \subseteq E_k$, the walk on F_k would exit first). Construct $F_{k+1} = \mathrm{Add}[\xi_{k+1}^0, F_k]$ if the particle is not killed. Note that $F_{k+1} \subseteq E_{k+1}$, since either ξ_{k+1}^0 exits F_k throughout a point of E_k, or it does through a point not in E_k, but in this case the coupling guarantees that the exiting point is in E_{k+1}. The total number of coupled particles, κ, follows a binomial distribution with parameters (k, η). □

The following lemma controls the behaviour of a standard IDLA with M points started with a Euclidean ball $B[n]$ already occupied. It closely follows the spirit of [1], but instead of pushing the precision to get the best almost sure bound, we only look at points that are far enough from the edge of the theoretical shape to keep the probability of inclusion exponentially close to 1.

Lemma 2.2. *For any $n \in \mathbb{R}^+$ and $N \in \mathbb{N}^*$, let us write $r = r_{n,N} = \frac{N}{b_n}$. Then we have*

$$\mathbb{P}\big(B\big[n\big(1 + r - r^{3/2}\big)^{1/d}\big] \subset DA_N\big(B[n]\big)\big) \geq 1 - \exp\big(-C_3 n r^{3/2}\big).$$

We will use this lemma in the window where $nr^{3/2}$ is large but r is small, so that $r - r^{3/2} \geq 0$ (otherwise the lemma is true but useless).

Proof. Our lemma is almost exclusively a consequence of the many ideas provided in [1]. Therefore, we refer the reader to the appendix, in which we give a guide to the modifications one needs to do in [1] to get this result. □

The next two lemmas propose stochastic dominations between standard IDLA and the uIDLA process. We start with a lemma that compares one step.

Lemma 2.3. *There exists a constant $C_4 > 0$ such that, if $B[n] \subset S \subset T$ then $A_1(T)$ stochastically dominates $DA_\delta(S)$, where δ is a Bernoulli variable with parameter $\frac{|B[n]|}{|T|}(1 - \frac{C_4}{n})$.*

Proof. First, remark that our new point falls inside $B[n]$ with probability $\frac{|B[n]|}{|T|}$. Once more, we consider the stopping time $\tau_A = \inf\{t : \xi^x(t) \notin A\}$ and the function

$$h_y(x) = \mathbb{P}_x\big(\xi^x(\tau_A) = y\big).$$

This function is harmonic in x on $B[n]$. We are now interested in an averaging property for this harmonic function; namely, is $h_y(0)$ close to $\frac{1}{|B[n]|}\sum_{x \in B[n]} h_y(x)$?

The study on this averaging property is linked to that of *quadrature domains* and the divisible sandpile model, and, in particular, one shape on which a relation is known between the two terms is the shape taken by the divisible sandpile after toplings, with all the initial mass started at the origin, as defined in [19]. Let $m(x)$ be the final mass distribution corresponding to an initial mass M at the origin, then we have, for all harmonic functions h,

$$Mh(0) = \sum_{x \in \mathbb{Z}^d} m(x)h(x).$$

Recall that the final mass distribution m is equal to 1 on a given shape, has value between 0 and 1 at distance one from this shape, and is zero at distance more than one of this shape. It is hence a consequence of Levine and Peres's shape theorem (see [19]) that there is a constant $c > 0$ depending only on the dimension such that

$$h_y(0) = \frac{1}{|B[n]|} \sum_{x \in \mathbb{Z}^d} m(x) h_y(x),$$

with $m(x) = 1$ on $B[n-c]$ and $m(x) = 0$ outside $B[n+c]$. Combining the facts that m has values between 0 and 1 everywhere; and that $\sum_y h_y(x) = 1$, allows to bound the error given by replacing m with $\mathbb{1}_B$. We get that there is a constant C_4 depending only on the dimension, such that:

$$\sum_{y \in \partial A} \left| h_y(0) - \frac{1}{|B[n]|} \sum_{x \in B[n]} h_y(x) \right| \leq \frac{C_4}{n}.$$

Hence, our two particles can be coupled with probability $\frac{|B[n]|}{|A|}(1 - \frac{C_2}{n})$, which yields the result. □

Assume E is some subset of our aggregate F. As F evolves, there is a natural increasing subset $E_n \subset A_n(F)$ which corresponds to E and is in fact a time change of an uIDLA started from E. Basically, one traces only particles which started on E_n and follows them only until they exit E_n. Further, it is not necessary to know anything about the structure of F, it is enough to know its size. Formally, the definition is as follows: Let $E_0 = E$. Next, for every n define

$$E_{n+1} := \begin{cases} \text{Add}[\xi_{n+1}^{X_n}, E_n] & \text{with (independent) probability } \frac{|E_n|}{|F|+n} \\ E_n & \text{otherwise} \end{cases}$$

where X_n is a point chosen uniformly on E_n, and $\xi_{n+1}^{X_n}$ is a random walk starting at X_n and independent of $\xi_1^{X_1}, \ldots, \xi_k^{X_{k-1}}$. Finally, the Bernoulli events which determine whether the point will be added or not are independent of the walks (and of one another). We see that the process depends only on the size of F and not on its structure. This leads to the following definition

Definition. For $E \subset \mathbb{Z}^d$ and $m \geq |E|$ we let $A_n(E; m)$ be the E_n defined in the previous paragraph, for some F with $|F| = m$. We call $A_n(E; m)$ the subset uIDLA.

Clearly $A_n(E; |E|)$ is the same as $A_n(E)$ and, in general, if $E \subset F$, then $A_n(F)$ stochastically dominates $A_n(E; |F|)$. A little more than that is, in fact, true:

Lemma 2.4. *For any sets $E \subseteq F$, we have that $A_n(F)$ stochastically dominates $DA_{F \setminus E}(A_n(E; |F|))$.*

Proof. We will colour $A_n(F)$ in 3 colours, blue, red and black, such that the blue part has the same distribution as $A_n(E; |F|)$, the union of the red and the blue has the same disribution as $DA_{F \setminus E}(A_n(E; |F|))$ and black is the rest. Here is the colouring scheme:

We start the process with $A_0 = F$ coloured as follows: E is coloured blue and $F \setminus E$ is coloured red. Suppose we already constructed (and coloured) A_n. We choose a vertex x of A_n randomly to start the random walk from.

- If x is blue, perform the random walk until the particle exits the blue set. When it does, the site where it lands is coloured blue. If there was already a particle at that site, "wake it up" – it continues walking according to the rules in the following clauses.
- Now assume we have a red particle walking (which can only happen if a red particle was woken by a blue one, as in the previous clause). Perform the random walk until the particle exits the union of the red and the blue. When it does, that site will be coloured red. If there is a black particle there, wake it up and let it continue walking according to the rule in the next (and last) clause.
- If x is red or black, let the new particle be black. Let it perform simple random walk until the it exits the entire aggregate, and colour that site black.

Thus, for example, a particle might start from a blue site, walk until reaching a red site, change that site to blue, continue walking until reaching a black site, change that site to red, and then walk until exiting. This ends the description of the colouring.

Now, the fact that the blue part of the aggregate has the same distribution as $A_n(E;|F|)$ is evident. The fact that the union of the red and the blue has the same distribution as $DA_{F\setminus E}(A_n(E;|F|))$ is also simple, because the red part starts with $F\setminus E$ and then each red particle does a random walk and ends outside the eventual blue part.

One might claim that, even though each red particle does simple random walk, they are stopped and woken up mixing up their order. It is well-known that this does not affect the distribution of the final aggregate. For the convenience of the reader, let us recall the argument. One attaches labels to each red particle, and when a particle with a lower label steps over a particle with a higher label, they exchange labels so that the higher label continue to walk. This, of course, does not change the red part, but now each label does simple random walk until its final resting point, and only then does the next label start to walk. So the union of the blue and the red part has indeed the same distribution as $DA_{F\setminus E}(A_n(E;|F|))$ and the lemma is proved. □

The following lemma is extracted from [8]. It states that a random walk has a small probability of passing through an area of small density, and will be used to couple our process with a First Passage Percolation process. Rather than refer to the proof of [8] which holds in a more general setting, we give a shorter proof specific to $\mathbb{Z}^d$. Recall that we defined $b_n = |B[n]|$ the volume of the Euclidean ball of radius n intersected with $\mathbb{Z}^d$.

Lemma 2.5. *Let $p>0$. There exists $\varepsilon>0$ such that for any $n,m\geq 1$ large enough,*

$$\mathbb{P}_x\big(\xi \text{ exits } S\cup B[m] \text{ through } \partial B[m+n]\big)\leq p$$

uniformly in $x\in B[m]$ and $S\subseteq B[m+n]$ satisfying $|S|\leq \varepsilon b_n$.

Proof. By Markov's property, it is sufficient to bound

$$\mathbb{P}_y\big(\xi \text{ exits } S\cap B_y[n/3] \text{ through } \partial B_y[n/3]\big)$$

for starting points $y\in S\cap B[m+\frac{2n}{3}]\setminus B[m+\frac{n}{3}]$. Similarly, by shifting y to zero and replacing $n/3$ by n, it is enough to prove that

$$\mathbb{P}_0\big(\xi \text{ exits } S\cap B[n] \text{ through } \partial B[n]\big)\leq p$$

uniformly in any set $S\subset B[n]$ such that $|S|\leq \varepsilon b_n$, for ε small enough (thus our new ε is multiplied by 3^d, which does not affect the rest of the proof).

Now, if $|S|\leq \varepsilon b_n$ then for some $r\leq n$ we must have that $S\cap\partial B[r]\leq C\varepsilon r^{d-1}$. By [18, Lemma 6.3.7], every $x\in\partial B[r]$ has probability $\leq Cr^{1-d}$ that random walk started from 0 will exit $B[r]$ at x. Summing over $x\in S\cap\partial B[r]$ gets that the probability that random walk started from 0 will exit $B[r]$ at S is less than $C\varepsilon$. Thus for ϵ small enough, ξ exits S before reaching $\partial B[n]$ with probability greater than $1-p$. □

3. Stability of the Euclidean ball

3.1. *Inner stability of the ball*

In this section (Section 3) we show that, if you start a uIDLA from a large ball, it remains an approximate ball, with high probability. We first (Section 3.1) show inner stability, i.e. that the aggregate *contains* a ball of the approximately correct size. In a formula,

$$A_{b_m-b_n}\big(B[n]\big)\supseteq B\big[m\big(1-Cn^{-1/4}\big)\big]$$

with high probability. In other words, the only error is the missing $Cn^{-1/4}$ in the diameter.

It will be convenient, though, to formulate the claim slightly more generally: if $B[n]\subseteq S$ then $A_M(S)$ contains a ball of the correct size. We will use the notation $A_M(E;N)$ introduced on page 395 – recall that $A_M(E;N)$ is the way E evolves when you embed it in some set of size N, add M particles in a uIDLA fashion to that set, and examine only particles that landed on E. We first formulate a lemma for adding a relatively small number of particles, an $n^{-1/2}$ proportion:

Lemma 3.1. *There exist $\delta_2, C_5>0$ such that for any $M\geq b_n$,*

$$\mathbb{P}\big(B\big[n\big(1+n^{-1/2}-C_5n^{-3/4}\big)^{1/d}\big]\subseteq A_{Mn^{-1/2}}\big(B[n];M\big)\big)\geq 1-C\exp\big(-n^{\delta_2}\big).$$

Remark in particular that the probability does not depend on M.

Proof. The definition of $A_{Mn^{-1/2}}(B[n]; M)$ gives that it is the same as $A_K(B[n])$ where K is a random variable which stochastically dominates a binomial distribution with $Mn^{-1/2}$ trials and probability $b_n/(M + Mn^{-1/2})$ for success.

Recall that Lemma 2.3 says that adding a single particle to uIDLA stochastically dominates adding a single particle to standard IDLA, with an appropriate probability. Applying Lemma 2.3 K times gives that $A_K(B[n])$ stochastically dominates a standard IDLA with initial set $B[n]$ and with a random number L of particles (started at the origin), where L stochastically dominates a binomial distribution with K trials and probability $(b_n/(b_n + K))(1 - C_4/n)$ for success.

Combining both facts shows that $A_{Mn^{-1/2}}(B[n]; M)$ stochastically dominates standard IDLA started from a ball with the number of particles L following a binomial distribution $\mathrm{Bin}(s, p)$ with

$$s = Mn^{-1/2} \quad \text{and} \quad p = \frac{b_n}{M(1+n^{-1/2})^2}\left(1 - \frac{C_4}{n}\right).$$

Since $b_n = \omega_d n^d + O(n^{d-1})$, the classical Bernstein bound directly yields that L satisfies:

$$\left|L - \frac{n^{-1/2} b_n}{(1+n^{-1/2})^2}\left(1 - \frac{C_4}{n}\right)\right| \le n^{d-3/4}$$

with probability larger than $1 - C\exp(-2n^{d-1})$ (note that $d \ge 2$). We now assume that this bound for L is verified.

We then apply Lemma 2.2 with $B[n]$ already occupied and L new particles started at the origin. With the notations of the lemma, $r_{n,N} \sim n^{-1/2}$ and we get that the ball of radius $n(1 + r - r^{3/2})^{1/d}$ is included in the cluster $DA_L(B[n])$ with probability at least $1 - \exp(-C_3 n^{1/4})$. We estimate

$$\begin{aligned} 1 + r - r^{3/2} &\ge 1 + \frac{n^{-1/2}}{(1+n^{-1/2})^2}\left(1 - \frac{C_4}{n}\right) - \frac{n^{d-3/4}}{b_n} - Cn^{-3/4} \\ &\ge 1 + n^{-1/2} - C_5 n^{-3/4} \end{aligned}$$

where the first inequality is the lower bound on L, and these inequalities hold with probability $1 - C\exp(-n^{\delta_2})$. □

The case where the number of particles we add is proportional to the volume (or more) is a corollary:

Corollary 3.1. *There exist $\delta_3, C_6 > 0$ such that for any $M \ge b_n$,*

$$\begin{aligned} &\mathbb{P}\big(\forall a \ge 1 B\big[na^{1/d}\big(1 - C_6 n^{-1/4}\big)\big] \subseteq A_{(a-1)M}\big(B[n]; M\big)\big) \\ &\quad \ge 1 - C\exp\big(-n^{\delta_3}\big). \end{aligned}$$

Proof. Examine first the case that $a \le 2$. We apply the previous lemma repeatedly K times, i.e. define

$$\begin{aligned} &S_0 = B[n], \qquad M_0 = M, \\ &S_{i+1} = A_{M_i n^{-1/2}}(S_i; M_i), \qquad M_{i+1} = M_i\big(1 + n^{-1/2}\big), \end{aligned}$$

with K chosen in such a way that we obtain additional $|S|$ particles. Since each time we add $M_i n^{-1/2}$ particles and $M_i \ge M$, we deduce that $K \le n^{1/2}$ Therefore, with probability larger than

$$1 - n^{1/2}\exp\big(-n^{\delta_2}\big)$$

the aggregate $A_{(a-1)M}(B[n]; M)$ contains the Euclidean ball of radius

$$\begin{aligned} na^{1/d}\big(1 - Cn^{-3/4}\big)^K &\ge na^{1/d}\big(1 - Cn^{-3/4}\big)^{n^{1/2}} \\ &\ge na^{1/d}\big(1 - Cn^{-1/4}\big). \end{aligned} \tag{3.1}$$

This takes care of a along a sequence. For a general $a \in [1, 2]$, we find some i such that $M_i < (a-1)M < M_{i+1}$ and the inequality still holds from monotonicity of the aggregate (we lose $Cn^{d-1/2}$ particles from the approximation, but this only changes the value of the constant in (3.1)).

For general a (i.e. $a > 2$) we repeat the last calculation for $2n$, $4n$ etc. We get that the claim holds for all a except for an event whose probability is smaller than

$$C\sum_{i=0}^{n}\bigl(2^i n\bigr)^{1/2}\exp\bigl(-2^i n^{\delta_2}\bigr).$$

Since this sum converges exponentially, we may bound it by $C\exp(n^{-\delta_3})$ for an appropriate δ_3. Similarly, the errors in (3.1) converge exponentially, so they only change the constant. So we get that the radius is bounded by

$$na^{1/d}\bigl(1 - C_6 n^{-1/4}\bigr)$$

for a suitable constant C_6. □

3.2. *Genealogical construction and outer stability*

Our aim in this section is to prove a converse to Corollary 3.1 for the outer stability of the ball. We begin by comparing the process started from any set $S \subset B[n]$ with the process started from $B[n]$, on an event of high probability. Here we are comparing uIDLA to another uIDLA (and not to standard IDLA, as in the previous section), so the argument is much simpler.

Lemma 3.2. *There exist $\delta_5 > 0$ and $C_7 > 0$ such that for any set S with $S \subseteq B[n]$, and for any $1 \le a \le 2$, there is a coupling of $A_{(a-1)|S|}(S)$ and $A_{(a-1)b_n(1+C_7|S|^{-1/4})}(B[n])$ such that*

$$\mathbb{P}\bigl(A_{(a-1)|S|}(S) \subseteq A_{(a-1)b_n(1+C_7|S|^{-1/8})}\bigl(B[n]\bigr)\bigr) \ge 1 - C\exp\bigl(-|S|^{\delta_5}\bigr).$$

Proof. We may assume $|S|$ is sufficiently large. Recall the definition of subset uIDLA on page 395 and the natural coupling of $A_i(S; b_n)$ and $A_i(B[n])$, with the property that $A_i(S; b_n) \subseteq A_i(B[n])$. Examine first the first $b_n|S|^{-1/2}$ particles added to $A_i(B[n])$. Each of these is added to $A_i(S; b_n)$ with probability at least $|S|/(b_n + b_n|S|^{-1/2})$. A Chernoff bound therefore shows that

$$\mathbb{P}\left(\bigl|A_{b_n|S|^{-1/2}}(S; b_n)\bigr| - |S| > \frac{|S|^{1/2}}{1+|S|^{-1/2}} - |S|^{3/8}\right) > 1 - C\exp\bigl(-c|S|^{1/8}\bigr).$$

By repeating this procedure at most $|S|^{1/2}$ times (here we use the assumption that $a \le 2$), we get that on an event of probability at least

$$1 - C|S|^{1/2}\exp\bigl(-C|S|^{1/8}\bigr) \ge 1 - C\exp\bigl(-|S|^{1/9}\bigr),$$

we have $|A_{(a-1)b_n(1+C_7|S|^{1/8})}(S; b_n)| > (a-1)|S|$. This finishes the lemma: we construct the coupling by letting $A_i(S) = A_{j(i)}(S; b_n)$ where $j(i)$ is the first time that $|A_{j(i)}(S; b_n)| = |S| + i$ and then with probability at least $1 - C\exp(-|S|^{1/9})$ we have $j((a-1)|S|) \le (a-1)b_n(1 + C_7|S|^{1/8})$ so

$$\begin{aligned} A_{(a-1)|S|}(S) &= A_{j((a-1)|S|)}(S; b_n) \subseteq A_{(a-1)b_n(1+C_7|S|^{1/8})}(S; b_n) \\ &\subseteq A_{(a-1)b_n(1+C_7|S|^{1/8})}\bigl(B[n]\bigr). \end{aligned}$$

As needed. □

We will now prove that the uIDLA started from a ball is contained in a suitable ball with high probability.

Proposition 3.1. *There exist $C_8 > 0$ such that for any $1 \le a \le 2$ the event*

$$A_{(a-1)b_n}\bigl(B[n]\bigr) \subseteq B\bigl[na^{1/d}\bigl(1 + C_8 n^{-1/5}\bigr)\bigr]$$

occurs with superpolynomially large probability.

Here and below, when we say about a sequence of events E_n that they "occur with superpolynomially large probability" we mean that there exists a function ϕ decreasing to 0 faster than any power of n such that $\mathbb{P}(E_n) > 1 - \phi(n)$. We might also use the phrase "$\mathbb{P}(E_n)$ grows superpolynomially" (and we do not insinuate by that the the probabilities increase as a function of n, just the bound above).

In order to prove this proposition, we first remark that as a consequence of Corollary 3.1, the ball of radius $na^{1/d}(1 - C_6 n^{-1/4})$ is included in the uIDLA cluster $A_{(a-1)b_n}(B[n])$ with stretched exponentially small probability. Hence we only need to control a number of particles of order $an^{d-1/4}$. However, these particles could in principle cover a thin spike that would reach very far. We know this cannot happen in regular IDLA, but in our case, a new particle may start on the furthermost point of the cluster, which complicates the situation. We therefore need to consider the genealogy of the particles in the process.

Recall that a *rooted tree* is a graph with no cycle and one marked point called the root. A *rooted forest* is a family of disjoint rooted trees.

We construct the uIDLA starting from a set S in a new fashion. Consider a rooted forest whose vertices are indexed by integers and constructed as follows. At time 0, $\mathcal{T}_0(S)$ is given by $|S|$ isolated sites indexed by $1, 2, \ldots, |S|$, which are the roots of the trees. At each step the vertex set of $\mathcal{T}_k$ is $A_k(S)$ and the edges of $\mathcal{T}_k$ are constructed inductively as follows: $\mathcal{T}_{k+1}$ has all the edges of $\mathcal{T}_k$ and one more, from the starting point of the random walk which constructed A_{k+1} to its end i.e. to $A_{k+1} \setminus A_k$. We will call this construction the *genealogical construction* of the uIDLA cluster, and $\mathcal{T}_k$ the *genealogical tree* encoding it. We will look closely at the forest structure of $\mathcal{T}_k$, not at its embedding in $\mathbb{Z}^d$: rather, we think of a particle in the cluster as having both a position in $\mathbb{Z}^d$ and a position in the *genealogical tree (or forest)* associated with the cluster.

We start by an elementary lemma which is a generalisation of [10, Lemma 2.1]. As in First Passage Percolation, we attribute to every edge of the forest a geometric random variable with parameter $1/2$, independent of the random variables of other edges. We define the *passage time* between two vertices as the sum of the random variables over edges on the geodesic between those two vertices (note that in this case, there is only one choice for the minimal path). The *reaching time* of a vertex is the passage time between the root and this vertex.

Lemma 3.3. *Let $n, h > 0$. Consider $\mathcal{T}_n(\{0\})$ constructed as above when starting from $A = \{0\}$. There exist $c, C > 0$ such that for any $h \geq C \log n$, then*

$$\mathbb{P}[\exists \text{ a vertex with reaching time larger than } h] \leq e^{-ch}.$$

Proof. Let us first consider a slightly different model. Let $\tilde{\mathcal{T}}_t$ be the tree obtained from the same rule as for $\mathcal{T}_n$, but in continuous time (meaning that a new edge appears on each vertex according to an exponential clock of mean 1). Rather than explicitly writing the coupling between $\tilde{\mathcal{T}}_t$ and $\mathcal{T}_n$, we embed $\tilde{\mathcal{T}}_t$ in our probability space so that it is independent from $\mathcal{T}_n$. This model is exactly the model studied in [10]. In particular, if $X_t(k)$ is the number of vertices at graph distance k from the root, Lemma 2.1 of [10] shows that

$$\mathbb{E}\big[X_t(k)\big] = \frac{t^k}{k!}.$$

Choose $C_9 > 1$ large enough and $c_7 > 0$ small enough so that $\mathbb{E}[X_t(k)] \leq e^{-c_7 k}$ for $k \geq C_9 t$. Let $\tilde{D}_t(h)$ be the number of sites with passage time larger than h in $\tilde{\mathcal{T}}_t$. We find

$$\mathbb{E}\big[\tilde{D}_t(h)\big] = \sum_{k=0}^{\infty} p_{k,h} \mathbb{E}\big[X_t(k)\big] = \sum_{k=0}^{\infty} p_{k,h} \frac{t^k}{k!},$$

where $p_{k,h}$ is the probability that the sum of k independent geometric random variables of mean 1/2 is larger than $h - k$. There exists $c_8 > 0$ such that $p_{k,h} \leq e^{-c_8 h}$ for any $k \leq h/3$. For simplicity, let us assume that $c_8 < 2/3$. By dividing the sum between $k \leq c_8 h/2$ and $k \geq c_8 h/2$, we find that for $h \geq 2C_9 t/c_8$,

$$\mathbb{E}\big[\tilde{D}_t(h)\big] = \sum_{k=0}^{c_8 h/2} p_{k,h} \frac{t^k}{k!} + \sum_{k=c_8 h/2}^{\infty} p_{k,h} \frac{t^k}{k!}$$

$$\leq \sum_{k=0}^{c_8 h/2} p_{k,h} \frac{t^k}{k!} + \sum_{k=c_8 h/2}^{\infty} \frac{t^k}{k!}$$

$$\leq \frac{c_8 h}{2} e^{c_8 h/2 C_9} e^{-c_8 h} + \frac{e^{-c_7 c_8 h/2}}{1 - e^{-c_7}}.$$

In the second line, we used that $p_{k,h} \leq 1$, and in the third line both the bound on $\frac{t^k}{k!} \leq e^{-c_7 k}$ obtained by assuming that $k \geq c_8 h/2 \geq C_9 t$, and the bounds $\frac{t^k}{k!} \leq e^t \leq e^{c_8 h/2 C_9}$ and $p_{k,h} \leq e^{-c_8 h}$ when $k \leq c_8 h/2$.

It only remains to go back from continuous time to discrete time. Let $D_n(h)$ be the number of sites with reaching time larger than h in $\mathcal{T}_n$. From our construction, conditionally on the event $\{\tilde{\mathcal{T}}_t$ has k sites$\}$, $\tilde{D}_t$ has the same law as D_k.

Since with probability at least $1/2$, the aggregate $\tilde{\mathcal{T}}_{2\log n}$ has more than n particles, we deduce that

$$\begin{aligned}
\mathbb{P}\big[D_n(h) > 0\big] &\leq \mathbb{E}\big[D_n(h)\big] \leq 2\mathbb{E}\big[D_n(h)\big]\mathbb{P}(\tilde{\mathcal{T}}_{2\log n} \text{ has more than } n \text{ sites}) \\
&\leq 2\sum_{k \geq n} \mathbb{E}\big[D_k(h)\big]\mathbb{P}(\tilde{\mathcal{T}}_{2\log n} \text{ has } k \text{ sites}) \\
&\leq 2\sum_{k \geq n} \mathbb{E}\big[\tilde{D}_{2\log n}(h) | \tilde{\mathcal{T}}_{2\log n} \text{ has } k \text{ sites}\big]\mathbb{P}(\tilde{\mathcal{T}}_{2\log n} \text{ has } k \text{ sites}) \\
&\leq 2\mathbb{E}\big[\tilde{D}_{2\log n}(h)\big] \leq \exp(-c_9 h)
\end{aligned}$$

for any $h \geq (4C_9/c_8)\log n$, and c_9 sufficiently small. □

We are now ready to prove Proposition 3.1. Recall that it stated that with superpolynomially large probability, $A_{(a-1)b_n}(B[n]) \subset B[a^{1/d} n(1 + C_8 n^{-1/9d})]$.

Proof. By Corollary 3.1, we know that with stretched exponential probability, $A_{(a-1)b_n}(B[n])$ contains $B[a^{1/d} n(1 - C_6 n^{-1/4})]$. But that leaves only $C_{10} n^{d-1/4}$ particles unaccounted for. Consequently, there are at most $C_{10} n^{3d/4}$ particles outside $B[a^{1/d} n]$ at the end of the construction and therefore also at every previous step.

Recall that Lemma 2.5 states that it is difficult to traverse any annulus $B[m] \setminus B[n]$ containing less than εb_{m-n} vertices. In our setting, this means that there exists some constant β_d such that for each of the annuli

$$R_k = B\big[n + (k+1)\beta_d n^{3/4}\big] \setminus B\big[n + k\beta_d n^{3/4}\big],$$

the conclusion of Lemma 2.5 holds in this annulus, with $p_{\text{Lemma 2.5}} = 1/2$, if it is filled with less than $C_{10} n^{d-1/4}$ particles (note that $3d/4 \leq d - 1/4$ since $d \geq 2$). Remark that β_d is a constant that depends only on the dimension.

Since all the R_k's are outside $B[n]$, each of them contains at most $C_{10} n^{d-1/4}$ particles at any point in the construction of the cluster. Hence, the number of annuli that a particle can cross between its starting point and its exit point is stochastically dominated by a geometric variable of parameter $1/2$, and all these geometric variables can be taken to be independent.

The above discussion shows that a single particle may not go further than $C \log n$ annuli from its starting point. To get from this a bound on the size of the aggregate is a question about the forest $\mathcal{T}$. Precisely, the maximum k that we consider is stochastically dominated by the maximum reaching time in the forest $\mathcal{T}_{(a-1)b_n}(B[n])$.

We now apply Lemma 3.3. Recall that it stated that for $\mathcal{T}_{b_n}(\{0\})$, the probability that it has a vertex with reaching time bigger than $\log(n)^2$ is smaller than $\exp(-c\log(n)^2)$. For every $x \in B[n]$, the corresponding tree in $\mathcal{T}_{(a-1)b_n}(B[n])$ is stochastically dominated by $\mathcal{T}_{b_n}(\{0\})$ (recall that $a \leq 2$) so we get that, with superpolynomially large probability, the reaching time of every x in every tree of $\mathcal{T}_{(a-1)b_n}(B[n])$ is smaller than $\log(n)^2$.

Now, the reaching time was defined using geometric random variables independent of the forest $\mathcal{T}_{(a-1)b_n}(B[n])$, so we can use the number of R_k crossed by the corresponding particles, because the events that "there are at most $C_{10} n^{d-1/4}$ particles outside $B[a^{1/d} n]$ and yet our particle crossed annulus i" have probability bounded above by $\frac{1}{2}$, independently of the tree structure.

We conclude that with superpolynomially large probability, the annulus $R_{\log(n)^2}$ is not reached by any particle, so that $A_{(a-1)b_n}(B[n]) \subseteq B[na^{1/d}(1 + \beta_d \log(n)^2 n^{-1/4}]$, which concludes the proof. □

Proposition 3.1 and Lemma 3.2 together imply the following corollary, which is a converse to Corollary 3.1:

Corollary 3.2. *There exist $C_{11} > 0$ such that for any set $S \subseteq B[n]$,*

$$\mathbb{P}\big(\forall a \geq 1 A_{(a-1)|S|}(S) \subseteq B\big(\big[na^{1/d}\big(1 + C_{11}n^{-1/5}\big]\big)\big)$$

grows superpolynomially in $|S|$.

This follows from Lemma 3.2, which states that

$$A_{(a-1)|S|}(S) \subset A_{(a-1)b_n(1+Cn^{-1/2})}\big(B[n]\big)$$

and Proposition 3.1 for $a \leq 2$. Iterating (as in the end of the proof of Corollary 3.1) gives the result for general a. We omit the details.

4. Proof of Theorem 1.1

4.1. *Inner bound*

Our proof requires rough initial bounds before better bounds can be proved. Our rough outer bound is the obvious remark that $A_n \subset B[n]$ because it is connected. For a rough inner bound, we have the following lemma:

Lemma 4.1. *There exists $\delta_6 > 0$ such that or all n big enough,*

$$\mathbb{P}\big(B[n/2] \subseteq A_{e^{b_n}}\big) \geq 1 - \exp\big(-n^{\delta_6}\big).$$

Proof. We consider the particles that start from the origin. At step k, the event that the new particle starts from the origin has probability $1/k$, hence the number of particles started from the origin by time e^{b_n} has expected value b_n and is bigger than $2b_n/3$ with probability $e^{-b_n/18}$ from a Chernoff bound. Classical IDLA results (in particular, the explicit bound in [2, paragraph 3.1.3] is much stronger than what is needed here) guarantee that the standard IDLA with $2b_n/3$ particles started from the origin covers at least $B([n/2])$ except on an event of stretched exponentially small probability. □

Now that we have a rough bound, we are in a position to improve it. The following proposition states that a rough inner bound can always be improved, provided we have an outer bound as well.

Proposition 4.1. *Let $\lambda > 1$ and $\varepsilon > 0$ be two parameters. Suppose that $B[n] \subseteq S \subseteq B[\lambda n]$ and that $|S| > (1+\varepsilon)b_n$. For a constant η_2 depending only on the dimension, and uniformly in S,*

$$B\left[3\lambda n\left(1 + \frac{\eta_2\varepsilon}{\lambda^d}\right)^{1/d}\right] \subseteq A_{((3\lambda)^d-1)|S|}(S)$$

with superpolynomially large (in εn) probability.

There are two steps in the proof. We first look at the growth of $B[n]$ while ignoring completely the sites in $S \setminus B[n]$. Then, we use sites of $S \setminus B[n]$ (which are not too far from the origin). These sites represent a tiny proportion of $A_{((3\lambda)^d-1)|S|}(S)$, but it is more than sufficient to counter the loss of the first step. Lemma 2.4 is crucial in this argument.

Proof. We know from Corollary 3.1 that the ball of radius $3\lambda n(1 - C_6 n^{-1/4})$ is included in the subset uIDLA aggregate $A_{((3\lambda)^d-1)|S|}(B[n]; |S|)$ with probability greater than $1 - \exp(-n^{\delta_3})$.

Lemma 2.4 now yields that the aggregate we are interested in stochastically dominates the one built by adding the particles of $S \setminus B[n]$ to the subset uIDLA aggregate $A_{((3\lambda)^d-1)|S|}(B[n]; |S|)$. In a formula,

$$\begin{aligned}
&\mathbb{P}\big(E \subseteq A_{((3\lambda)^d-1)|S|}(S)\big)\\
&\quad\geq \mathbb{P}\big(E \subseteq DA_{S\setminus B[n]}\big(A_{((3\lambda)^d-1)|S|}\big(B[b]; |S|\big)\big)\big)\\
&\quad\geq \mathbb{P}\big(E \subseteq DA_{S\setminus B[n]}\big(B\big[3\lambda n\big(1 - C_6 n^{-1/4}\big)\big]\big)\big) - \exp\big(-n^{\delta_3}\big)
\end{aligned}$$

(this holds for any set E but, for the curious, we will eventually use it for $E = B[3\lambda n(1 + \eta_2\varepsilon/\lambda^d)^{1/d}]$, i.e. the set from the statement of the lemma).

Next, since $S \subset B[\lambda n]$, all the points in $S \setminus B[n]$ are inside the half-radius of $B[3\lambda n(1 - C_6 n^{-1/4})]$, and we are in a position to apply Lemma 2.1. It yields the following:

$$\begin{aligned}&\mathbb{P}\big(E \subseteq DA_{S\setminus B[n]}\big(B\big[3\lambda n\big(1 - C_6 n^{-1/4}\big)\big]\big)\big)\\&\quad\geq \mathbb{P}\big(E \subseteq DA_{\kappa}\big(B\big[3\lambda n\big(1 - C_6 n^{-1/4}\big)\big]\big)\big),\end{aligned}$$

where κ is a random variable following a binomial law with $|S| - b_n > \varepsilon b_n$ trials and probability of success η, and $\eta > 0$ is the constant defined in Lemma 2.1.

Now, applying Chernoff's bound yields that $\kappa > \frac{3}{4}\eta\varepsilon b_n$ with probability higher than $1 - \exp(-c\varepsilon b_n)$. This means that the number of particles added is not too small. This fact, together with the inner bound for standard IDLA (from [2,12] once again), guarantees that with exponentially high probability, $DA_\kappa(B[3\lambda n(1 - C_6 n^{-1/4})])$ contains a ball of radius $3\lambda n(1 + \eta\varepsilon/(8 \cdot 3^{d-1}\lambda^d))^{1/d}$. The lemma thus holds with the value $\eta_2 = \eta/(8 \cdot 3^{d-1})$. □

This method for improving inner bounds enables us to prove the inner part of Theorem 1.1.

Proposition 4.2. *Let $d \geq 2$. There exists constants c_1, C_1 depending only on the dimension such that almost surely,*

$$B\big[n\big(1 - C_1 n^{-c_1}\big)\big] \subseteq A_{b_n},$$

for n large enough.

Proof. Lemma 4.1 and the remark before it provide us with the following bounds: for arbitrarily large m_0, with probability at least $1 - \exp(-m_0^{\delta_6})$, $B[m_0] \subseteq A_{e^{2b_{m_0}}} \subseteq B[e^{2b_{m_0}}]$.

Corollary 3.2 then guaranties that conditionally on the previous event, with superpolynomially (in m_0) large probability,

$$A_{(a^d-1)e^{2b_{m_0}}} \subseteq B\big[ae^{2b_{m_0}}\big(1 + C_{11}e^{-2b_{m_0}/9d}\big)\big]$$

for any a. In other words, for some $\tau = \tau(m_0)$, $A_n \subseteq B[\tau n^{1/d}]$ for all n. Let us also assume that τ is sufficiently large so that $b_{\tau n^{1/d}} \geq 2n$ (though it would have probably held even if we had not assumed it explicitly).

We now repeatedly apply Proposition 4.1, starting from m_0. Recall that it states that if $B[r] \subseteq A_M \subseteq B[\lambda r]$ and if $M > (1 + \varepsilon)b_r$, then with high probability, for some other r' and M' we have $B[r'] \subset A_{M'}$. So applying the proposition repeatedly gives sequences r_i and M_i such that, with high probability, $B[r_i] \subseteq A_{M_i}$. Let us list all relevant parameters:

$$\begin{aligned}r_0 &= m_0 \qquad r_{i+1} = 3\lambda_i r_i\left(1 + \frac{\eta_2\varepsilon_i}{\lambda_i^d}\right)^{1/d}\\M_0 &= e^{2b_{m_0}} \qquad M_{i+1} = (3\lambda_i)^d M_i\\\varepsilon_i &= M_i/b_{r_i} - 1 \qquad \lambda_i = \tau M_i^{1/d}/r_i\end{aligned}$$

The constant η_2 comes from Proposition 4.1, but we may assume that it is small enough, so let us assume $\eta_2 \leq \frac{1}{2}$.

Let us now analyse these parameters. We first note that λ_i is decreasing – indeed, $M_i^{1/d}$ is increased at each step by $3\lambda_i$ while r_i is increased by more. On the other hand, we always have $B[r_i] \subseteq A_{M_i}$ and $b_{\tau M_i^{1/d}} \geq 2M_i$ so

$$\lambda_i^d = \left(\frac{\tau M_i^{1/d}}{r_i}\right)^d = \frac{b_{\tau M_i^{1/d}}}{b_{r_i}} \geq \frac{2M_i}{M_i} = 2$$

so $\lambda_i \geq 2^{1/d}$ for all i.

More important is the behaviour of ε_i. Putting together the formulas for r_{i+1} and M_{i+1} gives

$$\varepsilon_{i+1} + 1 = \frac{M_{i+1}}{b_{r_{i+1}}} = \frac{(3\lambda_i)^d M_i}{(3\lambda_i)^d b_{r_i}(1 + \eta_2\varepsilon_i/\lambda_i^d)} = \frac{\varepsilon_i + 1}{1 + \eta_2\varepsilon_i/\lambda_i^d}$$

rearranging gives

$$\varepsilon_{i+1} \leq \varepsilon_i - \frac{c\varepsilon_i}{\lambda_i^d}$$

and since λ_i is bounded above, we get that ε_i decreases exponentially in i.

On the other hand, ε_i does not decrease too fast: since we assumed $\eta_2 \leq \frac{1}{2}$, and since $\lambda_i^d \geq 2$ we get that $\varepsilon_{i+1} \geq \min\{1, \frac{1}{2}\varepsilon_i\}$. Since $M_{i+1} \geq 3M_i$, we get that M_i increases faster than ε_i decreases. This is important because the bad event of Proposition 4.1 happens with probability superpolynomially small in $M_i\varepsilon_i$. Thus we have just shown that these bad events have summable probabilities, and further, the sum is superpolynomially small in m_0.

This establishes the proposition on a sequence. Indeed, since M_i increases no more than exponentially, then we get $\varepsilon_i \leq CM_i^{-c}$. So we get $B[r_i] \subseteq A_{b_{r_i}(1-r_i^{-c})}$, which is equivalent to the claim. To extend from a subsequence to all integers, we use Corollary 3.1. We get that between M_i and M_{i+1} the contained ball still follows the volume of the aggregate up to a polynomial probability, with superpolynomially large probability (in M_i). These probabilities may be summed.

All in all we get that for some c_{10} independent of m_0 and some $C(m_0)$, the event

$$B\big[n\big(1 - C(m_0)n^{-c_{10}}\big)\big] \subset A_{b_n} \quad \forall n$$

holds with superpolynomially large probability. This means that almost surely, it does indeed hold for some m_0, and this means that the proposition holds with $c_1 = \frac{1}{2}c_{10}$, and an arbitrary C_1. □

4.2. *Outer bound*

Proposition 4.3. *Almost surely,* $A_{b_n} \subseteq B[n(1 + Cn^{-c})]$.

Proof. The proof is identical to the proof of Proposition 3.1, but using the inner bound (Proposition 4.2) as a basis. Let us recall quickly the argument. Proposition 4.2 ensures that $B[n(1 - Cn^{1-c})] \subset A_{b_n}$ but that leaves only Cn^{d-c} particles unaccounted for (and possibly outside $B[n]$). Hence the same holds during the entire process up to time b_n. Lemma 2.5 then ensures that annuli of width Cn^{1-c} around $B[n]$ are difficult to cross for a random walker, hence none of the particles cross more than $\log n$ of them. Finally, Lemma 3.3 ensures that none of the trees of $\mathcal{T}_{b_n}(\{0\})$ has depth larger than $\log^2 n$, so no particle may end up further than $Cn^{1-c}\log^3 n$. This ends the proof. □

Appendix: Proof of Lemma 2.2

The following is a guide on how to read [1] and modify the authors' proof to obtain the desired result. Note than any reference used in the following is to be found inside [1]. First, we start with an already occupied region $B[n]$. The flashing process introduced by the authors is the same, except it only starts flashing after exiting $B[n]$, and the coupling they describe holds true in our setting. Therefore an interior bound like the one we wish to prove can be proven in the flashing process setting. Then, in Section 4.2, we will use $\xi = nr^{3/2}$ (instead of $\log(n)$) in Eq. (4.10). Moreover we follow the authors' recommandation to use a constant $h_k = h$ when dealing with IDLA.

Recall that $W_k(\mathcal{T})$ is the number of unsettled explorers (out of our N initial explorers) that stand in a cell $\mathcal{T}$ when the cluster is built up to radius r_k, and that $M_k(\mathcal{T})$ is the number of explorers that exit $B[r_k]$ through $\mathcal{T}$. Bounding the number of settled explorers that exit $B[r_k]$ through $\mathcal{T}$ by saying at most one can have settled on each site of $B[r_k - h] \setminus B[n]$, we get the usual equation:

$$W_k(\mathcal{T}) + L_k(\mathcal{T}) \geq M_k(\mathcal{T}),$$

with $L_k(\mathcal{T})$ the number of particles that exit $B[r_k]$ through $\mathcal{T}$ when one is started on each site of $B[r_k - h] \setminus B[n]$. Note that here we stray from the authors' proof since our L_k is not the same as theirs. Hence we need to check that we still have the desired value for $\mathbb{E}[M_k(\mathcal{T}) - L_k(\mathcal{T})]$. The computation is done in a subparagraph called *Step 1* which we emulate here. Recall that for an integer-valued function f, $M(f, r_k, \mathcal{T})$ is the number of particles that exit $B[r_k]$ through $\mathcal{T}$ when $f(x)$ particles are started at each point x independently. We follow the authors in also denoting $M(A, r_k, \mathcal{T}) = M(\mathbf{1}_A, r_k, \mathcal{T})$ when A is a subset of $\mathbb{Z}^d$.

$$\begin{aligned}
\mathbb{E}\big[M_k(\mathcal{T}) - L_k(\mathcal{T})\big] &= \mathbb{E}\big[M(N\mathbf{1}_0, r_k, \mathcal{T})\big] - \mathbb{E}\big[M\big(B[r_k - h] \setminus B[n], r_k, \mathcal{T}\big)\big] \\
&= \mathbb{E}\big[M\big((b_n + N - b_{r_k-h})\mathbf{1}_0, r_k, \mathcal{T}\big)\big] \\
&\quad + \mathbb{E}\big[M\big(B[n], r_k, \mathcal{T}\big)\big] - \mathbb{E}\big[M(b_n\mathbf{1}_0, r_k, \mathcal{T})\big] \\
&\quad + \mathbb{E}\big[M(b_{r_k-h}\mathbf{1}_0, r_k, \mathcal{T})\big] - \mathbb{E}\big[M\big(B[r_k - h], r_k, \mathcal{T}\big)\big]
\end{aligned}$$

The absolute values of the differences on the second and third lines are then each bounded using Corollary 5.4, so that we get for some constants C, κ depending only on the dimension,

$$\begin{aligned}\mathbb{E}\big[M_k(\mathcal{T}) - L_k(\mathcal{T})\big] &\geq (b_n + N - b_{r_k - h})\mathbb{P}_0\big(S(H_k) \in \mathcal{T}\big) - C \\ &\geq (b_n + N - b_{r_k - h})\frac{\kappa}{r_k^{d-1}}.\end{aligned}$$

Note that as long as $r_k - h \leq n(1 + r - r^{3/2})^{1/d}$, the first factor on the right hand side (that is, the difference in volumes) is bigger than $\kappa_d n^d r^{3/2}$ for a constant κ_d depending only on the dimension.

We then resume the course of the authors' proof, noting that the bound (4.14) does not concern us since the order of the left-hand-side is at most logarithmic (see (4.18)). Finally, we conclude in (4.15) by replacing the authors' $\log(n)$ with $nr^{3/2}$.

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