Upper bounds on the percolation correlation length

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Abstract

We study the size of the near-critical window for Bernoulli percolation on $\mathbb{Z}^d$. More precisely, we use a quantitative Grimmett-Marstrand theorem to prove that the correlation length, both below and above criticality, is bounded from above by $\exp(C/|p-p_c|^2)$. Improving on this bound would be a further step towards the conjecture that there is no infinite cluster at criticality on $\mathbb{Z}^d$ for every $d \geq 2$.

1 Introduction

1.1 Definition of the model and motivation

Fix an integer $d \geq 2$. Two vertices x and y of $\mathbb{Z}^d$ are said to be *neighbours* (denoted $x \sim y$) if $\|x-y\|_2 = 1$. In such case, $\{x,y\}$ is called an *edge* of $\mathbb{Z}^d$. The set of edges is denoted by $E(\mathbb{Z}^d)$. For $n \geq 1$, introduce the box $\Lambda_n := \{-n,\ldots,n\}^d$ and its (vertex) boundary $\partial\Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$. Also, we define $\mathrm{Slab}_n^d := \mathbb{Z}^2 \times \{-n,\ldots,n\}^{d-2}$.

A percolation configuration $\omega = (\omega(e) : e \in E(\mathbb{Z}^d))$ is an element of $\{0,1\}^{E(\mathbb{Z}^d)}$. If $\omega(e) = 1$, the edge e is said to be *open*, otherwise it is said to be *closed*. Two vertices x and y are said to be *connected in S* (in ω) if there exists a path $x = v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_k = y$ of vertices in S such that $\omega(\{v_i, v_{i+1}\}) = 1$ for every $0 \leq i < k$. We write $x \stackrel{S}{\longleftrightarrow} y$ (if $S = \mathbb{Z}^d$, we simply drop S from the notation) and $x \stackrel{S}{\longleftrightarrow} \infty$ if $x \stackrel{S}{\longleftrightarrow} \partial\Lambda_n$ holds for any $n \geq 1$. A *cluster* is a maximal set of vertices that are connected together in ω.

For $p \in [0,1]$, consider the Bernoulli bond percolation measure $\mathbb{P}_p$ on $\{0,1\}^{\mathbb{E}(\mathbb{Z}^d)}$ under which the variables $\omega(e)$ with $e \in E(\mathbb{Z}^d)$ are i.i.d. Bernoulli random variables with parameter p. Define $p_c = p_c(d) \in (0,1)$ such that $\mathbb{P}_p[0 \leftrightarrow \infty]$ is 0 (resp. strictly positive) if $p < p_c$ (resp. $p > p_c$). The main open question in percolation theory is to understand the behaviour at criticality, i.e. when p is equal to p_c, and in particular to prove that there does not exist an infinite cluster at p_c.

Conjecture 1. *For every $d \geq 2$, $\mathbb{P}_{p_c}[0 \leftrightarrow \infty] = 0$.*

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This conjecture has been solved for $d = 2$ [Har60] and for $d \geq 11$ [FvdH17]; see also [HS90]. The result is also known on graphs of the form $\mathbb{Z}^2 \times G$ with G finite; see [DST15]. On transitive graphs with rapid growth, additional tools are available, and the following cases are known: non-amenable graphs [BLPS99], graphs with exponential growth [Hut], and recently some graphs with stretched-exponential growth [HH].

A natural scheme to attack the conjecture on $\mathbb{Z}^d$ is to find a $\delta > 0$ and a sequence of events $\mathscr{E}_n$ depending on edges in Λ_n only, such that *for any* p,

$$\exists n > 0 \text{ s.t. } \mathbb{P}_p[\mathscr{E}_n] > 1 - \delta \quad \iff \quad \mathbb{P}_p[0 \leftrightarrow \infty] > 0. \tag{$\star$}$$

If such a sequence exists, the set of p such that $\mathbb{P}_p[0 \leftrightarrow \infty] > 0$ is an open set since it is the union of the open sets (indexed by n) $\{p : \mathbb{P}_p[\mathscr{E}_n] > 1 - \delta\}$ (this set is open since $p \mapsto \mathbb{P}_p[\mathscr{E}_n]$ is continuous).

Of course, this strategy is tempting, but the main difficulty is that the $\Longrightarrow$ and the $\Longleftarrow$ implications involved in ($\star$) are difficult to prove simultaneously. One may for instance easily check the $\Longrightarrow$ implication by asking a lot on $\mathscr{E}_n$, but then the $\Longleftarrow$ one becomes difficult, and vice-versa. To illustrate this trade-off phenomenon, let us give a few examples of possible sequences $(\mathscr{E}_n)$, going from the strongest criterion (meaning the one for which the $\Longrightarrow$ implication is the easiest to prove) to the weakest one (meaning the one for which $\Longrightarrow$ is the hardest).

Example 1. Let $\mathscr{E}_n$ be the event that $\Lambda_{n/10}$ is connected to $\partial\Lambda_n$ and that the second largest cluster in Λ_n has radius smaller than $n/10$. In such case, a coarse-graining argument similar to [AP96] implies the $\Longrightarrow$ implication easily. Proving $\Longleftarrow$ is still open in particular because of the difficulty to exclude the existence of many large clusters avoiding each other.

Example 2. Let $\mathscr{E}_n$ be the intersection of the events that $(\pm n, 0) + \Lambda_{n/2}$ are connected in Λ_{2n} and that there exists at most one cluster in Λ_{2n} going from $(\pm n, 0) + \Lambda_{n/2}$ to $(\pm n, 0) + \partial\Lambda_n$. A coarse-graining argument may be used to prove $\Longrightarrow$ but $\Longleftarrow$ remains open due to the same reason as the previous condition.

In general, uniqueness of clusters going from one area to another one is a key difficulty in these problems. This might be related to the fact that in high dimensions Λ_n indeed hosts many disjoint clusters in p_c, see [A97]. In order to circumvent this difficulty, one can make different choices for $\mathscr{E}_n$.

Example 3. Let $\mathscr{E}_n$ be the same event as in the second example, but with $(\pm n, 0) + \Lambda_{n/2}$ replaced by $(\pm n, 0) + \Lambda_{u_n}$, with u_n much smaller than $n/2$. In such case, one may expect that the smaller the u_n, the easier (resp. harder) $\Longrightarrow$ (resp. $\Longleftarrow$) becomes. This is, of course, assuming that the main difficulty in proving **P2** lies in the requirement that there is at most one cluster between $(\pm n, 0) + \Lambda_{u_n}$ and $(\pm n, 0) + \partial\Lambda_n$, but this assumption seems to be verified in the existing results in that direction.

Recently, a paper of Cerf [Cer15] provided a beautiful insight on how big u_n must be taken to have that with large probability, the box of size n contains at most one cluster going from Λ_{u_n} to $\partial\Lambda_n$. We will come back to this later in the introduction, but let us mention the result right now.

Given $1 \leq m \leq n$, consider the set of clusters in the configuration restricted to the box Λ_n, and define $A_2(m, n)$ to be the event that there exists at least two disjoint such clusters intersecting both Λ_m and $\partial\Lambda_n$.

Proposition 1 (Cerf [Cer15])**.** *Let $d \geq 2$. There exists $\alpha = \alpha(d) \in (0,1)$ such that for any $p \in [0,1]$ and n large enough,*

$$\mathbb{P}_p\left[A_2(n^\alpha, n)\right] \leq \frac{1}{n^\alpha}.$$

Let us finish by a last example, which is very simple but interesting for the discussion that follows.

Example 4. Let $\mathscr{E}_n$ be the event that the box Λ_N is connected to $\{n\} \times \{-n, \dots, n\}^{d-1}$, with $N = N(\delta) > 0$ independent of n. Here, $\Longleftarrow$ follows easily from the ergodicity of $\mathbb{P}_p$ but again the $\Longrightarrow$ implication seems difficult to obtain.

The search for a good sequence of events $\mathscr{E}_n$ has been at the heart of attempts to prove the conjecture. An interesting development was made in [GM90]. In this paper, the author considered the sequence of events $\mathscr{E}_n$ defined in the fourth example. As mentioned above, the $\Longrightarrow$ seems extremely difficult to derive. Nevertheless, Grimmett and Marstrand introduced a clever renormalisation scheme allowing to prove the following weaker version of the implication: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every n,

$$\mathbb{P}_p[\mathscr{E}_n] > 1 - \delta \quad \Longrightarrow \quad \mathbb{P}_{p+\varepsilon}[0 \xleftrightarrow{\mathrm{Slab}_n^d} \infty] > 0$$

(recall that $\mathrm{Slab}_n^d = \mathbb{Z}^2 \times \{-n, \dots, n\}^{d-2}$). In words, the implication can be proved if one allows some sprinkling. As suggested in [GM90], if one could get rid of the sprinkling by ε in the previous statement, then the conjecture would follow.

The goal of this paper is to prove a quantitative version of the Grimmett-Marstrand argument by bounding the critical point of Slab_n^d in terms of n. In the language of the Grimmett-Marstrand theorem, we will be interested in how small ε can be taken as a function of n. We believe that improving how small ε can be taken is a good intermediate problem for the conjecture. Getting bounds is non-trivial and requires some understanding of the critical phase. As a consequence, each improvement on the existing bounds should shed a new light on the critical behaviour.

There is a quantity which is intimately related to $p_c(\mathrm{Slab}_n^d)$, called the *correlation length*, which appears repeatedly in physics. In order to have a statement which is independent of the Grimmett-Marstrand theorem, we choose to first state our main result in terms of the correlation length.

1.2 An upper bound on the correlation length

For $p < p_c$, [AB87, Men86, DT15b] proved that $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$ decays exponentially fast. A sub-multiplicativity argument (see e.g. [Gri99, Section 6.2]) yields the existence of the *correlation length* ξ_p defined by

$$\xi_p := \lim_{n\to\infty} -\frac{n}{\log \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]}.$$

For $p > p_c$, the correlation length is also defined, but the formula is slightly modified:

$$\xi_p := \lim_{n\to\infty} -\frac{n}{\log \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty]}.$$

The justification of the definition above runs as follows. Grimmett and Marstrand [GM90] showed that for any $p > p_c$, there exists $n \geq 1$ such that

$$\mathbb{P}_p\big[0 \xleftrightarrow{\mathrm{Slab}_n^d} \infty\big] > 0.$$

This fact implies that $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_N, 0 \not\leftrightarrow \infty]$ decays exponentially fast in N, see [CCN87]. Finally, [CCG$^+$89] contains a justification of the existence of the limit.

Our main result, which follows readily from Theorem 4 below (see next section) and the argument of [CCN87], is the following.

Theorem 2. *Let $d \geq 3$. There exists $C = C(d) > 0$ such that for any $p \neq p_c$,*

$$\xi_p \leq \exp(C|p - p_c|^{-2}).$$

The results below and above p_c are different in nature (even though the same proof gives both), a point which will become clearer when we discuss the proof in the next section. In particular, the use of [GM90] to connect slabs and the correlation length mentioned above is used only for $p > p_c$.

Our bound on ξ_p is far from the truth. Conjecturally, one has $\xi_p = |p - p_c|^{-\nu + o(1)}$, where $o(1)$ tends to 0 as $p \to p_c$ ($p \neq p_c$) and ν is given by

$$\nu = \begin{cases} \frac{4}{3} & \text{if } d = 2, \\ 0.87(1) & \text{if } d = 3, \\ 0.69(1) & \text{if } d = 4, \\ 0.56(1) & \text{if } d = 5, \\ \frac{1}{2} & \text{if } d \geq 6. \end{cases}$$

The predictions for $d = 3, 4, 5$ are numerical while the prediction for $d = 2$ is based on Conformal Field Theory, Quantum gravity or Coulomb gas formalism, and the prediction for $d \geq 6$ on the fact that the model should have a mean-field behaviour. For site percolation on the triangular lattice, $\xi_p = |p - p_c|^{-4/3 + o(1)}$ was proved in [SW01] using the conformal invariance of the model proved in [Smi01] and scaling relations obtained by Kesten in [Kes87] (such scaling relations were proved under the hyper-scaling hypothesis [BCKS99] which is expected to be valid for $d \leq 5$). In fact, the Russo-Seymour-Welsh theory [Rus78, SW78] and [Kes87] classically imply that there exists $C > 0$ such that $\xi_p \leq |p - p_c|^{-C}$ for Bernoulli bond percolation on $\mathbb{Z}^2$. For $d \geq 19$, $\xi_p = |p - p_c|^{-1/2 + o(1)}$ was proved in [HS90, H90] for $p < p_c$.

1.3 A quantitative Grimmett-Marstrand theorem

As mentioned in the previous section, the bound on the correlation length is related to bounds on $p_c(\mathrm{Slab}_n^d)$. It does not come as a surprise that one of the main results of this paper is therefore a quantitative version of the result of Grimmett and Marstrand. Recall the definition of $A_2(m, n)$ preceding Proposition 1.

Theorem 3. *Fix $d \geq 3$. There exists a constant $C = C(d) > 0$ such that the following holds. Assume that for some $p \in [0, 1]$ and some $\varepsilon > 0$, there exist $1 \leq k \leq K \leq n \leq N < \infty$ such that $K \leq \varepsilon n$ and*

(a) $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_N] \geq \varepsilon$,

(b) $\mathbb{P}_p[\Lambda_k \leftrightarrow \partial\Lambda_N] \geq 1 - \exp(-\frac{1}{\varepsilon})$,

(c) $\mathbb{P}_q[A_2(k, K)] \leq \exp(-\frac{1}{\varepsilon})$ *and* $\mathbb{P}_q[A_2(n, N)] \leq \exp(-\frac{1}{\varepsilon})$ *for every* $q \geq p$.

Then

$$\mathbb{P}_{p+C\varepsilon}[0 \xleftrightarrow{\mathrm{Slab}^d_{2N}} \infty] \geq \tfrac{\varepsilon}{2}.$$

Observe that when working with the events $\mathscr{E}_n$ defined in the fourth example of the previous section, one usually shows that $\mathbb{P}_p[\mathscr{E}_n]$ tends to 1 as n tends to infinity with an explicit speed of convergence. Combined with the previous result, this enables us to optimize on n to find the smallest sprinkling possible. In particular, we get the following result.

Theorem 4. *Fix $d \geq 3$. There exists a constant $C = C(d) > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_{p_n + \frac{C}{\sqrt{\log n}}}[0 \xleftrightarrow{\mathrm{Slab}^d_n} \infty] \geq \frac{1}{2\sqrt{\log n}},$$

where $p_n < p_c$ is the smallest p such that $\xi_p = n$. In particular, we have that

$$p_c(\mathrm{Slab}^d_n) < p_c + \frac{C}{\sqrt{\log n}}.$$

Let us sketch the proof of Theorem 4. The strategy is to apply Theorem 3 with

$$\begin{aligned} p &= p_n + O(1/\sqrt{\log n}), \\ \varepsilon &= 1/\sqrt{\log n}, \\ (k, K, n, N) &= (n^{\alpha^3}, n^{\alpha^2}, n^{\alpha}, n), \end{aligned}$$

where α is given by Proposition 1. By definition of α, **(c)** is satisfied when n is large enough. In order to obtain **(a)**, we rely on a differential inequality introduced in [DT15a] (based on ideas of Chayes & Chayes, [CC86]), which will enable us to prove the following proposition.

Proposition 5. *For every n large enough,*

$$\mathbb{P}_{p_n + \frac{1}{\sqrt{\log n}}}[0 \leftrightarrow \partial\Lambda_n] \geq \frac{1}{\sqrt{\log n}}. \tag{1}$$

In order to obtain **(b)**, we will rely on the following proposition.

Proposition 6. *For every $\beta > 0$, there exists $c = c(\beta, d) \geq 2$ such that for every n large enough,*

$$\mathbb{P}_{p_n + \frac{c}{\sqrt{\log n}}}[\Lambda_{n^\beta} \leftrightarrow \partial\Lambda_n] \geq 1 - e^{-\sqrt{\log n}}.$$

The proof relies on a general sharp threshold result on Boolean functions (see [Tal94] or [BKK+92]). Based on this result, we show that the logarithmic derivative of the probability $\mathbb{P}_p[\Lambda_{n^\alpha} \leftrightarrow \partial\Lambda_n]$ is always larger than $O(\log n)$ (see Lemma 7). This $O(\log n)$ lower bound, which comes from the abstract sharp threshold result, is the barrier to our strategy and explains our $C/\sqrt{\log n}$ upper bound on the critical window. Any improvement in the proposition above would yield better bounds on the critical window and the correlation length.

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2 Proof of Proposition 5

Given a finite set S containing 0, and a parameter $p \in [0,1]$, define

$$\varphi_p(S) := \sum_{\substack{x \sim y \\ x \in S,\, y \notin S}} p\,\mathbb{P}_p[0 \stackrel{S}{\longleftrightarrow} x].$$

Fix $n \geq 1$. Let us recall two relations between this quantity and the one-arm probability, established in [DT15a]. First, for every $S \subset \Lambda_n$ containing 0, the last displayed equation of Section 1.4 of [DT15a] gives the upper bound

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{nk}] \leq \varphi_p(S)^{k-1}. \tag{2}$$

for every $k \geq 1$. Also, the quantity $\varphi_p(S)$ can be used to bound the derivative of the one-arm probability. Lemma 1.4 of [DT15a] states that for every $p \in [0,1]$,

$$\frac{d}{dp}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \geq \frac{1}{p(1-p)} \cdot \Big[\inf_{0 \in S \subset \Lambda_n} \varphi_p(S)\Big] \cdot (1 - \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]). \tag{3}$$

The proof of Proposition 5 can be easily derived from the two equations above. If for some $p \in [0,1]$, there exists a subset S of Λ_n with $\varphi_p(S) < \frac{1}{e}$, then one deduces immediately from (2) that

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_k] \leq e^{-\lfloor k/n \rfloor - 1},$$

which implies that $\xi_p < n$. As a consequence, $\varphi_{p_n}(S) \geq \frac{1}{e}$ for any set S included in Λ_n containing 0. Since $\varphi_p(S)$ is increasing in p, we have $\varphi_p(S) \geq \frac{1}{e}$ for any $p \geq p_n$, and the differential inequality (3) gives that for every $p \geq p_n$,

$$\frac{d}{dp}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \geq \tfrac{4}{e}(1 - \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]). \tag{4}$$

Now, set $p'_n := p_n + \frac{1}{\log n}$. Either $\mathbb{P}_{p'_n}[0 \leftrightarrow \partial\Lambda_n] > 1 - \frac{e}{4}$, or integrating (4) between p_n and p'_n gives (1). This concludes the proof.

3 Proof of Proposition 6

Fix $0 < \beta < \alpha$ (where α is chosen such that the statement of Proposition 1 holds). We rely on the following sharp threshold statement.

Lemma 7. *For any $\delta > 0$, there exists $c = c(\delta, \beta, d) > 0$ such that for every $p \in [\delta, 1-\delta]$ and every n large enough,*

$$\frac{f'(p)}{f(p)(1-f(p))} \geq c \log n, \quad \textit{where } f(p) = \mathbb{P}_p\left[\Lambda_{n^\beta} \leftrightarrow \partial\Lambda_n\right].$$

Before proving this statement, let us see how it implies Proposition 6. Set $p_n' := p_n + \frac{2}{\log n}$ and observe that Proposition 5 implies that

$$\mathbb{P}_{p_n'}\left[\Lambda_{n^\beta} \leftrightarrow \partial\Lambda_n\right] \geq \mathbb{P}_{p_n'}\left[0 \leftrightarrow \partial\Lambda_n\right] \geq \frac{1}{\log n}.$$

Integrating the differential inequality obtained from the previous lemma between p_n' and $p_n' + C/\sqrt{\log n}$ gives $\mathbb{P}_{p_n'}[\Lambda_{n^\beta} \leftrightarrow \Lambda_n] \geq 1 - \exp(-\sqrt{n})$, as needed, provided that $C = C(\beta, \delta)$ is chosen large enough. In order to conclude the proof of Proposition 6, it therefore suffices to show the lemma.

Before diving into the proof, recall the following definitions. An event A is *increasing* if it is stable by opening edges. The edge e is *pivotal for the event A* if the configurations ω^e and ω_e defined by

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \neq e \\ 1 & \text{if } f = e, \end{cases} \qquad \text{and } \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e \\ 0 & \text{if } f = e \end{cases}$$

satisfy $\omega^e \in A$ and $\omega_e \notin A$.

Proof of Lemma 7. Fix n large enough and set $m = n^\beta$ for simplicity of notation. We use the following standard sharp threshold result for Boolean functions (see e.g. [BKK+92] or [Tal94]): for any $\delta > 0$, there exists a constant $c' = c'(\delta) > 0$ such that for any increasing event A depending on a finite set E of edges, and any $p \in [\delta, 1-\delta]$,

$$\frac{d}{dp}\mathbb{P}_p\left[A\right] \geq c' \log\left(\frac{1}{\max\{\mathbb{P}_p\left[e \text{ pivotal for } A\right] : e \in E\}}\right) \cdot \mathbb{P}_p\left[A\right](1 - \mathbb{P}_p\left[A\right]). \tag{5}$$

We will establish that for every edge $e \in E$,

$$\mathbb{P}_p[e \text{ is a closed pivotal for } \Lambda_m \leftrightarrow \partial\Lambda_n] \leq \frac{1}{m^{\alpha/4}}. \tag{6}$$

The proof follows by first using that the status of an edge is independent of the event that it is pivotal and then applying (5) to the event $A = \Lambda_m \leftrightarrow \partial\Lambda_n$.

In order to prove (6), fix an edge $e \in E$ and distinguish between two cases, depending whether the edge e is close to $\partial\Lambda_m \cup \partial\Lambda_n$ or not. Write d for the L^∞-distance between the edge e and $\partial\Lambda_m \cup \partial\Lambda_n$.

If $d \geq m^{1/4}$, then observe that a translated version of the event $A_2(m^{\alpha/4}, m^{1/4})$ must occur around the edge e when the edge is a closed pivotal. Therefore, Proposition 1 implies that (6) holds.

The more difficult case is when $d \leq m^{1/4}$. Let us first assume that e is at a distance smaller than $m^{1/4}$ of Λ_m. Then, consider a translation τ by a vector in $\Lambda_{m^{1/4}}$ such that e belongs to the translate $\tau\Lambda_m$ of Λ_m by τ. Set $I := \lfloor \frac{1}{2}\, m^{3/4} \rfloor$ and for $0 \leq i < I$ define the edges $e_i = \tau^i e$ (where τ^i denotes the i-th iterate of τ). It will be important in the proof that for every $i < j$, the two endpoints of the edge e_i belong to $\tau^j \Lambda_m$. Define also the event

$$B_i := \{e_i \text{ is a closed pivotal for } \tau^i\Lambda_m \leftrightarrow \tau^i\partial\Lambda_n\}.$$

Writing M for the number of indices i for which B_i occurs, translation invariance and the Cauchy-Schwarz inequality imply

$$(I \cdot \mathbb{P}_p[B_0])^2 = \mathbb{E}_p[M]^2 \le \mathbb{E}_p[M^2] = \mathbb{E}_p[M] + 2\sum_{i<j} \mathbb{P}_p[B_i \cap B_j]. \tag{7}$$

Let us bound probabilities on the right-hand side. Fix $i < j$ and assume that $B_i \cap B_j$ occurs. Then, we claim that there must exist two disjoint clusters in $\Lambda_{n/2}$ crossing the annulus between Λ_{2m} and $\Lambda_{n/2}$. Indeed, one extremity x_i of e_i must be connected to the boundary of $\tau^i\Lambda_n$, and one extremity x_j of e_j must be connected to the boundary of $\tau^j\Lambda_n$. The fact that e_j is a *closed* pivotal implies in particular that $\tau^j\Lambda_m \not\leftrightarrow \tau^j\Lambda_n$ and hence, since x_i belongs to $\tau^j\Lambda_m$, it is not connected to the boundary of $\tau^j\Lambda_n$ so that the clusters of x_i and x_j in the box $\Lambda_{n/2}$ must be disjoint. For n large enough, we have $2m \le (n/2)^\alpha$ and Proposition 1 implies that

$$\mathbb{P}_p[B_i \cap B_j] \le \frac{1}{2m}.$$

Plugging this estimate in (7) and using the trivial bound $M \le I$, we obtain

$$\mathbb{P}_p[B_0]^2 \le \frac{1}{I} + \frac{1}{m} \le \frac{1}{\sqrt{m}},$$

provided n is large enough. This completes the proof in this case.

The exact same reasoning also works if one assumes that the edge e is within distance $m^{1/4}$ of the boundary of Λ_n. Consider a translation τ by a vector in $\Lambda_{m^{1/4}}$ such that e does *not* belong to $\tau\Lambda_n$. One can define the edges e_i and the events B_i as above. In this case, for $i < j$, the edge e_i does not belong to $\tau^j\Lambda_n$ and the same reasoning as above concludes the proof. □

Remark. The previous lemma can be obtained in an alternative fashion using Section 3 of [DCRT]. We believe that the present solution is simpler in the case of Bernoulli percolation and may have further applications.

4 Proof of Theorem 3

4.1 Strategy of the proof

We prove that the assumptions **(a)**–**(b)**–**(c)** guarantee that at $p+\varepsilon$, the origin is connected to infinity inside the slab Slab^d_{2N} with probability larger than $c\varepsilon$.

We will perform a renormalisation procedure similar to the one used in the work of Grimmett and Marstrand [GM90]. In the proof of Grimmett and Marstrand, the renormalisation scheme uses a "seeding" procedure, where a seed is defined as a large box, all the edges of which are open. In their proof, an infinite cluster is created in a thick slab by propagating a cluster from one seed to another. This uses the fact that any seed is connected to several other seeds in a box of size N, provided N large enough. Since creating a seed has an exponential cost, this "seeding" procedure requires to take N extremely large, which would provide us with even worse bounds if we make it quantitative: an earlier version of our arguments that used seeds had the bound

$$\xi(p) < e^{e^{e^{e^{|p-p_c|^{-1}}}}}.$$

We will therefore modify the Grimmett-Marstrand argument by avoiding the use of seeds. This motivates the condition **(b')** below which will replace condition **(b)**. In the following statement, the constant c depends on d only.

(b') For every connected set $S \ni 0$ with a diameter larger than n,

$$\mathbb{P}_p\left[S \leftrightarrow F(N)\right] \geq 1 - 2\exp[-c/\varepsilon],$$

where $F(N) := \{(x_1, \ldots, x_d) \in \partial\Lambda_N \,:\, x_1 = N, x_2 \geq 0, \ldots, x_d \geq 0\}$.

Remark. The introduction of facets is purely technical step and should not worry the reader. Indeed, the probability of connecting to a facet is easily compared to the probability of connecting to the boundary of the box. To this end, divide $\partial\Lambda_N$ into $d2^d$ facets $F_1, \ldots, F_{d2^d}$ where F_i is the intersection of $\partial\Lambda_N$ with one of the $d2^d$ quadrants of $\mathbb{Z}^d$. Using the Harris-FKG inequality (sometimes called "the square root trick" when used in this way, see [Gri99]) together with **(b)**, we find that

$$\mathbb{P}_p[\Lambda_k \xleftrightarrow{\Lambda_N} F(N)] \geq 1 - \exp[-1/(\varepsilon d 2^d)]. \tag{8}$$

The condition **(b')** can be understood as a strengthening of the condition **(b)** where the box Λ_k is replaced by arbitrary sufficiently large sets, and the boundary of Λ_N is replaced by one of its facets. Using this condition, we will be able to construct an infinite cluster in Slab^d_{2N} by propagating it using local connections. Heuristically, if the cluster of the origin is connected to a large box Λ away from 0, then it must contain a large set, which is sufficient to propagate this cluster to other boxes neighbouring Λ. The condition **(b')** was introduced in the work of Martineau and Tassion [MT13], where it was established using abstract measurability arguments. The main contribution here is to make it quantitative.

Proposition 8. *Let $p \in [0,1]$, there exists $C > 0$ such that for every $\varepsilon > 0$ and $k \leq K \leq n \leq N$ such that $K \leq \varepsilon n$ as well as* **(a)** *and* **(b')** *hold,*

$$\mathbb{P}_{p+C\varepsilon}[0 \xleftrightarrow{\mathrm{Slab}^d_{2N}} \infty] \geq \varepsilon/2.$$

In the proof of Proposition 8, we will use a renormalisation based on an exploration procedure of the cluster of the origin. At each step of the exploration, we will have some negative information coming from previously explored closed edges at the boundary of the cluster. This negative information constitutes a major complication, and this is why p needs to be increased further, as in the original argument of Grimmett and Marstrand.

The proof of the theorem is now organized as follows. In Section 4.2, we prove that **(b')** is implied by **(a)**, **(b)** and **(c)**. This reduces the proof of the theorem to the proof of Proposition 8. This proof is done in Section 4.3.

4.2 Proof that conditions (a), (b) and (c) imply (b')

Below, the constants c_i depend on d only.

Let $p \in [0,1]$, $\varepsilon > 0$ and $k \le K \le n \le N$ be such that $K \le \varepsilon n$ and the three conditions (a), (b) and (c) hold. Fix a connected set S containing 0 with a diameter at least n. Without loss of generality, we may assume $S \subset \Lambda_n$

Consider a family of points $x_1, \dots, x_\ell \in S$ such that the boxes $B_i'' := x_i + \Lambda_K$ are all disjoint and included in Λ_n. Also, introduce the smaller box $B_i' := x_i + \Lambda_k$. Note that we may choose $\ell \ge c_1/\varepsilon$ such points.

For every $i \in \{1, \dots, \ell\}$, define the two events

$$\begin{aligned} E_i &:= \{x_i \longleftrightarrow \partial B_i''\} \cap \{\exists \text{ unique cluster in } B_i'' \text{ from } B_i' \text{ to } \partial B_i''\}, \\ E_i' &:= \{B_i' \leftrightarrow \partial \Lambda_N\}. \end{aligned}$$

Let X be the number of indices i for which E_i does not occur. By translation invariance and conditions (a) and (c),

$$\begin{aligned} \mathbb{P}_p[E_i] &\ge \mathbb{P}_p[x_i \longleftrightarrow \partial B_i''] - \mathbb{P}_p[A_2(k,K)] \\ &\ge \varepsilon - \exp(-1/\varepsilon) \ge \varepsilon/2. \end{aligned}$$

Large deviations estimates for Bernoulli random variables (recall that boxes B_i' are disjoint) show that

$$\mathbb{P}_p[X \le \ell/2] \le e^{-c_2 \ell}.$$

Now, (b) implies that for every i,

$$\mathbb{P}_p[E_i'] \ge 1 - \exp[-1/(d2^d\varepsilon)].$$

Indeed, find a facet F of $x_i + \Lambda_N$ outside Λ_{N-1} and then apply (8) (with 0 shifted to x_i) to get

$$\mathbb{P}_p[E_i'] \ge \mathbb{P}_p[B_i' \leftrightarrow F] \overset{(8)}{\ge} 1 - \exp[-1/(d2^d\varepsilon)].$$

Let Y be the number of indices i for which E_i' does not occur. By Markov's inequality we have

$$\mathbb{P}_p[Y \ge \ell/2] \le \frac{2}{\ell} \cdot \mathbb{E}_p[Y] \le 2\exp[-1/(d2^d\varepsilon)]$$

Since $E_i \cap E_i'$ implies the existence of a path from x_i to $\partial\Lambda_N$, and therefore from S to $\partial\Lambda_N$, we obtain that

$$\begin{aligned} \mathbb{P}_p[S \leftrightarrow \partial\Lambda_N] &\ge \mathbb{P}_p[X > \ell/2, Y < \ell/2] \\ &\ge 1 - \mathbb{P}_p[X \le \ell/2] - \mathbb{P}_p[Y \ge \ell/2] \\ &\ge 1 - Ce^{-c_3/\varepsilon}. \end{aligned}$$

It remains to replace the boundary of $\partial\Lambda_N$ in the equation above by the facet $F(N)$. Yet, because we assumed $S \subset \Lambda_n$,

$$\mathbb{P}_p[S \overset{\Lambda_N}{\longleftrightarrow} F(N)] \ge \mathbb{P}_p[\{\Lambda_n \leftrightarrow F(N)\} \cap \{S \leftrightarrow \partial\Lambda_N\} \cap A(n,N)^c] \ge 1 - Ce^{-c_4/\varepsilon}$$

thanks to (b) (again in the form (8)) and (c). This concludes the proof.

4.3 Proof of Proposition 8

The proposition is proved by a renormalisation argument. We couple a growing exploration process on the slab with a growing exploration process on a rescaled version of the square lattice. One will need a simple condition for a growing exploration process on $\mathbb{Z}^2$ to contain an infinite cluster. Therefore, before moving to the proof, we describe a particular type of exploration process on $\mathbb{Z}^2$ and give a sufficient condition for the existence of an infinite connected component.

Fix an arbitrary ordering of the edges of $\mathbb{Z}^2$. Let $\{0\} = A_0 \subset A_1 \subset A_2 \ldots$ and $\emptyset = B_0 \subset B_1 \subset B_2 \ldots$ be two growing sequences of subsets of $\mathbb{Z}^2$. We say that the sequence $X_t = (A_t, B_t)$ is an *exploration sequence* if for every $t \geq 0$,

$$\begin{aligned} &X_{t+1} = X_t \text{ if there is no edge connecting } A_t \text{ to } (A_t \cup B_t)^c, \\ &X_{t+1} = (A_t \cup \{x_t\}, B_t) \text{ or } X_{t+1} = (A_t, B_t \cup \{x_t\}) \text{ otherwise,} \end{aligned}$$

where x_t is the endpoint in $(A_t \cup B_t)^c$ of the minimal edge connecting A_t to $(A_t \cup B_t)^c$. A typical example of a random exploration sequence results from the exploration of the cluster of the origin in a site percolation process on $\mathbb{Z}^2$. In this case, the set A_t corresponds to the open sites discovered after t steps of exploration and B_t is the discovered part of the (closed) boundary of the cluster.

We say that an exploration sequence *percolates* if the set $\cup_{t\geq 0} A_t$ is infinite. The following lemma, proved in [GM90, Lemma 1], gives a sufficient condition for a random exploration sequence to percolate.

Lemma 9. *Let p_c^{site} be the critical parameter of Bernoulli site percolation on $\mathbb{Z}^2$. Let $X_t = (A_t, B_t)$ be a random exploration sequence and assume that there exists some $q > p_c^{\mathrm{site}}$ such that for every $t \geq 0$,*

$$\mathbb{P}\big(B_{t+1} = B_t \mid X_0, \ldots, X_t\big) \geq q \ \text{a.s.},$$

then the process X percolates with probability larger than a constant $c = c(q) > 0$ that can be taken arbitrarily close to 1 provided that q is close enough to 1.

We now move to the proof of Proposition 8. For every $x \in \mathbb{Z}^2$, set $\Lambda_x = Nx + \Lambda_N$ and $\widetilde{\Lambda}_x = Nx + \Lambda_{2N}$. Below, we will refer to Λ_0, which will mean the box of size N centered at 0.

Let ω be a Bernoulli percolation of parameter p in Slab_{2N}^d and for every $x \in \mathbb{Z}^2$, let ω^x be a $\lambda\varepsilon$-percolation on $\widetilde{\Lambda}_x$, where λ is some constant to be fixed later. We assume that ω and the ω^x's are independent of each other. We will prove that the origin is connected to infinity in

$$\omega_{\mathrm{total}} := \omega \vee \big(\vee_{x\in\mathbb{Z}^2}\, \omega^x \big)$$

with a probability which is larger than $\varepsilon/2$ (the notation $\vee$ stands for the maximum, or the union of the open edges if one prefers). This will conclude the proof since ω_{total} is stochastically dominated by a $(p + 25 \cdot \lambda\varepsilon)$-percolation (each edge of the slab appears in at most 25 boxes $\widetilde{\Lambda}_x$).

To prove this claim, define an increasing sequence of percolation configuration $(\omega_t)_{t\geq 0}$ in the slab, coupled with a random exploration sequence $X_t = (A_t, B_t)$ in $\mathbb{Z}^2$. Given a percolation configuration ω in the slab, let $\mathscr{C}(\omega)$ be the set of vertices that are connected inside $\mathbb{Z}^2 \times \{-2N, \ldots, 2N\}^{d-2}$ to 0 by a path of ω.

Definition. Set $X_0 = (A_0, B_0) := (\{0\}, \emptyset)$ and $\omega_0 = \omega$. For every $t \geq 0$, let ω_{t+1} and X_{t+1} be constructed from ω_t and X_t as follows. If there is no edge connecting A_t to $(A_t \cup B_t)^c$, define $X_{t+1} = X_t$. Otherwise, let $x = x_t$ be the extremity in $(A_t \cup B_t)^c$ of the minimal edge connecting A_t to $(A_t \cup B_t)^c$ and define

$$\omega_{t+1} := \omega_t \vee \omega^x,$$
$$X_{t+1} := \begin{cases} (A_t \cup \{x\}, B_t) & \text{if } 0 \leftrightarrow \Lambda_x \text{ in } \omega_{t+1}, \\ (A_t, B_t \cup \{x\}) & \text{otherwise.} \end{cases}$$

By construction, we have the following two properties:

(i) $\omega_\infty := \bigvee_{t \geq 0} \omega_t \leq \omega_{\text{total}}$,

(ii) if (X_t) percolates, then 0 is connected to infinity in ω_∞.

We now wish to prove a third property which, when combined with the previous two and **(a)**, concludes the proof.

(iii) $\mathbb{P}[X \text{ percolates} \mid 0 \leftrightarrow \partial\Lambda_0 \text{ in } \mathscr{C}(\omega_0)] \geq 1/2$.

The proof relies on an application of Lemma 9. In order to apply this lemma, let us fix $q > p_c^{\text{site}}(\mathbb{Z}^2)$ in such a way that $c(q) \geq 1/2$ and try to prove

$$\mathbb{P}\big(B_{t+1} = B_t \mid X_0, \ldots, X_t\big) \geq q \text{ a.s.}$$

Since $B_{t+1} = B_t$ as soon as there is no edge connecting A_t to $(A_t \cup B_t)^c$, we can focus on the case where the minimal edge e connecting A_t to $(A_t \cup B_t)^c$ is well defined, and therefore its endpoint x in $(A_t \cup B_t)^c$ also is. In this case, we have $B_{t+1} = B_t$ if 0 is connected to Λ_x in ω_{t+1}. Since $X_0, \ldots, X_t$ and the event that x is well defined are measurable with respect to $\mathscr{C}(\omega_0), \ldots, \mathscr{C}(\omega_t)$, it suffices to show that for any admissible $C_0, C_1, \ldots, C_t$, we have

$$\mathbb{P}\big(\Lambda_0 \leftrightarrow \Lambda_x \text{ in } \omega_{t+1} \mid \mathscr{C}(\omega_0) = C_0, \ldots, \mathscr{C}(\omega_t) = C_t\big) \geq q \text{ a.s.,} \tag{9}$$

which is equivalent to showing that for every admissible C_t,

$$\mathbb{P}\big(C_t \leftrightarrow \Lambda_x \text{ in } \omega_t \vee \omega^x \mid \omega_t|_{\partial_E C_t} \equiv 0\big) \geq q \text{ a.s.}$$

Now, observe that any admissible C_t must intersect $\Lambda_{x'}$, where x' is the endpoint of e in A_t (when $t = 0$, we further use that the radius of C_0 is at least N). Let y be a vertex of $C_t \cap \Lambda_{x'}$. Since at least one of the facets of $y + \Lambda_N$ is included in Λ_x, condition **(b')** (applied after shifting 0 to y) implies

$$\mathbb{P}_p[C_t \overset{\widetilde{\Lambda}_x}{\longleftrightarrow} \Lambda_x] \geq 1 - e^{-c/\varepsilon}.$$

Since ω^x is independent of ω_t, Lemma 10 below shows that (9) holds, provided the constant λ is large enough. This concludes the proof of Item (iii) and therefore of the proposition. $\square$

For the next (and last) lemma, it will be convenient to have a notation for the edge boundary restricted to a fixed set. Fix therefore a set $R \subset \mathbb{Z}^d$, and define

$$\Delta A = \big\{\{x,y\} \subset R \,:\, |x-y| = 1,\, x \in A,\, y \in R \setminus A\big\}.$$

Lemma 10. *For any $\delta, c > 0$, there exists $\lambda > 0$ such that for any $p \in [\delta, 1-\delta]$ and $\varepsilon > 0$, as well as any $A, B \subset R$, $\mathbb{P}_p[A \overset{R}{\longleftrightarrow} B] \geq 1 - \exp(-c/\varepsilon)$ implies that*

$$\mathbb{P}\big[A \overset{R}{\longleftrightarrow} B \text{ in } \omega \vee \tilde{\omega} \,\big|\, \omega(e) = 0, \forall e \in \Delta A\big] \geq 1 - \delta,$$

where ω is a Bernoulli percolation configuration satisfying $\mathbb{P}[\omega(e) = 1] \geq p$ for every e, and $\tilde{\omega}$ a Bernoulli percolation of parameter $\lambda\varepsilon$ which is independent of ω.

Proof. If $A \cap B \neq \emptyset$, the result is obvious. We therefore assume $A \cap B = \emptyset$. Also, introduce the event E that $\omega(e) = 0$ for all $e \in \Delta A$ and the set W defined by

$$W = \big\{\{x,y\} \in \Delta A \text{ and } y \overset{R \setminus A}{\longleftrightarrow} B \text{ in } \omega\big\}.$$

Any path from A to B in R must use at least one edge of W. Consequently, for any $t \in \mathbb{N}$, we have $\mathbb{P}_p[A \overset{R}{\not\leftrightarrow} B] \geq (1-p)^{t-1}\, \mathbb{P}_p[|W| < t]$. Then, using that $|W| \geq t$ is independent of the event E, we deduce that

$$\begin{aligned}
\mathbb{P}[A \overset{R}{\longleftrightarrow} B \text{ in } \omega \vee \tilde{\omega} \mid E] &\geq \mathbb{P}[\exists e \in W : \tilde{\omega}(e) = 1, W \geq t \mid E] \\
&\geq (1 - (1-\lambda\varepsilon)^t)\mathbb{P}[W \geq t] \\
&\geq (1 - (1-\lambda\varepsilon)^t)\Big(1 - \frac{\mathbb{P}_p[A \overset{R}{\not\leftrightarrow} B]}{(1-p)^{t-1}}\Big) \\
&\geq (1 - (1-\lambda\varepsilon)^t)\Big(1 - \frac{\exp(-c/\varepsilon)}{(1-p)^{t-1}}\Big).
\end{aligned}$$

Choosing $\lambda = \lambda(\delta, c)$ large enough, the result follows by optimizing on t. □

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