

Stable spectrum for pseudo-Riemannian locally symmetric spaces

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Abstract

Let $X = G/H$ be a reductive symmetric space with $\text{rank } G/H = \text{rank } K/K \cap H$, where K (resp. $K \cap H$) is a maximal compact subgroup of G (resp. of H). We investigate the discrete spectrum of certain Clifford–Klein forms $\Gamma \backslash X$, where Γ is a discrete subgroup of G acting properly discontinuously and freely on X : we construct an infinite set of joint eigenvalues for “intrinsic” differential operators on $\Gamma \backslash X$, and this set is stable under small deformations of Γ in G . *To cite this article: F. Kassel, T. Kobayashi, C. R. Acad. Sci. Paris, Ser. I 348 (2010).*

Résumé

Spectre stable pour les variétés pseudo-riemanniennes localement symétriques. Soit $X = G/H$ un espace symétrique réductif vérifiant $\text{rang } G/H = \text{rang } K/K \cap H$, où K (resp. $K \cap H$) est un sous-groupe compact maximal de G (resp. de H). Nous étudions le spectre discret de certaines formes de Clifford–Klein $\Gamma \backslash X$, où Γ est un sous-groupe discret de G agissant librement et proprement sur X : nous construisons un ensemble infini de valeurs propres pour les opérateurs différentiels “intrinsèques” sur $\Gamma \backslash X$, et cet ensemble est stable par petites déformations de Γ dans G . *Pour citer cet article : F. Kassel, T. Kobayashi, C. R. Acad. Sci. Paris, Sér. I 348 (2010).*

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Soit $X = G/H$ un espace symétrique, où G est un groupe de Lie réductif connexe non compact et H la composante neutre du groupe des points fixes de G par un certain automorphisme involutif σ . L'espace X est naturellement muni d'une métrique pseudo-riemannienne G -invariante. Une *forme de Clifford–Klein* de X est un quotient $X_\Gamma = \Gamma \backslash X$ où Γ est un sous-groupe discret de G agissant librement et proprement sur X ; c'est une variété complète localement modelée sur X . Soit $\mathbb{D}(X)$ l'algèbre des opérateurs

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différentiels G -invariants sur X . Tout élément $D \in \mathbb{D}(X)$ (par exemple le laplacien) induit un opérateur différentiel D_Γ sur X_Γ . Le spectre discret $\text{Spec}_d(X_\Gamma)$ de X_Γ est l'ensemble des morphismes d'algèbres $\lambda : \mathbb{D}(X) \rightarrow \mathbb{C}$ pour lesquels il existe une fonction $f \in L^2(X_\Gamma)$ non nulle vérifiant $D_\Gamma f = \lambda(D)f$ pour tout $D \in \mathbb{D}(X)$ au sens des distributions. Soit $K = G^\theta$ un sous-groupe compact maximal de G , où θ est une involution de Cartan commutant avec σ . Notre résultat principal concerne les formes de Clifford–Klein de X qui sont *standard*, au sens où Γ est inclus dans un sous-groupe réductif de G agissant proprement sur X .

Théorème 0.1 *Supposons $\text{rang } G/H = \text{rang } K/K \cap H$. Le spectre discret $\text{Spec}_d(X_\Gamma)$ est infini pour toute forme de Clifford–Klein compacte standard X_Γ de X ; de plus, il existe une partie infinie de $\text{Spec}_d(X_\Gamma)$ qui est stable par petites déformations de Γ dans G . Ceci reste vrai lorsque Γ est convexe cocompact dans un sous-groupe réductif de G de rang réel 1.*

Dans la situation du théorème 0.1, il existe un voisinage $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ de l'inclusion naturelle tel que pour tout $\varphi \in \mathcal{U}$ le quotient $X_{\varphi(\Gamma)} = \varphi(\Gamma) \backslash X$ soit une forme de Clifford–Klein de X , compacte si X_Γ l'est : cela résulte de [5] (propreté) et [8] (compacité). Le théorème 0.1 affirme que, quitte à réduire le voisinage \mathcal{U} , il existe un ensemble infini qui est inclus dans $\text{Spec}_d(X_{\varphi(\Gamma)})$ pour tout $\varphi \in \mathcal{U}$. L'étude des petites déformations de formes de Clifford–Klein dans ce cadre général remonte à l'article [10].

Soit $\mathfrak{j}_\mathbb{C}$ un sous-espace abélien semi-simple maximal de l'ensemble des points fixes de $-\sigma$ dans l'algèbre de Lie complexifiée $\mathfrak{g}_\mathbb{C}$ de G , soit $\mathfrak{j}_\mathbb{C}^*$ son dual, et soit W le groupe de Weyl de $\mathfrak{j}_\mathbb{C}$ dans $\mathfrak{g}_\mathbb{C}$. Le spectre discret de toute forme de Clifford–Klein de X s'identifie naturellement à une partie de $\mathfrak{j}_\mathbb{C}^*/W$. Sous les hypothèses du théorème 0.1 on peut supposer que $\mathfrak{j}_\mathbb{C} = \mathfrak{j} \otimes_\mathbb{R} \mathbb{C}$ pour un certain sous-espace abélien maximal \mathfrak{j} de $\sqrt{-1}\mathfrak{k}$, où \mathfrak{k} est l'algèbre de Lie de K . Fixons un système $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ de racines positives de $\mathfrak{j}_\mathbb{C}$ dans $\mathfrak{g}_\mathbb{C}$, ce qui définit une chambre de Weyl positive \mathfrak{j}_+^* de \mathfrak{j}^* . Soient $\rho \in \mathfrak{j}^*$ et $\rho_c \in \mathfrak{j}^*$ les demi-sommes respectives des racines de $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ et $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \cap \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, et soit Λ_+ l'intersection de \mathfrak{j}_+^* avec le réseau de \mathfrak{j} engendré par les plus hauts poids des représentations irréductibles de K ayant des vecteurs $(K \cap H)$ -invariants non nuls. Pour tout $\lambda \in \mathfrak{j}_+^*$ nous notons $d(\lambda)$ la “distance pondérée” naturelle de λ aux murs de \mathfrak{j}_+^* (voir le paragraphe 2). Avec ces notations, voici une version plus précise du théorème 0.1.

Théorème 0.2 *Sous les hypothèses du théorème 0.1, il existe une constante $R > 0$ et un voisinage $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ de l'inclusion naturelle tels que $\{\lambda \in 2\rho_c - \rho + \Lambda_+ : d(\lambda) \geq R\} \subset \text{Spec}_d(X_{\varphi(\Gamma)})$ pour tout $\varphi \in \mathcal{U}$.*

Nous donnons une liste d'espaces symétriques X auxquels nos théorèmes s'appliquent, et décrivons explicitement une partie infinie du spectre discret stable des formes de Clifford–Klein compactes standard de $X = \text{SO}(2, 4)/\text{U}(1, 2)$ (en utilisant [11]) et de l'espace anti-de Sitter $X = \text{AdS}^3 = \text{SO}(2, 2)/\text{SO}(1, 2)$ (voir (2)). Rappelons que les variétés anti-de Sitter (c'est-à-dire lorentziennes de courbure constante < 0) compactes de dimension 3 sont les formes de Clifford–Klein compactes de AdS^3 , à revêtement fini, isométrie et renormalisation près [7], [13]. Nous démontrons un résultat analogue aux théorèmes 0.1 et 0.2 pour toutes ces formes de Clifford–Klein compactes, même celles qui ne sont pas standard.

Théorème 0.3 *Le spectre discret de toute variété anti-de Sitter compacte de dimension 3 est infini, et contient une partie infinie qui est stable par petites déformations de la structure anti-de Sitter.*

Pour démontrer nos résultats, nous construisons des fonctions propres sur les formes de Clifford–Klein de X à partir de fonctions propres sur X construites par Flensted-Jensen [2]. Nous donnons des estimées asymptotiques uniformes de ces dernières, en fonction de la projection $\nu : G \rightarrow \overline{\mathfrak{b}}_+$ associée à une décomposition $G = KBH$ (voir le paragraphe 3). Nous relierons la projection ν à la projection $\mu : G \rightarrow \overline{\mathfrak{a}}_+$ associée à une décomposition de Cartan $G = KAK$ où $A \supset B$ (voir le paragraphe 3), et utilisons les estimées de [5] et [6] sur la restriction de μ à Γ et à ses déformés. Les détails seront publiés ultérieurement.

1. A general program

Let $X = G/H$ be a reductive symmetric space, where G is a connected noncompact reductive linear Lie group and $H = (G^\sigma)_0$ the identity component of the set of fixed points of G under some involutive automorphism σ . We note that X naturally carries a G -invariant pseudo-Riemannian metric, which is induced by the Killing form of the Lie algebra \mathfrak{g} of G if G is semisimple. A *Clifford–Klein form* of X is a quotient $X_\Gamma = \Gamma \backslash X$ where Γ is a discrete subgroup of G acting properly discontinuously and freely on X ; it is a complete manifold locally modelled on X . Any G -invariant differential operator D on X (such as the Laplacian) induces a differential operator D_Γ on X_Γ , and the map $D \mapsto D_\Gamma$ is an injective \mathbb{C} -algebra homomorphism from the ring $\mathbb{D}(X)$ of G -invariant differential operators on X into the ring of differential operators on X_Γ . We may think of its image as the set of “intrinsic” differential operators on X_Γ . The *discrete spectrum* $\text{Spec}_d(X_\Gamma)$ of X_Γ is the set of \mathbb{C} -algebra homomorphisms $\lambda : \mathbb{D}(X) \rightarrow \mathbb{C}$ such that the set $L^2(X_\Gamma, \mathcal{M}_\lambda)$ of weak solutions $f \in L^2(X_\Gamma)$ to the system

$$D_\Gamma f = \lambda(D)f \quad \text{for all } D \in \mathbb{D}(X) \quad (\mathcal{M}_\lambda)$$

is nontrivial. In this note we wish to initiate the following general program.

A) Construct elements of $L^2(X_\Gamma, \mathcal{M}_\lambda)$, *i.e.* joint eigenfunctions on X_Γ corresponding to $\text{Spec}_d(X_\Gamma)$.

B) Understand the behavior of $\text{Spec}_d(X_\Gamma)$ under small deformations of Γ in G .

Problem B builds on the fact that for certain Clifford–Klein forms X_Γ , the proper discontinuity of the action of Γ on X is preserved under small deformations of Γ . The study of small deformations of Clifford–Klein forms in such a general setting was initiated in [10].

2. Main results

Let θ be a Cartan involution of G commuting with σ and let $K = G^\theta$ be the corresponding maximal compact subgroup of G , with Lie algebra \mathfrak{k} . In this note, we assume that

$$\text{rank } G/H = \text{rank } K/K \cap H, \quad (1)$$

where $\text{rank } G/H$ (resp. $\text{rank } K/K \cap H$) is the dimension of a maximal semisimple abelian subspace in the set of fixed points of $-d\sigma$ in \mathfrak{g} (resp. \mathfrak{k}). We investigate Problems A and B for an important class of Clifford–Klein forms X_Γ , namely those that are *standard*, in the sense that Γ is contained in some closed reductive subgroup L of G acting properly on $X = G/H$. Note that if such an X_Γ is compact, then Γ must be a uniform lattice in L and the action of L on X must be cocompact. Here is our main result.

Theorem 2.1 *Under the rank assumption (1), the discrete spectrum $\text{Spec}_d(X_\Gamma)$ is infinite for any standard compact Clifford–Klein form X_Γ of X ; furthermore, there is an infinite subset of $\text{Spec}_d(X_\Gamma)$ that is stable under small deformations of Γ in G . The same conclusion holds when Γ is convex cocompact in some reductive subgroup L of G with $\text{rank}_{\mathbb{R}} L = 1$.*

Underlying Theorem 2.1 is the existence, due to [5] (properness) and [8] (compactness), of a neighborhood $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ of the natural inclusion such that $X_{\varphi(\Gamma)} = \varphi(\Gamma) \backslash X$ is a Clifford–Klein form of X for all $\varphi \in \mathcal{U}$, with $X_{\varphi(\Gamma)}$ compact if X_Γ is. Theorem 2.1 states that, after possibly replacing \mathcal{U} by some smaller neighborhood, there is an infinite set that is contained in $\text{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathcal{U}$.

Recall that a discrete subgroup Γ of a reductive group L with $\text{rank}_{\mathbb{R}} L = 1$ is said to be *convex cocompact* if it acts cocompactly on the convex hull of its limit set in the Riemannian symmetric space of L , this limit set being nonempty. Convex cocompact groups include uniform lattices, but also discrete groups of infinite covolume such as Schottky groups.

In order to describe the *stable discrete spectrum* of X_Γ in Theorem 2.1, let us briefly recall the structure of $\mathbb{D}(X)$ (see [4] for more details) and introduce some notation. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of \mathfrak{g} into eigenspaces of $d\sigma$, with respective eigenvalues $+1$ and -1 , and let $\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} + \mathfrak{q}_\mathbb{C}$ be its complexification. Fix a maximal semisimple abelian subspace $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{q}_\mathbb{C}$ and let W be the Weyl group of the restricted root system $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $\mathfrak{j}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. The \mathbb{C} -algebra $\mathbb{D}(X)$ is naturally isomorphic to the subalgebra $S(\mathfrak{j}_\mathbb{C})^W$ of W -fixed points in the symmetric algebra $S(\mathfrak{j}_\mathbb{C})$; it is a polynomial ring in $r := \dim_\mathbb{C} \mathfrak{j}_\mathbb{C} = \text{rank } G/H$ generators. In particular, the set of \mathbb{C} -algebra homomorphisms from $\mathbb{D}(X)$ to \mathbb{C} naturally identifies with $\mathfrak{j}_\mathbb{C}^*/W$, where $\mathfrak{j}_\mathbb{C}^*$ is the dual vector space of $\mathfrak{j}_\mathbb{C}$. For any Clifford–Klein form X_Γ of X , we see $\text{Spec}_d(X_\Gamma)$ as a subset of $\mathfrak{j}_\mathbb{C}^*/W$. Under the rank hypothesis (1), we may assume that $\mathfrak{j}_\mathbb{C}$ is the complexification of a maximal abelian subspace \mathfrak{j} of $\sqrt{-1}\mathfrak{k}$, on which all restricted roots $\alpha \in \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ take real values. We endow \mathfrak{j}^* with a W -invariant inner product $\langle \cdot, \cdot \rangle$, fix a basis Ψ of $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, defining a system $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of positive roots, and let \mathfrak{j}_+^* be the corresponding positive Weyl chamber in \mathfrak{j}^* , defined by $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Psi$. For $\lambda \in \mathfrak{j}_+^*$, we consider the natural “weighted distance” from λ to the walls of \mathfrak{j}_+^* given by $d(\lambda) = \min_{\alpha \in \Psi} \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$. Let ρ (resp. ρ_c) be half the sum of roots in $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ (resp. in $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \cap \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$), counted with multiplicities, and let Λ_+ be the intersection of \mathfrak{j}_+^* with the lattice of \mathfrak{j}^* generated by all highest weights of irreducible representations of K with nonzero $(K \cap H)$ -fixed vectors. With this notation, here is a more precise statement of Theorem 2.1.

Theorem 2.2 *In the setting of Theorem 2.1, there are a constant $R > 0$ and a neighborhood $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ of the natural inclusion such that $\{\lambda \in 2\rho_c - \rho + \Lambda_+ : d(\lambda) \geq R\} \subset \text{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathcal{U}$.*

We explicitly construct eigenfunctions $f \in L^2(X_{\varphi(\Gamma)}, \mathcal{M}_\lambda)$ for all $\lambda \in 2\rho_c - \rho + \Lambda_+$ with $d(\lambda)$ large enough and all homomorphisms φ that are close enough to the natural inclusion of Γ in G . These eigenfunctions actually satisfy (\mathcal{M}_λ) as functions of class C^N where N is the maximal degree of r generators of $\mathbb{D}(X)$, and they belong to $L^p(X_{\varphi(\Gamma)})$ for all $1 \leq p \leq \infty$.

Using [12, Cor. 3.3.7], we see that Theorems 2.1 and 2.2 apply in particular to the following triples (G, H, L) , where $n, p, q \geq 1$. In Example (vi), the group G_0 may be $\text{SO}(p, 2q)$, $\text{SU}(p, q)$, $\text{Sp}(p, q)$, $\text{Sp}(n, \mathbb{R})$, $\text{SO}^*(2n)$, or certain exceptional groups, and $\text{Diag}(G_0)$ denotes the diagonal of $G_0 \times G_0$.

	G	H	L
(i)	$\text{SO}(2, 2n)$	$\text{SO}(1, 2n)$	$\text{U}(1, n)$
(ii)	$\text{SO}(2, 4n)$	$\text{U}(1, 2n)$	$\text{SO}(1, 4n)$
(iii)	$\text{SO}(4, 4n)$	$\text{SO}(3, 4n)$	$\text{Sp}(1, n)$
(iv)	$\text{U}(2, 2n)$	$\text{U}(1) \times \text{U}(1, 2n)$	$\text{Sp}(1, n)$
(v)	$\text{SO}(8, 8)$	$\text{SO}(7, 8)$	$\text{Spin}(1, 8)$
(vi)	$G_0 \times G_0$	$\text{Diag}(G_0)$	$G_0 \times \{1\}$
(vii)	$\text{SO}(2, 2n) \times \text{SO}(2, 2n)$	$\text{Diag}(\text{SO}(2, 2n))$	$\text{SO}(1, 2n) \times \text{U}(1, n)$
(viii)	$\text{SO}(4, 4n) \times \text{SO}(4, 4n)$	$\text{Diag}(\text{SO}(4, 4n))$	$\text{SO}(3, 4n) \times \text{Sp}(1, n)$
(ix)	$\text{U}(2, 2n) \times \text{U}(2, 2n)$	$\text{Diag}(\text{U}(2, 2n))$	$\text{U}(1, 2n) \times \text{Sp}(1, n)$
(x)	$\text{SO}(8, 8) \times \text{SO}(8, 8)$	$\text{Diag}(\text{SO}(8, 8))$	$\text{SO}(7, 8) \times \text{Spin}(1, 8)$
(xi)	$\text{SO}(4, 4) \times \text{SO}(4, 4)$	$\text{Diag}(\text{SO}(4, 4))$	$\text{SO}(4, 1) \times \text{Spin}(4, 3)$
(xii)	$\text{SO}(4, 3) \times \text{SO}(4, 3)$	$\text{Diag}(\text{SO}(4, 3))$	$\text{SO}(4, 1) \times \text{G}_{2(2)}$

(xiii)	$\mathrm{SO}^*(8) \times \mathrm{SO}^*(8)$	$\mathrm{Diag}(\mathrm{SO}^*(8))$	$\mathrm{U}(3, 1) \times \mathrm{Spin}(1, 6)$
(xiv)	$\mathrm{SO}^*(8) \times \mathrm{SO}^*(8)$	$\mathrm{Diag}(\mathrm{SO}^*(8))$	$(\mathrm{SO}^*(6) \times \mathrm{SO}^*(2)) \times \mathrm{Spin}(1, 6)$

Note that in Examples (vii) to (xiv), which are of the form $(G, H, L) = (G_0 \times G_0, \mathrm{Diag}(G_0), H_0 \times L_0)$, if Γ_{H_0} (resp. Γ_{L_0}) is a uniform lattice of H_0 (resp. of L_0), then the compact Clifford–Klein form $(\Gamma_{H_0} \times \Gamma_{L_0}) \backslash G/H$ identifies with $\Gamma_{L_0} \backslash G_0 / \Gamma_{H_0}$ and is locally modelled on G_0 . In Examples (ii), (vii), (xi) and (xii), small nonstandard deformations of standard compact Clifford–Klein forms of X can be obtained using a *bending construction* due to Johnson and Millson (see [5]). In Example (vi), small nonstandard deformations also exist for $G_0 = \mathrm{SO}(1, 2n)$ or $\mathrm{SU}(1, n)$ (see [10]).

An infinite subset of the stable discrete spectrum for standard compact Clifford–Klein forms of X may be found explicitly for $X = \mathrm{SO}(2, 4)/\mathrm{U}(1, 2)$ and for the 3-dimensional *anti-de Sitter space* $X = \mathrm{AdS}^3 = \mathrm{SO}(2, 2)/\mathrm{SO}(1, 2)$. By [7] and [13], the 3-dimensional compact anti-de Sitter manifolds (*i.e.* the 3-dimensional compact Lorentz manifolds with constant negative curvature) are the compact Clifford–Klein forms of AdS^3 , up to finite covering, isometry, and renormalization of the metric. Using [6], we prove that Theorems 2.1 and 2.2 actually hold true for *all* these compact Clifford–Klein forms, not only standard ones.

Theorem 2.3 *The discrete spectrum of any 3-dimensional compact anti-de Sitter manifold M is infinite. Explicitly, there is an integer n_0 such that the discrete spectrum of the Laplacian Δ_M on M satisfies*

$$\mathrm{Spec}_d(\Delta_M) \supset \left\{ \frac{1}{2}n(n+1) : n \in \mathbb{N}, n \geq n_0 \right\}, \quad (2)$$

and (2) still holds after a small deformation of the anti-de Sitter structure of M .

Here we are using the normalization of the Lorentz metric by the Killing form. Theorem 2.3 holds more generally for any 3-dimensional anti-de Sitter manifold M satisfying some convex cocompactness property. We note that here $\mathbb{D}(X)$ is generated by the Laplacian, so that $\mathrm{Spec}_d(M)$ identifies with $\mathrm{Spec}_d(\Delta_M) \subset \mathbb{C}$. Since M is a Lorentz manifold, Δ_M is a hyperbolic operator. We may compare (2) with the following easy computation:

$$\mathrm{Spec}_d(\mathbb{P}^3(\mathbb{R}), \Delta_{\mathbb{P}^3(\mathbb{R})}) = \left\{ -\frac{1}{2}n(n+1) : n \in \mathbb{N} \right\}.$$

Theorems 2.1, 2.2, and 2.3 follow from a more general result that we prove for triples (G, H, Γ) that satisfy (1) and two conditions on the image of Γ by some Cartan projection of G (see Definition 3.3).

3. Ideas of proofs

For simplicity we assume that G is a real form of a simply connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let (H^d, G^d, K^d) be the dual triple of (K, G, H) , *i.e.* the triple of connected reductive Lie groups with the same complexified Lie algebras and such that G^d/K^d is a Riemannian symmetric space. There is an injective homomorphism [2] from the set $\mathcal{A}_K(X)$ of K -finite analytic functions on $X = G/H$ into the set $\mathcal{A}_{H^d}(G^d/K^d)$ of H^d -finite analytic functions on G^d/K^d . For $\lambda \in 2\rho_{\mathbb{C}} - \rho + \Lambda_+$, Flensted-Jensen [2] introduced the function $\psi_{\lambda} \in \mathcal{A}_K(X)$ whose image in $\mathcal{A}_{H^d}(G^d/K^d)$ is given by

$$g^d K^d \mapsto \int_{K^d \cap H^d} e^{-\langle \lambda + \rho, \zeta((g^d)^{-1}\ell) \rangle} d\ell,$$

where $G^d = K^d A^d N^d$ is an Iwasawa decomposition of G^d with $A^d = \exp \mathfrak{j}$, and $\zeta : G^d \rightarrow \mathfrak{j}$ is given by $g^d \in K^d e^{\zeta(g^d)} N^d$ for all $g^d \in G^d$. Assuming (1), he proved that $\psi_{\lambda} \in L^2(X, \mathcal{M}_{\lambda})$ for $d(\lambda)$ large enough. To establish Theorems 2.1, 2.2, and 2.3 we prove the following, where \bar{x} is the image of $x \in X$ in $X_{\varphi(\Gamma)}$.

Proposition 3.1 *In the setting of Theorems 2.1 or 2.3, there is a constant $R > 0$ and a neighborhood $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ of the natural inclusion such that for all $\lambda \in 2\rho_c - \rho + \Lambda_+$ with $d(\lambda) \geq R$ and all $\varphi \in \mathcal{U}$, the eigenfunction*

$$\psi_\lambda^{\varphi(\Gamma)} : \bar{x} \mapsto \sum_{\gamma \in \Gamma} \psi_\lambda(\varphi(\gamma) \cdot x)$$

on $X_{\varphi(\Gamma)}$ is well-defined, nonzero, L^p for all $1 \leq p \leq \infty$, and of class C^m whenever $d(\lambda) \geq (m+1)R$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with θ and let \mathfrak{b} be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Fix a system $\Sigma^+(\mathfrak{g}^{\sigma\theta}, \mathfrak{b})$ of positive restricted roots of \mathfrak{b} in the set of fixed points of $d(\sigma\theta)$ in \mathfrak{g} , and let $\overline{\mathfrak{b}_+} \subset \mathfrak{b}$ be the corresponding closed positive Weyl chamber, so that the decomposition $G = K \exp(\overline{\mathfrak{b}_+})H$ holds. We define a map $\nu : G \rightarrow \overline{\mathfrak{b}_+}$ by $g \in Ke^{\nu(g)}H$ for all $g \in G$. Proposition 3.1 relies on the following uniform asymptotic estimates for ψ_λ , which we establish by building on the work of Flensted-Jensen [2] and Matsuki-Oshima [14]. Here $\|\cdot\|$ denotes any fixed norm on \mathfrak{b} .

Lemma 3.2 *Under the rank assumption (1), there is a constant $\varepsilon > 0$ such that for any $\lambda \in 2\rho_c - \rho + \Lambda_+$,*

- (i) $\psi_\lambda(eH) = 1$ and $|\psi_\lambda(gH)| \leq \cosh(\varepsilon\|\nu(g)\|)^{-d(\lambda+\rho)}$ for all $g \in G$,
- (ii) for any $D \in \mathbb{D}(X)$, the function $g \mapsto D\psi_\lambda(gH)e^{\varepsilon d(\lambda+\rho)\|\nu(g)\|}$ is bounded on G .

To deduce Proposition 3.1 from Lemma 3.2, we consider a maximal abelian subspace \mathfrak{a} of \mathfrak{p} containing \mathfrak{b} . We fix a system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of positive restricted roots and let $\overline{\mathfrak{a}_+} \subset \mathfrak{a}$ be the corresponding closed positive Weyl chamber, so that the Cartan decomposition $G = K \exp(\overline{\mathfrak{a}_+})K$ holds. By the *properness criterion* of Benoist [1] and Kobayashi [9], the Cartan projection $\mu : G \rightarrow \overline{\mathfrak{a}_+}$, defined by $g \in Ke^{\mu(g)}K$ for all $g \in G$, controls the properness of the action of any closed subgroup of G on G/H . We introduce the following two conditions, where $\|\cdot\|$ denotes any norm on \mathfrak{a} extending that of \mathfrak{b} , inducing a distance $\text{dist}_\mathfrak{a}$ on \mathfrak{a} , and $\ell_F : \Gamma \rightarrow \mathbb{N}$ is the word length with respect to F .

Definition 3.3 *Let $c, C > 0$. A subgroup Γ of G with finite generating subset F is said to satisfy*

- the angle condition with constants (c, C) if $\text{dist}_\mathfrak{a}(\mu(\gamma), \mu(H)) \geq c\|\mu(\gamma)\| - C$ for all $\gamma \in \Gamma$,
- the QI condition with constants (c, C) if $\|\mu(\gamma)\| \geq c\ell_F(\gamma) - C$ for all $\gamma \in \Gamma$.

In the setting of Theorems 2.1 or 2.3, there are constants $c, C > 0$ and a neighborhood $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ of the natural inclusion such that for all $\varphi \in \mathcal{U}$, the group $\varphi(\Gamma)$ satisfies both the angle condition with constants (c, C) (by [5] and [6]) and the QI condition with constants (c, C) (as was first proved by Guichard [3]). Proposition 3.1 is a consequence of this fact, of Lemma 3.2, and of the following inequality.

Lemma 3.4 *There is a constant $C_0 > 0$ such that $\|\nu(g)\| \geq C_0 \text{dist}_\mathfrak{a}(\mu(g), \mu(H))$ for all $g \in G$.*

Detailed proofs will appear elsewhere.

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