Spectral analysis on pseudo-Riemannian locally symmetric spaces

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Abstract: We summarize recent results initiating spectral analysis on pseudo-Riemannian locally symmetric spaces $\Gamma \backslash G/H$, beyond the classical setting where $H$ is compact (e.g. theory of automorphic forms for arithmetic $\Gamma$) or $\Gamma$ is trivial (e.g. Plancherel-type formula for semisimple symmetric spaces).

Key words: Locally symmetric space; pseudo-Riemannian manifold; discontinuous group; Laplacian; invariant differential operator; branching law; spherical variety.

1. Introduction A pseudo-Riemannian manifold is a smooth manifold $M$ equipped with a smooth, nondegenerate symmetric bilinear tensor $g$ of signature $(p, q)$. It is called Riemannian if $q = 0$, and Lorentzian if $q = 1$. As in the Riemannian case, the metric $g$ induces a Radon measure on $M$ and a second-order differential operator

$$\Box = \text{div grad}$$

called the Laplacian. It is a symmetric operator on the Hilbert space $L^2(X)$. The Laplacian $\Box$ is not an elliptic differential operator if $p, q > 0$.

A semisimple symmetric space $X$ is a homogeneous space $G/H$ where $G$ is a semisimple Lie group and $H$ an open subgroup of the group of fixed points of $G$ under some involutive automorphism. The manifold $X$ carries a $G$-invariant pseudo-Riemannian metric induced by the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. The group $G$ acts on $X$ by isometries, and the $C_\infty$-algebra $\mathcal{D}(X)$ of $G$-invariant differential operators on $X$ is commutative.

In this note we consider quotients $X_\Gamma = \Gamma \backslash X$ of a semisimple symmetric space $X = G/H$ by discrete subgroups $\Gamma$ of $G$ acting properly discontinuously and freely on $X$ ("discontinuous groups for $X"$). Such quotients are called pseudo-Riemannian locally symmetric spaces. They are complete $(G, X)$-manifolds in the sense of Ehresmann and Thurston, and they inherit a pseudo-Riemannian structure from $X$. Any $G$-invariant differential operator $D$ on $X$ induces a differential operator $D_\Gamma$ on $X_\Gamma$ via the covering map $\pi_\Gamma: X \to X_\Gamma$. For instance, the Laplacian $\Box$ on $X$ is $G$-invariant, and $(\Box_X)_\Gamma = \Box_{X_\Gamma}$. We think of

$$\mathcal{P} := \{D_\Gamma : D \in \mathcal{D}_G(X)\}$$

as the set of "intrinsic differential operators" on the locally symmetric space $X_\Gamma$. It is a subalgebra of the $C$-algebra $\mathcal{D}(X_\Gamma)$ of differential operators on $X_\Gamma$:

$$\mathcal{D}_G(X) \ni f \mapsto f \in \mathcal{D}(X_\Gamma), \quad D \mapsto D_\Gamma.$$  

For a $C$-algebra homomorphism $\lambda: \mathcal{D}_G(X) \to \mathbb{C}$, we denote by $C^\infty(X_\Gamma; \mathcal{M}_\lambda)$ the space of smooth functions $f$ on $X_\Gamma$ (joint eigenfunctions) satisfying the following system of partial differential equations:

$$(\mathcal{M}_\lambda) \quad D_\Gamma f = \lambda(D) f \quad \text{for all } D \in \mathcal{D}_G(X).$$

Let $L^2(X_\Gamma; \mathcal{M}_\lambda)$ be the space of square-integrable functions on $X_\Gamma$ satisfying $(\mathcal{M}_\lambda)$ in the weak sense. It is a closed subspace of the Hilbert space $L^2(X_\Gamma)$. 

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We are interested in the following problems.

**Problems 1.** For intrinsic differential operators on $X_T = \Gamma\backslash G/H$,

1. construct joint eigenfunctions on $X_T$;
2. find a spectral theory on $L^2(X_T)$.

In the classical setting where $H$ is a maximal compact subgroup $K$ of $G$, i.e. $X_T$ is a Riemannian locally symmetric space, a rich and deep theory has been developed over several decades, in particular, in connection with automorphic forms when $\Gamma$ is arithmetic. For compact $H$, the spectral decomposition of $L^2(X_T)$ is closely related to a disintegration of the regular representation of $G$ on $L^2(\Gamma\backslash G)$:

\[
L^2(\Gamma\backslash G) \simeq \int_G m_\Gamma(\pi) \pi \, d\sigma(\pi),
\]

where $d\sigma$ is a Borel measure on the unitary dual $\hat{G}$ and $m_\Gamma: \hat{G} \to \mathbb{N} \cup \{\infty\}$ a measurable function called multiplicity. There is a natural isomorphism

\[
L^2(X_T) \simeq L^2(\Gamma\backslash G)^H
\]

and the Hilbert space $L^2(X_T)$ is decomposed as

\[
L^2(X_T) \simeq \int_{(\hat{G})_H} m_\Gamma(\pi) \pi^H \, d\sigma(\pi),
\]

where $\pi^H$ denotes the space of $H$-invariant vectors in the representation space of $\pi$ and

\[
(\hat{G})_H := \{ \pi \in \hat{G} : \pi^H \neq \{0\} \}.
\]

Since the center $\mathfrak{z}(\mathfrak{g}_C)$ of the enveloping algebra $U(\mathfrak{g}_C)$ acts on the space of smooth vectors of $\pi$ as scalars for every $\pi \in \hat{G}$, the decomposition (1.4) respects the actions of $\mathbb{D}(X)$ and $\mathfrak{z}(\mathfrak{g}_C)$ via the natural $\mathbb{C}$-algebra homomorphism $d\ell: \mathfrak{z}(\mathfrak{g}_C) \to \mathbb{D}(X)$. This homomorphism is surjective e.g. if $G$ is a classical group.

The situation changes drastically beyond the aforementioned classical setting, namely, when $H$ is not compact anymore. New difficulties include:

1. (Representation theory) By the ergodicity theorem of Howe–Moore [7], if $H$ is noncompact, then $L^2(\Gamma\backslash G)^H = \{0\}$, and so (1.3) fails:

\[
L^2(X_T) \neq L^2(\Gamma\backslash G)^H
\]

and the irreducible decomposition (1.2) of the regular representation $L^2(\Gamma\backslash G)$ of $G$ does not yield a spectral decomposition of $L^2(X_T)$.

2. (Analysis) In contrast to the usual Riemannian case (see [23]), the Laplacian $\Box_{X_T}$ is not elliptic anymore, and thus even the following subproblems of Problem 1.(2) are open in general for $X_T = \Gamma\backslash G/H$ with $H$ noncompact.

**Questions 2.**

1. Does the Laplacian $\Box_{X_T}$, defined on $C^\infty_c(X_T)$, extend to a self-adjoint operator on $L^2(X_T)$?
2. Does $L^2(X_T; \mathcal{M}_\lambda)$ contain real analytic functions as a dense subspace?
3. Does $L^2(X_T)$ decompose discretely into a sum of subspaces $L^2(X_T; \mathcal{M}_\lambda)$ when $X_T$ is compact?

**2. Standard quotients** We observe that a discrete group of isometries on a pseudo-Riemannian manifold $X$ does not always act properly discontinuously on $X$, and the quotient space $X_T = \Gamma\backslash X$ is not necessarily Hausdorff. In fact, some semisimple symmetric spaces $X$ do not admit infinite discontinuous groups of isometries (Calabi–Markus phenomenon [2, 12]), and thus it is not obvious a priori whether there are interesting examples of pseudo-Riemannian locally symmetric spaces $X_T$ beyond the classical Riemannian case.

Fortunately, there exist semisimple symmetric spaces $X = G/H$ admitting “large” discontinuous groups $\Gamma$ such that $X_T$ is compact or of finite volume. Let us recall a useful idea for finding such $X$ and $\Gamma$.

Suppose a Lie subgroup $L$ of $G$ acts properly on $X$. Then the action of any discrete subgroup $\Gamma$ of $L$ on $X$ is automatically properly discontinuous, and this action is free whenever $\Gamma$ is torsion-free. Moreover, if $L$ acts cocompactly (e.g. transitively) on $X$, then $\text{vol}(X_T) < +\infty$ if and only if $\text{vol}(\Gamma\backslash L) < +\infty$.

**Definition 3** (Standard quotient $X_T$). A quotient $X_T = \Gamma\backslash X$ of $X = G/H$ by a discrete subgroup of $G$ is called standard if $\Gamma$ is contained in a reductive subgroup $L$ of $G$ acting properly on $X$.

A criterion on triples $(G, L, H)$ of reductive Lie
groups for \( L \) to act properly on \( X = G/H \) was established in [12], and a list of irreducible symmetric spaces \( G/H \) admitting proper and cocompact actions of reductive subgroups \( L \) was given in [19]. Recently, Tojo [24] announced that the list in [19] exhausts all such triples \((L,G,H)\) with \( L \) maximal.

3. Construction of discrete spectrum

Let \( X = G/H \) be a semisimple symmetric space. Let \( j \) be a maximal semisimple abelian subspace in the orthogonal complement of \( h \) in \( g \) with respect to the Killing form, and \( W \) the Weyl group for the root system \( \Sigma(\mathfrak{g},i_C) \). The Harish-Chandra isomorphism \( \Psi: S(i_C)^W \cong \mathbb{D}_G(X) \) (see [6]) induces a bijection

\[
\Psi^*: \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \cong i_C/W.
\]

The dimension of \( j \) is called the rank of the symmetric space \( X = G/H \). Let \( K \) be a maximal compact subgroup of \( G \) such that \( H \cap K \) is a maximal compact subgroup of \( H \). Assume that \( G \) is connected without compact factor and that the following rank condition is satisfied:

\[
\text{rank } G/H = \text{rank } K/(H \cap K).
\]

Then we can take \( j \) as a subspace of \( t \). We fix compatible positive systems \( \Sigma^+(\mathfrak{g}_C,i_C) \) and \( \Sigma^+(\mathfrak{k}_C,i_C) \), denote by \( \rho \) and \( \rho_c \) the corresponding half sums of positive roots counted with multiplicities, and set

\[
\Lambda := 2\rho_c - \rho + \mathbb{Z}\text{-span}\{\text{highest weights of } (\hat{K})_{H\cap K}\}.
\]

For \( C \geq 0 \), we consider the countable set

\[
\Lambda_C := \{\lambda \in \Lambda : \langle \lambda, \alpha \rangle > C \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}_C,i_C)\}.
\]

**Fact 4** (Flensted-Jensen [5]). *If the rank condition (3.2) holds, then there exists \( C > 0 \) such that*

\[
L^2(X; \mathcal{M}_\lambda) \neq \{0\} \quad \text{for all } \lambda \in \Lambda_C.
\]

In fact one can take \( C = 0 \) [20]. We now turn to locally symmetric spaces \( X_\Gamma \):

**Theorem 5** ([8], [9, Th. 1.5]). *Under the rank condition (3.2), for any standard quotient \( X_\Gamma \) with \( \Gamma \) torsion-free, there exists \( C_\Gamma > 0 \) such that*

\[
L^2(X; \mathcal{M}_\lambda) \neq \{0\} \quad \text{for all } \lambda \in \Lambda_{C_\Gamma}.
\]

Thus the discrete spectrum \( \text{Spec}_d(X_\Gamma) \), which is by definition the set of \( \lambda \in \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \) such that \( L^2(X; \mathcal{M}_\lambda) \neq \{0\} \), is infinite.

Theorem 5 applied to \((G \times \{1\}, G \times G, \text{Diag } G)\) instead of \((L,G,H)\) (group manifold case) implies:

**Example 6.** Suppose \( \text{rank } G = \text{rank } K \). For any torsion-free discrete subgroup \( \Gamma \) and any discrete series representation \( \pi_\lambda \) of \( G \) with sufficiently regular Harish-Chandra parameter \( \lambda \),

\[
\text{Hom}_G(\pi_\lambda, L^2(\Gamma \backslash G)) \neq \{0\}.
\]

This sharpens and generalizes the known results asserting that if \( \Gamma \) is an arithmetic subgroup of \( G \), then (3.3) holds after replacing \( \Gamma \) by a finite-index subgroup \( \Gamma' \) (possibly depending on \( \pi_\lambda \)), see Borel–Wallach [1], Clozel [3], DeGeorge–Wallach [4], Kazhdan [11], Rohlfs–Speh [21], and Savin [22].

**Remark 7.** (1) Theorem 5 extends to a more general setting where \( X_\Gamma \) is not necessarily standard: namely, the conclusion still holds as soon as the action of \( \Gamma \) on \( X \) satisfies a strong properness condition called sharpness [9, Th. 3.8].

(2) The rank condition (3.2) is necessary for \( \text{Spec}_d(X) \) to be nonempty (see Matsuki–Oshima [20]), in which case Fact 4 applies. On the other hand, \( \text{Spec}_d(X_\Gamma) \) may be nonempty even if (3.2) fails. This leads us to the notion of discrete spectrum of type I and II, see Definition 12 below.

4. Spectral decomposition of \( L^2(X_\Gamma) \)

In this section, we discuss spectral decomposition on standard quotients \( X_\Gamma \). We do not impose the rank condition (3.2), but require that \( L_C \) act spherically on \( X_C \), i.e. a Borel subgroup of \( L_C \) has an open orbit in \( X_C \). To be precise, our setting is as follows:

**Setting 8.** We consider a symmetric space \( X = G/H \) with \( G \) noncompact and simple, a reductive subgroup \( L \) of \( G \) acting properly on \( X \) such that \( X_C = G_C/H_C \) is \( L_C \)-spherical, and a torsion-free discrete subgroup \( \Gamma \) of \( L \).

For compact \( H \), we can take \( L = G \). However, our main interest is for noncompact \( H \), in which case \( L \neq G \) in the setting 8.
In Theorems 9 and 10 below, we allow the case where \( \text{vol}(X_{\Gamma}) = +\infty \).

Theorem 9 (Spectral decomposition). In the setting \( \mathcal{S} \), there exist a measure \( d\mu \) on \( \text{Hom} := \text{Hom}_{\mathcal{C}-\text{alg}}(\mathbb{D}_G(X), \mathbb{C}) \) and a measurable family \( (\mathcal{F}_{\lambda})_{\lambda \in \text{Hom}} \) of linear maps, with

\[
\mathcal{F}_{\lambda} : C^\infty_c(X_{\Gamma}) \rightarrow C^\infty(X_{\Gamma}; \mathcal{M}_\lambda),
\]

such that any \( f \in C^\infty_c(X_{\Gamma}) \) can be expanded into

\[
f = \int_{\text{Hom}} \mathcal{F}_{\lambda} f \; d\mu(\lambda),
\]

with a Parseval–Plancherel type formula

\[
\|f\|^2_{L^2(X_{\Gamma})} = \int_{\text{Hom}} \|\mathcal{F}_{\lambda} f\|^2_{L^2(X_{\Gamma})} \; d\mu(\lambda).
\]

The measure \( d\mu \) can be described via a “transfer map” discussed in Section 5, see (5.4). In particular, we see that (4.1) is a discrete sum if \( X_{\Gamma} \) is compact, answering Question 2.3 in our setting. The proof of Theorem 9 gives an answer to Questions 2.(1)–(2):

Theorem 10. In the setting \( \mathcal{S} \),

1. the pseudo-Riemannian Laplacian \( \Box_{X_{\Gamma}} \) defined on \( C^\infty_c(X_{\Gamma}) \) is essentially self-adjoint on \( L^2(X_{\Gamma}) \);
2. any \( L^2 \)-eigenfunction of the Laplacian \( \Box_{X_{\Gamma}} \) can be approximated by real analytic \( L^2 \)-eigenfunctions.

Theorem 11. In the setting \( \mathcal{S} \), the discrete spectrum \( \text{Spec}_{d}(X_{\Gamma}) \) is infinite whenever \( \Gamma \) is cocompact or arithmetic in the subgroup \( L \).

Let \( D'(X) \) be the space of distributions on \( X \), endowed with its standard topology. Let \( p^*_\Gamma : L^2(X_{\Gamma}) \rightarrow D'(X) \) be the pull-back by the projection \( p_\Gamma : X \rightarrow X_{\Gamma} \). For \( \lambda \in \text{Spec}_{d}(X_{\Gamma}) \), we denote by \( L^2(X_{\Gamma}; \mathcal{M}_\lambda)_{\mathcal{H}} \) the preimage under \( p^*_\Gamma \) of the closure in \( D'(X) \) of \( L^2(X_{\Gamma}; \mathcal{M}_\lambda) \), and by \( L^2(X_{\Gamma}; \mathcal{M}_\lambda)_{\mathcal{H}} \) its orthogonal complement in \( L^2(X_{\Gamma}; \mathcal{M}_\lambda) \).

Definition 12. For \( i = I \) or \( II \), the discrete spectrum of type \( i \) of \( X_{\Gamma} \) is the subset \( \text{Spec}_{d}(X_{\Gamma})_I \) of \( \text{Spec}_{d}(X_{\Gamma}) \) consisting of those elements \( \lambda \) such that \( L^2(X_{\Gamma}; \mathcal{M}_\lambda)_I \neq \{0\} \).

By construction, \( \text{Spec}_{d}(X_{\Gamma})_I \) is contained in \( \text{Spec}_{d}(X) \), hence it is nonempty only if (3.2) holds (Remark 7.(2)); in this case \( \text{Spec}_{d}(X_{\Gamma})_I \) is actually infinite for standard \( X_{\Gamma} \) by Theorem 5. On the other hand, Theorem 11 has the following refinement.

Theorem 13. In the setting \( \mathcal{S} \), \( \text{Spec}_{d}(X_{\Gamma})_I \) is infinite whenever \( \Gamma \) is cocompact or arithmetic in \( L \).

Example 14. Let \( M \) be a 3-dimensional compact standard anti-de Sitter manifold. Then both \( \text{Spec}_{d}(X_{\Gamma})_I \) and \( \text{Spec}_{d}(X_{\Gamma})_II \) are infinite, and

\[
\text{Spec}_{d}(X_{\Gamma})_I \subset [0, +\infty), \quad \text{Spec}_{d}(X_{\Gamma})_II \subset (-\infty, 0].
\]

5. Transfer maps

In Section 1 we considered spectral analysis on locally symmetric spaces \( X_{\Gamma} \) through the algebra \( \mathcal{P} (\simeq \mathbb{D}_G(X)) \) of intrinsic differential operators on \( X_{\Gamma} \). For standard quotients \( X_{\Gamma} \) with \( \Gamma \subset L \), another \( \mathcal{C} \)-algebra \( \mathcal{Q} \) of differential operators on \( X_{\Gamma} \) is obtained from the center \( \mathfrak{z}(\mathcal{C}) \) of the enveloping algebra \( U(\mathfrak{i}_c) \): indeed, \( \mathfrak{z}(\mathcal{C}) \) acts on smooth functions on \( X \) by differentiation, yielding a \( \mathcal{C} \)-algebra of \( L \)-invariant differential operators on \( X_{\Gamma} \), hence a \( \mathcal{C} \)-algebra of differential operators on \( X_{\Gamma} = \Gamma \backslash X \) since \( \Gamma \subset L \). In general, there is no inclusion relation between \( \mathcal{P} \) and \( \mathcal{Q} \). In order to compare the roles of \( \mathcal{P} \) and \( \mathcal{Q} \), we highlight a natural homomorphism \( \mathfrak{z}(\mathfrak{g}_C) \rightarrow \mathcal{P} \) and a surjective one \( d\ell : \mathfrak{z}(\mathfrak{i}_c) \rightarrow \mathcal{Q} \). Loosely speaking, the algebras \( \mathfrak{z}(\mathfrak{g}_C) \) and \( \mathfrak{z}(\mathfrak{i}_c) \) separate irreducible representations of the groups \( G \) and \( L \), respectively, hence it is important to understand how irreducible representations of \( G \) behave when restricted to the subgroup \( L \) (branching problem) in order to utilize the algebra \( \mathcal{Q} \) for the spectral analysis on \( X_{\Gamma} \) via the algebra \( \mathcal{P} \) (see [16, 17]). We shall return to this point in Theorem 15 below.

Suppose a reductive subgroup \( L \) acts properly and transitively on \( X = G/H \). Then \( L_H := L \cap H \) is compact. We may assume that \( L_K := L \cap K \) is a maximal compact subgroup of \( L \) containing \( L_H \), after possibly replacing \( L \) by some conjugate. Then the locally pseudo-Riemannian symmetric space \( X = \Gamma \backslash G/H \) fibers over the Riemannian locally symmetric space \( Y_{\Gamma} = \Gamma \backslash L/L_K \) with fiber \( F := L_K/L_H \):

\[
F \rightarrow X_{\Gamma} \rightarrow Y_{\Gamma}.
\]

To expand functions on \( X_{\Gamma} \) along the fiber \( F \),
we define an endomorphism $p_\tau$ of $C^\infty(X_\Gamma)$ by
\[
(p_\tau f)(\cdot) := \frac{1}{\dim \tau} \int_K f(\cdot) \, \text{Trace} \, \tau(k) \, dk
\]
for every $\tau \in \mathcal{K}_L$. Then $p_\tau$ is an idempotent, namely, $p_\tau^2 = p_\tau$. The $\tau$-component of $C^\infty(X_\Gamma)$ is defined by
\[
C^\infty(X_\Gamma)_\tau := \text{Image}(p_\tau) = \text{Ker}(p_\tau - \text{id}).
\]
We note that $C^\infty(X_\Gamma)_\tau \neq \{0\}$ if and only if $\tau$ has a nonzero $L_H$-invariant vector, i.e. $\tau \in (\mathcal{K}_L)_L$. It is easy to see that the projection $p_\tau$ commutes with any element in $\mathcal{Q}$ ($\simeq \mathcal{D}_G(X)$), but not always with “intrinsic differential operators” $D_\tau \in \mathcal{P}$ ($\simeq \mathcal{D}_G(X)$), and consequently it may well happen that
\[
p_\tau(C^\infty(X_\Gamma;M_\lambda)) \not\subset C^\infty(X_\Gamma;M_\lambda).
\]
To make a connection between the two subalgebras $\mathcal{P}$ and $\mathcal{Q}$, we introduce a third subalgebra $\mathcal{R}$ of $\mathcal{D}(X_\Gamma)$, coming from the fiber $F$ in (5.1). Namely, $\mathcal{R}$ is isomorphic to the $\mathbb{C}$-algebra $\mathcal{D}_{\mathcal{L}_K}(F)$ of $\mathcal{L}_K$-invariant differential operators $D$ on $F$, and obtained by extending elements of $\mathcal{D}_{\mathcal{L}_K}(F)$ to $\mathcal{L}$-invariant differential operators on $X$, yielding differential operators on the quotient $X_\Gamma$.

Suppose now that we are in the setting $8$. The subgroup $L$ acts transitively on $X$ by [18, Lem. 4.2] and [13, Lem. 5.1]. Moreover, we can prove that
\[
\mathcal{Q} \subset \langle \mathcal{P}, \mathcal{R} \rangle
\]
where $\langle \mathcal{P}, \mathcal{R} \rangle$ denotes the subalgebra of $\mathcal{D}(X_\Gamma)$ generated by $\mathcal{P}$ and $\mathcal{R}$. This implies the following strong constraints on the restriction of representations:

**Theorem 15.** In the setting $8$, any irreducible $(\mathfrak{g}, K)$-module occurring in $C^\infty(X)$ is discretely decomposable as an $(1, L \cap K)$-module.

See [13, 14, 15] for a general theory of discretely decomposable restrictions of representations. See also [17] for a discussion on Theorem 15 when dropping the assumption that $L$ acts properly on $X$.

In addition to (5.2), the quotient fields of $\mathcal{P}$ and $\langle \mathcal{Q}, \mathcal{R} \rangle$ coincide [10], and we obtain the following:

**Theorem 16 (Transfer map).** In the setting $8$, for any $\tau \in (\mathcal{L}_K)_L$ there is an injective map $\nu(\cdot, \tau) : \text{Hom}_{\mathcal{C\text{-}alg}}(\mathcal{D}_G(X), \mathbb{C}) \hookrightarrow \text{Hom}_{\mathcal{C\text{-}alg}}(\mathcal{F}, \mathbb{C})$ such that for any $\lambda \in \text{Hom}_{\mathcal{C\text{-}alg}}(\mathcal{D}_G(X), \mathbb{C})$, any $f \in C^\infty(X_\Gamma;M_\lambda)$, and any $z \in \mathcal{F}$,
\[
df(z)(p_\tau f) = \nu(\lambda, \tau)(z) \, p_\tau f.
\]

We write $\lambda(\cdot, \tau)$ for the inverse map of $\nu(\cdot, \tau)$ on its image. We call $\nu(\cdot, \tau)$ and $\lambda(\cdot, \tau)$ transfer maps, as they “transfer” eigenfunctions for $\mathcal{P}$ to those for $\mathcal{Q}$, and vice versa, on the $\tau$-component $C^\infty(X_\Gamma)_\tau$.

For an explicit description of transfer maps, let
\[
\Phi^* : \text{Hom}_{\mathcal{C\text{-}alg}}(\mathcal{F}, \mathbb{C}) \rightarrow \mathcal{F}/W(\mathcal{F})
\]
be the Harish-Chandra isomorphism as in (3.1), where $W(\mathcal{F})$ denotes the Weyl group of the root system $\Delta(\mathbb{C}, t_c)$ with respect to a Cartan subalgebra $t_c$ in $\mathcal{C}$. We note that there is no natural inclusion relation between $j_c$ and $t_c$.

For each $\tau \in (\mathcal{L}_K)_L$, we find an affine map $S_\tau : j_c \rightarrow t_c$ such that the following diagram commutes:

\[
\begin{array}{ccc}
j_c^* & \rightarrow & t_c^* \\
\downarrow & & \downarrow \\
j_c^*/W & \rightarrow & t_c^*/W(\mathcal{F}) \\
\Phi^* & \rightarrow & \Phi^*
\end{array}
\]

Then a closed formula for the transfer map $\nu(\cdot, \tau)$ is derived from that of the affine map $S_\tau$ which was determined explicitly in [10] for the complexifications of the triples $(L, G, H)$ in the setting $8$.

Via the transfer maps, we can utilize representations of the subgroup $L$ efficiently for the spectral analysis on $X_\Gamma$, as follows. As in (1.2), let
\[
L^2(\Gamma \setminus L) \simeq \int_L m_\Gamma(\theta) \, d\sigma(\theta)
\]
be a disintegration of the regular representation $L^2(\Gamma \setminus L)$ of the subgroup $L$. Then the transform $\mathcal{F}_\lambda$ in Theorem 9 can be built naturally by using (5.3) and the expansion of $C^\infty_c(X_\Gamma)$ along the fiber $F$ in (5.1). Consider the map...
\[ \Lambda: (\widetilde{j})_{LH} \times (\widetilde{L})_{LH} \to \text{Hom}_{\mathbb{C}}(D_{G}(X), \mathbb{C}), \]

\( (\vartheta, \tau) \mapsto \lambda(\chi_{\vartheta}, \tau), \) where \( \chi_{\vartheta} \in \text{Hom}_{\mathbb{C}}(\mathfrak{g}(\mathbb{C}), \mathbb{C}) \)

is the infinitesimal character of \( \vartheta \in \widetilde{L}. \) Then the Plancherel measure \( d\mu \) on \( \text{Hom}_{\mathbb{C}}(D_{G}(X), \mathbb{C}) \)

in Theorem 9 can be defined by

\[ d\mu = \Lambda_{*}(d\sigma|_{(\widetilde{j})_{LH} \times (\widetilde{L})_{LH}}) \]

Detailed proofs of Theorems 9, 10, 11, 15, and 16 will appear elsewhere.

References


