SPECTRAL PROPERTIES OF A CLASS OF OPERATORS ASSOCIATED WITH MAPS IN ONE DIMENSION

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1. Generalities, definitions, and results.

Let f be a piecewise monotone map of the interval $[0,1]$ to itself, and g a function of bounded variation on $[0,1]$. Hofbauer, Keller [3], [4] and Rychlik [10] have studied operators $\mathcal{L}$ on functions of bounded variation, where

$$\mathcal{L}\,\Phi(x) = \sum_{y:fy=x} g(y)\Phi(y) \quad .$$

Among other things, they show that the essential spectral radius of $\mathcal{L}$ is in many cases strictly smaller than the spectral radius; there exist therefore isolated eigenvalues of finite multiplicity. The purpose of the present paper is to prove similar results for a more general class of operators forming an algebra (and therefore containing sums of operators like $\mathcal{L}$). An analogous extension was presented in [9] for operators associated with expanding maps.

Let Var φ denote the total variation of a function $\varphi : \mathbb{R} \mapsto \mathbb{C}$. Throughout what follows, X will be a finite union of bounded intervals*) of $\mathbb{R}$, and we write

$$\mathcal{B} = \mathcal{B}(X) = \{\varphi : \text{Var}\, \varphi < \infty \text{ and}$$

$$\varphi \text{ vanishes on } \mathbb{R} \backslash X\} .$$

Then, $\mathcal{B}$ is a Banach space with respect to the norm $\varphi \mapsto \text{Var}\, \varphi$.

For each ω in some countable set Ω, the following are supposed to be given:

(i) a connected subset $V_\omega \subset X$, i.e., V_ω is empty, reduced to a point, or is an interval of $\mathbb{R}$ (not necessarily open or closed);

(ii) a map $\psi_\omega : V_\omega \mapsto X$ such that $V_\omega \mapsto \psi_\omega V_\omega$ is a homeomorphism;

(iii) a function $\varphi_\omega \in \mathcal{B}$, vanishing on $X \backslash V_\omega$.

We assume that

$$\sum_\omega \text{Var}\, \varphi_\omega < \infty \tag{1.1}$$

and define linear operators $\mathcal{M}$, $\mathcal{M}'$ on $\mathcal{B}$ by

$$\left.\begin{aligned} \mathcal{M}\Phi(x) &= \sum_\omega \varphi_\omega(x)\, \Phi(\psi_\omega x) \\ \mathcal{M}'\Phi(x) &= \sum_\omega \varphi_\omega(\psi_\omega^{-1} x)\, \Phi(\psi_\omega^{-1} x) \end{aligned}\right\} \tag{1.2}$$

[We let $\varphi_\omega(x)\, \Phi(\psi_\omega x) = 0$ when $x \notin V_\omega$, $\varphi_\omega(\psi_\omega^{-1} x)\, \Phi(\psi_\omega^{-1} x) = 0$ when $x \notin \psi_\omega V_\omega$].

In view of (1.1), $\mathcal{M}$ and $\mathcal{M}'$ are bounded operators:

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*) One interval is all that is really needed, but it is convenient to allow several in view of the discussion in Appendix A.

$$\|\mathcal{M}\|,\|\mathcal{M}'\| \leq \sum_{\omega} \mathrm{Var}\ \varphi_\omega .$$

[This follows from $\|u\|_o \leq \frac{1}{2}$ Var u, Var(uv) $\leq \|u\|_o$ Var v + $\|v\|_o$ Var u $\leq$ Var u Var v , and Var(u∘ψ) ≤ Var u].

Note that the operators $\mathcal{M}$ and $\mathcal{M}'$ play symmetric roles, one being formally the transpose of the other.

The setup described above is fairly general, and includes in particular the situation where X is replaced by a 1-dimensional complex $\tilde{X}$ (see Appendix A). One usually considers the case where the ψ_ω are local inverses of a map f (see Appendix A). The present more general treatment has the advantage of maintaining the symmetry between $\mathcal{M}$ and $\mathcal{M}'$.

1.1. <u>Theorem</u>. *Let* $|\mathcal{M}|$, $|\mathcal{M}'|$ *denote the operators obtained when* φ_ω *is replaced by* $|\varphi_\omega|$ *in the definition of* $\mathcal{M}$, $\mathcal{M}'$, *and write* *)

$$R = \lim_{m\to\infty} (\||\mathcal{M}|^m 1\|_o)^{1/m}$$

$$R' = \lim_{m\to\infty} (\||\mathcal{M}'|^m 1\|_o)^{1/m} \qquad (1.3)$$

(these are the spectral radii of $|\mathcal{M}|$, $|\mathcal{M}'|$ acting on bounded functions $X \mapsto \mathbb{C}$, with the "uniform" norm $\|.\|_o$) .

(a) *The spectral radius of* $\mathcal{M}$, *acting on* $\mathcal{B}$, *is* ≤ max (R,R').

(b) *The essential spectral radius of* $\mathcal{M}$ *is* ≤ R' .

*) In (1.3) and later formulae one may replace 1 by the characteristic function of X (which is in $\mathcal{B}$, unlike 1) ; we let $\|\Phi\|_o = \sup_{x\in\mathbb{R}} |\Phi(x)|$.

1.2. Theorem. *If* $\varphi_\omega \geq 0$ *for all* ω *, then the spectral radius of* $\mathcal{M}$ *is* $\geq R$ *. If furthermore* $R' < R$ *, then* R *is an eigenvalue of* $\mathcal{M}$ *, and there is a corresponding eigenfunction* $\Phi_R \geq 0$ *.*

These theorems are proved in Section 2.

Let us say that two functions with bounded variation are equivalent if they differ on a countable set, and let $\mathcal{B}_c$ be the quotient Banach space. It is of interest that the proofs of the above theorems also apply if $\mathcal{B}$ is replaced by $\mathcal{B}_c$.

1.3. Proposition. *Let* $\pi : \mathcal{B} \mapsto \mathcal{B}_c$ *be the quotient map and* $\mathcal{M}_c$ *the operator on* $\mathcal{B}_c$ *such that* $\mathcal{M}_c\pi = \pi\mathcal{M}$ *. If* $|\lambda| > R'$ *and* E^λ *,* E_c^λ *are the generalized eigenspaces of* $\mathcal{M}$ *,* $\mathcal{M}_c$ *to the eigenvalue* λ *, then* π *maps* E^λ *bijectively to* E_c^λ *.*

Let Φ_c denote the image of Φ in B_c . If $(\mathcal{M} - \lambda)^n \Phi = 0$, then $(\mathcal{M}_c - \lambda)^n \Phi_c = 0$. Suppose $\Phi_c = 0$ and write $\Psi = (\mathcal{M} - \lambda)^{n-1}\Phi$.

Then

$$
\begin{aligned}
\operatorname{Var} \Psi &= \frac{1}{2}\sum_x |\Psi(x)| = \frac{1}{2}|\lambda|^{-m} \sum_x |(\mathcal{M}^m\Psi)(x)| \\
&\leq \frac{1}{2}|\lambda|^{-m} \sum_x (|\mathcal{M}|^m|\Psi|)(x) \\
&= \frac{1}{2}|\lambda|^{-m} \sum_x (|\mathcal{M}'|^m 1)(x)\cdot|\Psi(x)| \\
&\leq \frac{1}{2}|\lambda|^{-m} \| |\mathcal{M}'|^m 1 \|_o \sum_x |\Psi(x)| \\
&= |\lambda|^{-m} \| |\mathcal{M}'|^m 1 \|_o \operatorname{Var} \Psi .
\end{aligned}
$$

If $\lambda > R'$, this implies $\text{Var}\ \Psi = 0$, i.e., $\Psi = 0$. By induction also $\Phi = 0$. Therefore $E^\lambda \mapsto E_c^\lambda$ is injective.

Let E (resp. E_c) be the sum of the E^λ (resp. E_c^λ) with $|\lambda| > R' + \varepsilon$ for given $\varepsilon > 0$. Take Φ such that $\Phi_c \in E_c$, and write $\Phi = \Phi' + \Phi''$ where $\Phi' \in E$ and Φ'' corresponds to the part of the spectrum of $\mathcal{M}$ in $\{\lambda : |\lambda| \le R' + \varepsilon\}$, i.e.,

$$\lim \| \mathcal{M}^m \Phi'' \|^{1/m} \le R' + \varepsilon .$$

Then

$$\lim \| \mathcal{M}_c^m \Phi''_c \|^{1/m} \le R' + \varepsilon$$

hence Φ''_c corresponds to the part of the spectrum of $\mathcal{M}_c$ in $\{\lambda : |\lambda| \le R' + \varepsilon\}$. But since $\Phi_c , \Phi'_c \in E_c$, also $\Phi''_c \in E_c$, hence $\Phi''_c = 0$, i.e., $\Phi_c = \Phi'_c$. This shows that $E \mapsto E_c$ is surjective. Therefore all maps $E^\lambda \mapsto E_c^\lambda$ are surjective, concluding the proof of the proposition.

In the case where the Ψ_ω are local inverses of a single piecewise monotone map f , Baladi and Keller [1] have been able to relate the zeros of a ζ-function counting f-periodic points (with certain weights) and the spectrum of $\mathcal{M}$. Such a relation is well-known in other situations*). If one tries to obtain an analogous result in our case, one is confronted with the following technical problem : relate the condition $\psi J \cap J \ne \emptyset$, and the choice of a fixed point of the homeomorphism ψ in the interval J . This can be done in various ways. In [1], forward expansiveness of f is assumed. Here we shall arrange that ψ be orientation reversing, so that $\psi J \cap J \ne \emptyset$ if and only if ψ has a fixed point in J , and the latter is unique. This leads to the Conjecture 1.4 below.

*) See [8] for a review, and the references given there, in particular to the important work of Haydn [2], and Tangerman [11]. See also [9] for the case where the ψ_ω are not local inverses of a single map f .

We need at this point some definitions. Write $\varepsilon_\omega = \pm 1$ depending on whether ψ_ω is increasing or decreasing (an arbitrary choice is made if V_ω is reduced to a point), and define

$$\mathcal{M}^\varepsilon \Phi(x) = \sum_\omega \varepsilon_\omega \varphi_\omega(x) \; \Phi(\psi_\omega x) \quad .$$

Let also

$$\zeta_m^- = \sum_{\omega_1 \ldots \omega_m}^- \varphi_{\omega_m}(x(\bar{\omega})) \, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x(\bar{\omega})) \ldots$$
$$\ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\bar{\omega}))$$

where Σ^- extends over those m-tuples such that $\psi_{\omega_1} \cdots \psi_{\omega_m} V_{\omega_m} \cap V_{\omega_m} \neq \emptyset$, and $\prod_1^m \varepsilon_{\omega_k} = -1$; the unique fixed point of $\psi_{\omega_1} \cdots \psi_{\omega_m}$ is denoted by $x(\bar{\omega})$.

1.4. <u>Conjecture</u>. *The power series*

$$\zeta^-(z) = \exp 2 \sum_{m=1}^\infty \frac{z^m}{m} \zeta_m^-$$

extends to a meromorphic function for $|z| < 1/R'$, *and the order* *) *of* ζ^- *at* λ^{-1} *is* $n^\varepsilon(\lambda) - n(\lambda)$ *where* $n(\lambda)$ *and* $n^\varepsilon(\lambda)$ *are the multiplicities* **) *of* λ *as eigenvalues of* $\mathcal{M}$ *and* $\mathcal{M}^\varepsilon$ respectively.

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*) We define the order of the meromorphic function ζ at z_0 to be the unique $n \in \mathbf{Z}$ such that $(z-z_0)^{-n}\zeta(z)$ is holomorphic and non zero at z_0 .

**) The multiplicity of an eigenvalue λ is the dimension of the corresponding generalized eigenspace. If λ is not an eigenvalue, we say that λ has multiplicity 0 .

A special case of this conjecture is proved in Appendix A as a corollary of the work of Baladi and Keller [1]. (This extends slightly a result of Milnor and Thurston [5]).

Probably one can, along the lines of [9], give a fairly general definition of ζ-functions in the present setup so that $1/\zeta(z)$ is holomorphic for $|z| < 1/R'$ and the zeros of $1/\zeta$ are the inverses of the eigenvalues of $\mathfrak{M}$ with the same multiplicity. The definition of ζ would however involve some *arbitrary choices* due to the nonuniqueness of fixed points for the maps $\psi_{\omega_1} \cdots \psi_{\omega_m}$.

2. Proof of Theorems 1.1. and 1.2.

2.1. Estimating Var Φ .

We shall *demi-measure* on $\mathbb{R}$ a triple $\sigma = (\sigma_0, \sigma_-, \sigma_+)$ of complex measures with compact support in $\mathbb{R}$ such that σ_0 is nonatomic and $\sigma_\pm$ are atomic. We may thus write

$$\sigma_\pm = \sum_k a_{k\pm}\, \delta_{x(k)}$$

where $\delta_{x(k)}$ is the unit mass at $x(k)$. (Intuitively, we may think of $\sigma_\pm$ as consisting of masses $a_{k\pm}$ just to the right or left of $x(k)$). The norm of the demi-measure σ is the quantity

$$\int |\sigma| = \int |\sigma_0| + \sum_k (|a_{k+}| + |a_{k-}|) \quad .$$

We define now two different products $\sigma\Phi$ and $\Phi\sigma$ of the demi-measure σ and the function Φ with bounded variation and compact support $\mathbb{R} \mapsto \mathbb{C}$. Both $\sigma\Phi$ and $\Phi\sigma$ are demi-measures:

$$\sigma\Phi = (\Phi\sigma_o\, , \sum_k a_{k-}\ \Phi(x(k))\delta_{x(k)}\, ,$$

$$\sum_k a_{k+}\Phi(x(k)+)\delta_{x(k)})$$

$$\Phi\sigma = (\Phi\sigma_o\, , \sum_k a_{k-}\ \Phi(x(k)-)\delta_{x(k)}\, ,$$

$$\sum_k a_{k+}\ \Phi(x(k))\delta_{x(k)})\ ,$$

where $\Phi(x(k)\pm)$ are the limits of $\Phi(x)$ when $x \to x(k)$ from right or left; these limits exist because Φ has bounded variation. The maps $(\Phi,\sigma) \mapsto \sigma\Phi, \Phi\sigma$ are bilinear, and we have the associativity properties

$$\left.\begin{array}{rcl} \sigma(\Phi_1\Phi_2) & = & (\sigma\Phi_1)\Phi_2 \\ \Phi_1(\sigma\Phi_2) & = & (\Phi_1\sigma)\Phi_2 \\ (\Phi_1,\Phi_2)\sigma & = & \Phi_1(\Phi_2\,\sigma) \end{array}\right\}$$

Note also that

$$\int |\sigma\Phi| \le \|\Phi\|_o \int |\sigma|\ \ ,\ \ \int |\Phi\sigma| \le \|\Phi\|_o \int |\sigma|\ .$$

Let $\Phi : \mathbb{R} \mapsto \mathbb{C}$ have bounded variation and compact support. The derivative of Φ in the sense of distributions is then a measure ρ which does not change if Φ is modified at a point. We can also define a demi-measure $\nabla\Phi = (\rho_o,\rho_-,\rho_+)$ where ρ_o is the nonatomic part of ρ , and

$$\rho_- = \sum_k (\Phi(x(k)) - \Phi(x(k)-))\ \delta_{x(k)}$$

$$\rho_+ = \sum_k (\Phi(x(k)+) - \Phi(x(k)))\ \delta_{x(k)}\ .$$

With this definition, we have

$$\mathrm{Var}\ \Phi = \int |\nabla \Phi|$$

and the crucial property

$$\nabla(\Phi_1\Phi_2) = \Phi_1(\nabla\Phi_2) + (\nabla\Phi_1)\Phi_2$$

which is readily verified.

If $\psi : \mathbb{R} \mapsto \mathbb{R}$ is a homeomorphism, and σ a demi-measure, there is a naturally defined direct image $\psi\sigma = (\psi\sigma_o , \psi\sigma_{\mp}) , \psi\sigma_{\pm})$, where the choice of signs depends on whether ψ is an increasing or decreasing function. We have

$$\psi(\Phi_1\sigma\Phi_2) = (\Phi_1 \circ \psi)\,(\psi\sigma)\,(\Phi_2 \circ \psi)\,, \qquad \nabla(\Phi \circ \psi) = \psi\nabla\Phi$$

if ψ is increasing, and

$$\psi(\Phi_1\sigma\Phi_2) = (\Phi_2 \circ \psi)\,(\psi\sigma)\,(\Phi_1 \circ \psi)\,, \qquad \nabla(\Phi \circ \psi) = -\,\psi\nabla\Phi$$

if ψ is decreasing. Note also that

$$\int |\psi\sigma| = \int |\sigma|\ .$$

2.2. <u>The spectral radius of $\mathfrak{M}$.</u>

We may write

$$\mathrm{Var}\ \mathfrak{M}^m\Phi = \int |\nabla\, \mathfrak{M}^m\Phi|$$

$$\leq \sum_{k=1}^{m} \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m}) \ldots (\nabla\varphi_{\omega_k} \circ \psi_{k+1} \circ \ldots \circ \psi_{\omega_m}) \ldots$$

$$(\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \ldots \circ \psi_{\omega_m})\,(\Phi \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_m})|$$

$$+ \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m}) \ldots (\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \ldots \circ \psi_{\omega_m})(\nabla\Phi \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_m})|\ .$$

Since every $\psi_{\omega_{k+1}} \circ \ldots \circ \psi_{\omega_m}$ preserves or reverses orientation, we have

$$\mathrm{Var}\, \mathcal{M}^m \Phi \leq \sum_{k=1}^{m} \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m} \circ \psi_{\omega_m}^{-1} \circ \ldots \circ \psi_{\omega_{k+1}}^{-1}) \ldots (\nabla \varphi_{\omega_k}) \ldots$$

$$(\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \ldots \psi_{\omega_k}) (\Phi \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_k})|$$

$$+ \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m} \circ \psi_{\omega_m}^{-1} \circ \ldots \circ \psi_{\omega_1}^{-1}) \ldots (\varphi_{\omega_1} \circ \psi_{\omega_1}^{-1}) (\nabla \Phi)|$$

+ symmetric terms

where the factors to the left and to the right of $\nabla \varphi_{\omega_k}$ or $\nabla \Phi$ are interchanged in the "symmetric terms". Therefore, this is

$$\leq \sum_{k=1}^{m} \int \sum_{\omega_k} |(|\mathcal{M}'|^{m-k} 1)(\nabla \varphi_{\omega_k}) (|\mathcal{M}|^{k-1} |\Phi|)|$$

$$+ \int |(|\mathcal{M}'|^{m} 1) \nabla \Phi|$$

+ symmetric terms

$$\leq 2 \sum_{k=1}^{m} \|(|\mathcal{M}'|^{m-k} 1)\|_o (\sum_{\omega} \mathrm{Var}\, \varphi_{\omega}) \| |\mathcal{M}|^{k-1} 1 \|_o \|\Phi\|_o$$

$$+ 2 \|(|\mathcal{M}'|^{m} 1)\|_o \,\mathrm{Var}\, \Phi \,.$$

With respect to the norm Var ou $\mathcal{B}$ we have thus

$$\|\mathcal{M}^m\| \leq \mathrm{const} \left[\sum_{k=1}^{m} \| |\mathcal{M}'|^{m-k} 1 \|_o \| |\mathcal{M}|^{k-1} 1 \|_o + \| |\mathcal{M}'|^{m} 1 \|_o \right]$$

and the spectral radius of $\mathcal{M}$ is

$$\lim_{m\to\infty} (\|\mathfrak{M}^m\|)^{1/m} \le \max (R,R') .$$

Note that, since the ψ_{ω_k} are not defined everywhere, they are some inconsequential ambiguities in the formulae of the above proof.

2.3. <u>The operators $T^{(m)}$</u>.

If $\Lambda \subset \Omega$, let us write

$$\mathfrak{M}_\Lambda \Phi(x) = \sum_{\omega\in\Lambda} \varphi_\omega (x)\ \Phi\ (\psi_\omega\, x)\ .$$

For each integer $m \ge 1$ we select $\Lambda = \Lambda(m)$ finite such that

$$\|\mathfrak{M}^m - \mathfrak{M}^m_\Lambda\| \le R'^m \tag{2.1}$$

Given $\varepsilon > 0$ we may now choose finite sets $A,B \subset X$ such that $X\backslash A$ is a union of disjoint open intervals $J(b)$ of $\mathbb{R}$, and furthermore

(i) $b \in J(b)$,

(ii) the ranges of $\psi_{\omega_1} \cdots \psi_{\omega_k}$ (for $1 \le k \le m$, $\omega_1,\ldots,\omega_k \in \Lambda$) are unions of points $a \in A$ and intervals $J(b)$, $b \in B$,

(iii) φ_{ω_k} has variation $\le \varepsilon$ on $\psi^{-1}_{\omega_1} \cdots \psi^{-1}_{\omega_1} J(b)$ for each $J(b)$ in the range of $\psi_{\omega_1} \cdots \psi_{\omega_k}$.

If $\Phi \in \mathfrak{B}$, we define

$$T^{(m)}\Phi(x) = \begin{cases} \Phi(a) & \text{for } x = a \in A \\ \Phi(b) & \text{for } x \in J(b) \\ 0 & \text{for } x \notin X \end{cases}$$

By construction, $T^{(m)}$ is a linear operator on $\mathfrak{B}$, with $\|T^{(m)}\| \le 1$, and

$$\sum_{b\in B} \|(\Phi - T^{(m)}\Phi)|J(b)\|_o \le \text{Var}\ \Phi \tag{2.2}$$

$$\mathrm{Var}\,(\Phi - T^{(m)}\,\Phi) \le 2\,\mathrm{Var}\,\Phi \tag{2.3}$$

2.4. The essential spectral radius of $\mathcal{M}$.

Since $\mathcal{M}_\Lambda^m\, T^{(m)}$ has finite rank, Nussbaum's formula [6] implies that the essential spectral radius of $\mathcal{M}$ is

$$\le \limsup_{m\to\infty} (\| \mathcal{M}^m - \mathcal{M}_\Lambda^m T^{(m)} \|)^{1/m} .$$

We shall prove that

$$\limsup_{m\to\infty} (\| \mathcal{M}_\Lambda^m - \mathcal{M}_\Lambda^m\, T^{(m)} \|)^{1/m} \le R' e^{\delta} \tag{2.4}$$

with arbitrarily small $\delta > 0$ for suitable $T^{(m)}$. In view of (2.1) this will show that the spectral radius of $\mathcal{M}$ is $\le R'$.

We begin with an ancillary estimate. For $1 \le \ell \le m$, we have

$$\| |\mathcal{M}_\Lambda|^\ell\, |\Phi - T^{(m)}\Phi| \|_0$$

$$\le \sup_x \sum_{\omega_1 \ldots \omega_\ell \in \Lambda} |\varphi_{\omega_\ell}(x) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_\ell} x)|$$

$$|\Phi(\psi_{\omega_1} \ldots \psi_{\omega_\ell} x) - (T^{(m)}\Phi)\, (\psi_{\omega_1} \ldots \psi_{\omega_\ell} x)| .$$

For fixed x consider the points $\psi_{\omega_1} \ldots \psi_{\omega_\ell} x$ which are in a given interval $J(b)$. For all $k \le \ell$ and $\omega_k \in \Lambda$ we have

$$|\varphi_{\omega_k}(\psi_{\omega_{k+1}} \ldots \psi_{\omega_\ell} x) - \varphi_{\omega_k}(\psi_{\omega_k}^{-1} \ldots \psi_{\omega_1}^{-1} b)| \le \varepsilon$$

by assumption, hence

$$|||\mathcal{M}_\Lambda|^\ell|\Phi-T^{(m)}\Phi|||_o$$

$$\leq \sum_b \sum_{\omega_\ell} \varphi_{\omega_\ell\varepsilon}(\psi\omega_\ell^{-1} \cdots \psi\omega_1^{-1} b) \ldots \sum_{\omega_1} \varphi_{\omega_1\varepsilon}(\psi\omega_1^{-1} b)\|(\Phi-T^{(m)}\Phi)|J(b)\|_o$$

where we have written $\varphi_{\omega\varepsilon}(x) = |\varphi_\omega(x)| + \varepsilon$ when $x \in \Lambda_\omega$, $\varphi_{\omega\varepsilon}(x) = 0$ otherwise.

Therefore, using also (2.2) we have

$$|||\mathcal{M}_\Lambda|^\ell|\Phi - T^{(m)}\Phi|||_o$$

$$\leq \|(\mathcal{M}'_{\Lambda\varepsilon})^\ell 1\|_o \sum_b \|(\Phi-T^{(m)}\Phi)|J(b)\|_o$$

$$\leq \|(\mathcal{M}'_{\Lambda\varepsilon})^\ell 1\|_o \operatorname{Var}\Phi \tag{2.5}$$

where $\mathcal{M}_{\Lambda\varepsilon}$ is obtained if we replace φ_ω by $\varphi_{\omega\varepsilon}$ in the definition of $\mathcal{M}_\Lambda$.

We can get an upper bound to the first factor of the right-hand side by using the upper semi-continuity of the spectral radius (for the $\|\ \|_o$ norm) . Indeed, given $\delta > 0$, we may assume that ε has been chosen so small that

$$\|(\mathcal{M}'_{\Lambda\varepsilon})^\ell 1\|_o \leq C(R' e^{\delta/2})^\ell \tag{2.6}$$

for some constant $C > 0$.

We compute now

$$\operatorname{Var}(\mathcal{M}_\Lambda^m - \mathcal{M}_\Lambda^m T^{(m)})\Phi$$

$$= \int |\nabla \mathcal{M}_\Lambda^m(\Phi-T^{(m)}\Phi)|$$

$$\leq \sum_{k=1}^{m} \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m}) \ldots (\nabla \varphi_{\omega_k} \circ \psi_{\omega_{k+1}} \circ \ldots \circ \psi_{\omega_m})$$

$$\ldots (\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \ldots \circ \psi_{\omega_m})((\Phi - T^{(m)}\Phi) \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_m})|$$

$$+ \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m}) \ldots (\varphi_{\omega_1} \circ \psi_{\omega_2} \ldots \psi_{\omega_m}) (\nabla (\Phi - T^{(m)}\Phi) \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_m})|$$

$$\leq \sum_{k=1}^{m} \sum_{\omega_1 \ldots \omega_m} \int (\varphi_{\omega_m} \circ \psi_{\omega_m}^{-1} \circ \ldots \circ \psi_{\omega_{k+1}}^{-1}) \ldots (\nabla \varphi_{\omega_k})$$

$$\ldots (\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \ldots \circ \psi_{\omega_k}) ((\Phi - T^{(m)}\Phi) \circ \psi_{\omega_1} \circ \ldots \circ \psi_{\omega_k})|$$

$$+ \sum_{\omega_1 \ldots \omega_m} \int |(\varphi_{\omega_m} \circ \psi_{\omega_m}^{-1} \circ \ldots \psi_{\omega_1}^{-1}) \ldots (\varphi_{\omega_1} \circ \psi_{\omega_1}^{-1})(\nabla (\Phi - T^{(m)}\Phi))|$$

+ symmetric terms

$$\leq \sum_{k=1}^{m} \int \sum_{\omega_k} |(|\mathfrak{M}'_\Lambda|^{m-k} 1)(\nabla \varphi_{\omega_k})(|\mathfrak{M}_\Lambda|^{k-1} |\Phi - T^{(m)}\Phi|)|$$

$$+ \int |(|\mathfrak{M}'_\Lambda|^{m} 1) \nabla (\Phi - T^{(m)}\Phi)|$$

+ symmetric terms

$$\leq 2 \sum_{k=1}^{m} \| |\mathfrak{M}'_\Lambda|^{m-k} 1 \|_o \, (\sum_{\omega} \mathrm{Var}\ \varphi_\omega \| |\mathfrak{M}_\Lambda|^{k-1} |\Phi - T^{(m)}\Phi| \|_o$$

$$+ 2 \| |\mathfrak{M}'_\Lambda|^{m} 1 \|_o \ \mathrm{Var}(\Phi - T^{(m)}\Phi)$$

In view of (2.5) and (2.3) we have therefore

$$\mathrm{Var}(\mathcal{M}_\Lambda^m - \mathcal{M}_\Lambda^m T^{(m)})\Phi$$

$$\leq \text{const} \left[\sum_{k=1}^{m} \| |\mathcal{M}'_\Lambda|^{m-k} 1 \|_o \cdot \| (\mathcal{M}'_{\Lambda\varepsilon})^{k-1} 1 \|_o \mathrm{Var}\,\Phi + \| |\mathcal{M}'|^m 1 \|_o \mathrm{Var}\,\Phi \right] .$$

Using (1.3) and (2.6), this yields

$$\| \mathcal{M}_\Lambda^m - \mathcal{M}_\Lambda^m T^{(m)} \| \leq \text{const. } (m+1)\ (R' e^{\delta/2})^m$$

$$\leq \text{const } (R' e^{\delta})^m \ .$$

From this (2.4) follows, concluding the proof of Theorem 1.1.

2.5. Proof of Theorem 1.2.

If $\varphi_\omega \geq 0$, we have

$$\lim_{m\to\infty} \| \mathcal{M}^m \|^{1/m} = \lim_{m\to\infty} \| |\mathcal{M}|^m \|^{1/m}$$

$$\geq \lim_{m\to\infty} (\| |\mathcal{M}|^m 1 \|)^{1/m} \geq \lim_{m\to\infty} (\| |\mathcal{M}|^m 1 \|_o)^{1/m} = R \tag{2.7}$$

so that the spectral radius of $\mathcal{M}$ is $\geq R$. In particular if $R' \leq R$, Theorem 1.1. (a) shows that the spectral radius of $\mathcal{M}$ is equal to R .

We assume now that $R' < R$ and prove (following [8] Section 4.9) that R is an eigenvalue of $\mathcal{M}$, and has an eigenfunction $\Phi_R \geq 0$. We may write

$$1 = \Psi + \sum_j \Psi_j \tag{2.8}$$

where, for each j , λ_j is an eigenvalue of $\mathcal{M}$ with $|\lambda_j| = R$, and Ψ_j is in the corresponding generalized eigenspace; Ψ is such that

$$\lim_{m\to\infty} \frac{\| \mathcal{M}^m \Psi \|}{\tilde{\lambda}^m} = 0 \tag{2.9}$$

with $0 < \tilde{\lambda} < R$. In view of (2.7) ,

$$\lim_{m\to\infty} \frac{1}{m} \log \|\mathcal{M}^m 1\| = \log R$$

and therefore the Ψ_j do not all vanish. Write the restriction of $\mathcal{M}$ to the generalized eigenspaces corresponding to the λ_j in Jordan normal form. It is then readily seen that there is an integer $k \geq 0$ such that

$$\lim_{m\to\infty} \frac{1}{\lambda_j^m \cdot m^k} \mathcal{M}^m \Psi_j = \Phi_j \tag{2.10}$$

and

$$\mathcal{M}\Phi_j = \lambda_j \Phi_j$$

for all j, and $\Phi_j \neq 0$ for some j. From (2.8) we get

$$0 \leq \frac{\mathcal{M}^m 1}{R^m} = \frac{\mathcal{M}^m \Psi}{R^m} + \sum_j \left(\frac{\lambda_j}{R}\right)^m \frac{\mathcal{M}^m \Psi_j}{\lambda_j^m} .$$

Using (2.9) and (2.10) this gives

$$\sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi_j \geq -\varepsilon(m) \tag{2.11}$$

where $\varepsilon(m) \to 0$ when $m \to \infty$. Note that the sum is finite, and that $|\lambda_j/R| = 1$ for all j. Therefore (2.11) can hold only if R is an eigenvalue, say $R = \lambda_o$, while $\Phi_o \geq 0$ and Φ_o does not vanish identically.

A. Appendix

We call *compact interval* a set homeomorphic to $[0,1] \subset \mathbb{R}$. A 1-*dimensional complex* $\tilde{X}$ will then be a finite union $\tilde{X} = \cup_{i\in I} \bar{X}_i$ of compact intervals which are disjoint except possibly for their endpoints. In particular, we may take $\tilde{X}$ to be the interval $[0,1]$ or the circle $S^1 = \mathbb{R}/\mathbb{Z}$. We denote by $\tilde{\mathfrak{B}}$ the Banach space of functions $\varphi : X \mapsto \mathbb{C}$ with bounded variation, with the norm

$$\|\varphi\| = \|\varphi\|_o + \operatorname{var} \varphi$$

where $\operatorname{var} \varphi = \sum_i \operatorname{var}(\varphi|\bar{X}_i)$, i.e., $\operatorname{var} \varphi$ is the sum of the variations on the compact intervals $\bar{X}_i$.

Choose now $X_i \subset \bar{X}_i$ such that (X_i) is a partition of $\tilde{X}$, and identify the X_i with disjoint subsets of $\mathbb{R}$. Writing $\cup X_i = X$, we can identify $\tilde{\mathfrak{B}}$ and $\mathfrak{B}(X)$ and we see that the norms $\|\cdot\|$ and $\operatorname{Var} \cdot$ are equivalent. Theorems 1.1 and 1.2 apply therefore directly to the situation where X is replaced by $\tilde{X}$ and $\mathfrak{B}$ by $\tilde{\mathfrak{B}}$.

We may say that $f : \tilde{X} \mapsto \tilde{X}$ is *piecewise monotone* if (for suitable choice of the X_i), each $f|X_i$ is a homeomorphism $X_i \mapsto fX_i$. Let also $g : \tilde{X} \mapsto \mathbb{C}$ be given, and suppose that g has bounded variation. Define

$$\mathfrak{M}\, \Phi(x) = \sum_{y:fy=x} g(y)\Phi(y)$$

$$\mathfrak{M}'\Phi(y) = g(y)\Phi(fy) .$$

These operators are the same as those defined in section 1 if we make a suitable choice of data. Let indeed $\Omega = I \times I$, let $V_{(i,j)} = X_i \cap fX_j$, let $\psi_{(i,j)}$ be the local inverse $V_{(i,j)} \mapsto X_j$ of f , let finally $\varphi_{(i,j)} = g \circ \psi_{(i,j)}$ on $V_{(i,j)}$ and 0 on $X \backslash V_{(i,j)}$. The properties (i), (ii), (iii) of section 1 are then satisfied, and the operators $\mathfrak{M}$, $\mathfrak{M}'$ are the same as those defined by (1.2) . Note that in the present situation

$$R' = \lim_{m \to \infty} \sup_x |g(f^{m-1}x) \dots g(fx)\ g(x)|^{1/m} .$$

With the above notation we formulate now and prove a special case of Conjecture 1.4.

A.1. <u>Proposition</u>. *Let* $f : \tilde{X} \mapsto \tilde{X}$ *be a piecewise monotone map, and* $g : \tilde{X} \mapsto \mathbb{C}$ *a piecewise constant function. Specifically, we assume that the finite partition of* $\tilde{X}$ *in connected sets* X_i *is such that* f *is monotone and* g *constant on each* X_i *. If* $x \in X_i$ *, let* $\varepsilon(x) = \pm 1$ *depending on* *whether* $f|X_i$ *is increasing or decreasing (arbitrary choice if* X_i *is a point). Define*

$$\mathcal{M}^{\varepsilon}\Phi(x) = \sum_{y:fy=x} \varepsilon(y)g(y)\Phi(y)$$

$$\mathrm{Fix}^{-} f^m = \left\{ x \in X : f^m x = x \text{ and } \prod_{k=0}^{m-1} \varepsilon(f^k x) = -1 \right\}$$

$$\zeta^{-}(z) = \exp 2 \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}^{-} f^m} \prod_{k=0}^{m-1} g(f^k x) .$$

Then , $\zeta^{-}(z)$ *is meromorphic for* $|z| < 1/R'$ *, and the order of* ζ^{-} *at* λ *is* $n^{\varepsilon}(\lambda) - n(\lambda)$ *where* $n(\lambda)$ *and* $n^{\varepsilon}(\lambda)$ *are the multiplicities of* λ *as eigenvalues of* $\mathcal{M}$ *and* $\mathcal{M}^{\varepsilon}$ *respectively.*

If $g = 1$, we see that $\zeta^{-}(z)$ is meromorphic for $|z| < 1$ as noted earlier by Milnor and Thurston [5] (see also Preston [7] for this theory).

We shall obtain a proof of the Proposition by following the analysis of Baladi and Keller [1]. First, we "double" a countable set of points of X (preimages of the endpoints of X_i for iterates of f). In this manner, X is replaced by an ordered set $\hat{X}$, and f by a continuous map $\hat{f} : \hat{X} \mapsto \hat{X}$. The sets X_i are replaced by $\hat{X}_i$, and $(\hat{X}_i)$ is now a partition of $\hat{X}$ into compact sets. If we extend g by continuity, the operators $\mathcal{M}, \mathcal{M}^{\varepsilon}$ are essentially unchanged by the above replacements, and ζ^{-} is changed at most trivially.

Next, we introduce an equivalence relation $\sim$ on $\hat{X}$, where $x \sim y$ means that $\hat{f}^k x$ and $\hat{f}^k y$ belong to the same $\hat{X}_{i(k)}$ for all $k \geq 0$. The equivalence classes are either points or closed intervals (a countable number of them). Let $\tilde{X}$ be the quotient $\hat{X}/\sim$ (obtained by collapsing a countable family of intervals to points). The map $\hat{f}$ passes to the quotient, giving a map $\tilde{f} : \tilde{X} \mapsto \tilde{X}$, and $\tilde{f}$ is positively expansive by construction. Note that g , being constant on each X_i , corresponds to a function $\tilde{g}$ on $\tilde{X}$. Let $\tilde{\zeta}$ and $\tilde{\zeta}^{\varepsilon}$ be defined by

$$\tilde{\zeta}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, \tilde{f}^m} \prod_{k=0}^{m-1} \tilde{g}(\tilde{f}^k x)$$

$$\tilde{\zeta}^{\varepsilon}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, \tilde{f}^m} \prod_{k=0}^{m-1} \tilde{\varepsilon}\, \tilde{g}(\tilde{f}^k x) \ .$$

Then

$$\zeta^{-}(z) = \tilde{\zeta}(z)/\tilde{\zeta}^{\varepsilon}(z) \ .$$

We know by [1] that the orders of $\tilde{\zeta}$ and $\tilde{\zeta}^{\varepsilon}$ at λ^{-1} are minus the multiplicities of λ as eigenvalue of $\tilde{\mathfrak{M}}$ and $\tilde{\mathfrak{M}}^{\varepsilon}$ respectively. It remains to show that these are also the multiplicities as eigenvalues of $\mathfrak{M}$ and $\mathfrak{M}^{\varepsilon}$. If Φ is an eigenfunction of $\mathfrak{M}$ to the eigenvalue λ, with $|\lambda| > R'$, we have

$$\Phi = \lambda^{-n}\, \mathfrak{M}^m\, \Phi \ .$$

On an equivalence class for $\sim$, the variation of $\mathfrak{M}^m \Phi$ is $\leq$ const . $(R'e^{\delta})^m$. var Φ, and Φ is thus constant. Therefore, Φ corresponds to an eigenfunction of $\tilde{\mathfrak{M}}$. The converse is clear. A similar argument applies to generalized eigenfunctions. Therefore, the multiplicity of λ is the same with respect to $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$, and similarly for $\mathfrak{M}^{\varepsilon}$ and $\tilde{\mathfrak{M}}^{\varepsilon}$. This concludes the proof of the proposition.

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