DYNAMICAL ZETA FUNCTIONS:

WHERE DO THEY COME FROM AND WHAT ARE THEY GOOD FOR ? *

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Abstract: The properties and usefulness of dynamical zeta functions associated with maps and flows are discussed, and they are compared with the more traditional number-theoretic zeta functions.

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0 Dynamical zeta functions

Let (M, f) be a dynamical system, i.e., M is a space and $f : M \to M$ a map. Let also $g : M \mapsto \mathbb{C}$ be a function and consider the formal power series

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x)$$

where $\mathrm{Fix}\, f^m$ is the set of fixed points of $f^m = f \circ \cdots \circ f$. The *dynamical zeta function* $\zeta(z)$ is a natural object from a combinatorial point of view, as is the corresponding zeta function for a flow (f^t).*)Under suitable conditions, the sum over $\mathrm{Fix}\, f^m$ in the definition of ζ converges, and $\zeta(z)$ is a meromorphic function in a certain domain. In this talk we shall discuss dynamical zeta functions in a relatively informal way. We shall try to explain where they come from and what they are good for. We shall not try to show how their properties are proved, but note that some properties of zeta functions are notoriously hard to prove ! In a sense the study of dynamical zeta functions is part of the theory of dynamical systems, but we shall see that it is also intimately related to statistical mechanics.

1 Where do they come from ?

1.1 Riemann zeta function

At the beginning there is the Riemann zeta function:

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\ &= \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1} . \end{aligned}$$

The representation as a product over primes reflects the unique factorization of integers into primes, and is due to Euler. Of course Euler is earlier than Riemann, but the function is associated with Riemann rather than Euler because of Riemann's detailed work on the analytic properties of ζ. In particular, writing

$$\xi(s) = \pi^{-s/2}\, \Gamma(s/2)\zeta(s)$$

we have the *functional equation* $\xi(s) = \xi(1 - s)$. The Riemann zeta function has been used by Hadamard and de la Vallée-Poussin to prove the *prime number theorem* (if $\pi(x)$ is the number of primes $\leq x$, then $\pi(x) \sim x/\log x$). In view of other arithmetic applications, various other number-theoretic zeta functions have been introduced after that of Riemann. For a survey of this fascinating field we refer to the article *Zeta functions* in the Encyclopedic Dictionary of Mathematics [15]. This brings us somehow to the next item.

*) At this level of generality, these objects were first introduced in Ruelle [21], [22], [23].

1.2 Weil conjectures

Let V be a nonsingular projective algebraic variety of dimension n over a finite field k with q elements. The variety V is thus defined by homogenous polynomial equations with coefficients in k for $n+1$ variables $x_0, \ldots, x_n$. These variables are in the algebraic closure $\bar{k}$ of k, and constitute the homogeneous coordinates of a point of V. The variety V is invariant under the *Frobenius map* $F : (x_0, \ldots, x_n) \mapsto (x_0^q, \ldots, x_n^q)$. Arithmetic considerations lead one to introduce a zeta function which counts the points of V with coordinates in the different finite extensions of the field k, or equivalently points of V which are fixed under F^m for some $m \geq 1$:

$$Z(z, V) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \text{ card Fix } F^m .$$

Conjectures made by Weil [34] about this function led to a lot of work by many people including A. Weil himself, B. Dwork, A. Grothendieck and finally P. Deligne [5] who concluded the proof of the conjectures. Here are (roughly) the results.

The function Z is rational:

$$Z = \prod_{\ell=0}^{2n} P_\ell(z)^{(-1)^{\ell+1}} .$$

The zeros of the polynomial P_ℓ have absolute value $q^{-\ell/2}$. The P_ℓ have a cohomological interpretation: P_ℓ is roughly the characteristic polynomial associated with the action of F on ℓ-dimensional cohomology.

1.3 Selberg zeta function

Given a compact surface of curvature -1, the Selberg zeta function (see [30]) is related to

$$\zeta(s) = \prod_{\gamma \in P} \left(1 - e^{-s\,T(\gamma)}\right)^{-1} \tag{1}$$

where P is the set of closed geodesics and $T(\gamma)$ is the length of γ. Preferably we interpret P as the set of (minimal) periodic orbits γ for the geodesic flow, and $T(\gamma)$ is the period of γ.

In fact the Selberg zeta function is $Z(s) = \left[\prod_{n=0}^{\infty} \zeta(s+n)\right]^{-1}$. It is an entire analytic function, and satisfies a functional equation. Furthermore its zeros can be analysed, and the "nontrivial zeros" are related to the eigenvalues of the Laplace-Beltrami operator on the compact surface of curvature -1 from which we started. (This yields a relation between the "classical mechanics" of the geodesic flow and the "quantum mechanics" of the Laplace operator). The localization of the "nontrivial zeros" corresponds to the *Riemann hypothesis* which asserts that the nontrivial zeros of the Riemann zeta function are located on the line $\text{Re}s = \frac{1}{2}$ (this, of course, remains unproved).

1.4 Dynamical zeta functions

Inspired by the zeta function of an algebraic variety over a finite field, Artin and Mazur [1] defined

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{card} \operatorname{Fix} f^m \ . \tag{2}$$

for a C^1 diffeomorphism f of a compact manifold, and proved that, for a dense set of such diffeomorphisms, ζ has nonvanishing radius of convergence.

The Artin-Mazur zeta function counts periodic points for a diffeomorphism, and the same definition was adopted later by Milnor and Thurston to count periodic points for a piecewise monotone map of the interval.

Instead of the discrete dynamical system generated by a map f, let us consider a flow (f^t). Denoting by $T(\gamma)$ the minimal period of a periodic orbit γ, it is natural to define a zeta function by (1). This form corresponds in fact closely to the Euler product formula for the Riemann zeta function.

Following Artin-Mazur and Selberg we have thus obtained natural definitions (2) and (1) for dynamical zeta functions associated with maps or flows. Knowledge of equilibrium statistical mechanics induces one however to make more general definitions, as follows.

A. Map case

Given a map $f : M \to M$ and a function $g : M \to$ matrices, we let

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \operatorname{Tr} \prod_{k=0}^{m-1} g\left(f^k x\right) \ .$$

In the particular case where $g = e^A$, $A : M \to \mathbb{C}$, we have

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \exp \sum_{k=0}^{m-1} A\left(f^k x\right) .$$

We have, in these formulae, replaced simple counting by counting with weights, which is the essence of equilibrium statistical mechanics.

B. Flow case

Given a flow $(f^t) : M \to M$ and a function $B : M \to \mathbb{R}$ or $\mathbb{C}$, we let

$$\zeta(s) = \prod_{\gamma \in P} \left[1 - \exp\left(-\int_0^{T(\gamma)} \left(s - B(f^t x_\gamma)\right) dt\right)\right]^{-1} \ . \tag{3}$$

There are extensions of this formula, which we shall not consider here.

The zeta functions for maps and flows look different, but the zeta function for a *special flow* (suspension) reduces to the form given for a map. In fact, starting from (1) one arrives not at (2) but at the more general form given above, which makes such a generalization more or less unavoidable.

The above generalizations were introduced in [21], [22], based on the ideas of statistical mechanics. Let me indicate another natural way of obtaining the general expression for the dynamical zeta function of a map.

Let S be the set of functions $X : M \to \mathbb{Z}_+$ such that $X = X \circ f$ and $|X| = \sum_{x\in M} X(x) < \infty$ (i.e., X is f-invariant and vanishes except at finitely many points). Each $X \in S$ has a unique decomposition

$$X = \sum_{\gamma} n_\gamma \, X_\gamma$$

where X_γ is the characteristic function of the f-periodic orbit γ, and the n_γ are integers ≥ 0 (only a finite number being > 0). This unique (additive) decomposition is analogous to the unique (multiplicative) decomposition of an integer into primes, and the analogy with the Riemann zeta function suggests to define

$$\zeta(z) = \sum_{X\in S} z^{|X|} = \prod_{\gamma\in P} \left(1 - z^{|\gamma|}\right)^{-1}$$

(where $-\log z$ replaces s, and $e^{|X|}$ replaces n), which is precisely the definition (2). But number theory suggest to consider more generally zeta functions of the form

$$\zeta(z) = \sum_{X\in S} g(X)\, z^{|X|}$$

where $g(X+Y) = g(X).g(Y)$, at least when X and Y are "relatively prime" (i.e., $X \to g(X)$ is a "multiplicative function"). This allows to represent ζ as a Euler product over prime periodic orbits. Given $g : M \to$ matrices, define

$$g(X) = \prod_{\gamma} \operatorname{Tr} \prod_{k} g\left(f^k x_\gamma\right)$$

where $X = \sum_{\gamma} n_\gamma \, X_\gamma$, x_γ is an arbitrary point of γ and the product over k extends from 0 to $n_\gamma\, |X_\gamma| - 1$; then $X \to g(X)$ is indeed multiplicative. In particular, if $g = e^A$, $A : M \to \mathbb{C}$, we have

$$\sum_{X\in S} g(X)\, z^{|X|} = \prod_{\gamma\in P} \left(1 - g(X_\gamma)\, z^{|\gamma|}\right)^{-1} .$$

2 What are they good for?

As the case of the Selberg zeta function shows, there are some dynamical zeta functions which have number-theoretic interest. This is an active area of research, which we can only mention in passing, see Sarnak [28], [29], Fried [6], Series [31], Pollicott [19], Cartier and Voros [4], Mayer [12], [13], [14], etc). In general, however, the dynamical zeta functions are different from those having number-theoretic interest, and are for instance not known to satisfy a functional equation. This is compensated by the fact that dynamical zeta functions have relations with statistical mechanics (entropy, pressure, Gibbs states, equilibrium states) while such relations are not known for number-theoretic zeta functions.

I shall now discuss some selected examples, the selection corresponding of course to my own interests. As a background reference to much of the material below, see the excellent monograph of Parry and Pollicott [17].

2.1 Expanding/contracting maps (See Tangerman [33], Haydn [9], Ruelle [25], [26])

Let $f : m \to M$ be of class C^r (M a compact Riemann manifold, $r > 0$ not necessarily integer), and assume that f is expanding by a factor $> \theta^{-1}$ (for some $\theta \in (0,1)$). Consider the *transfer operator* $\mathcal{L}_0$ such that

$$\mathcal{L}_0 \, \Phi(x) = \sum_{y : fy = x} g(y) \, \Phi(y)$$

with g of class C^r. Then one can define a *Fredholm determinant* $\det(1 - z\mathcal{L}_0)$ analytic for $|z| < \theta^{-r} \, \exp(-P(\log|g|))$, where the *pressure* $P(A)$ is the max of $h(\rho) + \rho(A)$ taken over ergodic measures ρ, $h(\rho)$ being the entropy. There is a similar Fredholm determinant associated with transfer operators $\mathcal{L}_k$ acting on k-forms. The zeta function (2) is of the form

$$\zeta(z) = \prod_{k \geq 0} \left(\det(1 - z\mathcal{L}_k)\right)^{(-1)^{k+1}}$$

and meromorphic for $|z| < \theta^{-r} \, \exp(-P(\log|g|))$. The Hölder version of this result is due to Haydn, the C^k version (in somewhat weaker form) to Tangerman. These results can be extended in various ways, in particular to the definition of Fredholm determinants for a *linear* class of operators [26]. This extension of the theory of Fredholm (and Grothendieck [7]) to kernels with δ-singularities may be expected to have applications in physics. For a different type of applications see [2] and references given there.

2.2 Hyperbolic maps

We discuss here an Axiom A diffeomorphism f in the sense of Smale [32], restricted to a basic set K. We assume that the contraction (expansion) in the contracting (expanding) direction is by a factor $< \theta (> \theta^{-1})$ with $0 < \theta < 1$. The "counting" zeta function

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \text{ card } Fix \, f^m$$

is rational, as conjectured by Smale [32], and proved by Guckenheimer [8] and Manning [11], but no general cohomological interpretation is known.

If A is a real α-Hölder continuous function, the inverse

$$\frac{1}{\zeta(z)} = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \exp \sum_{k=0}^{m-1} A(f^k x)$$

has a zero at $e^{-P(A)}$ and converges for

$$|z| \leq e^{-P(A)} \theta^{-\alpha/2} .$$

This extends earlier results of Ruelle [24] and Pollicott [18], and can be proved by a method due to Haydn [9].

2.3 Hyperbolic flows

We consider an Axiom A flow in the sense of Smale [32], restrict it to a basic set, and assume mixing. (An example is the geodesic flow on a compact Riemann manifold with sectional curvature < 0 everywhere, non constant in general). Given a real α-Hölder function B, we have defined the zeta function $\zeta(s)$ by (3). One can show that this function is analytic for $Re\ s > P(B)$, where $P(B)$ is the "pressure of the function B with respect to the flow", $\zeta(s)$ has a simple pole at $P(B)$, no other pole or zero with $Re\ s = P(B)$, and is meromorphic for $Re\ s > P(B) - \frac{1}{2} \log\ \theta$. (This is an extension of earlier results of Ruelle, and Pollicott [20]).

Parry and Pollicott [16], considering the case $B = 0$, realized that they could apply to the study of the distribution of the periods $T(\gamma)$ the same method that is used with the Riemann zeta function to prove the prime number theorem. (This method is based on the Wiener-Ikehara Tauberian theorem, see for instance [10]). The result is

$$\begin{aligned} \pi(x) &= (\#\text{closed orbits of minimal period } \leq x) \\ &\sim \frac{e^{hx}}{hx} \end{aligned}$$

where h is the topological entropy $P(0)$ of the flow. In the special case of the geodesic flow on a manifold of negative curvature one recovers an earlier result of Margulis on the distribution of the lengths of closed geodesics.

2.4 Piecewise monotone maps of the interval

Baladi and Keller [3] have obtained for the zeta function of these maps a result in the same spirit as for hyperbolic maps.

2.5 Open problems

One also expects analyticity results for zeta functions associated with rational maps of the Riemann sphere, see [27].

A fascinating possibility, which is probably a red hering, is that the Lee-Yang circle theorem may be somehow connected with zeta functions (see [23], the idea has later also been hit upon by B. Julia).

An unexplored topic is braids. Let $N(n)$ be the number of inequivalent braids (on k pieces of string) for which the minimal number of crossings is n. It is easily seen that

$$H = \lim_{n\to\infty} \frac{1}{n} \log N(n)$$

exists, and is > 0 if $k \geq 3$. This suggests doing statistical mechanics with braids (the ideas of quantum groups provide interactions) and one can also study zeta functions. I do for instance not know if one can replace $N(n)$ by a "number of crossings with periodic boundary conditions" and retain the limit H.

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