THERMODYNAMIC FORMALISM FOR MAPS SATISFYING POSITIVE EXPANSIVENESS AND SPECIFICATION

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Dedicated to Michael Fisher

Abstract. The relations between equilibrium state, Gibbs state, and eigenvector of the (adjoint) transfer operator are described for maps satisfying positive expansiveness and specification. In particular we show how the variational principle defining an equilibrium state can be converted into eigenvalue equations for the transfer operator and its adjoint. The results presented here are largely based on the work of N.T.A. Haydn and the author relating equilibrium and Gibbs state for homeomorphisms satisfying expansiveness and specification.

Introduction and definitions.

Let M be a compact space, $f : M \mapsto M$ a continuous map, and $A : M \to \mathbb{R}$ a function (typically, but not necessarily continuous). There are several probability measures on M that can be associated with this setup. The relations between these measures were studied by Haydn and Ruelle [6] *) under the assumption that M is metrizable, and f is a homeomorphism satisfying expansiveness and specification. In the present paper we consider the closely similar case of a map f satisfying positive expansiveness and specification. Several results closely related to those of [6] hold here also (and we can mostly refer to [6] for the proofs). The novelty is the relation obtained with eigenfunctions of the transfer operator **) $\mathcal{L}$ acting on functions and defined by

$$\mathcal{L}\Phi(x) = \sum_{y:fy=x} e^{A(y)}.\Phi(y),$$

*) For earlier results see Haydn [5].

**) Transfer operators originate in statistical mechanics, see Ruelle [9]. They are sometimes called Perron-Frobenius operators, but there is no historical justification for this terminology.

and its adjoint $\mathcal{L}^*$ acting on measures. One recovers well-known results if (M,f) is a shift of finite type and A is Hölder continuous (see Bowen [3], Ruelle [10]), but the treatment presented here is more general and natural, and can probably serve as a basis for wider generalizations.

In Section 1 we list the necessary definitions. In Section 2 we state the main results. Sections 3 and 4 are devoted to the proofs.

1. Generalities and definitions.

Let M be compact metrizable, and choose a metric d compatible with the topology. A continuous map $f : M \to M$ is *positively expansive* if there is $\epsilon > 0$ (called an expansive constant) such that

$$d(f^k x, f^k y) \leq \epsilon \text{ for all } k \geq 0$$

implies $x = y$. If $0 < \delta \leq \epsilon$, then δ is again an expansive constant.

Define the "Bowen ball"

$$B_x(\epsilon, n) = \{y \in M : d(f^k x, f^k y) \leq \epsilon \text{ for } k = 0, \ldots, n-1\}.$$

We say that x, y are (ϵ, n)-*close* if $y \in B_x(\epsilon, n)$. If f is positively expansive with expansive constant ϵ, then, for any $\delta > 0$, we may choose n such that

$$(y \in B_x(\epsilon, n)) \Rightarrow d(x, y) \leq \delta$$

(proof : by compactness). A subset E of M is (ϵ, n)-*separated* if distinct points of E are never (ϵ, n)-close; such a set E is necessarily finite.

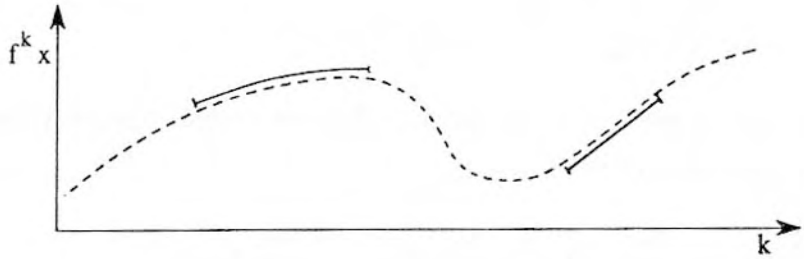
We say that f satisfies *specification* if for every $\epsilon > 0$ there is an integer $p = p(\epsilon) > 0$ such that, given ℓ points $x_1, \ldots, x_\ell \in M$ and integers $n_1, \ldots, n_\ell > 0$, $p_1, \ldots, p_\ell \geq p$, there exists $z \in M$ such that

$$d(f^{m(j-1)+i} z,\ f^i x_j) \leq \epsilon$$

for $i = 0, \ldots, n_j$ and $j = 1, \ldots, \ell$, where $m(j) = n_1 + p_1 + \ldots + n_j + p_j$.

If f is positively expansive, it suffices to verify the specification property for one choice of ϵ such that ϵ is an expansive constant. We may then say that f satisfies positive expansiveness and specification with the constants ϵ and p.

It is convenient to picture an orbit of f by plotting $f^k x$ as a function of k :



Positive expansiveness implies that, if two orbits are ϵ-close on $[\ell, \ell + n)$ for suitable $n = n(\delta)$, they are δ-close at ℓ. Specification means that one can interpolate between pieces of orbits on intervals separated by at least p.

Given $A : M \to \mathbb{R}$, let

$$K_A(\delta, n) = \sup\left\{\left|\sum_{k=0}^{n-1} [A(f^k x) - A(f^k y)]\right| : x, y \text{ are } (\delta, n)\text{-close}\right\}.$$

Choosing an expansive constant ϵ, we write

$$K_A = \sup_n K_A(\epsilon, n)$$

and define

$$\mathcal{V} = \{A : K_A < \infty\}$$

$\mathcal{V}$ is a Banach space for the norm $A \mapsto \|A\|_0 + K_A$ (where $\|A\|_0 = \sup_n |A(x)|$), and different choices of expansive constants ϵ give equivalent norms. (Note that the elements of $\mathcal{V}$ are not assumed to be continuous functions, but are bounded).

Let $\mathbf{E}$ be the set of probability measures on M; $\mathbf{E}$ is a convex weakly compact subset of the dual $\mathcal{C}(M)^*$ of the Banach space of continuous functions on M (with the norm $\| \cdot \|_0$). The f-invariant probability measures form a closed convex subset $\mathbf{I}$ of $\mathbf{E}$. The *entropy* $h : \mathbf{I} \mapsto \mathbb{R} \cup \{\infty\}$ is an affine function (see e.g. Billingsley [1] for the definition). If f is positively expansive, then h is finite and u.s.c. (upper semicontinuous). The *pressure* $p : \mathcal{C}_{\mathbb{R}}(M) \to \mathbb{R} \cup \{\infty\}$ may be defined by

$$P(A) = \sup_{\sigma \in \mathbf{I}} (h(\sigma) + \sigma(A)) \tag{1.1}$$

and we say that $\rho \in \mathbf{I}$ is an *equilibrium state* for A if $h(\rho) + \rho(A) = P(A)$. If the entropy is u.s.c. (in particular if f is positively expansive) there exists an equilibrium state.

If $A \in \mathcal{V}$ there is $A_n \in \mathcal{C}_{\mathbb{R}}(M)$ (for each $n > 0$) such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} A \circ f^k - A_n \right\|_0 \leq \frac{1}{n} K_A.$$

For $\sigma \in \mathbf{I}$, the limit of $\sigma(A_n)$ when $n \to \infty$ exists. [This is easily checked if one notices that $\sigma(A_n)$ is bounded and that, if $n = km + r$,

$$\left| \sigma(A_n) - \frac{km}{n}\sigma(A_m) - \frac{r}{n}\sigma(A_r) \right| \leq \frac{k+2}{n} K_A \;].$$

We may thus *define*

$$\sigma(A) = \lim_{n \to \infty} \sigma(A_n)$$

and the map $\sigma \mapsto \sigma(A)$ is weakly continuous on $\mathbf{I}$. We may therefore also define the pressure $P(A)$ and equilibrium states for A when $A \in \mathcal{V} + \mathcal{C}_{\mathbb{R}}(M)$.

The measure-theoretic definition (1.1) of the pressure is equivalent (by Walters' theorem [12]) to a topological definition, of which we shall use the following special case :

If f is a positively expansive map, with expansive constant 2ϵ, the pressure is given by

$$P(A) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(A, E_n)$$

where E_n is any maximal (ϵ, n)-separated set, and

$$Z_n(A, E) = \sum_{x \in E} \exp \sum_{k=0}^{n-1} A(f^k x).$$

(We may take $A \in \mathcal{V} + \mathcal{C}_{\mathbb{R}}(M)$).

If furthermore f satisfies specification, and $A \in \mathcal{V}$, then

$$|\log Z_n(A, E_n) - nP(A)| \leq a \tag{1.2}$$

where a depends on A, ϵ, but not on n (see [2] Lemma 2, or [6] Proposition 1.5 (b)).

Let us now assume that $f : M \to M$ satisfies positive expansiveness. We say that $x, y \in M$ are (positively) *conjugate* if $\lim_{k \to \infty} d(f^k x, f^k y) = 0$ and that they are n-conjugate if $f^n x = f^n y$. The points x, y are conjugate if and only if they are n-conjugate for some n (by positive expansiveness). We say that (U, φ) is a *conjugating homeomorphism* if U is a compact subset of M and $\varphi : U \mapsto \varphi U$ is a homeomorphism such that φx is n-conjugate to x for some n. We say that a probability measure μ on M is a *Gibbs state* for $A \in \mathcal{V}$ if, for every conjugating homeomorphism (U, φ),

$$\frac{d\varphi\mu}{d\mu} = \exp \sum_{k=0}^{\infty} [A \circ f^k \circ \varphi^{-1} - A \circ f^k].$$

We say that μ is a *quasi-Gibbs state* for $A \in \mathcal{V}$ if there is $C > 0$ such that, for every (U, φ),

$$\frac{d\varphi\mu}{d\mu} \leq C \exp \sum_{k=0}^{\infty} [A \circ f^k \circ \varphi^{-1} - A \circ f^k].$$

For compact $U \subset M$ such that $f|U$ is injective we denote by f_U^{-1} the inverse of the homeomorphism $U \to fU$. Let us say that a probability measure μ is a *conformal state* for A if, for every compact U such that $f|U$ is injective, we have

$$f(\mu|U) = [\exp(A \circ f_U^{-1} - P(A)].\mu|(fU)$$

i.e.,

$$\frac{d(f\mu)}{d\mu} = \exp(A \circ f_U^{-1} - P(A)]$$

on fU. If U is compact and $f^n|U$ injective, this implies also

$$\frac{d(f^n\mu)}{d\mu} = \exp\left(\sum_{k=0}^{n-1} A \circ f^k \circ (f^n)_U^{-1} - nP(A)\right)$$

on f^nU. Different versions of the above definition of conformal states have been considered successively (among others) by Keane [7], Sullivan [11], Denker and Urbański [4]. We shall see later that, with our assumptions, conformal states coincide with Gibbs states.

In order to introduce the transfer operator $\mathcal{L}_A$ we shall now assume that A is a Borel function in $\mathcal{V}$ (hence bounded), and that f satisfies positive expansiveness and specification.

Given $X \in M$, let $Y_1, Y_2, \ldots$ be the preimages of X by $f : fY_i = X$. We have $d(Y_i, Y_j) > \epsilon$ by positive expansiveness, and therefore there are only a finite number of preimages, say k. There is at least one preimage (because $f^p M = M$, by specification and positive expansiveness, and therefore f is onto). We may choose $\delta > 0$ (by continuity of f, and compactness of M) such that if $fy = x$ and $d(x, X) \leq \delta$, then $d(y, Y_i) \leq \frac{\epsilon}{2}$ for some i. Therefore, there are closed subsets V_i of $B_X(\delta) = \{x : d(x, X) \leq \delta\}$ and continuous maps $\psi_i : V_i \to B_{Y_i}(\frac{\epsilon}{2})$ such that the $\psi_i(x)$ are the preimages of $x \in B_X(\delta)$. We have thus

$$\mathcal{L}_A\Phi(x) = \sum_{y:fy=x} e^{A(y)}\Phi(y) = \sum_{i=1}^{k} \chi_i(x). \exp A(\psi_i(x)).\Phi(\psi_i(x))$$

where χ_i is the characteristic function of V_i, and we assume that $d(x, X) \leq \delta$. This shows that if Φ is a bounded Borel function, then $\mathcal{L}_A\Phi$ is a bounded Borel function.

2.1. Lemma. *$\mathcal{L}_A$ maps the space of bounded Borel functions onto itself.*

Given a bounded Borel function Ψ, we have to show that there exists Φ such that $\mathcal{L}_A\Phi = \Psi$. We may assume that the support of Ψ is contained in $B_X(\delta)$ and write $\Psi = \sum_{i=1}^k \chi_i.\Psi_i$ where support $\Psi_i \subset V_i$. Take

$$\Phi = \begin{cases} [\exp(-A \circ \psi_i)].(\Psi_i \circ f) & \text{on} \quad \psi_i V_i \\ 0 & \text{on} \quad M \backslash \cup_i \psi_i V_i \end{cases}$$

Then

$$\mathcal{L}_A\Phi = \sum_i \chi_i.(\exp A \circ \psi_i).(\Phi \circ \psi_i)$$

$$= \sum_i \chi_i.\Psi_i = \Psi$$

as announced. □

If μ is a measure with support in $B_X(\delta)$ we write

$$\mathcal{L}_A^*\mu = \sum_{i=1}^k (\exp A).\psi_i(\chi_i.\mu)$$

where $\psi_i(\chi_i.\mu)$ is the direct image of the measure $\chi_i.\mu$ by the continuous map ψ_i (the dot indicates product of a measure by a function). We have thus

$$(\mathcal{L}_A^*\mu)(\Phi) = \mu(\mathcal{L}_A\Phi). \tag{1.3}$$

The restriction that $\operatorname{supp}\mu \subset B_X(\delta)$ is removed by linearity, preserving (1.3) for an arbitrary measure μ and bounded Borel function Φ. Since we may take Φ continuous, (1.3) determines completely $\mathcal{L}_A^*$, which is a bounded linear map on measures on M; $\mathcal{L}_A^*$ is injective in view of Lemma 2.1. If there is a positive measure μ such that $\mathcal{L}_A^*\mu = \lambda^*\mu$ with $\mu \neq 0$, then $\mathcal{L}_A$ defines an operator on $L^p(\mu)$ for $1 \leq p \leq \infty$. [Note that if $\frac{1}{p} + \frac{1}{q} = 1$, we have $|\mathcal{L}_A(\Phi\Psi)| \leq (\mathcal{L}_A|\Phi|^p)^{\frac{1}{p}}(\mathcal{L}_A|\Psi|^q)^{\frac{1}{q}}$. In particular, if $\mathcal{L}_A 1 \leq C^{q/p}$, we obtain $|\mathcal{L}_A\Phi|^p \leq C\mathcal{L}_A|\Psi|^p$, hence $\mu(|\mathcal{L}_A\Psi|^p) \leq C\lambda^*\mu(|\Phi|^p)$. The cases $p = 1, \infty$ are easy].

2. Results.

Our main results are contained in the following theorem.

2.1. **Theorem.** *Let M be a compact metric space, and $f : M \to M$ an continuous map satisfying positive expansiveness and specification. Given a Borel function $A \in \mathcal{V}$, the following hold*

(a) *There is a unique f-invariant quasi-Gibbs state ρ for A, and ρ is also the only equilibrium state for A.*

(b) *There is a unique Gibbs state μ for A, which is also the unique conformal state, and (up to a multiplicative constant) the only eigenvector of $\mathcal{L}_A^*$ acting on measures :*

$$\mathcal{L}_A^* \mu = \lambda^* \mu$$

Furthermore, $\lambda^* = \exp P(A)$ where $P(A)$ is the pressure of A.

(c) *There is a unique $\Phi \in L^1(\mu)$ such that $\rho = \Phi\mu$, and Φ is (up to a multiplicative constant) the only eigenfunctions ≥ 0 of $\mathcal{L}_A$ acting on $L^1(\mu)$:*

$$\mathcal{L}_A \Phi = \lambda \Phi, \qquad 0 \leq \Phi \in L^1(\mu).$$

Furthermore, $\lambda = \exp P(A)$, and $\log \Phi$ is essentially bounded.

(d) *Let $\Psi \in L^p(\mu)$, $1 \leq p < \infty$, and write $\Psi' = \Psi - \mu(\Psi) \cdot \Phi$. Then*

$$\lim_{n \to \infty} e^{-nP(A)} \mathcal{L}_A^n \Psi' = 0$$

in $L^k(\mu)$.(*)

3. Equilibrium, Gibbs, and quasi-Gibbs states.

In this section we discuss the parts of Theorem 2.1 that do not involve the transfer operator $\mathcal{L}_A$ or the operator $\mathcal{L}_A^*$. These parts are, basically, corollaries of the proofs in Haydn and Ruelle [6]. The main difference between the situation considered in [6] (homeomorphisms satisfying expansiveness and specification) and the situation considered here (maps satisfying positive expansiveness and specification) is that here the definition of a Gibbs state is not f-invariant.

From now on we assume that the system (M, f) satisfies positive expansiveness and specification, and we take 4ϵ to be an expansive constant. The main idea of the proofs of

(*) In particular $e^{-P(A)\mathcal{L}_A}$ is an almost periodic operator on $L^P(\mu)$ in the sense of Ljubich [8].

the results indicated below is of the following general nature. Using positive expansiveness and specification one compares two (ϵ, n)-separated sets and establishes a bounded-to-one map of the first to the second, where the "bounded" is independent of n. To an element x of the first set corresponds a weight

$$\exp \sum_{k=0}^{n-1} A(f^k x)$$

and if x is mapped to y this is replaced by

$$\exp \sum_{k=0}^{n-1} A(f^k y).$$

The points x, y are chosen to be (ϵ, n)-close and, since $A \in \mathcal{V}$, the ratio of the weights has a bound independent of n. In this manner one obtains inequalities with bounded coefficients which constitute the essential content of this section.

3.1. Theorem. *If $A \in \mathcal{V}$, the following conditions for the probability measure μ on M are equivalent.*

(a) *μ is a quasi-Gibbs state*

(b) *There is $c > 0$ such that for all $x \in M$, $n \geq 0$,*

$$\exp\left[\sum_{k=0}^{n-1} A(f^k x) - nP(A) - c\right] \leq \mu(B_x(\epsilon, n)) \leq \exp\left[\sum_{k=0}^{n-1} A(f^k x) - nP(A) + c\right]$$

where $P(A)$ denotes the pressure. (In proving (a) $\Rightarrow$ (b) one can choose c to depend on μ only through the constant C in the definition of the quasi-Gibbs state μ).

(c) *Given a quasi-Gibbs state ν, the measures μ and ν are equivalent, and $d\mu/d\nu$, $d\nu/d\mu$ are essentially bounded. (In proving (a) $\Rightarrow$ (c) one can choose the bounds on $d\mu/d\nu$, $d\nu/d\mu$ to depend on μ only through C.*

The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are proved like Proposition 2.1 and Corollary 2.2 of [6]; (c) $\Rightarrow$ (a) is immediate. □

3.2. Proposition. (a) *For every $A \in \mathcal{V}$ there exists a quasi-Gibbs state μ.*

(b) *If μ is quasi-Gibbs, then $f\mu$ is quasi-Gibbs.*

(c) *There exists an f-invariant quasi-Gibbs state ρ; ρ is ergodic and therefore unique.*

(a) is proved like Proposition 2.4 of [6] by taking for μ any (vague) limit when $n \to \infty$ of

$$\mu_n = \left[\sum_{x \in E_n} \exp \sum_{k=0}^{n-1} B(f^k x)\right]^{-1} \sum_{x \in E_n} (\exp \sum_{k=0}^{n-1} B(f^k x))\delta_x$$

where E_n is a maximal (ϵ, n)-separated set for each integer $n > 0$.

Given a quasi-Gibbs state μ and a n-conjugating homeomorphism (U, φ) we shall study the Radon-Nikodym derivative $d(\varphi f^k \mu)/d(f^k \mu)$, with $k > 0$. The graph of (U, φ) is a finite union of subsets of sets of the form

$$\Gamma = \{(x, y) \in B_X(\epsilon, n) \times B_Y(\epsilon, n) : x, y \text{ are } n\text{-conjugate}\}.$$

For our purposes we may thus assume that the graph of (U, φ) is Γ above.

By expansiveness and specification we know that $fM = M$. If $f^k v = y = \varphi x$ there is u such that $f^k u = x$, and $u \in B_v(\epsilon, k - p)$ if $k \geq p$. By positive expansiveness, the map $v \mapsto u$ is at most N^p-to-one if N is the cardinal of a maximal ϵ-separated set. Therefore

$$[\varphi(f^k \mu)](dy) \leq (f^k \mu)(dy) \ . \ C^* \exp \sum_{k=0}^{\infty} [A(f^k x) - A(f^k y)]$$

where $C^* = C.N^p \exp(K + 2p\|A\|_0)$. In particular, this proves (b).

The implication (a) $\Rightarrow$ (c) of Theorem 3.1 now proves that $f^k \mu = \Phi_k . \mu$ where $\Phi_k \in L^\infty(\mu)$, and $0 < \alpha < \Phi_k < \alpha^{-1}$ for some constant α independent of k. If Φ is a weak limit of $\frac{1}{r} \sum_{k=0}^{r-1} \Phi_k$, then $\Phi . \mu = \rho$ is an f-invariant quasi-Gibbs state. If ρ were not ergodic, then there would exist a continuous function $\Phi : M \to \mathbb{R}$ such that

$$0 < \rho(S) < 1$$

with

$$S = \{x : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(f^k x) \geq 0\}.$$

Let (U, φ) be a conjugating homeomorphism, then $\varphi(U \cap S) = \varphi U \cap S$. Therefore if χ is the characteristic function of S, then $\chi . \rho / \rho(\chi)$ and $(1 - \chi) . \rho / \rho(1 - \chi)$ would be mutually singular quasi-Gibbs states, in contradiction with the implication (a) $\Rightarrow$ (c) of Theorem 3.1. This concludes the proof of (c). $\square$

3.3. Theorem. *Let $A \in \mathcal{V}$.*

(a) *There is a unique invariant quasi-Gibbs state ρ for A, and ρ is also the only equilibrium state for A.*

(b) *There is a unique Gibbs state μ for A.*

(c) *There is a unique $\Phi \in L^1(\mu)$ such that $\rho = \Phi\mu$; in fact $\log \Phi \in L^\infty(\mu)$.*

Proposition 3.2 (c) shows that there is an f-invariant quasi-Gibbs state ρ, and ρ is ergodic. An invariant quasi-Gibbs state is an equilibrium state (as in the proof of Theorem 2.5 in [6]). To prove (a) we have thus only to extend Bowen's result (Lemma 8 of [2]) that there is a unique equilibrium state for A from the case of a homeomorphism satisfying expansiveness and specification to the case of a map satisfying positive expansiveness and specification. This is done by checking Bowen's proof, or by using the inverse limit construction to go from a map to a homeomorphism.

A Gibbs state μ is constructed from a quasi-Gibbs state as in the proof of Theorem 2.5 of [6]*). If μ, μ' were distinct Gibbs states, one would obtain mutually singular Gibbs states by normalizing $|\mu - \mu'| \pm (\mu - \mu')$, in contradiction with theorem 3.1 (c). This proves (b).

Since ρ and μ are both quasi-Gibbs states, (c) follows from Theorem 3.1 (c). □

4. Eigenfunctions of $\mathcal{L}_A^*$ and $\mathcal{L}_A$.

In this section we complete the proof of Theorem 2.1. We assume that (M, f) satisfies positive expansiveness and specification, and that $A \in \mathcal{V}$, A Borel.

4.1. Lemma

(a) *Write*

$$L_n(x) = \sum_{y: f^n y = x} \exp \sum_{k=0}^{n-1} A(f^k x).$$

Then

$$|\log L_n(x) - nP(A)| \leq b$$

where b depends on A but not on n, x.

*) Proof courtesy of Dennis Sullivan.

(b) *Suppose that 6ϵ is an expansive constant, and let E_n be a maximal (ϵ, n)-separated set. Define the measures*

$$\mu_n = \left[\sum_{x\in E_n} \exp\sum_{k=0}^{n-1} A(f^k x)\right]^{-1} \sum_{x\in E_n} \left(\exp\sum_{k=0}^{n-1} A(f^k x)\right)\delta_x$$

$$\sigma_x^n = [\exp nP(A)]^{-1} \sum_{y:f^n y=x} \left(\exp\sum_{k=0}^{n-1} A(f^k x)\right)\delta_y$$

where δ_x is the unit mass at x. There exists then $D > 1$, depending on ϵ but not on the choice of n, E_n and x, such that for any continuous function $\Phi \geq 0$,

$$\frac{1}{D}(\mu_n(\Phi) - d_n) \leq \sigma_x^n(\Phi) \leq D(\mu_n(\Phi) + d_n)$$

where $d_n = \max\{|\Phi(x) - \Phi(y)| : y \in B_x(6\epsilon, n)\}$.

The proof is of the same sort as that of Proposition 1.5 in [6]. The idea is to replace a measure $\sum c_i\delta_{x(i)}$ by $\sum c_i\delta_{y(i)}$ where positive expansiveness and specification imply that $y(i) \in B_{x(i)}(3\epsilon, n)$, say. Then $\sum c_i\delta_{y(i)}$ is replaced by $\sum d_i\delta_{y(i)}$ where $0 < c_i \leq d_i$. We can then conclude that $\sum c_i\delta_{x(i)}$ is vaguely close to a measure $\leq \sum d_i\delta_{y(i)}$. This yields the announced inequalities by integrating $\sum c_i\delta_{x(i)}$ and $\sum d_i\delta_{y(i)}$ with the function 1 or $\Phi \geq 0$. The details are left to the reader. □

4.2. Proposition. *We define $\mathcal{L}_A^*$, acting on measures, as in Section 1. The following conditions on a measure $m \neq 0$ are equivalent*

(a) $\frac{m}{m(1)}$ *is the Gibbs state μ*

(b) $\frac{m}{m(1)}$ *is a conformal state*

(c) m *satisfies*

$$\mathcal{L}_A^* m = \lambda^* m \tag{4.1}$$

(in which case we have in fact $\lambda^* = \exp P(A)$).

We first show that (c) implies (a) (and $\lambda^* = \exp P(A)$). Let (U, φ) be an n-conjugating homeomorphism. For the characterization of Gibbs states we may replace U by small subsets. We may thus assume that f^n restricted to $f^n U$ has inverses

$$\psi : f^n U \to U \qquad \psi' : f^n U \to \varphi U$$

such that

$$\varphi = \psi'\psi^{-1} = \psi' f^n|U.$$

If Φ is an arbitrary Borel function vanishing outside of φU, we have

$$(\mathcal{L}_A^n(\Phi\circ\varphi))(x) = \left[\exp\sum_{k=0}^{n-1} A(f^k\psi x)\right].\Phi(\psi' x)$$

$$= \left[\exp\sum_{k=0}^{n-1} A(f^k\psi' x)\right]\cdot\left[\exp\sum_{k=0}^{n-1}\left(A(f^k\varphi^{-1}\psi' x) - A(f^k\psi' x)\right)\right].\Phi(\psi' x)$$

$$= \left(\mathcal{L}_A^n\left(\left[\exp\sum_{k=0}^{n-1}(A\circ f^k\circ\varphi^{-1} - A\circ f^k)\right].\Phi\right)\right)(x)$$

i.e., we have the identity

$$\mathcal{L}_A^n(\Phi\circ\varphi) = \mathcal{L}_A^n\left(\left[\exp\sum_{k=0}^{n-1}\left(A\circ f^k\circ\varphi^{-1} - A\circ f^k\right)\right].\Phi\right) \tag{4.2}$$

Assume now that (4.1) holds. We have then also $\mathcal{L}_A^{*n}m = \lambda^{*n}m$. Using (1.3) and (4.2) we obtain

$$\lambda^{*n}m(\Phi\circ\varphi) = (\mathcal{L}_A^{*n}m)(\Phi\circ\varphi) = m(\mathcal{L}_A^n(\Phi\circ\varphi))$$

$$= m\left(\mathcal{L}_A^n\left(\left[\exp\sum_{k=0}^{n-1}(A\circ f^k\circ\varphi^{-1} - A\circ f^k)\right].\Phi\right)\right)$$

$$= \lambda^{*n}m\left(\left[\exp\sum_{k=0}^{n-1}\left(A\circ f^k\circ\varphi^{-1} - A\circ f^k\right)\right].\Phi\right)$$

We have $\lambda^*\neq 0$ by Lemma 2.1, and since Φ is arbitrary, this gives

$$(\varphi m)(dx) = \left[\exp\sum_{k=0}^{n-1}\left(A\circ f^k\circ\varphi^{-1} - A\circ f^k\right)\right].m(dx)$$

i.e., m satisfies the linear conditions defining a Gibbs state for A. The same is true for the real and imaginary parts of m, and the positive and negative part of those. This implies that m is a multiple of the unique Gibbs state μ. In particular $\mathcal{L}_A^*\mu = \lambda^*\mu$, hence $\lambda^{*n} = \mu(\mathcal{L}_A^n 1)$, hence $\lambda^* = \exp P(A)$ by Lemma 4.1 (a).

We now prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If μ is a Gibbs state we may define a measure $\mu^*\geq 0$ by

$$\mu^*|fU = [exp(-A\circ f_U^{-1})].f(\mu|U)$$

for every compact set U such that $f|U$ is injective. The consistency of this definition (when $fU = fU'$ with $U\neq U'$) follows from the definition of Gibbs states. This also implies that

μ^* is, up to normalization, again a Gibbs state; since there is only one Gibbs state we obtain

$$\mu^* = \frac{\mu}{e^P}$$

for some $P \in \mathbb{R}$. Therefore the condition on a probability measure ν that

$$f(\nu|U) = [\exp(A \circ f_U^{-1} - P)].\nu|fU \tag{4.3}$$

for every compact set U such that $f|U$ is injective is satisfied when ν is the Gibbs state μ. Suppose now that (4.3) holds; if Ψ has support in U we find that

$$\nu(\mathcal{L}_A\Psi) = \nu[(exp(A \circ f_U^{-1}).(\Psi \circ f_U^{-1})] = e^P.\nu(\Psi)$$

and this extends by linearity to arbitrary Ψ, so that

$$\mathcal{L}_A^*\nu = e^P\nu. \tag{4.4}$$

As already noted this implies (by Lemma 4.1 (a)) that $e^P = \exp P(A)$. Taking $P = P(A)$ in (4.3) we see that the Gibbs state μ is a conformal state for A, i.e., (a) $\Rightarrow$ (b). Furthermore (4.4) shows that (b) $\Rightarrow$ (c). □

4.3. Proposition. *The function Φ such that $\rho = \Phi.\mu$ is (up to a multiplicative constant) the only eigenfunction ≥ 0 of $\mathcal{L}_A$ acting on $L^1(\mu)$:*

$$\mathcal{L}_A\Phi = \lambda\Phi, \quad 0 \leq \Phi \in L^1(\mu).$$

Furthermore $\lambda = \exp P(A)$.

We already know from Theorem 3.3 (c) that there exists $\Phi \in L^1(\mu)$ such that $\rho = \Phi.\mu$, and that $\log\Phi \in L^\infty(\mu)$. Furthermore, we have defined $\mathcal{L}_A$ as an operator on $L^1(\mu)$ at the end of Section 1.

If $\Phi' \in L^1(\mu)$, the product $\Phi'.\mu$ is an invariant measure if and only if Φ' is proportional to Φ. [This is because $\Phi'.\mu = (\Phi'/\Phi).\rho$, and ρ is ergodic].

Writing $\exp P(A) = \lambda^*$ we have, for arbitrary $\Psi \in L^\infty(\mu)$,

$$\begin{aligned}
&\mu[(\mathcal{L}_A\Phi' - \lambda\Phi').\Psi] \\
=&\mu[\mathcal{L}_A(\Phi'.(\Psi \circ f)) - \lambda\Phi'.\Psi] \\
=&(\mathcal{L}_A^*\mu)(\Phi'.(\Psi \circ f)) - \lambda\mu(\Phi'.\Psi) \\
=&\lambda^*\mu(\Phi'.(\Psi \circ f)) - \lambda\mu(\Phi'.\Psi) \\
=&\lambda^*(f(\Phi'.\mu))(\Psi) - \lambda(\Phi'.\mu)(\Psi)
\end{aligned}$$

Therefore, $\mathcal{L}_A\Phi' = \lambda\Phi'$ in $L^1(\mu)$ is equivalent to

$$f(\Phi'.\mu) = \frac{\lambda}{\lambda^*}(\Phi'.\mu).$$

If $0 \leq \Phi' \in L^1(\mu)$, and if Φ' is not almost everywhere zero, this implies $\lambda/\lambda^* = 1$, so that

$$\mathcal{L}_A\Phi' = \lambda\Phi' \qquad \text{for some } \lambda$$

$$\Leftrightarrow f(\Phi'.\mu) = (\Phi'.\mu) \quad \text{and} \quad \lambda = \lambda^*.$$

The identity $f(\Phi'.\mu) = (\Phi'.\mu)$ means that $\Phi'.\mu$ is an f-invariant measure, and we have seen that this is equivalent to Φ' proportional to Φ. Therefore if $0 \leq \Phi' \in L^1(\mu)$ (and Φ' not identically 0),

$$\mathcal{L}_A\Phi' = \lambda\Phi' \qquad \text{for some } \lambda$$

$\Leftrightarrow$ Φ' is proportional to Φ, and $\lambda = \exp P(A)$. This concludes the proof. □

4.4. Proposition. *If $\Psi \in L^p(\mu)$, $1 \leq p < \infty$, we have $\Psi = \alpha\Phi + \Psi'$, where $\alpha = \mu(\Psi)$, and*

$$\lim_{n\to\infty} \|e^{-nP(A)}\mathcal{L}_A^n\Psi'\|_p = 0.$$

We may identify $\log\Phi$ with a bounded Borel function and define

$$B = A + \log\Phi - \operatorname{Log}\Phi \circ f - P(A).$$

We have then $B \in \mathcal{V}$ and

$$\mathcal{L}_B X = \frac{1}{\lambda\Phi}\ \mathcal{L}_A(\Phi.X)$$

so that $\mathcal{L}_B^*\rho = \rho$ and $\mathcal{L}_B 1 = 1$ in $L^p(\rho)$. It suffices now to show that if $X \in L^p(\rho)$, then $X = \alpha.1 + Y$, where $\alpha = \rho(X)$ and $\mathcal{L}_B^n Y \to 0$ in $L^p(\rho)$ when $n \to \infty$.

We write $\mathcal{L}_B = \mathcal{L}$, and note that $\mathcal{L}$ is norm reducing in $L^p(\rho)$.

[If $\frac{1}{p} + \frac{1}{q} = 1$, the Hölder inequality gives $(\mathcal{L}|Z|)(x) = \sum_y e^{B(y)}|Z|(y) = \sum_y e^{\frac{1}{q}B(y)}e^{\frac{1}{p}B(y)}|Z|(y) \leq (\sum_y e^{B(y)}(|Z|(y))^p)^{\frac{1}{p}}$ because $\sum_y e^{B(y)} = 1$. Therefore $\rho(|\mathcal{L}Z|^p)$ $\leq \rho((\mathcal{L}|Z|)^p) \leq \rho(\mathcal{L}|Z|^p) = \rho(|Z|^p)$].

Given $\Delta > 0$ and integers $n_1, n_2, \ldots > 0$, we may construct continuous functions $Y_1, Y_2, \ldots$ such that

(i) $\quad \|Y - Y_1\|_p \leq \frac{\Delta}{2}, \quad \|Y_{k+1} - \mathcal{L}^{n_k}Y_k\|_p \leq \frac{\Delta}{2^{k+1}}$

(ii) $\quad \rho(Y_k) = 0$

(iii) $\qquad \|Y_{k+1}\|_\infty \leq \|\mathcal{L}^{n_k} Y_k\|_\infty + \frac{\Delta}{2^{k+1}}$

[Appropriate $\mathcal{L}^{n_k} Y_k$ by a continuous function in the L^p-norm, then add a constant to achieve $\rho(Y_{k+1}) = 0$].

Define

$$\sigma_x^n = \sum_{y : f^n y = x} \left[\exp \sum_{k=0}^{n-1} B(f^k y) \right] .\delta_y$$

Then

$$(\mathcal{L}^n Z)(x) = \sigma_x^n(Z)$$

and the σ_x^n are probability measures for ρ-almost all x [because $\sigma_x^n \geq 0$ and $\sigma_x^n(1) = (\mathcal{L}^n 1)(x) = 1$ in $L^p(\rho)$]. Write

$$Y_k^\pm = \frac{1}{2}(|Y_k| \pm Y_k)$$

so that $Y_k^\pm \geq 0,\ Y_k = Y_k^+ - Y_k^-,\ \rho(Y_k^+) = \rho(Y_k^-)$.

Let now 8ϵ be an expansive constant and E_n be a maximal (ϵ, n)-separated set for each positive integer n. Let also μ be a limit when $n \to \infty$ of the measures

$$\mu_n = \left[\sum_{x \in E_n} \exp \sum_{k=0}^{n-1} B(f^k x) \right]^{-1} \sum_{x \in E_n} \left(\exp \sum_{k=0}^{n-1} B(f^k x) \right) \delta_x .$$

We know by the proof of Proposition 3.2 (a) that μ is a quasi-Gibbs state. In particular (Theorem 3.1 (c)) we have

$$C^{-1}\rho \leq \mu \leq C\rho$$

for some $C \geq 1$. The integers $n_1, n_2, \ldots$ are chosen successively so that

$$\frac{1}{2}\ \mu(Y_k^\pm) \leq \mu_{n_k}(Y_k^\pm) \leq 2\mu(Y_k^\pm)$$

and

$$d_{n_k} = \max\{|Y_k^\pm(x) - Y_k^\pm(y)| : y \in B_x(6\epsilon, n_k)\} \leq \frac{1}{4C}\ \rho(Y_k^\pm).$$

We have then

$$\frac{1}{4C}\ \rho(Y_k^\pm) \leq \mu_{n_k}(Y_k^\pm) - d_{n_k}$$

$$\mu_{n_k}(Y_k^\pm) + d_{n_k} \leq 4C\rho(Y_k^\pm).$$

In view of Lemma 4.1 (b) this implies

$$\frac{1}{4CD}\ \rho(Y_k^\pm) \leq \sigma_x^{n_k}(Y_k^\pm) \leq 4CD\rho(Y_k^\pm).$$

Therefore, writing $\beta = (4CD)^{-2}$, and using (ii), we have

$$\mathcal{L}^{n_k} Y_k^{\mp} \geq \beta \mathcal{L}^{n_k} Y_k^{\pm}$$

and since

$$\mathcal{L}^{n_k} Y_k = \mathcal{L}^{n_k} Y_k^{+} - \mathcal{L}^{n_k} Y_k^{-}$$

we obtain

$$-(1-\beta)\mathcal{L}^{n_k} Y_k^{-} \leq \mathcal{L}^{n_k} Y_k \leq (1-\beta)\mathcal{L}^{n_k} Y_k^{+}$$

hence

$$\|\mathcal{L}^{n_k} Y_k\|_\infty \leq (1-\beta)\|Y_k\|_\infty.$$

Together with (iii) this implies

$$\lim_{k\to\infty} \|Y_k\|_\infty = 0.$$

We may write

$$\mathcal{L}^{n_1+\ldots+n_k} Y = \mathcal{L}^{n_1+\ldots+n_k}(Y - Y_1)$$

$$+\mathcal{L}^{n_2+\ldots+n_k}(\mathcal{L}^{n_1} Y_1 - Y_2) + \ldots + (\mathcal{L}^{n_k} Y_k - Y_{k+1}) + Y_{k+1}$$

hence, using (i),

$$\|\mathcal{L}^{n_1+\ldots+n_k} Y\|_p < \Delta + \|Y_{k+1}\|_\infty.$$

Since $\|Y_{k+1}\|_\infty \to 0$ and Δ is arbitrarily small, we have $\|\mathcal{L}^{n_1+\ldots+n_k} Y\|_p \to 0$, hence

$$\lim_{n\to\infty} \|\mathcal{L}^n Y\|_p = 0$$

because $\mathcal{L}$ is norm reducing. □

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