SHARP DETERMINANTS AND KNEADING OPERATORS FOR HOLOMORPHIC MAPS

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October 1995

In an earlier paper [1] two of us studied *generalized transfer operators* $\mathcal{M}$ associated to maps of intervals of $\mathbf{R}$. In particular the analyticity of a *sharp determinant* $\mathrm{Det}^{\#}(1-z\mathcal{M})$ was analyzed on the basis of spectral properties of $\mathcal{M}$ and by relating $\mathrm{Det}^{\#}(1-z\mathcal{M})$ to the determinant of $1+\mathcal{D}(z)$ for a suitable *kneading operator* $\mathcal{D}(z)$.

It is a natural idea to try to replace monotone maps of intervals of $\mathbf{R}$ by holomorphic diffeomorphisms of domains of $\mathbf{C}$ and to transpose the results of [1] to this new situation. This program of transposition to the complex has been carried through in part, and the present preprint shows our results. In a suitable function-theoretical setting one can analyze the spectral properties of the generalized transfer operator $\mathcal{M}$ and relate formally the sharp determinant $\mathrm{Det}^{\#}(1-z\mathcal{M})$ to the determinant of $1+\mathcal{D}(z)$ where $\mathcal{D}(z)$ is a kneading operator. Unfortunately the trace of $\mathcal{D}(z)$ might diverge, and only regularized determinants like $\mathrm{Det}_3(1+\mathcal{D}(z))$ make sense a priori. The present preprint leaves unfinished the task of either bounding $\mathrm{Tr}\,\mathcal{D}(z)$ or making suitable subtractions in the definition of $\mathrm{Det}^{\#}(1-z\mathcal{M})$. Nevertheless it has appeared useful to record the existing non trivial results obtained, in view of their later utilisation.

The present preprint is a result of extensive discussions between the authors. It consists of three parts. The first part contains an identity between formal power series. The second part is about spectral properties, and the third part (traces and determinants) concerns bounds on $\mathrm{Tr}\,\mathcal{D}(z)^2$ and higher traces.

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1. AN IDENTITY BETWEEN POWER SERIES.

1.1. Sharp trace and sharp determinants for transfer operators and related operators.

Let E be a compact subset of the complex plane which we assume fixed throughout. We shall consider the algebra $\mathcal{A}$ of *transfer operators* $\mathcal{M}$ defined on the vector space of complex measures by

$$\mathcal{M}\Phi(x) = \sum_{\omega\in\Omega} g_\omega(x)\,\Phi(\psi_\omega(x))\,, \tag{1.1}$$

where Ω is a finite set, each $g_\omega : \mathbf{C} \to \mathbf{C}$ is C^∞ with nonempty compact support, and each ψ_ω is a holomorphic diffeomorphism from an open set $\Lambda_\omega \subseteq E$ onto its image $\psi_\omega\Lambda_\omega \subset E$, with $\operatorname{supp} g_\omega \subseteq \Lambda_\omega$. (At the cost of additional assumptions the smoothness requirement on g_ω could be somewhat weakened — see also the last subsection of this section — and the setting may be generalized to the case where Ω is countable, or where $\sum_\omega$ is replaced by an integral.) In later sections, we shall restrict $\mathcal{M}$ to Banach subspaces of the space $\mathcal{B}_{00}(E)$ of measures with support in E. Note for the moment that $\mathcal{M}$ in fact maps the space of measures into $\mathcal{B}_{00}(E)$.

We start by the important observation that the representation (1.1) of an operator $\mathcal{M} \in \mathcal{A}$ is essentially unique (because the ψ_ω are holomorphic). To make this remark more precise, we say that a representation $\mathcal{M}\Phi = \sum_{i\in\mathcal{I}}\sum_{\omega\in\Omega_i} g_\omega\,\Phi\circ\psi_\omega$ is finer than $\mathcal{M}\Phi = \sum_{i\in\mathcal{I}} g_i\,\Phi\circ\psi_i$ if each ψ_ω is a restriction of the corresponding ψ_i to some subset, and $\sum_{\omega\in\Omega_i} g_\omega = g_i$. Two representations are called equivalent if there is a third one which is finer than both. The precise claim is now that any two representations of $\mathcal{M} \in \mathcal{A}$ are equivalent. (To prove this, write the difference of the two representations and then use the fact that it represents the zero operator on $\mathcal{B}_{00}(E)$ in particular on Φ which are Dirac measures, noting that whenever ψ_ω and $\psi'_{\omega'}$ do not coincide on some open connected set, the set of $x \in (\Lambda_\omega \cap \Lambda'_{\omega'})$ with $\psi_\omega(x) = \psi'_{\omega'}(x)$ is finite, and that when they do coincide a partition of unity may be used to produce an associated refinement.)

We define the *sharp trace* of $\mathcal{M} \in \mathcal{A}$ to be the Cauchy principal value of the integral

$$\operatorname{Tr}^\# \mathcal{M} = \sum_{\omega\in\Omega} \int_{\mathbf{C}} \bar\partial(g_\omega(x))\,\sigma(\psi_\omega(x) - x)\,dx\ , \tag{1.2}$$

where $\bar\partial f(x) = 1/2(\frac{\partial}{\partial x_1} f(x_1 + ix_2) + i\frac{\partial}{\partial x_2} f(x_1 + ix_2))$ in the sense of distributions, the measure dx is Lebesgue measure on the complex plane, and

$$\sigma(x) = \begin{cases} \frac{1}{\pi x} & x \neq 0 \\ 0 & x = 0\,. \end{cases}$$

(We shall also use the notation $\partial f(x) = 1/2(\frac{\partial}{\partial x_1} f(x_1 + ix_2) - i\frac{\partial}{\partial x_2} f(x_1 + ix_2))$.) Observe that the terms in (1.1) with ψ_ω the identity do not contribute to the sharp trace. Obviously, two equivalent representations of $\mathcal{M}$ produce the same value for the sharp trace, so that $\operatorname{Tr}^\# \mathcal{M}$ does not depend on the choice of a representaion by the above remarks.

Note that if $\mathcal{M}$ is such that for all ω

$$\partial\psi_\omega(x) \neq 1 \quad \text{whenever} \quad \psi_\omega(x) = x \quad \text{for} \quad x \in \operatorname{supp} g_\omega\,, \tag{1.3}$$

it would suffice to assume that the g_ω are C^1 in order to define the sharp trace. (Warning: Property (1.3) is not preserved by taking powers $\mathcal{M}^n$ of $\mathcal{M}$.) See the end of this section for an explicit formula for the sharp trace as a sum over fixed points of the ψ_ω when the "simple fixed points property" (1.3) holds.

We now extend the domain of definition of the sharp trace. For this, we introduce the linear operator

$$S\Phi(x) = \int_{\mathbf{C}} \sigma(x-y)\,\Phi(y)\,dy\,, \tag{1.4}$$

which sends distributions with compact support to distributions, measures with compact support to measures, and C^∞ functions with compact support to C^∞ functions. Note for further use the property that $\bar\partial S$ is the identity map on compactly supported distributions, and in particular on measures supported in E. (To prove this, use that $\bar\partial\sigma$ is the Dirac mass at 0, a proof of this well-known equality may be found, e.g., in [3, p. 34].) A consequence of $\bar\partial S\Phi = \Phi$ is that S maps measures with compact support to measures without atoms. Observe that $\mathcal{M} \in \mathcal{A}$ maps any atomless measure to an atomless measure in $\mathcal{B}_{00}(E)$.

We write $\mathcal{A}^S$ for the algebra of operators which are linear combinations of finite alternating products of transfer operators $\mathcal{A}$ and operators S, with at least one factor S, and denote by $\mathcal{A}^S_L$, respectively $\mathcal{A}^S_R$ those elements of $\mathcal{A}^S$ such that the leftmost (respectively rightmost) factor is different from S. Operators in $\mathcal{A}^S_L$ act on $\mathcal{B}_{00}(E)$, and operators in $\mathcal{A}^S_R$, or more generally $\mathcal{A}^S$, map $\mathcal{B}_{00}(E)$ to measures.

Lemma 1.1. *For an operator $\mathcal{K}$ in $\mathcal{A}^S$ there is a unique kernel $\mathcal{K}_{xy} : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ such that:*

(1.5 a) *$\mathcal{K}\Phi(x) = \int_{\mathbf{C}} \mathcal{K}_{xy}\,\Phi(y)\,dy$ for all $\Phi \in \mathcal{B}_{00}(E)$,*

(1.5 b) *$\mathcal{K}_{xy}$ is a finite sum of terms $h(x)\cdot\tilde h(y)\cdot\sigma(\psi x - \tilde\psi y)$ where h, $\tilde h$ are C^∞ functions, ψ, $\tilde\psi$ are local holomorphic diffeomorphisms on open sets Λ, respectively $\tilde\Lambda$, containing the supports of h, respectively $\tilde h$, and h, respectively $\tilde h$, has compact support if $\mathcal{K} \in \mathcal{A}^S_L$, respectively $\mathcal{A}^S_R$.*

Proof of Lemma 1.1: We consider a summand in $\mathcal{K}$ and prove the result by induction on the number of factors in the summand, starting the decomposition from the right. The existence of a representation (1.5.a.b) is clear for S, and also for $S\mathcal{M}$ and $\mathcal{M}S$ with $\mathcal{M} \in \mathcal{A}$, since

$$\begin{aligned} S\mathcal{M}\Phi(x) &= \int_{\mathbf{C}} \sigma(x-y) \sum_{\omega\in\Omega} g_\omega(y)\,\Phi(\psi_\omega(y))\,dy \\ &= \int_{\mathbf{C}} \sum_{\omega\in\Omega} \chi_{\psi_\omega(\Lambda_\omega)}\cdot\sigma(x-\psi_\omega^{-1}(z))g_\omega(\psi_\omega^{-1}(z))\,|\partial(\psi_\omega^{-1})(z)|^2\,\Phi(z)\,dz \end{aligned} \tag{1.6}$$

$$\mathcal{M}S\Phi(x) = \sum_{\omega\in\Omega} g_\omega(x) \int_{\mathbf{C}} \sigma(\psi_\omega(x)-y)\,\Phi(y)\,dy\,.$$

The kernel representation property (1.5.a.b) is preserved when multiplying on the left by $\mathcal{M} \in \mathcal{A}$:

$$\mathcal{M}\mathcal{K}\Phi(x) = \int_{\mathbf{C}} \sum_{\omega \in \Omega} g_\omega(x)\, \mathcal{K}_{\psi_\omega(x) y} \Phi(y)\, dy\,. \tag{1.7}$$

Multiplication to the left by S is slightly more delicate. By Fubini, we have

$$S\mathcal{K}\Phi(x) = \int_{\mathbf{C}} \int_{\mathbf{C}} \sigma(x-y)\, \mathcal{K}_{yz}\, \Phi(z)\, dz\, dy = \int_{\mathbf{C}} \int_{\mathbf{C}} \sigma(x-y)\, \mathcal{K}_{yz}\, dy\, \Phi(z)\, dz\,. \tag{1.8}$$

Let us now study the kernel $\int_{\mathbf{C}} \sigma(x-y)\, \mathcal{K}_{yz}\, dy$ of $S\mathcal{K}$ defined by (1.8), using the induction assumption (1.5.a.b) on $\mathcal{K}_{xy}$. First note that, since $\bar\partial \sigma$ is the Dirac measure at zero, we have for any holomorphic diffeomorphism Ψ:

$$\begin{aligned} \bar\partial(\sigma \circ \Psi)(x) &= \delta_0 \circ \psi \cdot \overline{\partial \Psi}(x) \\ &= \delta_{\Psi^{-1}(0)}\, |\partial(\Psi^{-1})(0)|^2\, \overline{\partial\Psi}(\Psi^{-1}(0)) \\ &= \frac{\delta_{\Psi^{-1}(0)}}{\partial\Psi(\Psi^{-1}(0))}\,. \end{aligned} \tag{1.9}$$

Therefore, for any C^∞ functions h, $\tilde h$, with h compactly supported (recall that no composition S^2 is allowed), any local holomorphic diffeomorphism $\psi : \Lambda \to \mathbf{C}$, and any points $u \in \psi(\Lambda)$, $x \in \Lambda$, with $\psi(x) \neq u$, we get, since $h = \bar\partial S h$

$$\begin{aligned} &\int_{\mathbf{C}} h(y)\, \sigma(x-y)\, \sigma(\psi(y)-u)\, dy = \\ &\qquad (Sh)(x)\, \sigma(\psi(x)-u) - \frac{(Sh)(\psi^{-1}(u))\, \sigma(x-\psi^{-1}(u))}{\partial\psi(\psi^{-1}(u))}\,. \end{aligned} \tag{1.10}$$

To check that (1.10) makes sense, we observe that since Sh is holomorphic outside of the support of h, and since $Sh(z)$ goes to zero as $|z| \to \infty$ by construction, Sh must have support contained in the support of h (and in particular in Λ). Formula (1.10) for $u = \tilde\psi(z)$ defines a summand of the new kernel $(S\mathcal{K})_{xz}$ except on a set of points (x, z) of zero two-dimensional complex Lebesgue measure: for fixed z (ψ, $\tilde\psi$), there are at most finitely many x such that $\psi(x) = \tilde\psi(z)$ (and similarly, at most finitely many such z for each x, ψ, $\tilde\psi$). Note also that the factor $\tilde h(z)$ implicit in both sides of (1.10) has compact support if $\mathcal{K} \in \mathcal{A}^S_R$.

We *define* the left-hand side of (1.10) to be zero when $\psi(x) = u$, in other words we extend formula (1.10) to such points x, u. (Note that the Cauchy principal value of the left-hand-side of (1.10) does *not* always vanish when $\psi(x) = u$. An easy counterexample can be obtained with ψ the identity map, assuming that $S\,\partial h$ does not vanish at x.) Our definition is legitimate in that it is consistent with the action of $S\mathcal{K}$ on $\mathcal{B}_{00}(E)$ (because the ambiguity only concerns a set of two-dimensional complex Lebesgue measure zero).

(Although we shall not need this, we note that when applying (1.10) inductively at an intermediate S factor, the choice for the case $\psi(x) = u$ we just made is unconsequential for the kernel since the concerned set is of zero measure and will be washed out by next integration.)

By induction we thus prove the existence of a representation (1.5.a.b) for $\mathcal{K} \in \mathcal{A}^S$. If $\mathcal{K}$, $\mathcal{K}' \in \mathcal{A}^S$ and $\mathcal{K}\Phi = \mathcal{K}'\Phi$ for all $\Phi \in \mathcal{B}_{00}(E)$ then the kernels $\mathcal{K}_{xy}$ and $\mathcal{K}'_{xy}$ in (1.5) must coincide as functions on $\mathbf{C} \times \mathbf{C}$. (Using Φ the Dirac mass at an arbitrary point y_0 one sees that the difference of the densities of the image measures $\mathcal{P}_{y_0}(x) = \mathcal{K}_{xy_0} - \mathcal{K}'_{xy_0}$ vanishes for Lebesgue almost all x. Since $\mathcal{P}_{y_0}$ is C^∞ except at finitely many points where it vanishes, it must vanish identically.) $\square$

If $\mathcal{K} \in \mathcal{A}^S_L + \mathcal{A}^S_R$ has kernel $\mathcal{K}_{xy}$ as in (1.5.a.b) then the function $\mathcal{K}_{xx}$ is a finite sum of terms $\hat{h}(x)\sigma(\Psi(x))$ where $\hat{h}(x)$ is C^∞, with compact support, and $\Psi(x)$ is holomorphic (and therefore vanishes to finite order if not identically). In particular the Cauchy principal value of $\int_\mathbf{C} \mathcal{K}_{xx}\, dx$ is well defined for $\mathcal{K} \in \mathcal{A}^S_L + \mathcal{A}^S_R$. Therefore, we may define the *sharp trace* of $\mathcal{K} \in \mathcal{A}^S_L + \mathcal{A}^S_R$ to be this Cauchy principal value:

$$\mathrm{Tr}^\# \mathcal{K} = \int_\mathbf{C} \mathcal{K}_{xx}\, dx\,. \tag{1.11}$$

Before we proceed with our extension of the domain of the definition of the sharp trace, we note that if $\mathcal{M} + \mathcal{K}$, with $\mathcal{M} \in \mathcal{A}$ and $\mathcal{K} \in \mathcal{A}^S_L + \mathcal{A}^S_R$, vanishes as an operator mapping $\mathcal{B}_{00}(E)$ to measures, then $\mathcal{M}$, and therefore also $\mathcal{K}$, must vanish. (To show this, consider the Dirac measure δ_{x_0} at x_0 in $\mathbf{C}$. If $\mathcal{M} \neq 0$, then the measure $\mathcal{M}\delta_{x_0}$ has atoms for x_0 well-chosen in function of a given representation (1.1). However, $\mathcal{K}\delta_{x_0}$ is an atomless measure for any x_0 by above considerations.) We now extend the definition linearly to $\mathcal{A}' = \mathcal{A} \oplus (\mathcal{A}^S_L + \mathcal{A}^S_R)$, using (1.2). By the uniqueness statement in Lemma 1.1, we have that $\mathrm{Tr}^\# \mathcal{K}$ only depends on $\mathcal{K} \in \mathcal{A}'$ as an operator mapping $\mathcal{B}_{00}(E)$ to measures.

Although $\mathcal{A}'$ is not an algebra, both $\mathcal{A}'_L = \mathcal{A} \oplus \mathcal{A}^S_L$ and $\mathcal{A}'_R = \mathcal{A} \oplus \mathcal{A}^S_R$ are algebras. In particular, if $\mathcal{K} \in \mathcal{A}'_L$ (or $\mathcal{A}'_R$), then $\mathcal{K}^m \in \mathcal{A}'_L$ (respectively $\mathcal{A}'_R$) for all $m \geq 1$.

We note the following "almost-trace properties" of $\mathrm{Tr}^\#$:

Lemma 1.2.

(1) *Let* $\mathcal{K}$ *belong to* $\mathcal{A}^S$. *Then* $\mathrm{Tr}^\# \mathcal{K}\mathcal{M} = \mathrm{Tr}^\# \mathcal{M}\mathcal{K}$ *for any* $\mathcal{M} \in \mathcal{A}$.

(2) *Let* $\mathcal{K}$ *belong to* $\mathcal{A}^S_L \cap \mathcal{A}^S_R$. *Then* $\mathrm{Tr}^\# S\mathcal{K} = \mathrm{Tr}^\# \mathcal{K}S$.

In Lemma 1.6, we shall see that that $\mathrm{Tr}^\# \mathcal{M}_1\mathcal{M}_2 = \mathrm{Tr}^\# \mathcal{M}_2\mathcal{M}_1$ for $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{A}$.

Proof of Lemma 1.2: For the first claim, we use (1.6–1.7) (and the analogue of (1.7) for $\mathcal{K}\mathcal{M}$) and write

$$\begin{aligned}\mathrm{Tr}^\# \mathcal{K}\mathcal{M} &= \int_\mathbf{C} \sum_{\omega\in\Omega} \chi_{\psi_\omega(\Lambda_\omega)} \mathcal{K}_{x\psi_\omega^{-1}x}\, g_\omega(\psi_\omega^{-1}(x))\, |\partial(\psi_\omega^{-1})(x)|^2\, dx \\ &= \int_\mathbf{C} \sum_{\omega\in\Omega} \mathcal{K}_{\psi_\omega(y)y}\, g_\omega(y)\, dy = \mathrm{Tr}^\# \mathcal{M}\mathcal{K}\,.\end{aligned}$$

In the above equalities, we used the fact that the standard change of variables formula holds for Cauchy principal values, whenever the change of variables is holomorphic. Since antiholomorphic maps are harmonic, this can be proved by induction on $m \ge 2$ as follows (g is a compactly supported C^∞ function, and ψ^{-1} is a holomorphic diffeomorphism on the support of g):

$$\begin{aligned}
(m-1)\int_{\mathbf{C}} \frac{1}{y^m}\, g(y)\, dy &= \int_{\mathbf{C}} \frac{1}{y^{m-1}}\, \partial g(y)\, dy \\
&= \int_{\mathbf{C}} \frac{1}{(\psi x)^{m-1}}\, \partial g(\psi x) \partial \psi(x)\, \bar\partial \bar\psi(x)\, dx \\
&= \int_{\mathbf{C}} \frac{1}{(\psi x)^{m-1}} \Big[\partial g(\psi x)\, \partial \psi(x)\, \bar\partial \bar\psi(x) + g(\psi x)\, \partial \bar\partial \bar\psi(x) \Big]\, dx \\
&= (m-1)\int_{\mathbf{C}} \frac{\partial \psi(x)}{(\psi x)^m}\, g(\psi x) \bar\partial \bar\psi(x)\, dx\,.
\end{aligned}$$

For the second claim, use (1.8) (and the equivalent formula for $\mathcal{K}S$) to get

$$\mathrm{Tr}^\# S\mathcal{K} = \int_{\mathbf{C}} \int_{\mathbf{C}} \sigma(x-y)\, \mathcal{K}_{yx}\, dy\, dx \qquad \mathrm{Tr}^\# \mathcal{K}S = \int_{\mathbf{C}} \int_{\mathbf{C}} \mathcal{K}_{xy}\, \sigma(y-x)\, dy\, dx\,,$$

where it is implicit (from the definition given after (1.10)) that we have suppressed from $\mathcal{K}_{xy}$ all terms of the form $h(x)\tilde h(y)\sigma(\psi x - \tilde\psi y)$ with ψ and $\tilde\psi$ identical on some open set, taking finer presentations if necessary (note that the suppressed terms are the same for both traces). To finish, apply Fubini. □

We can obviously extend additively the sharp trace from the vector space $\mathcal{A}'$ to the vector space $\mathcal{A}'' = \mathcal{A}''_L + \mathcal{A}''_R$ with $\mathcal{A}''_M = \mathcal{A}[[z]] \oplus \mathcal{A}^S_M[[z]]$ where $\mathcal{A}''_M[[z]]$ is the algebra of formal power series with coefficients in $\mathcal{A}^S_M$ for $M = L, R$. We use the same notation for this extension

$$\mathrm{Tr}^\# : \mathcal{A}'' \to \mathbf{C}[[z]]\,.$$

We define the *sharp determinant* of $\mathcal{K}(z) \in z\mathcal{A}''_L$ or $z\mathcal{A}''_R$ by the following formal power series:

$$\mathrm{Det}^\#(1-\mathcal{K}(z)) = \exp - \sum_{m=1}^{\infty} \frac{1}{m} \mathrm{Tr}^\#(\mathcal{K}(z))^m \in 1 + \mathbf{C}[[z]]\,. \tag{1.12}$$

1.2. The kneading operator $\mathcal{D}(z)$ and the main identity.

We associate to $\mathcal{M} \in \mathcal{A}$ an operator $\mathcal{N} = \mathcal{N}(\mathcal{M}) \in \mathcal{A}$ defined by

$$\mathcal{N}\Phi(x) = \sum_{\omega \in \Omega} \bar\partial g_\omega(x)\, \Phi(\psi_\omega(x))\,, \tag{1.13}$$

which sends measures to measures supported in E. (Note that $\mathcal{N}(\mathcal{M})$ only depends on $\mathcal{M}$ as an operator on $\mathcal{B}_{00}(E)$.)

Finally, we define the *formal kneading operator* $\mathcal{D} = \mathcal{D}(z) \in \mathcal{A}_L^S[[z]]$ associated to $\mathcal{M} \in \mathcal{A}$ by

$$\mathcal{D}(z) = z\,\mathcal{N}\,(1 - z\mathcal{M})^{-1}\,S = \sum_{k=1}^{\infty} z^k\,\mathcal{N}\,\mathcal{M}^{k-1}\,S\,. \tag{1.14}$$

The point is that for $|z|^{-1}$ larger than the spectral radius of $\mathcal{M}$ on $\mathcal{B}_{00}(E)$, we may view $\mathcal{D}(z)$ as acting on $\mathcal{B}_{00}(E)$, and we shall see in Section 3, restricting to a Banach subspace of $\mathcal{B}_{00}(E)$, that the operator $\mathcal{D}^2(z)$ is almost trace-class so that one should be able to relate its sharp determinant to a classical regularized determinant. We hope that this informal comment shows the importance of the main result of this section:

Proposition 1.3. *For any $\mathcal{M}$ in $\mathcal{A}$, and $\mathcal{D}(z)$ defined by* (1.13–1.14), *we have the following identity between formal power series*

$$\mathrm{Det}^{\#}(1 + \mathcal{D}(z)) = \frac{1}{\mathrm{Det}^{\#}(1 - z\mathcal{M})}\,. \tag{1.15}$$

One way to show Proposition 1.3 would be to adapt the proof of the formally identical result on transfer operators acting on functions of bounded variation on $\mathbf{R}$ in [1, Proposition 3.1] (using the fact that $1/(\pi x)$ is the complex analogue of the function $(1/2)\,\mathrm{sgn}\,(x)$ used there). We shall give here a more streamlined proof. We first need a lemma and its corollary:

Lemma 1.4. *For any $\mathcal{M}$ in $\mathcal{A}$ and the associated operator $\mathcal{N}(\mathcal{M}) \in \mathcal{A}$, we have*

$$\mathrm{Tr}^{\#}\,\mathcal{M} = \mathrm{Tr}^{\#}\,\mathcal{N}S = \mathrm{Tr}^{\#}\,S\mathcal{N}\,.$$

Proof of Lemma 1.4: The second equality is a consequence of the first claim of Lemma 1.2, so that it suffices to check the first one. By definition

$$\mathcal{N}S\Phi(x) = \sum_{\omega \in \Omega} \int_{\mathbf{C}} \bar\partial g_\omega(x)\,\sigma(\psi_\omega x - y)\,\Phi(y)\,dy\,.$$

Therefore

$$\mathrm{Tr}^{\#}\,\mathcal{N}S = \sum_{\omega \in \Omega} \int_{\mathbf{C}} \bar\partial g_\omega(x)\,\sigma(\psi_\omega x - x)\,dx = \mathrm{Tr}^{\#}\,\mathcal{M}\,. \ \Box$$

Corollary 1.5. *For any $\mathcal{M} \in \mathcal{A}$ and the associated operator $\mathcal{N}(\mathcal{M}) \in \mathcal{A}$, the operator $\widetilde{\mathcal{M}} = \mathcal{M} - S\mathcal{N} \in \mathcal{A}_R^S$ has the property that $\mathrm{Tr}^{\#}(\widetilde{\mathcal{M}})^\ell = 0$ for all integer $\ell \geq 1$.*

Proof of Corollary 1.5: We only need to check that

$$\widetilde{\mathcal{M}^\ell}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}^{\ell+1}}\,. \tag{1.16}$$

in the sense of operators from $\mathcal{B}_{00}(E)$ to measures, for all $\ell \geq 1$. (Indeed, using the representation (1.5) for $(\widetilde{\mathcal{M}})^\ell \in \mathcal{A}_R^S$ and $\widetilde{\mathcal{M}^\ell} \in \mathcal{A}_R^S$, it follows that $\mathrm{Tr}^\#(\widetilde{\mathcal{M}})^\ell = \mathrm{Tr}^\# \widetilde{\mathcal{M}^\ell}$ which vanishes by Lemma 1.4 applied to $\mathcal{M}^\ell$.) We write $\mathcal{N}_\ell$ for the transfer operator associated to $\mathcal{M}^\ell$, so that $\widetilde{\mathcal{M}^\ell} = \mathcal{M}^\ell - S\mathcal{N}_\ell$. We also associate to $\mathcal{M} \in \mathcal{A}$ an operator $\mathcal{M}_{0,1} \in \mathcal{A}$ defined by

$$\mathcal{M}_{0,1}\Phi(x) = \sum_{\omega\in\Omega} g_\omega(x)\, \overline{\partial\psi_\omega}(x)\, \Phi(\psi_\omega(x))\,, \tag{1.17}$$

noting that for $\ell \geq 1$ the chain rule implies $(\mathcal{M}^\ell)_{0,1} = (\mathcal{M}_{0,1})^\ell$ (we simply write $\mathcal{M}^\ell_{0,1}$). Since each ψ_ω is holomorphic we have by the Leibniz rule, and because $\bar\partial S = S\bar\partial$ is the identity on compactly supported distributions, for all $\ell \geq 1$:

$$\widetilde{\mathcal{M}^\ell} = \mathcal{M}^\ell - S\mathcal{N}_\ell = S\mathcal{M}^\ell_{0,1}\bar\partial\,, \tag{1.18}$$

in the sense of operators from $\mathcal{B}_{00}(E)$ to measures. Using again that $\bar\partial S$ is the identity on distributions with compact support, we thus get

$$\widetilde{\mathcal{M}^\ell}\widetilde{\mathcal{M}} = S\mathcal{M}^\ell_{0,1}\bar\partial S\mathcal{M}_{0,1}\bar\partial = S\mathcal{M}^\ell_{0,1}\mathcal{M}_{0,1}\bar\partial = \widetilde{\mathcal{M}^{\ell+1}}\,. \ \square$$

Proof of Proposition 1.3: Let $M_i = L$ or R, $i = 1, 2, 3$. We first note that whenever $\mathcal{K}_1$ and $\mathcal{K}_2$ are two elements of $\mathcal{A}''_{M_1} \cup \{S\}$ with $\mathcal{K}_1\mathcal{K}_2 \in \mathcal{A}''_{M_2}$, $\mathcal{K}_2\mathcal{K}_1 \in \mathcal{A}''_{M_3}$, and $\mathrm{Tr}^\#(\mathcal{K}_1\mathcal{K}_2)^m = \mathrm{Tr}^\#(\mathcal{K}_2\mathcal{K}_1)^m$ for all $m \geq 1$ then

$$\mathrm{Det}^\#(1 + \mathcal{K}_1\mathcal{K}_2) = \mathrm{Det}^\#(1 + \mathcal{K}_2\mathcal{K}_1)\,. \tag{1.19}$$

If, additionally, $\mathcal{K}(\vec{\jmath}) = \mathcal{K}_1^{j_m}\mathcal{K}_2^{j_{m-1}} \cdots \mathcal{K}_2^{j_0}$ is in $\mathcal{A}''_L$ or $\mathcal{A}''_R$ for all integer $m \geq 1$, and $j_\ell \geq 0$ for $0 \leq \ell \leq m$, with $\sum_{0\leq\ell\leq m} j_\ell \geq 1$, and if $\mathrm{Tr}^\# \mathcal{K}(\vec{\jmath})$ is invariant under circular permutations of $\vec{\jmath}$, then

$$\mathrm{Det}^\#(1 + \mathcal{K}_1)\ \mathrm{Det}^\#(1 + \mathcal{K}_2) = \mathrm{Det}^\#((1 + \mathcal{K}_1)\,(1 + \mathcal{K}_2))\,. \tag{1.20}$$

(See, e.g., [5, Appendix A] for a proof.) We now have by Lemma 1.2, (1.19–1.20) and Corollary 1.5:

$$\begin{aligned}\mathrm{Det}^\#(1 + \mathcal{D}(z))\ \mathrm{Det}^\#(1 - z\mathcal{M}) &= \mathrm{Det}^\#(1 + z\mathcal{N}(1 - z\mathcal{M})^{-1}S)\ \mathrm{Det}^\#(1 - z\mathcal{M}) \\ &= \mathrm{Det}^\#(1 + zS\mathcal{N}(1 - z\mathcal{M})^{-1})\ \mathrm{Det}^\#(1 - z\mathcal{M}) \\ &= \mathrm{Det}^\#(1 - z\mathcal{M} + zS\mathcal{N}) \\ &= \mathrm{Det}^\#(1 - z\widetilde{\mathcal{M}}) = 1\,. \ \square\end{aligned}$$

As a consequence of our computations, we are able to prove:

Lemma 1.6. *For any $\mathcal{M}$, $\mathcal{P}$ in $\mathcal{A}$ we have* $\mathrm{Tr}^\#(\mathcal{M}\mathcal{P}) = \mathrm{Tr}^\#(\mathcal{P}\mathcal{M})$.

Proof of Lemma 1.6: We shall use Lemma 1.4, which tells us that

$$\mathrm{Tr}^\#(\mathcal{M}\mathcal{P}) = \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{M}\mathcal{P}) \quad \text{and} \quad \mathrm{Tr}^\#(\mathcal{P}\mathcal{M}) = \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{P}\mathcal{M}) \, ,$$

with $\mathcal{N}(\mathcal{Q})$ as in (1.13). By definition and the Leibniz rule we have

$$\mathcal{N}(\mathcal{M}\mathcal{P}) = \mathcal{N}(\mathcal{M})\mathcal{P} + \mathcal{M}_{0,1}\mathcal{N}(\mathcal{P}) \, , \tag{1.21}$$

with $\mathcal{M}_{0,1}$ defined in (1.17). Now, using once more that $\bar\partial S$ is the identity on $\mathcal{B}_{00}(E)$ to apply (1.18), we get the following equality between operators from $\mathcal{B}_{00}(E)$ to measures:

$$\begin{aligned} S\mathcal{M}_{0,1}\mathcal{N}(\mathcal{P}) &= S\mathcal{M}_{0,1}\bar\partial S\mathcal{N}(\mathcal{P}) \\ &= (\mathcal{M} - S\mathcal{N}(\mathcal{M}))\, S\mathcal{N}(\mathcal{P}) \\ &= \mathcal{M}S\mathcal{N}(\mathcal{P}) - S\mathcal{N}(\mathcal{M})S\mathcal{N}(\mathcal{P}) \, . \end{aligned} \tag{1.22}$$

Putting (1.21) and (1.22) together, and applying Lemma 1.2 (1) yields

$$\begin{aligned} \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{M}\mathcal{P}) &= \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{M})\mathcal{P} + \mathrm{Tr}^\# \, \mathcal{M}S\mathcal{N}(\mathcal{P}) - \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{M})S\mathcal{N}(\mathcal{P}) \\ &= \mathrm{Tr}^\# \, \mathcal{P}S\mathcal{N}(\mathcal{M}) + \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{P})\mathcal{M} - \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{P})S\mathcal{N}(\mathcal{M}) \\ &= \mathrm{Tr}^\# \, S\mathcal{N}(\mathcal{P}\mathcal{M}) \, , \end{aligned}$$

as desired. □

1.3. The kneading operators $\mathcal{D}_{(r)}$.

We introduce yet another linear operator sending compactly supported distributions (or measures) to distributions (measures),

$$S_{(r)}\Phi(x) = \int_{\mathbf{C}} \sigma_r(x-y)\, \Phi(y)\, dy \, ,$$

for integer $r \geq 0$, where we set $\sigma_0(x) = \sigma(x)$, $\sigma_1(x) = \log(|x|)/\pi$ for $x \neq 0$, and $\sigma_1(0) = 0$, and generally σ_r for $r \geq 2$ a solution of $\partial\sigma_r = \sigma_{r-1}$ which we choose to be continuous and zero at the origin (for example, $\sigma_2(x) = x(\log|x| - 1)$). By definition $\partial^r \sigma_r = \sigma$. Note also that $\sigma_0 \chi_E$ is in L^p for all $p < 2$, $\sigma_1 \chi_E$ is in L^q for all $q < \infty$, where χ_E denotes the characteristic function of E, and σ_r is continuous for all $r \geq 2$. Introducing the notation

$$\bar S^r \Phi = \bar\partial S_{(r)} \Phi \, , \tag{1.23}$$

for $\Phi \in \mathcal{B}_{00}(E)$, and integer $r \geq 0$ it is not difficult to check that $\partial^r \bar S^r = \bar S^r \partial^r$ is the identity on $\mathcal{B}_{00}(E)$.

For $\mathcal{M} \in \mathcal{A}$ we now define the associated operators $\mathcal{D}_{(r)}(z) \in z\partial^r \mathcal{A}[[z]] S_{(r)}$ for integers $r \geq 0$, following [6], by

$$\mathcal{D}_{(r)}(z) = \partial^r \mathcal{D}(z) \bar S^r = z\partial^r \, \mathcal{N} \, (1 - z\mathcal{M})^{-1} \, S_{(r)} \, . \tag{1.24}$$

(Recall that $\mathcal{N} = \mathcal{N}(\mathcal{M})$ was defined in (1.13).) Clearly, $\mathcal{D}_{(0)}(z) = \mathcal{D}(z)$.

By definition of $S_{(r)}$, if z^{-1} is outside of the spectrum of $\mathcal{M}$ acting on the space $\mathcal{B}_{r1}$ of distributions Φ and such that $\bar\partial \partial^r \Phi$ is a measure (use that $\sigma_r \in \mathcal{B}_{r1}$), the operator $\mathcal{D}_{(r)}(z)$ has a chance to be well-behaved when acting on a suitable Banach subspace of $\mathcal{B}_{00}(E)$. The main point (see Lemma 1.7 below) is that the sharp determinant is preserved by the modifications (1.24), where we first need to extend the sharp determinant to $\partial^r \mathcal{A}[[z]] S_{(r)}$, i.e., to define the sharp trace of powers of operators in $\partial^r \mathcal{A}[[z]] S_{(r)}$. To do this, we first write $\mathcal{K} = \partial^r \mathcal{M} S_{(r)}$ (for $\mathcal{M} \in \mathcal{A}$) in kernel form:

$$
\begin{aligned}
\mathcal{K}\Phi(x) &= \partial^r \mathcal{M} S_{(r)}\, \Phi(x) \\
&= \partial_x^r \Big(\sum_\omega g_\omega(x) \int_{\mathbf{C}} \sigma_r(\psi_\omega x - y)\, \Phi(y)\, dy \Big) \\
&= \int_{\mathbf{C}} \sum_\omega \sum_{s=0}^{r} \binom{r}{s} \big(\partial_x^s g_\omega\big)(x)\, \partial_x^{r-s} \big(\sigma_r(\psi_\omega x - y)\big)\, \Phi(y)\, dy \\
&= \int \mathcal{K}_{xy} \Phi(y)\, dy \,,
\end{aligned}
\tag{1.25}
$$

with $\mathcal{K}_{xy}$ a finite sum of terms $h(x) \cdot \sigma_t(\psi x - y)$, for $0 \le t \le r$, where h is a C^∞ function with compact support, and ψ is a local holomorphic diffeomorphism on an open set Λ, containing the supports of h. When constructing inductively a kernel $\mathcal{L}_{xz}$ for $\mathcal{L}$ a finite product of operators of the type (1.25), we are confronted with integrals of the type

$$
\int_{\mathbf{C}} h(y) \sigma_r(u - y)\, \sigma_s(\psi y - v)\, dy \tag{1.26}
$$

where h is a C^∞ function with compact support, and ψ a local holomorphic diffeomorphism on an open set Λ containing the support of h, and the integers r and s are both nonnegative. If $r = s = 0$, we *define* the new kernel by the same rule as in (1.10), thus collapsing the two σ_0 in (1.26) and producing two terms with just one σ_0 factor. If $r \cdot s = 0$ but $r + s > 0$, since $\bar S^t h$ is a well defined C^∞ function supported in E for $t \ge 0$ (just conjugate the arguments following (1.10)) we may ∂_y-integrate by parts in (1.26) (using that $\partial^t \bar S^t$ is the identity) and thus choose on which side (left or right) we want the σ_0 factor to be. Applying these remarks, we end up for $\mathcal{L}_{xz}$ with a sum of terms of two types: either a single

$$
h_0(x)\, h_1(z)\, \sigma_0(\psi_1(x) - \tilde\psi_1(z)) \tag{1.27}
$$

or an integral

$$
\begin{aligned}
h_0(x)\, h_{j+1}(z) \int_{\mathbf{C} \times \cdots \times \mathbf{C}} & h_1(y_1)\, \sigma_{s_1}(\psi_1 x - \tilde\psi_1 y_1)\, h_2(y_2)\, \sigma_{s_2}(\psi_2 y_1 - \tilde\psi_2 y_2) \\
& \cdots \sigma_{s_j}(\psi_j y_{j-1} - \tilde\psi_j z)\, dy_1 \cdots dy_{j-1}
\end{aligned}
\tag{1.28}
$$

where $j \ge 2$, and all the s_i except perhaps the second one s_2 are strictly positive (the h_i are C^∞ with compact support, and the ψ_i, $\tilde\psi_i$ local holomorphic diffeomorphisms on open sets as usual). Because of the properties of the σ_i, the Cauchy principal value of $\int \mathcal{L}_{xx}\, dx$ is thus well defined and we may set $\mathrm{Tr}^\# \mathcal{L}$ to be this value. The result can now be stated:

Lemma 1.7. *For $\mathcal{M} \in \mathcal{A}$ and $\mathcal{D}_{(r)}(z)$ defined by (1.24), we have for all integers $r \geq 0$ $\mathrm{Det}^\#(1 + \mathcal{D}_{(r)}(z)) = \mathrm{Det}^\#(1 + \mathcal{D}(z))$.*

Proof of Lemma 1.7: By definition of the sharp determinant, we must check that for any $\mathcal{P} \in \mathcal{A}_R^S \cap \mathcal{A}_L^S$ and integer $r \geq 1$

$$\mathrm{Tr}^\# \mathcal{P}S = \mathrm{Tr}^\# \partial^r \mathcal{P} S \bar{S}^r \,. \tag{1.29}$$

Clearly, it suffices to verify that for all operators $\mathcal{K} = \partial^s \mathcal{P} S \bar{S}^{s-1}$ with $\mathcal{P} \in \mathcal{A}_R^S \cap \mathcal{A}_L^S$ and integer $s \geq 1$ we have

$$\mathrm{Tr}^\# \mathcal{K}\bar{S} = \mathrm{Tr}^\# \bar{S}\mathcal{K}. \tag{1.30}$$

(Both sides of (1.30) are well defined since $\mathcal{K}\bar{S} = \partial^s \mathcal{P} S \bar{S}^s$ and $\bar{S}\mathcal{K} = \partial^{s-1} \mathcal{P} S \bar{S}^{s-1}$.) We first observe that since $\bar{S}\partial = \partial \bar{S}$ is the identity on $\mathcal{B}_{00}(E)$, we have

$$\bar{S}\Phi(x) = \int_{\mathbf{C}} \bar{\sigma}(x-y)\, \Phi(y)\, dy\,,$$

with $\bar{\sigma}(x) = \overline{\sigma(x)}$ having the property that $\partial\bar{\sigma}$ is the Dirac mass at 0. By (1.25–1.28) we know that the operator $\mathcal{K}$ appearing in (1.30) acting on $\mathcal{B}_{00}(E)$ can be written in kernel form with $\mathcal{K}_{xy}$ a finite sum of terms

$$h(x) \cdot \tilde{h}(y) \cdot \sigma_{-1}(\psi x - \tilde{\psi} y) \tag{1.31}$$

where $\sigma_{-1} = \partial\sigma$ (apply ∂_x to terms (1.27)) and terms

$$\int_{\mathbf{C}\times\cdots\times\mathbf{C}} h_1(y_1)\, \sigma_{s_1-1}(\psi_1 x - \tilde{\psi}_1 y_1) \cdots \sigma_{s_j}(\psi_j y_{j-1} - \tilde{\psi}_j z)\, dy_1 \cdots dy_{j-1} \tag{1.32}$$

with $s_1 - 1, s_2 \geq 0$ and the other $s_i > 0$ (apply ∂_x to (1.28)). If (1.32) contains two σ_0 factors we proceed as described above to define its value. Using this kernel $\mathcal{K}_{xy}$, and (1.30) we write

$$\mathcal{K}\bar{S}\Phi(x) = \int_{\mathbf{C}}\int_{\mathbf{C}} \mathcal{K}_{xy}\bar{\sigma}(y-z)\, \Phi(z)\, dz\, dy \quad \bar{S}\mathcal{K}\Phi(x) = \int_{\mathbf{C}}\int_{\mathbf{C}} \bar{\sigma}(x-y)\mathcal{K}_{yz}\, \Phi(z)\, dz\, dy\,. \tag{1.33}$$

We claim that we can use the expression $\int \mathcal{K}_{xy}\, \bar{\sigma}(y-z)\, dy$ from (1.33) for the kernel of $\mathcal{K}\bar{S}$ without problems (and similarly for $\bar{S}\mathcal{K}$). The only summands of $\mathcal{K}_{xy}$ which require some care are those of the form (1.31), for which we may legitimately use (since $\partial\bar{\sigma}$ is the Dirac mass at the origin)

$$\int \tilde{h}(y)\, \sigma_{-1}(\psi x - \tilde{\psi} y)\, \bar{\sigma}(y-z)\, dy =$$

$$= \frac{\tilde{h}(z)}{\tilde{\psi}'(z)}\, \sigma_0(\psi x - \tilde{\psi} z) + \int \frac{\partial\tilde{h}(y)\tilde{\psi}'(y) - \tilde{h}(y)\tilde{\psi}''(y)}{(\tilde{\psi}'(y))^2}\, \sigma_0(\psi x - \tilde{\psi} y)\, \bar{\sigma}(y-z)\, dy\,.$$

(A similar computation exists for $\bar{S}\mathcal{K}$.) Finally, we just apply Fubini and get:

$$\mathrm{Tr}^\# \, \mathcal{K}\bar{S} = \int_{\mathbf{C}} \int_{\mathbf{C}} \mathcal{K}_{xy} \bar{\sigma}(y-x) \, dy \, dx = \int_{\mathbf{C}} \int_{\mathbf{C}} \bar{\sigma}(x-y) \mathcal{K}_{yx} \, dy \, dx = \mathrm{Tr}^\# \, \bar{S}\mathcal{K} \,,$$

as required. □

1.4. The half-adjoint, a functional equation, and the kneading operator $\widehat{\mathcal{D}}(z)$.

We associate with $\mathcal{M}$ another operator in $\mathcal{A}$ (the *half-adjoint* of $\mathcal{M}$) defined by

$$\widehat{\mathcal{M}}\Phi(x) = \sum_{\omega \in \Omega} \frac{g_\omega(\psi_\omega^{-1} x)}{\partial \psi_\omega(\psi_\omega^{-1}(x))} \, \Phi(\psi_\omega^{-1} x) \,. \tag{1.34}$$

(We assume that $\mathrm{supp}\, g_\omega \subset \psi_\omega \Lambda_\omega$.) Note that $\widehat{\mathcal{M}}$ is independent of the representation of $\mathcal{M}$ of the form (1.1). Clearly $\widehat{}$ is an involution. Since each ψ_ω is a local holomorphic diffeomorphism,

$$\begin{aligned} \mathrm{Tr}^\# \, \widehat{\mathcal{M}} &= \int_{\mathbf{C}} \bar{\partial} \left(\frac{g_\omega}{\partial \psi_\omega} \circ \psi_\omega^{-1} \right) \sigma(\psi_\omega^{-1}(x) - x) \, dx \\ &= \int_{\mathbf{C}} \bar{\partial} \left(\frac{g_\omega}{\partial \psi_\omega} \right) (\psi_\omega^{-1}(x)) \, \overline{\partial \psi_\omega^{-1}}(x) \; \sigma(\psi_\omega^{-1}(x) - x) \, dx \\ &= \int_{\mathbf{C}} \frac{\bar{\partial} g_\omega}{\partial \psi_\omega} (\psi_\omega^{-1}(x)) \, \overline{\partial \psi_\omega^{-1}}(x) \; \sigma(\psi_\omega^{-1}(x) - x) \, dx \\ &= \int_{\mathbf{C}} \frac{\bar{\partial} g_\omega}{\partial \psi_\omega} (y) \, \overline{\partial \psi_\omega^{-1}}(\psi_\omega(y)) \, |\partial \psi_\omega(y)|^2 \, \sigma(y - \psi_\omega y) \, dy \\ &= -\, \mathrm{Tr}^\# \, \mathcal{M} \,. \end{aligned} \tag{1.35}$$

Since $\widehat{\mathcal{M}_1 \mathcal{M}_2} = \widehat{\mathcal{M}}_2 \widehat{\mathcal{M}}_1$ we have $\mathrm{Tr}^\# (\widehat{\mathcal{M}})^n = -\, \mathrm{Tr}^\# \, \mathcal{M}^n$ for all integers $n \geq 1$. We therefore obtain the functional equation

$$\mathrm{Det}^\# (1 - z\mathcal{M}) \; \mathrm{Det}^\# (1 - z\widehat{\mathcal{M}}) = 1 \,. \tag{1.36}$$

Replacing $\mathcal{M}$ by $\widehat{\mathcal{M}}$ and $\mathcal{N}(\mathcal{M})$ by $\mathcal{N}(\widehat{\mathcal{M}})$ in (1.9) we may define an operator $\widehat{\mathcal{D}}(z) \in \mathcal{A}''$. Using the functional equation (1.36) and the above properties of $\widehat{}$, we may now write a more complete version of Proposition 1.3:

$$\mathrm{Det}^\# (1 + \mathcal{D}(z)) = \mathrm{Det}^\# (1 - z\widehat{\mathcal{M}}) = \frac{1}{\mathrm{Det}^\# (1 - z\mathcal{M})} = \frac{1}{\mathrm{Det}^\# (1 + \widehat{\mathcal{D}}(z))} \,. \tag{1.37}$$

We also define operators $\widehat{\mathcal{D}}_{(r)}(z)$ for integers $r \geq 0$ by $z \partial^r \mathcal{N}(\widehat{\mathcal{M}})(1 - z\widehat{\mathcal{M}})^{-1} S_{(r)}$ and note as in Lemma 1.7 that $\mathrm{Det}^\# (1 + \widehat{\mathcal{D}}_{(r)}(z)) = \mathrm{Det}^\# (1 + \widehat{\mathcal{D}}(z))$ for all r.

1.5. Simple fixed points: a formula for the sharp trace of a transfer operator.

We assume for a moment that condition (1.3) holds for $\mathcal{M}$, but relax the smoothness assumption on the g_ω, requiring only that they are C^1 with support in E. With these assumptions, each $\bar\partial g_\omega(x)\,\sigma(\psi_\omega(x)-x)$ is in fact integrable. Recalling (1.9), and observing that there are only finitely many points $x \in \mathbf{C}$ with $\psi_\omega(x) = x$ (because (1.3) excludes the case where ψ_ω is the identity map), we get the explicit formula:

$$\mathrm{Tr}^{\#}\,\mathcal{M} = -\sum_{\omega\in\Omega}\sum_{x:\psi_\omega(x)=x} \frac{g_\omega(x)}{\partial\psi_\omega(x)-1}\,. \tag{1.38}$$

When (1.3) does not hold, i.e. when $\partial\psi_\omega(x_0) = 1$ for some $x_0 = \psi_\omega(x_0)$ in the support of g_ω, then either ψ_ω is the identity, or the tangency is of finite order, i.e., $\partial^k\psi_\omega(x_0) \neq 0$ for some $k \geq 2$. In the second case, one may thus use ∂ integration by parts to obtain a formula for the principal value of $\int \bar\partial g_\omega(x)\,\sigma(\psi_\omega x - x)\,dx$. The formula will involve in general the derivatives of $\partial^j g_\omega$ at x_0 of order $0 \leq j \leq k-1$. We just consider a simple example, where $\psi_\omega(x) = x + \alpha x^2$ with $\alpha \in \mathbf{C}$ ($\alpha \neq 0$) and g_ω is C^2. Then:

$$\begin{aligned}\int_{\mathbf{C}} \bar\partial g_\omega(x)\,\sigma(\psi_\omega x - x)\,dx &= \int_{\mathbf{C}} \bar\partial g_\omega(x)\,\sigma(\alpha x^2)\,dx = \int_{\mathbf{C}} \partial\bar\partial g_\omega(x)\,\sigma(\alpha x)\,dx \\ &= \int_{\mathbf{C}} \bar\partial\partial g_\omega(x)\,\sigma(\alpha x)\,dx = -\frac{\partial g_\omega(0)}{\alpha}\,.\end{aligned}$$

2. SPECTRAL PROPERTIES

2.1. The spaces $\mathcal{B}_{\mathbf{KL}}$.

For $K, L \geq 0$, let $\mathcal{B}_{KL}$ be the Banach space of distributions Φ with support in a fixed bounded set $B \subset \mathbf{C}$ and $\partial^K\,\bar\partial^L\Phi =$ a measure. We write $\|\Phi\|_{KL} =$ mass of $|\partial^K\,\bar\partial^L\Phi|$. We shall generally use a functional notation for the elements Φ of $\mathcal{B}_{KL}$. Note that $\partial^K\,\bar\partial^L$ is a canonical isomorphism of $\mathcal{B}_{KL}$ to a subspace of the space of measures with support in B.

If δ_0 is the unit mass at $0 \in \mathbf{C}$, the equation $\partial^u\bar\partial^v\varphi = \delta_0$ has a fundamental solution φ_{uv} such that $\varphi_{01}(z) = \frac{1}{\pi z}$, $\varphi_{10}(z) = \frac{1}{\pi\bar z}$, $\varphi_{02}(z) = \frac{\bar z}{\pi z}$, $\varphi_{11}(z) = \frac{1}{\pi}\log|z|$, $\varphi_{20}(z) = \frac{z}{\pi z}, \ldots \varphi_{21}(z) = \frac{z}{\pi}(\log|z|+1)$, etc. Therefore

$\mathcal{B}_{KL} \subset L^p(B)$	for all $p < 2$ if $K + L = 1$
$\mathcal{B}_{KL} \subset L^\infty(B)$	if $K + L = 2$ and $KL = 0$
$\mathcal{B}_{KL} \subset L^q(B)$	for all $q < \infty$ if $K = L = 1$
$\mathcal{B}_{KL} \subset \mathcal{C}_0(B)$	(continuous functions on $\mathbf{C}$ vanishing outside of B) if $K + L \geq 3$.

In fact one can prove that $\mathcal{B}_{20} \cap \mathcal{B}_{11} \cap \mathcal{B}_{02} \subset \mathcal{C}_0(B)$ [2].

Note that $\mathcal{B}_{K'L'} \subset \mathcal{B}_{KL}$ if $K' \geq K,\ L' \geq L$. Also, since $\partial^u\bar\partial^v\mathcal{B}_{KL} \subset \mathcal{B}_{K-u,L-v}$ for $u \leq K,\ v \leq L$, we have ready information on spaces in which these derivatives lie.

2.2. The operators $\mathcal{M}_{\mathbf{k}\boldsymbol{\ell}}, \widehat{\mathcal{M}}_{\mathbf{k}\boldsymbol{\ell}}$.

For $k, \ell \in \mathbf{Z}$, let

$$\mathcal{M}_{k\ell}\Phi = \sum_{\in\Omega} g_\omega.(\psi'_\omega)^k.(\bar{\psi}'_\omega)^\ell.(\Phi\circ\psi_\omega)$$
$$\widehat{\mathcal{M}}_{k\ell}\Phi = \sum_{\omega\in\Omega} (g_\omega\circ\psi_\omega^{-1}).(\psi'_\omega\circ\psi_\omega^{-1})^k.(\bar{\psi}'_\omega\circ\psi_\omega^{-1})^\ell.(\Phi\circ\psi_\omega^{-1}).$$

We assume that Ω is finite, $g_\omega \in \mathcal{C}_0(B)$, and that ψ_ω is an invertible holomorphic map $\Lambda_\omega \to \psi_\omega\Lambda_\omega$ with open $\Lambda_\omega \supset \operatorname{supp} g_\omega$ and $\Lambda_\omega,\ \psi_\omega\Lambda_\omega \subset B$. [At the cost of making more complicated assumptions below, one could take Ω countable infinite, or replace $\sum_\omega$ by an integral, or assume only that g_ω vanishes outside Λ_ω].

Let us say that the family (g_ω) is $\mathcal{B}_{KL}$ *adapted* if $g_\omega \in \mathcal{B}_{KL}$ and $K+L \geq 3$. If $K+L=2$ we require $\partial^K\bar{\partial}Lg_\omega \in L^r$ for some $r>1$. If $K+L=1$ we require $\partial^K\bar{\partial}^L g_\omega \in L^r$ for some $r>2$. Note that if $K' \leq K,\ L' \leq L$ and (g_ω) is $\mathcal{B}_{KL}$ adapted, then (g_ω) is also $\mathcal{B}_{K'L'}$ adapted.

With the above definition, if (g_ω) is $\mathcal{B}_{KL}$ adapted then $\mathcal{M}_{k\ell},\ \widehat{\mathcal{M}}_{k\ell} : \mathcal{B}_{KL} \to \mathcal{B}_{KL}$ are bounded operators for all k, ℓ. [This is readily checked by expanding $\partial^K\bar{\partial}^L\mathcal{M}_{KL}\Phi$ and using the properties of the spaces $\mathcal{B}_{KL}$ obtained in Section 2.1].

We may also let $\mathcal{M}_{k\ell}$ act on $\mathcal{C}_0(B)$; we let $R_{k\ell}, \widehat{R}_{k\ell}$ be the corresponding spectral radii:

$$R_{k\ell} = \lim_{m\to\infty}\left(\|(\mathcal{M}_{k\ell})^m\|_\infty\right)^{1/m}$$
$$\widehat{R}_{k\ell} = \lim_{m\to\infty}\left(\|(\widehat{\mathcal{M}}_{k\ell})^m\|_\infty\right)^{1/m}.$$

Let us assume that (g_ω) is $\mathcal{B}_{KL}$ adapted and $\Phi \in \mathcal{B}_{KL}$. We may write

$$\partial^K\bar{\partial}^L(\mathcal{M}_{k\ell}\Phi) = \mathcal{M}^*\partial^K\bar{\partial}^L\Phi + \mathcal{M}'\partial^K\bar{\partial}^L\Phi$$

where $\mathcal{M}^*$ and $\mathcal{M}'$ act on measures. The operator $\mathcal{M}^*$ is given in functional notation by

$$(\mathcal{M}^*\Phi)(x) = \sum_\omega g_\omega\cdot(\psi'_\omega)^{k+K}(\bar{\psi}'_\omega)^{\ell+L}(\Psi\circ\psi_\omega)$$

i.e., formally, $\mathcal{M}^* = \mathcal{M}_{k+K,\ell+L}$. We turn now to the definition of $\mathcal{M}'$. If we act repeatedly on the measure Ψ by convolution with the fundamental solution φ_{01} or φ_{10} and multiplication by a smooth function X with compact support and equal to 1 on B we obtain terms Ψ_{uv} with $u \leq K,\ v \leq L$ such that when $\Psi = \partial^K\bar{\partial}^L\Phi$, the Ψ_{uv} are the lower order derivatives $\partial^u\bar{\partial}^v\Phi$. We obtain $\mathcal{M}'\Psi$ by adding the $\Psi_{uv}\circ\psi_\omega$ multiplied by appropriate derivatives of $g_\omega,\ \psi'_\omega,\ \bar{\psi}'_\omega$ and summing over ω. For $K+L-u-v = 1, 2, \geq 3$ the maps $\Psi \to \Psi_{uv}$ are found to be compact from measures to L^p (with $p<2$), L^q (with $q<\infty$) or $\mathcal{C}_0$ $(\operatorname{supp} X)$; from this it follows that $\mathcal{M}'$ (acting on measures) is compact.

Similarly

$$\partial^K \bar\partial^L(\widehat{\mathcal M}_{k\ell}\Phi) = \widehat{\mathcal M}^*\partial^K \bar\partial^L\Phi + \widehat{\mathcal M}'\partial^K \bar\partial^L\Phi$$

where $\mathcal M^*$ and $\mathcal M'$ act on measures; $\widehat{\mathcal M}^*$ is given formally by $\widehat{\mathcal M}^* = \widehat{\mathcal M}_{k-K,\ell-L}$ and $\widehat{\mathcal M}'$ is compact.

2.3. Theorem. *Let (g_ω) be $\mathcal B_{KL}$ adapted. Then the essential spectral radius of $\mathcal M_{k\ell}$ (resp. $\widehat{\mathcal M}_{k\ell}$) acting on $\mathcal B_{KL}$ is $\leq \widehat R_{k+K-1,\ell+L-1}$ (resp. $\leq R_{k-K+1,\ell-L+1}$).*

Indeed, a direct computation shows that $\mathcal M^*$ (respectively $\widehat{\mathcal M}^*$) is the adjoint of $\widehat{\mathcal M}_{k+K-1,\ell+L-1}$ (resp. $\mathcal M_{k-K+1,\ell-L+1}$) acting on continuous functions. Therefore the spectral radius of $\mathcal M^*$ (resp. $\widehat{\mathcal M}^*$) is $\leq \widehat R_{k+K-1,\ell+L-1}$ (resp. $\leq R_{k-K+1,\ell-L+1}$). The theorem results from the fact that a compact perturbation does not change the essential spectral radius, and from the fact that $\partial^K \bar\partial^L$ maps $\mathcal B_{KL}$ isometrically into the measures, replacing $\mathcal M_{k\ell}$ (resp. $\widehat{\mathcal M}_{k\ell}$) by $\mathcal M^* + \mathcal M'$ (resp. $\widehat{\mathcal M}^* + \widehat{\mathcal M}'$). □

2.4. Theorem. *Let (g_ω) be $\mathcal B_{KL}$ adapted and $K + L \geq 2$.*

(a) *The spectral radius of $\mathcal M_{k\ell}$ (resp. $\widehat{\mathcal M}_{k\ell}$) is $\leq \max(\widehat R_{k+K-1,\ell+L-1}, R_{k\ell})$ (resp. $\leq \max(R_{k-K+1,\ell-L+1}, \widehat R_{k\ell})$.*

(b) *The generalized eigenfunctions corresponding to eigenvalues λ satisfying $|\lambda| > \widehat R_{k+K-1,\ell+L-1}$ (resp. $> R_{k-K+1,\ell-L+1}$) are continuous.*

Let ρ be the spectral radius of $\mathcal M_{k\ell}$ acting on $\mathcal B_{KL}$. It suffices to prove (a) under the assumption $\rho > \widehat R_{k+K-1,\ell+L-1}$. This assumption implies that $\mathcal M_{k\ell}$ has an eigenvalue λ with $|\lambda| = \rho$ and a corresponding eigenfunction $\Phi \in \mathcal B_{KL}$. If (b) holds then Φ is bounded hence $\rho = |\lambda| \leq R_{k\ell}$, proving (a).

Since $\mathcal B_{KL} \subset \mathcal C_0(B)$ when $K+L \geq 3$, it suffices to prove (b) for $K+L = 2$. Let λ, with $|\lambda| > \widehat R_{k+K-1,\ell+L-1}$, be an eigenvalue of $\mathcal M_{k\ell}$ acting on $\mathcal B_{KL}$, and Φ be a corresponding eigenfunction. We have thus

$$(\mathcal M^* - \lambda)\partial^K \bar\partial^L\Phi = -\mathcal M'\partial^K \bar\partial^L\Phi$$

and the right hand side is in $L^r(B)$ for some $r > 1$ because $\mathcal M_{k\ell}$ is $\mathcal B_{KL}$ adapted. The spectral radius of $\mathcal M^*$ acting on L' is $\leq \widehat R_{k+K-1,\ell+L-1}$ (as noted in the proof of theorem 2.3) hence $< |\lambda|$, and $\mathcal M^*$ is also bounded on L^∞. Using the Riesz-Thorin interpolation theorem we therefore see that the spectral radius of $\mathcal M^*$ acting on L^r becomes $< |\lambda|$ if r is sufficiently close to 1. We can now conclude that

$$\partial^K \bar\partial^L\Phi = -(\mathcal M^* - \lambda)^{-1}\mathcal M'\partial^K \bar\partial^L\Phi$$

is in L^r, and Φ is thus in $\mathcal C_0(B)$. Generalized eigenfunctions are treated in similar manner, and this completes the verification of (b).

The case of $\widehat{\mathcal M}_{k\ell}$ is handled by the same arguments. □

Remark. Let (g_ω) be $\mathcal{B}_{KL}$ adapted, with $K+L=1$. If Φ is a (generalized) eigenfunction of $\mathcal{M}_{k\ell}$ to an eigenvalue λ with $|\lambda| > \widehat{R}_{k+K-1,\ell+L-1}$, we have as above $\partial^K\,\bar\partial^L\Phi \in L^r$ for some $r>1$. Therefore $\Phi \in L^p$ for some $p>2$.

2.5. Theorem. *Let $g_\omega \geq 0$, (g_ω) be $\mathcal{B}_{KL}$ adapted with $K+L\geq 2$, and $\widehat{R}_{k+K-1,k+L-1} < R_{kk}$ (resp. $R_{k-K+1,k-L+1} < \widehat{R}_{kk}$). Then R_{kk} is an eigenvalue of $\mathcal{M}_{kk}$ acting on $\mathcal{B}_{KL}$ (resp. $\widehat{R}_{kk}$ is an eigenvalue of $\widehat{\mathcal{M}}_{kk}$ acting on $\mathcal{B}_{KL}$) and there is a corresponding eigenfunction $\Phi \geq 0$.*

A very similar result has been proved in a related situation [4]. We give here again a complete proof.

We shall use the notation $\mathcal{M}=\mathcal{M}_{kk}$, $R=R_{kk}$, $||\cdot|| = ||\cdot||_{KL}$. We can take χ smooth such that χ takes values in $[0,1]$, has support in B, and is 1 on the $\psi_\omega \operatorname{supp} g_\omega$.

The spectral radius ρ of $\mathcal{M}$ acting on $\mathcal{B}_{KL}$ is the same as the spectral radius of the operator $\mathcal{M}^*+\mathcal{M}'$ acting on the space $\partial^K\,\bar\partial^L\mathcal{B}_{KL}$ of measures (the operators $\mathcal{M}^*$, $\mathcal{M}'$ are defined as earlier). Therefore

$$\begin{gathered}\text{spectral radius of } \mathcal{M}^*+\mathcal{M}' \\ \text{acting on } \partial^K\,\bar\partial^L\mathcal{B}_{KL}\cap L^1 \leq \rho.\end{gathered}$$

Writing

$$\begin{aligned}\sigma &= \limsup_{m\to\infty} ||\mathcal{M}^m\chi||^{1/m} \\ &= \limsup_{m\to\infty} \left(\left\|(\mathcal{M}^*+\mathcal{M}')^m\partial^K\,\bar\partial^L\chi\right\|_1\right)^{1/m}\end{aligned}$$

we also have $\sigma \leq \rho$.

Because (g_ω) is $\mathcal{B}_{KL}$ adapted, $\mathcal{M}'$ is bounded $L^1 \to L^r$ for some $r>1$ and so $\mathcal{M}^*+\mathcal{M}'$ is bounded $L^r \to L^r$. Therefore, given $\varepsilon > 0$, we can find (using the Hölder inequality) $s>1$ such that

$$\limsup_{m\to\infty} \left(\left\|(\mathcal{M}^*+\mathcal{M}')^m\partial^K\,\bar\partial^L\chi\right\|_s\right)^{1/m} < \sigma+\varepsilon.$$

We have (using the positivity of the g_ω)

$$\begin{aligned}R &= \lim_{m\to\infty} (\text{norm of } \mathcal{M}^m \text{ acting on } L^\infty)^{1/m} \\ &= \lim_{m\to\infty} (||\mathcal{M}^m\chi||_\infty)^{1/m} \\ &\leq \limsup_{m\to\infty} (\text{const}\,.\left\|\partial^K\,\bar\partial^L(\mathcal{M}^m\chi)\right\|_s)^{1/m} \\ &= \limsup_{m\to\infty} \left(\left\|(\mathcal{M}^*+\mathcal{M}')^m\partial^K\,\bar\partial^L\chi\right\|_s\right)^{1/m} \\ &< \sigma+\varepsilon.\end{aligned}$$

Hence $R \leq \sigma \leq \rho$ and, if $R > \widehat{R}_{k+K-1,k+L-1}$, Theorem 2.4 gives $\rho \leq R$. Therefore $\sigma=\rho=R$ and

$$\limsup_{m\to\infty} ||\mathcal{M}^m\chi||^{1/m} = R.$$

Replacing everywhere lim sup by lim inf we see that

$$\lim_{m\to\infty} ||\mathcal{M}^m \chi||^{1/m} = R \tag{1}$$

We may write

$$\chi = \Psi + \sum_j \Psi_j \tag{2}$$

where, for each j, λ_j is an eigenvalue of $\mathcal{M}$ (acting on $\mathcal{B}$) with $|\lambda_j| = R$ and Ψ_j is in the corresponding generalized eigenspace; Ψ is such that

$$\lim_{m\to\infty} \frac{||\mathcal{M}^m \Psi||}{\widetilde{\lambda}^m} = 0$$

with $0 < \widetilde{\lambda} < R$. In view of (1) the Ψ_j do not all vanish. Write the restriction of $\mathcal{M}$ to the generalized eigenspaces corresponding to the λ_j in Jordan normal form: it is then readily seen that there is an integer $k \geq 0$ such that

$$\lim_{m\to\infty} \frac{1}{\lambda_j^m m^k} \mathcal{M}^m \Psi_j = \Phi_j$$

and

$$\mathcal{M}\Phi_j = \lambda \Phi_j$$

for all j, and $\Phi_j \neq 0$ for some j. From (2) we get

$$0 \leq \frac{\mathcal{M}^m \chi}{R^m m^k} = \frac{\mathcal{M}^m \Psi}{R^m m^k} + \sum_j \left(\frac{\lambda_j}{R}\right)^m \frac{\mathcal{M}^m \Psi_j}{\lambda_j^m m^k}.$$

Therefore

$$\sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi_j \geq -\varphi_m \tag{3}$$

where both sides of this inequality are real functions $\in \mathcal{B}$ and

$$\varphi_m = \frac{\mathcal{M}^m \Psi}{R^m m^k} + \sum_j \left(\frac{\lambda_j}{R}\right)^m \left(\frac{\mathcal{M}^m \Psi_j}{\lambda_j^m m^k} - \Phi_j\right).$$

Note that $\varphi_m \to 0$ in $\mathcal{B}$, hence in L^q for $q < \infty$.

Let $\langle\quad\rangle_m$ denote the average $\lim\limits_{M\to+\infty} \frac{1}{M} \sum\limits_{m=0}^{M-1}$, and write $\lambda_0 = R$ (with $\Psi_0 = \Phi_0 = 0$ if λ_0 is not an eigenvalue of $\mathcal{M}$). For arbitrary real α, β, we have

$$\left\langle (1 + \sin(m\,\alpha + \beta)) \sum_j \left(\frac{\lambda j}{R}\right)^m \Phi_j \right\rangle_m$$
$$\geq -\big\langle (1 + \sin(m\,\alpha + \beta))\varphi_m \big\rangle_m$$

hence

$$\Phi_0 + \left\langle \sin(m\alpha+\beta) \sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi_j \right\rangle_m \geq 0.$$

If we had $\Phi_0 = 0$, taking $\beta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$ would give

$$\left\langle e^{im\alpha} \sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi_j \right\rangle_m = 0$$

implying that $\Phi_j = 0$ for all j contrary to our assumptions. Therefore $\lambda_0 = R$ is an eigenvalue and (taking $\beta = 0, \pi$)

$$\Phi_0 \pm \left\langle \sin m\,\alpha \sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi j \right\rangle_m \geq 0$$

so that $\Phi_0 \geq 0$, and Φ_0 is not identically 0. □

3. TRACES AND DETERMINANTS – EXISTENCE.

In this section we want to justify the existence of the traces and determinants of some of the operators which arise in this paper. We begin with some attention to the function spaces on which we shall work.

Given a compact set $E \subseteq \mathbf{C}$, let $\mathcal{B}(E) = \mathcal{B}_{0,2}(E)$ denote the space of distributions f on $\mathbf{C}$ such that $\operatorname{supp} f \subseteq E$ and $\overline{\partial}^2 f$ (taken in the sense of distributions) is a finite measure on $\mathbf{C}$. We define the $\mathcal{B}$-norm of f by

$$||f||_{\mathcal{B}} = \int_{\mathbf{C}} |\overline{\partial}^2 f|, \tag{3.1}$$

and we use this norm for all the spaces $\mathcal{B}(E)$.

Lemma 3.2. *If $f \in \mathcal{B}(E)$, then f is in fact represented by a bounded measurable function (which we also denote by f) and we have that*

$$||f||_\infty \leq \frac{2}{\pi} \int_{\mathbf{C}} |\overline{\partial}^2 f|. \tag{3.3}$$

To see this we first recall that

$$\overline{\partial}\,(\frac{1}{x}) = \pi\,\delta_0 \quad \text{and} \quad \overline{\partial}^2(\frac{\overline{x}}{x}) = \pi\,\delta_0 \tag{3.4}$$

in the sense of distributions on $\mathbf{C}$, where δ_0 denotes the Dirac mass at the origin. The first equation is well-known, and the second equation follows from the first. (Note that $\frac{1}{x}$ is locally integrable on $\mathbf{C}$.)

Let $f \in \mathcal{B}(E)$ be given, and define F by

$$F(x) = \int_{\mathbf{C}} \frac{1}{\pi} \frac{\overline{x} - \overline{y}}{x - y} \, \overline{\partial}^2 f(y). \tag{3.5}$$

This is a bounded measurable function on $\mathbf{C}$, since $\overline{\partial}^2 f$ is a finite measure, and we have that

$$\|F\|_\infty \leq \frac{1}{\pi} \int_{\mathbf{C}} |\overline{\partial}^2 f|. \tag{3.6}$$

We also have that $\overline{\partial}^2 F = \overline{\partial}^2 f$ (in the sense of distributions), because of (3.4). Thus $F - f$ is holomorphic, and it is therefore constant, since F is bounded and f has compact support. This constant is bounded by $\|F\|_\infty$, and (3.3) follows. This proves Lemma 3.2.

Lemma 3.7. *If $E \subseteq \mathbf{C}$ is compact and $f \in \mathcal{B}(E)$, then $\overline{\partial} f$ (taken in the sense of distributions) is an integrable function on $\mathbf{C}$ which is given by*

$$\overline{\partial} f(x) = \int_{\mathbf{C}} \frac{1}{\pi} \frac{1}{x - y} \, \overline{\partial}^2 f(y). \tag{3.8}$$

Also, $f \mapsto \overline{\partial} f$ defines a bounded linear operator from $\mathcal{B}(E)$ into $L^p(E)$ for all $p < 2$.

Indeed, if $f \in \mathcal{B}(E)$ and F is defined by (3.5), then we saw above that $F - f$ is constant. Thus $\overline{\partial} f = \overline{\partial} F$. This permits us to derive (3.8) from (3.5) by standard arguments in distribution theory. Of course $\overline{\partial} f \equiv 0$ on $\mathbf{C} \backslash E$, since $\operatorname{supp} f \subseteq E$. The last part, about $\overline{\partial} f \in L^p(E)$, follows from the fact that $\frac{1}{x} \in L^p_{loc}$ for all $p < 2$. This proves Lemma 3.7.

In the next lemmas we accumulate some facts about linear operators acting on $\mathcal{B}(E)$.

Lemma 3.9. *Let E be a compact subset of $\mathbf{C}$. Suppose that*

$$\begin{aligned} &g \text{ is a continuous function on } \mathbf{C}, \\ &\text{the second derivatives of } g \text{ are locally integrable,} \\ &\text{and } \nabla g \in L^p_{loc} \text{ for some } p > 2. \end{aligned} \tag{3.10}$$

(All these derivatives of g are taken in the sense of distributions.) Then $f \mapsto g\, f$ defines a bounded linear operator on $\mathcal{B}(E)$. If the support of g is contained in some compact set $W \subseteq \mathbf{C}$, then $f \mapsto g\, f$ defines a bounded linear operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$.

According to the Leibniz rule we have that

$$\overline{\partial}^2 (gf) = (\overline{\partial}^2 g)\, f + 2(\overline{\partial}\, g)\, (\overline{\partial}\, f) + g\, (\overline{\partial}^2 f). \tag{3.11}$$

Since we are working with distributional derivatives we should be a little bit careful. This identity would be a standard tautology from distribution theory if g were C^∞. In general the right hand side makes sense as a measure because of (3.10) and Lemmas 3.2 and 3.7, and it can be derived from the case where g is C^∞ by a standard approximation argument.

Lemma 3.9 follows easily from the identity (3.11) and the bounds in Lemmas 3.2 and 3.7.

Recall that if $\eta \in L^1(\mathbf{C})$ and if $h \in L^p(\mathbf{C})$ for some $p, 1 \leq p \leq \infty$, then $\eta * h \in L^p(\mathbf{C})$ and

$$\|\eta * h\|_p \leq \|\eta\|_1 \, \|h\|_p.$$

Here $\eta * h$ denotes convolution between η and h. The next lemma provides an analogous result for the $\mathcal{B}$ spaces.

Lemma 3.12. *Suppose that g satisfies (3.10) and has support contained in the compact set W, and let θ be a locally integrable function on $\mathbf{C}$. Then $f \mapsto g\,(\theta * f)$ defines a bounded linear operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$ for every compact set $E \subseteq \mathbf{C}$. This operator is also compact, and in fact it can be approximated in the operator norm by finite rank operators of the form $f \mapsto \sum_i g\, a_i \langle f, b_i \rangle$, where $\{a_i\}, \{b_i\}$ are finite sequences of polynomials, and $\langle \cdot, \cdot \rangle$ denotes the standard pairing of functions on $\mathbf{C}$.*

Consider first the part about the boundedness of the operator. The main point is that convolution commutes with differentiation, so that $\overline{\partial}^2 \theta * f = \theta * (\overline{\partial}^2 f)$, etc. This permits us to control $\theta * f$ and its first two $\overline{\partial}$ derivatives in terms of f and its first two $\overline{\partial}$ derivatives, and then we can get $g\,(\theta * f)$ into $\mathcal{B}(W)$ using the same argument as in Lemma 3.9.

Now consider the second part, about compactness. Given any compact set K in $\mathbf{C}$, we can approximate θ by polynomials in $L^1(K)$. By choosing K correctly we can approximate our operator $f \mapsto g\,(\theta * f)$ in the operator norm $\mathcal{B}(E) \to \mathcal{B}(W)$ by operators of the same form, but with θ replaced by polynomials. This uses the fact that if $\eta \in L^1_{loc}$, then the operator norm of $f \mapsto g\,(\eta * f)$, as an operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$, is controlled by the $L^1(K)$ norm of η if we choose K correctly. This fact can be checked using the argument of the preceding paragraph. A simple computation shows that $f \mapsto g\,(\theta * f)$ is a finite rank operator of the desired form when θ is a polynomial.

This proves Lemma 3.12.

Let us call a linear operator acting on functions on $\mathbf{C}$ a transfer operator if it can be expressed in the form

$$\mathcal{M}\Phi(x) = \sum_{\omega \in \Omega} g_\omega(x)\, \Phi \circ \psi_\omega(x). \tag{3.13}$$

As before we assume that Ω is finite, that each g_ω is at least a continuous function with compact support, that each ψ_ω is a holomorphic diffeomorphism from an open set $\Lambda_\omega \subseteq \mathbf{C}$ onto its image, and that $\Lambda_\omega \supseteq \operatorname{supp} g_\omega$. Although $\Phi \circ \psi_\omega(x)$ is not defined for all x, it is defined on the support of g_ω, and we can simply interpret $\Phi \circ \psi_\omega(x)$ to be 0 when x does not lie in the domain of ψ_ω.

Note that these transfer operators are well-defined on L^1_{loc} functions, for instance, because the ψ_ω's are diffeomorphisms (on their domains) and preserve sets of Lebesgue measure 0 in particular. We shall frequently use the fact that transfer operators are bounded on L^∞.

Lemma 3.14. *Let E and W be compact subsets of $\mathbf{C}$. If $\mathcal{M}$ is a transfer operator as in (3.13), if each g_ω satisfies (3.10), and if $\operatorname{supp} g_\omega \subseteq W$ for each ω, then $\mathcal{M}$ defines a bounded linear operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$.*

In order to deal with this in a reasonable way it is helpful to establish first a technical fact about membership in $\mathcal{B}(W)$.

Sublemma 3.15. *An integrable function f on $\mathbf{C}$ lies in $\mathcal{B}(W)$ if and only if there is a sequence of smooth functions $\{f_j\}$ such that $f_j \to f$ in the L^1 norm,* $\sup_j \|f_j\|_{\mathcal{B}} < \infty$*, and the supports of the f_j's shrink to a subset of W, in the sense that for each open set $U \supseteq W$ there is an l such that* $\operatorname{supp} f_j \subseteq U$ *when $j > l$.*

The proof of this is standard. If $f \in \mathcal{B}(W)$, then we can get a sequence $\{f_j\}$ as above by convolving f with an approximation to the identity with supports shrinking to $\{0\}$. The resulting functions will have supports shrinking to a subset of W, and the $\mathcal{B}$ norms will remain bounded because derivatives commute with convolution, and because the convolution of an L^1 function with a measure is again an L^1 function whose L^1 norm is bounded by the product of the L^1 norm of the original function and the total variation of the measure. Conversely, suppose that f is integrable and that there exists an approximating sequence $\{f_j\}$ as in the sublemma. Then we have that $\overline{\partial}^2 f_j \to \overline{\partial}^2 f$ in the sense of distributions. Our assumptions on $\{f_j\}$ imply a uniform mass bound on $\overline{\partial}^2 f_j$, and hence $\overline{\partial}^2 f$ must be a finite measure. Since f is certainly supported in W we get that $f \in \mathcal{B}(W)$, as desired. This proves Sublemma 3.15.

Let us come back now to the proof of Lemma 3.14. If we assume that Φ is smooth, or if we compute formally, then we can get that $\|\mathcal{M}\Phi\|_{\mathcal{B}} \le C\|\Phi\|_{\mathcal{B}}$ for some constant C which does not depend on Φ. This uses a Leibniz computation like (3.11) and our assumptions on the g_ω's to reduce the problem to one of having estimates for $\Phi \circ \psi_\omega$ and its $\overline{\partial}$ derivatives on the support of g_ω, in the same way as in the proof of Lemma 3.9. The holomorphicity of the ψ_ω's permits us to compute $\overline{\partial}$ derivatives of $\Phi \circ \psi_\omega$ in terms of $\overline{\partial}$ derivatives of Φ, with extra multiplicative factors coming from the derivatives of the ψ_ω's. Composition with ψ_ω and multiplication by its derivatives do not disturb integrability properties, at least if we remain within the compact subset $\operatorname{supp} g_\omega$ of the domain of ψ_ω, as we do here. Thus we can bound $\|\mathcal{M}\Phi\|_{\mathcal{B}} = \|\overline{\partial}^2 \mathcal{M}\Phi\|_1$ in terms of $\|\Phi\|_{\mathcal{B}}$, using also Lemmas 3.2 and 3.7. Once we have this bound for all smooth functions Φ with compact support we can use Sublemma 3.15 to get that $\mathcal{M}$ actually maps $\mathcal{B}(E)$ into $\mathcal{B}(W)$. This proves Lemma 3.14.

Another operator that we shall be interested in is the operator S defined by

$$Sf(x) = \int_{\mathbf{C}} \frac{1}{\pi} \frac{1}{x-y} f(y)\, dy. \tag{3.16}$$

This operator sends integrable functions with compact support to locally integrable functions on $\mathbf{C}$, or, more generally, it sends distributions with compact support to distributions. It is the inverse to $\overline{\partial}$, in the sense that

$$\overline{\partial}\,(Sf) = f \quad \text{and} \quad S(\overline{\partial}\, f) = f \tag{3.17}$$

for all distributions f on $\mathbf{C}$ with compact support. These equations come down to the first formula in (3.4), by the standard tricks in distribution theory. (That is, the distributional interpretation of (3.17) converts these equations, via duality, to their counterparts for test functions, which can then be reduced to (3.4).)

We shall also want to have a nice one-parameter family of regularizations of S. Fix a C^∞ function ν on $\mathbf{C}$, once and for all, with

$$\nu(x) = 1 \text{ when } |x| \geq 1 \quad \text{and} \quad \nu(x) = 0 \text{ when } |x| \leq \frac{1}{2}. \tag{3.18}$$

Define an operator S_t for $t > 0$ by

$$S_t f(x) = \int_{\mathbf{C}} \frac{1}{\pi} \frac{1}{x-y} \nu(\frac{x-y}{t}) f(y)\, dy. \tag{3.19}$$

Again this operator acts on functions or distributions with compact support. Notice that

$$\begin{aligned} \frac{1}{x-y} \nu(\frac{x-y}{t}) &= \frac{1}{x-y} \qquad \text{when } |x-y| \geq t, \\ &= 0 \qquad \text{when } |x-y| \leq t/2. \end{aligned} \tag{3.20}$$

Thus S_t approximates S but its kernel has no singularity.

In practice one should think of t as being small, or at least not large, and for simplicity we shall restrict our attention to $t \leq 1$.

Lemma 3.21. *Suppose that g satisfies (3.10) and has support contained in the compact set W. Then $f \mapsto g\,(Sf)$ and $f \mapsto g\,(S_t f)$ define bounded linear operators from $\mathcal{B}(E)$ into $\mathcal{B}(W)$ for every compact set $E \subseteq \mathbf{C}$ and for every $t \in (0,1]$. Moreover, the second operator converges to the first one as $t \to 0$ in operator norm (defined using $\|\cdot\|_{\mathcal{B}}$). In fact, the operator norm of $gS - gS_t$ is $O(t)$.*

The boundedness of these operators follows from Lemma 3.12, because they are given in terms of convolutions of locally integrable functions. The convergence in the operator norm follows from the fact that $\frac{1}{x}\nu(\frac{x}{t})$ converges to $\frac{1}{x}$ as $t \to 0$ in L^1 on any compact subset of $\mathbf{C}$. In fact

$$\int_{\mathbf{C}} |\frac{1}{x} - \frac{1}{x}\nu(\frac{x}{t})|\, dx = \text{ constant} \cdot t,$$

as one can easily check. This implies that $gS - gS_t$ has norm $= O(t)$, because the norm of the operator described in Lemma 3.12 is controlled by the L^1 norm of the function θ given there. This proves Lemma 3.21.

One of our main goals in this section is to show that certain operators constructed from S are trace class. Let us recall the definition.

If X and Y are Banach spaces, then we say that a linear operator $T : X \to Y$ is trace class (or nuclear) if it can be represented as a sum of rank one operators,

$$Tu = \sum_j \lambda_j\, w_j \langle u, v_j \rangle, \tag{3.22}$$

where the w_j's are elements of the unit ball of Y, the v_j's are elements of the unit ball of the dual space of X, $\langle \cdot, \cdot \rangle$ denotes the pairing between X and its dual space, and the λ_j's are scalars which satisfy $\sum_j |\lambda_j| < \infty$. The infimum of $\sum_j |\lambda_j|$ over all such representations of T is called the trace norm of T.

Note that the composition of a bounded operator and a trace class operator (in either order) is trace class.

Unfortunately operators like $g\,S$ are not trace class, the singularity in the kernel is too strong. We shall see that products of S's and commutators with S's can give rise to trace class operators in a natural way. Or almost anyway; we shall see that products of S_t's and commutators with S_t's give rise to trace class operators whose trace norm is bounded uniformly in t.

Let us begin with a general criterion for an integral operator to be trace class.

Lemma 3.23. *Let X be a Banach space and let E be a compact subset of $\mathbf{C}$. Suppose that $H : E \to X$ is continuous (using the norm topology for X). Define an operator $T : L^\infty(E) \to X$ by*

$$Tf = \int_{\mathbf{C}} H(y)\, f(y)\, dy. \tag{3.24}$$

Then T is trace class, and

$$\text{trace norm } T \le |E| \cdot \sup_{y \in E} \|H(y)\|_X, \tag{3.25}$$

where $|E|$ denotes the Lebesgue measure of E.

This result is pretty standard, but let us prove it for the sake of completeness. We need to approximate T by finite rank operators, and so we approximate the integral in (3.24) by Riemann sums.

Incidentally, H is uniformly continuous on E, since E is compact, and this ensures that there is no funny business in defining the integral in (3.24). This will be made explicit in the argument that follows.

Fix a square Q_0 in $\mathbf{C}$ which contains E. For each $j = 0, 1, 2, \ldots$ let $\Delta(j)$ denote the obvious decomposition of Q_0 into 2^{2j} subsquares of Q_0, each of size 2^{-j} times the size of Q_0, with sides parallel to the same axes as for Q_0, and with the elements of $\Delta(j)$ having disjoint interiors. Thus when $j = 0$ we simply get Q_0 back again, when $j = 1$ we get the usual decomposition of Q_0 into four pieces, etc.

If Q is any square in $\mathbf{C}$, let $\gamma(Q)$ denote its center.

Given $j \ge 0$ define $T_j : L^\infty(E) \to X$ by

$$T_j f(x) = \sum_{Q \in \Delta(j)} H(\gamma(Q)) \int_{Q \cap E} f(y)\, dy. \tag{3.26}$$

Let us check that this finite-rank operator satisfies

$$\text{trace norm } T_j \le |E| \cdot \sup_{y \in E} \|H(y)\|_X. \tag{3.27}$$

This comes down to the observation that

$$f \mapsto \int_{Q\cap E} f(y)\,dy \quad \text{is a bounded linear functional on } L^\infty(E) \\ \text{with norm } \le |Q \cap E|. \tag{3.28}$$

Next let us check that

$$\lim_{j,k\to\infty} \text{ trace norm } (T_k - T_j) = 0. \tag{3.29}$$

Let j and k be large and given, with $k > j$. Notice that every $Q \in \Delta(k)$ has a unique "ancestor" $\widehat{Q}(j)$ in $\Delta(j)$, where $Q \subseteq \widehat{Q}(j)$ but the interior of Q is disjoint from all other elements of $\Delta(j)$. In other words the elements of $\Delta(k)$ can be obtained by subdividing the elements of $\Delta(j)$. This observation leads us to the formula

$$\begin{aligned}(T_k - T_j)f &= \sum_{Q\in\Delta(k)} H(\gamma(Q)) \int_{Q\cap E} f(y)\,dy - \sum_{R\in\Delta(j)} H(\gamma(R)) \int_{R\cap E} f(y)\,dy \\ &= \sum_{Q\in\Delta(k)} (H(\gamma(Q)) - H(\gamma(\widehat{Q}(j)))) \int_{Q\cap E} f(y)\,dy.\end{aligned} \tag{3.30}$$

This formula uses the fact that every element R of $\Delta(j)$ is the disjoint union of its descendants in $\Delta(k)$ (so that $\int_{R\cap E} f$ equals the sum of the corresponding integrals for the descendants Q of R in $\Delta(k)$). On the other hand the (uniform) continuity of H implies that

$$\lim_{j,k\to\infty} \max_{Q\in\Delta(k)} \|H(\gamma(Q)) - H(\gamma(\widehat{Q}(j))\|_X = 0. \tag{3.31}$$

It is easy to derive (3.29) from (3.30), (3.31), and (3.28).

Thus $\{T_j\}$ is a Cauchy sequence with respect to the trace norm. Of course $\{T_j\}$ converges to T in the operator norm. Standard arguments imply that T is trace class and that

$$\text{trace norm } T \le \liminf_{j\to\infty} \text{ trace norm } T_j \le |E| \cdot \sup_{y\in E} \|H(y)\|_X. \tag{3.32}$$

This completes the proof of Lemma 3.23.

In practice we shall want to apply Lemma 3.23 with $L^\infty(E)$ replaced with $\mathcal{B}(E)$, and we can do that freely because of Lemma 3.2. We shall also normally take X to be some $\mathcal{B}$ space.

Now we want to look at operators constructed from S, starting with the very smooth S_t's.

Lemma 3.33. *Let E and W be compact subsets of* $\mathbf{C}$, *and suppose that g is supported in W and satisfies (3.10). Then $f \mapsto g\,(S_t f)$ defines a trace class operator from $L^\infty(E)$ into $\mathcal{B}(W)$ for every $t \in (0,1]$, and the trace norm is $O(t^{-1})$.*

The hypothesis on g is not sharp, but it is adequate. We shall not try to get the sharpest results here.

As above, we can replace $L^\infty(E)$ with $\mathcal{B}(E)$ for free, because of Lemma 3.2.

Let us first observe that we can reduce to the case where g is smooth. Indeed, let χ be a smooth function with support contained in some compact set W' and with $\chi \equiv 1$ on W. Then we can view $g\,S_t : L^\infty(E) \to \mathcal{B}(W)$ as the composition of $\chi\,S_t : L^\infty(E) \to \mathcal{B}(W')$ and the operator of multiplication by g, which defines a bounded operator from $\mathcal{B}(W')$ into $\mathcal{B}(W)$ (Lemma 3.9). This permits us to reduce to the case where W, g are replaced with W', χ. Thus we may assume that g is smooth.

We want to apply Lemma 3.23. We can write $g\,S_t$ explicitly as

$$(g\,S_t)f(x) = \frac{1}{\pi}\int_{\mathbf{C}} g(x)\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\,f(y)\,dy. \tag{3.34}$$

The mapping

$$y \to g(x)\,\frac{1}{x-y}\,\nu(\frac{x-y}{t}) \tag{3.35}$$

defines a continuous map from $\mathbf{C}$ into $\mathcal{B}(W)$. This is easy to check, using the smoothness of $\frac{1}{x-y}\,\nu(\frac{x-y}{t})$. Therefore $g\,S_t$ defines a trace class map from $L^\infty(E)$ into $\mathcal{B}(W)$. We also get that

$$\begin{aligned}\text{trace norm } g\,S_t \;&\le\; |E|\cdot\sup_{y\in E}\,\|\,g(x)\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\,\|_{\mathcal{B}} \\ &\le\; |E|\cdot\sup_{y\in E}\int_{\mathbf{C}} |\,\overline{\partial}_x^2\{\,g(x)\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\}|\,dx\end{aligned} \tag{3.36}$$

We want to show that the right hand side is $O(t^{-1})$.

Notice that

$$\begin{aligned}|\,\overline{\partial}_x^2\{\,g(x)\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\}| &= \frac{1}{|x-y|}|\,\overline{\partial}_x^2\{\,g(x)\,\nu(\frac{x-y}{t})\}| \\ &\le \frac{2}{t}\,|\,\overline{\partial}_x^2\{\,g(x)\,\nu(\frac{x-y}{t})\}|,\end{aligned} \tag{3.37}$$

since $\frac{1}{x-y}$ is holomorphic in x away from the pole $x = y$, and because $\nu(\frac{x-y}{t})$ vanishes when $|x-y| \le t/2$, by (3.18). Therefore

$$\text{trace norm } g\,S_t \;\le\; |E|\cdot\sup_{y\in E}\int_{\mathbf{C}} \frac{2}{t}\,|\,\overline{\partial}_x^2\{\,g(x)\,\nu(\frac{x-y}{t})\}|\,dx. \tag{3.38}$$

Thus it suffices to show that

$$\sup_{0<t\le 1}\,\sup_{y\in E}\int_{\mathbf{C}} |\,\overline{\partial}_x^2\{\,g(x)\,\nu(\frac{x-y}{t})\}|\,dx < \infty. \tag{3.39}$$

Let us first make some preliminary observations. Using (3.18) and calculus we get that

$$\overline{\partial}_x\,\nu(\frac{x-y}{t}) = \overline{\partial}_x^2\,\nu(\frac{x-y}{t}) = 0 \quad \text{ when } |x-y| > t \text{ or } |x-y| < \frac{t}{2}, \tag{3.40}$$

$$\sup_{x,y\in\mathbf{C}} |\overline{\partial}_x\,\nu(\frac{x-y}{t})| = \sup_{x,y\in\mathbf{C}} t^{-1}\,|(\overline{\partial}\nu)(\frac{x-y}{t})| = O(t^{-1}), \qquad \text{and} \tag{3.41}$$

$$\sup_{x,y\in\mathbf{C}} |\overline{\partial}_x^2\,\nu(\frac{x-y}{t})| = \sup_{x,y\in\mathbf{C}} t^{-2}\,|(\overline{\partial}^2\nu)(\frac{x-y}{t})| = O(t^{-2}). \tag{3.42}$$

Hence

$$\int_{\mathbf{C}} |\overline{\partial}_x \, \nu(\frac{x-y}{t})| = O(t) \quad \text{and} \tag{3.43}$$

$$\int_{\mathbf{C}} |\overline{\partial}_x^2 \, \nu(\frac{x-y}{t})| = O(1). \tag{3.44}$$

To prove (3.39) one simply uses the Leibniz formula and remembers that g is smooth and has compact support. When both $\overline{\partial}_x$'s land on g the integral is bounded because ν is bounded. When one $\overline{\partial}_x$ lands on g and one lands on the ν we can control the integral using (3.43). When both $\overline{\partial}_x$'s land on ν we use (3.44).

This proves (3.39), and Lemma 3.33 follows.

Lemma 3.45. *Let E and W be compact subsets of $\mathbf{C}$, and let g and h be functions on $\mathbf{C}$ such that g satisfies (3.10),* $\operatorname{supp} g \subseteq W$*, and*

$$h \in C^1 \quad \textit{and} \quad \overline{\partial}^2 h \in L^p_{loc} \ \textit{for some } p > 2. \tag{3.46}$$

Then the operator $g\,[h, S_t] : L^\infty(E) \to \mathcal{B}(W)$ is trace class for each $t \in (0,1]$, where we identify g and h with their corresponding multiplication operators, and

$$\sup_{0<t\leq 1} \text{trace norm } g\,[h, S_t] < \infty. \tag{3.47}$$

To prove this we first observe that we may take g to be smooth. This follows from the same argument as in the corresponding step of the proof of Lemma 3.33.

To show that $g\,[h, S_t]$ is trace class we apply Lemma 3.23. We need to check that

$$y \to g(x)\,(h(x) - h(y))\,\frac{1}{x-y}\,\nu(\frac{x-y}{t}) \tag{3.48}$$

defines a continuous map from $\mathbf{C}$ into $\mathcal{B}(W)$, and we need to bound its supremum norm. The factor of $g(x)$ already ensures that these functions of x are supported in W, and so it suffices to show that

$$y \to \overline{\partial}_x^2 \{\, g(x)\,(h(x) - h(y))\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\} \tag{3.49}$$

defines a continuous map from $\mathbf{C}$ into the finite measures on $\mathbf{C}$ (continuity with respect to the total variation norm on the space of finite measures on $\mathbf{C}$), and to bound the total variations of these measures. In this case these measures are given by integrable functions, because the ν kills the singularity in $\frac{1}{x-y}$ and because of our assumptions (3.46) on h, and the continuity of (3.49) as a map into $L^1(\mathbf{C})$ is immediate. The issue is to control the L^1 norms, i.e., to prove that

$$\sup_{0<t\leq 1} \sup_{y\in E} \int_W |\overline{\partial}_x^2 \{\, g(x)\,(h(x) - h(y))\,\frac{1}{x-y}\,\nu(\frac{x-y}{t})\}|\,dx < \infty. \tag{3.50}$$

The computations are neither exciting nor difficult, but let us be a little bit careful. Notice first that we can pull the $\frac{1}{x-y}$ outside the $\overline{\partial}_x^2$, because $\frac{1}{x-y}$ is holomorphic in x away from y and because $\nu(\frac{x-y}{t})$ vanishes for x in a neighborhood of y (by (3.18)). Thus we can reduce to showing that

$$\sup_{0<t\leq 1} \sup_{y\in E} \int_W |\frac{1}{x-y}\,\overline{\partial}_x^2\{\, g(x)\,(h(x)-h(y))\,\nu(\frac{x-y}{t})\}|\,dx < \infty. \tag{3.51}$$

At this stage one simply has to use the Leibniz formula and treat the various terms. Consider first the term

$$\frac{1}{x-y}\,\{\, g(x)\,(\overline{\partial}_x^2 h(x))\,\nu(\frac{x-y}{t})\} \tag{3.52}$$

Since g and ν are bounded this is controlled by our assumption (3.46), Hölder's inequality, and the fact that $\frac{1}{u}$ lies in $L^q_{loc}(\mathbf{C})$ for all $q<2$. Next consider

$$\frac{1}{x-y}\,(\overline{\partial}_x h(x))\,\overline{\partial}_x\{\, g(x)\,\nu(\frac{x-y}{t})\}. \tag{3.53}$$

We can ignore $\overline{\partial}_x h(x)$, because it is bounded on the compact set W by (3.46). Thus we are faced with estimating

$$\int_W |\frac{1}{x-y}\{(\overline{\partial}_x g(x))\,\nu(\frac{x-y}{t}) + g(x)\,\overline{\partial}_x \nu(\frac{x-y}{t})\}|\,dx. \tag{3.54}$$

The $\overline{\partial}_x g(x)$ term is easily controlled because g is smooth, so that $\overline{\partial}_x g(x)$ is bounded on W, and because ν is bounded. For the remaining piece we use the boundedness of g and (3.40), (3.41) to get that

$$\sup_{t>0} \sup_{y\in E} \int_W |\frac{1}{x-y}\,\overline{\partial}_x \nu(\frac{x-y}{t})|\,dx < \infty. \tag{3.55}$$

Thus we conclude that (3.54) is bounded in t.

It remains to show that

$$\sup_{t>0} \sup_{y\in E} \int_W |\frac{h(x)-h(y)}{x-y}\,\overline{\partial}_x^2\{\, g(x)\,\nu(\frac{x-y}{t})\}|\,dx < \infty. \tag{3.56}$$

Notice first that

$$\sup_{x\in W} \sup_{y\in E} |\frac{h(x)-h(y)}{x-y}| < \infty, \tag{3.57}$$

since we are assuming that h is C^1. Therefore (3.56) reduces to

$$\sup_{t>0} \sup_{y\in E} \int_W |\overline{\partial}_x^2\{\, g(x)\,\nu(\frac{x-y}{t})\}|\,dx < \infty. \tag{3.58}$$

Using Leibniz again we are faced with three terms. The first is $(\overline{\partial}_x^2 g(x))\,\nu(\frac{x-y}{t})$. This is bounded on W since g is smooth and ν is bounded, and hence the integral is bounded. The second term is $(\overline{\partial}_x g(x))\,(\overline{\partial}_x \nu(\frac{x-y}{t}))$. The integral of this is bounded because of (3.43) and the boundedness of $\overline{\partial} g$. Similarly, we can control the contribution of the $g(x)\,\overline{\partial}_x^2 \nu(\frac{x-y}{t})$ term to (3.58) using (3.44). Thus we get (3.58), and hence (3.56).

This proves (3.50), as desired, and Lemma 3.45 follows.

It would be nice if the commutator between S_t and a transfer operator had bounded trace norm, at least under suitable conditions. There is a result like this, but unfortunately we cannot quite take the commutator, we have to convert one transfer operator into another, and we also get an annoying error term that comes from the truncation in S_t.

Given a transfer operator $\mathcal{M}$ as in (3.13), define $\mathcal{M}_{0,1}$ by

$$\mathcal{M}_{0,1}\Phi(x) = \sum_{\omega\in\Omega} g_\omega(x)\,\overline{\psi'_\omega}(x)\,\Phi\circ\psi_\omega(x). \tag{3.59}$$

Lemma 3.60. *Suppose that g_ω, ψ_ω, etc., are as in the paragraph after (3.13), and suppose also that each g_ω satisfies (3.46). Let E and W be compact sets in $\mathbf{C}$, and let χ be a smooth function on $\mathbf{C}$ which is supported in W. Then the operator $\chi\,(\mathcal{M}\,S_t - S_t\,\mathcal{M}_{0,1}) : \mathcal{B}(E) \to \mathcal{B}(W)$ is trace class for every $t \in (0,1]$, and we can write it as $T_t + U_t$, where T_t and U_t are trace class operators from $\mathcal{B}(E)$ into $\mathcal{B}(W)$ which satisfy*

$$\sup_{0<t\le 1} \text{ trace norm } T_t < \infty \qquad \textit{and} \tag{3.61}$$

$$\text{operator norm } U_t = O(t). \tag{3.62}$$

We are not asserting a uniform bound on the trace norms of the error terms U_t. The proof will give a bound of $O(t^{-1})$, but we shall not need this. These error terms are an unfortunate consequence of our truncations, they are not truly natural. In our applications of Lemma 3.60 these error terms will not be so bad, because there will be another factor of S_t, and the good estimate for the operator norm of the error term in (3.62) will balance out the bad estimate for the trace norm in Lemma 3.33.

Before we begin the proof of Lemma 3.60 in earnest let us dispense with some preliminary reductions.

We may as well assume that Ω has only one element ω, so that we can get rid of the sum in our transfer operator.

Let us show that we can replace the function g_ω with something smooth. Let h be a smooth function with compact support such that $h \equiv 1$ on $\operatorname{supp} g_\omega$ and $\operatorname{supp} h \subseteq \Lambda_\omega$ (the domain of ψ_ω). We may as well assume that $E \supseteq \operatorname{supp} h$, since otherwise we can simply enlarge E, which strengthens the conclusion of the lemma.

Let us write ψ for ψ_ω, to simplify the notation. Define a transfer operator $\mathcal{N}$ by

$$\mathcal{N}\,\Phi(x) = h(x)\,\Phi\circ\psi(x). \tag{3.63}$$

Also, as in (3.59), define $\mathcal{N}_{0,1}$ by

$$\mathcal{N}_{0,1}\,\Phi(x) = h(x)\,\overline{\psi}'(x)\,\Phi\circ\psi(x). \tag{3.64}$$

Thus $\mathcal{M} = g_\omega \mathcal{N}$ and $\mathcal{M}_{0,1} = g_\omega \mathcal{N}_{0,1}$. We can rewrite our operator as

$$\chi\,(\mathcal{M}\,S_t - S_t\,\mathcal{M}_{0,1}) = g_\omega\,\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1}) + \chi\,[g_\omega, S_t]\,\mathcal{N}_{0,1}. \tag{3.65}$$

The last term is trace class, with bounded norm, because of Lemma 3.45 (with g_ω playing the role that h did there), and because the transfer operator $\mathcal{N}_{0,1}$ is bounded on L^∞. Thus we need only show that the first term on the right hand side satisfies the conclusions of the lemma. For this we may forget about the g_ω in front, because multiplication by it defines a bounded linear operator on $\mathcal{B}(W)$, by Lemma 3.9.

In summary, it suffices to show that the operator $\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1}) : \mathcal{B}(E) \to \mathcal{B}(W)$ is trace class, and that we can write it as a sum of two trace class operators, where one has uniformly bounded trace norm and the other has operator norm which is $O(t)$.

We need to compute the kernel of $\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1})$. By definitions we have that

$$S_t\,\mathcal{N}_{0,1} f(x) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{x-y}\,\nu\Big(\frac{x-y}{t}\Big)\,h(y)\,\overline{\psi}'(y)\,f(\psi(y))\,dy. \tag{3.66}$$

We can make a change of variables to get

$$S_t\,\mathcal{N}_{0,1} f(x) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{x-\psi^{-1}(y)}\,\nu\Big(\frac{x-\psi^{-1}(y)}{t}\Big)\,h(\psi^{-1}(y))\,(\psi^{-1})'(y)\,f(y)\,dy. \tag{3.67}$$

(Don't forget about the jacobian.) We are abusing our notation somewhat here; when $y \notin \psi(\operatorname{supp} h)$ one should interpret $h(\psi^{-1}(y))$ and the whole integrand as being 0, and we shall follow this convention throughout these computations. We can now write

$$(\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1}))f(x) = \frac{1}{\pi}\int_{\mathbf{C}} K(x,y)\,f(y)\,dy, \tag{3.68}$$

where

$$\begin{aligned} K(x,y) = \;& \chi(x)\,\{h(x)\,\frac{1}{\psi(x)-y}\;\;\nu\Big(\frac{\psi(x)-y}{t}\Big) \\ & \qquad\qquad - \frac{(\psi^{-1})'(y)}{x-\psi^{-1}(y)}\,\nu\Big(\frac{x-\psi^{-1}(y)}{t}\Big)\,h(\psi^{-1}(y))\}. \end{aligned} \tag{3.69}$$

It is easy to see that

$$y \to K(x,y) \tag{3.70}$$

is a continuous map from E into $\mathcal{B}(W)$. Indeed, we have that $K(x,y)=0$ when $x \notin W$, since χ is assumed to be supported in W, and we also have that $K(x,y)$ is smooth, since χ and h are smooth and the ν kills the singularity in $\frac{1}{x-y}$. Therefore our operator $\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1})$ is trace class, by Lemma 3.23. From Lemma 3.23 we also get the estimate

$$\begin{aligned} \text{trace norm}\ \chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1}) &\leq |E|\,\frac{1}{\pi}\,\sup_{y\in E}\,\|K(\cdot,y)\|_{\mathcal{B}} \\ &\leq |E|\,\frac{1}{\pi}\,\sup_{y\in E}\int_W |\overline{\partial}_x^2 K(x,y)|\,dx. \end{aligned} \tag{3.71}$$

Unfortunately we shall not get a uniform bound (in t) for

$$\sup_{y\in E}\int_W |\overline{\partial}_x^2 K(x,y)|\,dx. \tag{3.72}$$

The problem stems from the two different ways in which ν appears in (3.69). Before we get to the heart of this we should deal with some preliminary issues. We should do some bookkeeping concerning the singularity of $K(x,y)$.

Let $\Lambda = \Lambda_\omega$ denote the domain of ψ, an open set which contains the compact set $\operatorname{supp} h$. Choose $r > 0$ so that

$$u \in \Lambda \quad \text{when } v \in \operatorname{supp} h \text{ and } |u - v| \leq 10r. \tag{3.73}$$

Of course r does not depend on t. Let H_i, M_i, $i = 1,2,3$, denote the compact sets defined by

$$H_i = \{u \in \mathbf{C} : \operatorname{dist}(u, \operatorname{supp} h) \leq i\,r\}, \qquad M_i = \psi(H_i). \tag{3.74}$$

We can find a constant $s > 0$ (depending on r but not t) so that $s < r$,

$$x \in \operatorname{supp} h \quad \text{and } |\psi(x) - y| \leq s \quad \text{imply } y \in M_1, \quad \text{and} \tag{3.75}$$

$$y \in \psi(\operatorname{supp} h) \text{ and } |x - \psi^{-1}(y)| \leq s \quad \text{imply } x \in H_1. \tag{3.76}$$

(Actually, (3.76) follows from (3.73), since $s < r$.)

The point of this parameter s is that we do not need to worry about $t \geq s$ and we have some useful localizations when $t < s$. The first assertion is made precise by the observation that

$$\sup_{s\leq t\leq 1}\,\sup_{y\in E}\int_W |\overline{\partial}_x^2 K(x,y)|\,dx < \infty. \tag{3.77}$$

Indeed, as long as t is bounded away from 0, everything in (3.69) is smooth, with uniform estimates. Thus we conclude that (3.71) is bounded for $t \geq s$, and so we automatically have the decomposition that we seek when $t \geq s$ (with no error term). ¿From now on we restrict ourselves to $t < s$.

Let Γ denote the operator $\pi\,\chi\,(\mathcal{N}\,S_t - S_t\,\mathcal{N}_{0,1})$. In the argument that follows we shall successively decompose Γ into pieces, peeling away the simplest terms until we get to the main part.

Let $\theta(y)$ be a smooth function on $\mathbf{C}$ which satisfies $\theta \equiv 1$ on M_1 and $\operatorname{supp}\theta \subseteq M_2$. Let us split Γ into $\Gamma_1 + \Gamma_2$, where $\Gamma_1(f) = \Gamma(\theta f)$ and $\Gamma_2(f) = \Gamma((1-\theta)f)$. Let us check that $\Gamma_2 : \mathcal{B}(E) \to \mathcal{B}(W)$ is trace class and that

$$\sup_{0<t<s} \text{ trace norm } \Gamma_2 < \infty. \tag{3.78}$$

The kernel of Γ_2 is just $K(x,y)\,(1-\theta(y))$. Lemma 3.23 implies that Γ_2 is trace class (since $K(x,y)$ is smooth) and that (3.78) will follow if we can prove that

$$\sup_{0<t<s} \sup_{y\in E} \int_W |\overline{\partial}_x^2 K(x,y)\,(1-\theta(y))|\,dx < \infty. \tag{3.79}$$

The formula (3.69) for $K(x,y)$ implies that

$$\overline{\partial}_x^2 K(x,y)\,(1-\theta(y)) = \overline{\partial}_x^2\{\chi(x)\,h(x)\,\frac{1}{\psi(x)-y} \quad \nu(\frac{\psi(x)-y}{t})\}\,(1-\theta(y)), \tag{3.80}$$

because $h(\psi^{-1}(y))$ vanishes on the support of $1-\theta$. (Remember our convention from (3.67).) The right side of (3.80) will be different from 0 only when $x \in \operatorname{supp} h$ and $y \notin M_1$. For these x, y we have that $|\psi(x)-y| > s$, because of (3.75). This means that $|\psi(x)-y| > t$ for the t's that we are considering, and so $\nu(\frac{\psi(\cdot)-y}{t}) \equiv 1$ on a neighborhood of such an x, by (3.18). Thus we may ignore it in (3.80). We conclude that (3.80) remains bounded when $x \in \operatorname{supp} h$, $y \notin M_1$, and $t < s$, since $|\psi(x)-y| > s$ in this case, and this keeps us away from the singularity. This proves (3.79) and therefore (3.78).

In order to finish the proof of Lemma 3.60, it suffices to show that

$$\begin{aligned}&\text{we can decompose } \Gamma_1 : \mathcal{B}(E) \to \mathcal{B}(W) \text{ into a sum of two operators}\\ &\text{for each } 0<t<s, \text{ where one is trace class with bounded trace norm}\\ &\text{and the other has operator norm } = O(t).\end{aligned} \tag{3.81}$$

(Remember the reductions which preceded (3.66).)

Let us split Γ_1 into two more pieces, corresponding to x near or far from $\operatorname{supp} h$. Let $\eta(x)$ be a smooth cut-off function on $\mathbf{C}$ which satisfies $\eta \equiv 1$ on a neighborhood of H_1 and $\operatorname{supp}\eta \subseteq H_2$. Define operators Γ_{11} and Γ_{12} by

$$\Gamma_{11}(f) = \eta\Gamma_1(f) = \eta\Gamma(\theta f), \quad \Gamma_{12}(f) = (1-\eta)\,\Gamma_1(f) = (1-\eta)\,\Gamma(\theta f). \tag{3.82}$$

Thus $\Gamma_1 = \Gamma_{11} + \Gamma_{12}$.

Let us check that Γ_{12} defines a trace class operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$ with trace norm which is uniformly bounded in t, $0 < t < s$. The kernel of Γ_{12} is just $(1-\eta(x))\,K(x,y)\,\theta(y)$. This kernel is smooth, and so we can apply Lemma 3.23 to conclude that it is trace class. To bound the trace norm it suffices to show that

$$\sup_{0<t<s} \sup_{y\in E} \int_W |\overline{\partial}_x^2\{(1-\eta(x))\,K(x,y)\,\theta(y)\}|\,dx < \infty. \tag{3.83}$$

The x's that are relevant for this integral all lie outside H_1 (since $\eta \equiv 1$ on a neighborhood of H_1), and in particular they all lie outside the support of h. For these x's we have that

$$\overline{\partial}_x^2\{(1-\eta(x))\,K(x,y)\,\theta(y)\} = \\ -\,\overline{\partial}_x^2\{(1-\eta(x))\,\chi(x)\,\frac{(\psi^{-1})'(y)}{x-\psi^{-1}(y)}\,\nu(\frac{x-\psi^{-1}(y)}{t})\,h(\psi^{-1}(y))\,\theta(y)\}. \tag{3.84}$$

This quantity vanishes unless $y \in \psi(\operatorname{supp} h)$, because of the $h(\psi^{-1}(y))$. For these x's and y's – i.e., $x \notin H_1$, $y \in \psi(\operatorname{supp} h)$ – we have that $|x - \psi^{-1}(y)| > s$, because of (3.76). Thus $|x - \psi^{-1}(y)| > t$ for the t's that are relevant to (3.83), and so $\nu(\frac{x-\psi^{-1}(y)}{t}) = 1$ for these x's and y's and on neighborhoods of these x's and y's, because of (3.18). This means that we can ignore the ν part of (3.84). What remains in (3.84) is uniformly bounded for the relevant x's and y's, because of our bound $|x - \psi^{-1}(y)| > s$ and the smoothness of the various functions. Therefore (3.83) holds, and we conclude that Γ_{12} defines a trace class operator with uniformly bounded norm.

We have now reduced Lemma 3.60 to the problem of showing that $\Gamma_{11} : \mathcal{B}(E) \to \mathcal{B}(W)$ can be decomposed into a sum of two operators for each $0 < t < s$, where one is trace class with bounded trace norm and the other has operator norm which is $O(t)$.

The kernel associated to Γ_{11} is given by $\eta(x)\,K(x,y)\,\theta(y)$. Define new kernels $J_i(x,y)$, $i = 1,2,3$, by

$$J_1(x,y) = \eta(x)\,\chi(x)\,h(x)\,\frac{1}{\psi(x)-y}\{\nu(\frac{\psi(x)-y}{t}) - \nu(\frac{x-\psi^{-1}(y)}{t})\}\,\theta(y), \tag{3.85}$$

$$J_2(x,y) = \eta(x)\,\chi(x)\,h(x)\,\{\frac{1}{\psi(x)-y} - \frac{(\psi^{-1})'(y)}{x-\psi^{-1}(y)}\}\,\nu(\frac{x-\psi^{-1}(y)}{t})\,\theta(y), \tag{3.86}$$

$$J_3(x,y) = \eta(x)\,\chi(x)\,\{h(x) - h(\psi^{-1}(y))\}\,\frac{(\psi^{-1})'(y)}{x-\psi^{-1}(y)}\,\nu(\frac{x-\psi^{-1}(y)}{t})\,\theta(y). \tag{3.87}$$

We can pretend that $\psi^{-1}(y)$ is always defined in these formulae, because θ is supported in $M_2 \subseteq \psi(\Lambda)$. One can compute directly from (3.69) that

$$\eta(x)\,K(x,y)\,\theta(y) = J_1(x,y) + J_2(x,y) + J_3(x,y). \tag{3.88}$$

These kernels $J_i(x,y)$ are all smooth and they all vanish when $x \in \mathbf{C}\backslash W$, and therefore they define trace class operators from $\mathcal{B}(E)$ into $\mathcal{B}(W)$, by Lemma 3.23. It remains to get estimates. It turns out that the trace norms of the operators which correspond to $J_2(x,y)$ and $J_3(x,y)$ are bounded, while $J_1(x,y)$ is our long-awaited error term.

Let us first check that the trace norm of the operator corresponding to $J_2(x,y)$ is bounded. In view of Lemma 3.23 it suffices to show that

$$\sup_{0<t<s}\ \sup_{y\in E} \int_W |\,\overline{\partial}_x^2 J_2(x,y)|\,dx < \infty. \tag{3.89}$$

The main point for this term is that

$$\frac{1}{\psi(x)-y} - \frac{(\psi^{-1})'(y)}{x-\psi^{-1}(y)} \tag{3.90}$$

is smooth (and even holomorphic) for $x \in \Lambda$ and $y \in \psi(\Lambda)$. To see this it is a little more pleasant to set $y = \psi(z)$, so that (3.90) becomes

$$\frac{1}{\psi(x) - \psi(z)} - \frac{\psi'(z)^{-1}}{x - z}. \tag{3.91}$$

This uses also the identity $(\psi^{-1})'(\psi(z)) = \psi'(z)^{-1}$. The smoothness (and holomorphicity) of (3.91) is a standard exercise. (The poles cancel.) For (3.89) we only care about x's in $\operatorname{supp} \eta$, which is contained in the compact subset H_2 of Λ, and we only care about the y's in $\operatorname{supp} \theta$, which is a compact subset of $\psi(\Lambda)$. Thus (3.89) only involves (3.90) on a compact subset of its domain. We conclude that we can write $J_2(x, y)$ as

$$J_2(x, y) = L(x, y)\, \nu(\frac{x - \psi^{-1}(y)}{t}), \tag{3.92}$$

where $L(x, y)$ is smooth and has compact support, and where $L(x, y)$ also does not depend on t. It is not hard to verify (3.89) using (3.92), the Leibniz rule, the boundedness of ν, and the estimates (3.43) and (3.44). Thus the operator that corresponds to $J_2(x, y)$ has bounded trace norm.

To check that the trace norm of the operator that corresponds to $J_3(x, y)$ is bounded it suffices to show that

$$\sup_{0<t<s} \sup_{y \in E} \int_W |\overline{\partial}_x^2 J_3(x, y)|\, dx < \infty, \tag{3.93}$$

because of Lemma 3.23. We can rewrite this as

$$\sup_{0<t<s} \sup_{y \in E \cap M_2} \int_W |\overline{\partial}_x^2 J_3(x, y)|\, dx < \infty, \tag{3.94}$$

since $\operatorname{supp} \theta \subseteq M_2$. Set

$$N(x, z) = \eta(x)\, \chi(x)\, \{h(x) - h(z)\}\, \frac{1}{x - z}\, \nu(\frac{x - z}{t})\, \theta(\psi(z)). \tag{3.95}$$

In order to prove (3.94) it suffices to show that

$$\sup_{0<t<s} \sup_{z \in H_2} \int_W |\overline{\partial}_x^2 N(x, z)|\, dx < \infty. \tag{3.96}$$

Let us check this. In passing from $J_3(x, y)$ to $N(x, z)$ we have made two changes. The first is that we dropped the $(\psi^{-1})'(y)$, which does not matter because it is bounded and pulls through the $\overline{\partial}_x^2$ (since it does not depend on x). The second change was to replace y with $\psi(z)$. This is again trivial from the perspective of x and $\overline{\partial}_x^2$, and we accomodated it in (3.96) by taking the supremum over $z \in H_2$ instead of $y \in E \cap M_2$. (Remember from (3.74) that $M_2 = \psi(H_2)$.) Thus (3.96) will imply (3.94). To prove (3.96) one can use exactly the same argument as used to prove (3.50). In fact (3.96) is practically the same as (3.50), except that we have different functions now. The present situation is a little simpler, because η, χ, and h are smooth. At any rate we get (3.96) and hence (3.93), and we conclude that the operator which corresponds to $J_3(x, y)$ has bounded trace norm.

It remains to deal with the operator J_1 given by

$$J_1 f(x) = \int_{\mathbf{C}} J_1(x, y)\, f(y)\, dy. \tag{3.97}$$

This will give us the error term U_t described in Lemma 3.60, and we want to show that it has norm $= O(t)$ as an operator from $\mathcal{B}(E)$ into $\mathcal{B}(W)$. Let us begin by rewriting (3.85) as

$$J_1(x, y) = \alpha(x)\, \frac{1}{\psi(x) - y} \{\nu(\frac{\psi(x) - y}{t}) - \nu(\frac{x - \psi^{-1}(y)}{t})\}\, \theta(y), \tag{3.98}$$

where $\alpha(x) = \eta(x)\, \chi(x)\, h(x)$. Thus $\alpha(x)$ is a smooth function with $\operatorname{supp} \alpha \subseteq W \cap \operatorname{supp} h$.

Set $\mu = \nu - 1$, where ν is as in (3.18). Thus μ is a C^∞ function on $\mathbf{C}$ such that

$$\mu(x) = 0 \text{ when } |x| \geq 1 \quad \text{and} \quad \mu(x) = 1 \text{ when } |x| \leq \frac{1}{2}. \tag{3.99}$$

We can rewrite (3.98) as

$$J_1(x, y) = \alpha(x)\, \frac{1}{\psi(x) - y} \{\mu(\frac{\psi(x) - y}{t}) - \mu(\frac{x - \psi^{-1}(y)}{t})\}\, \theta(y). \tag{3.100}$$

This is more convenient for localizing.

Let us now split the two μ terms apart. Set

$$J_{11}(x, y) = \alpha(x)\, \frac{1}{\psi(x) - y}\, \mu(\frac{\psi(x) - y}{t})\, \theta(y) \tag{3.101}$$

$$J_{12}(x, y) = \alpha(x)\, \frac{1}{\psi(x) - y}\, \mu(\frac{x - \psi^{-1}(y)}{t})\, \theta(y), \tag{3.102}$$

and let J_{11} and J_{12} denote the corresponding operators, as in (3.97), so that $J_1 = J_{11} - J_{12}$.

Sublemma 3.103. *$J_{11} : \mathcal{B}(E) \to \mathcal{B}(W)$ has operator norm $= O(t)$.*

To prove this we want to first peel off the ψ and the α into a separate operator. Define a transfer operator $\mathcal{L}$ by

$$\mathcal{L} f(x) = \alpha(x)\, f(\psi(x)). \tag{3.104}$$

Remember that $\operatorname{supp} \alpha \subseteq \operatorname{supp} h$ is a compact subset of the domain Λ of ψ, so that (3.104) makes sense. Set $b_t(w) = \frac{1}{w}\, \mu(\frac{w}{t})$, and define the multiplication operator Θ and the convolution operator B_t by $\Theta(f) = \theta f$ and $B_t f = b_t * f$. We have that

$$J_{11} = \mathcal{L} \circ B_t \circ \Theta, \tag{3.105}$$

as one can compute from (3.101).

Set $E' = \{u \in \mathbf{C} : \text{dist}(u, E) \leq s\}$. Note that

$$\text{supp } B_t f \subseteq E' \quad \text{when} \quad \text{supp } f \subseteq E. \tag{3.106}$$

This follows from (3.99) and the fact that we are restricting ourselves to $t < s$ (since (3.77)). The main point now is that B_t maps $\mathcal{B}(E)$ into $\mathcal{B}(E')$ with norm $\leq \|b_t\|_1$. Indeed, as in Lemma 3.12, we have that $\overline{\partial}^2(b_t * f) = b_t * (\overline{\partial}^2 f)$, and so our assertion about $B_t : \mathcal{B}(E) \to \mathcal{B}(E')$ reduces to (3.106) and the standard fact that the convolution of an L^1 function with a finite measure is an L^1 function, with the norm of the result being less than or equal to the product of the norms of the input. On the other hand we have that

$$\|b_t\|_1 = t\,\|b_1\|_1, \tag{3.107}$$

by an easy scaling argument, and $\|b_1\|_1 < \infty$ by direct computation. Thus we conclude that $B_t : \mathcal{B}(E) \to \mathcal{B}(E')$ has norm $= O(t)$.

To finish the proof of Sublemma 3.103 it suffices to observe that $\Theta : \mathcal{B}(E) \to \mathcal{B}(E)$ is bounded, because of Lemma 3.9, and that $\mathcal{L} : \mathcal{B}(E') \to \mathcal{B}(W)$ is bounded, because of Lemma 3.14 (since $\text{supp}\,\alpha \subseteq W$).

This completes the proof of Sublemma 3.103.

It remains to show that J_{12} has norm $= O(t)$ as an operator from $\mathcal{B}(E)$ to $\mathcal{B}(W)$. The argument will be similar in spirit to the proof of Sublemma 3.103, but the details will be messier because the formulae are less convenient.

Let us first work towards peeling off the α from J_{12}. Define $Q(x, y)$ by

$$Q(x, y) = \frac{1}{\psi(x) - y}\,\mu(\frac{x - \psi^{-1}(y)}{t})\,\theta(y) \tag{3.108}$$

when $(x, y) \in \Lambda \times \psi(\Lambda)$, $Q(x, y) = 0$ otherwise. Remember that $\theta(y) \neq 0$ implies that $y \in M_2$, a compact subset of $\psi(\Lambda)$, and notice that $y \in M_2 = \psi(H_2)$ and $\mu(\frac{x-\psi^{-1}(y)}{t}) \neq 0$ imply that $|x - \psi^{-1}(y)| < t$ and hence $x \in H_3$. (See (3.74), and remember that $t < s < r$.) Thus $Q(x, y)$ is smooth, since H_3 is a compact subset of Λ. For the record, we have that

$$Q(x, y) \neq 0 \quad \text{implies } y \in M_2 \text{ and } x \in H_3. \tag{3.109}$$

Define an operator Q by

$$Qf(x) = \int_{\mathbf{C}} Q(x, y)\, f(y)\, dy. \tag{3.110}$$

Thus $\text{supp}\,Qf \subseteq H_3$ for all f, by (3.109). Since multiplication by α defines a bounded operator from $\mathcal{B}(H_3)$ into $\mathcal{B}(W)$, by Lemma 3.9 (and the fact that $\text{supp}\,\alpha \subseteq W$), we are reduced to showing that $Q : \mathcal{B}(E) \to \mathcal{B}(H_3)$ has operator norm $= O(t)$. Since we know $\text{supp}\,Qf \subseteq H_3$ already, it suffices to show that

$$\overline{\partial}^2 \circ Q : \mathcal{B}(E) \to L^1(H_3) \quad \text{has operator norm } = O(t). \tag{3.111}$$

We would like to move the ψ^{-1} from inside $Q(x,y)$, essentially by making the change of variables $y = \psi(z)$. Define $P(x,z)$ by

$$P(x,z) = \frac{1}{\psi(x) - \psi(z)} \, \mu(\frac{x-z}{t}) \, \theta(\psi(z)) \, |\psi'(z)|^2 \tag{3.112}$$

when $(x,z) \in \Lambda \times \Lambda$, $P(x,z) = 0$ otherwise. Thus $P(x,z) = Q(x,\psi(z)) \, |\psi'(z)|^2$, modulo technicalities. Again this is a smooth function, and we have that

$$P(x,z) \neq 0 \quad \text{implies that } z \in H_2 \text{ and } x \in H_3, \tag{3.113}$$

by (3.109). We can rewrite (3.110) as

$$Qf(x) = \int_{\mathbf{C}} P(x,z) \, f(\psi(z)) \, dz. \tag{3.114}$$

For the purpose of computing distributional $\overline{\partial}$ derivatives the $\frac{1}{\psi(x)-\psi(z)}$ singularity in $P(x,z)$ and its counterpart in $Q(x,y)$ are slightly annoying. These singularities are integrable (see also (3.124) and (3.125) below), and so there is no problem with using these kernels to define our operators, but we should be slightly careful with differentiating them. To avoid any problem let us approximate them by more regular kernels. Given $\epsilon > 0$, $\epsilon < t$, set

$$P_\epsilon(x,z) = \frac{1}{\psi(x) - \psi(z)} \, \{\mu(\frac{x-z}{t}) - \mu(\frac{x-z}{\epsilon})\} \, \theta(\psi(z)) \, |\psi'(z)|^2 \tag{3.115}$$

when $(x,z) \in \Lambda \times \Lambda$, $P_\epsilon(x,z) = 0$ otherwise. The point here is that $P_\epsilon(x,z) = 0$ when $|x - z| < \epsilon/2$, because of (3.99), and so we have killed the singularity. Define operators Q_ϵ by

$$Q_\epsilon f(x) = \int_{\mathbf{C}} P_\epsilon(x,z) \, f(\psi(z)) \, dz. \tag{3.116}$$

We have that

$$Q_\epsilon f \to Qf \quad \text{as } \epsilon \to 0 \tag{3.117}$$

in the L^∞ norm for all bounded functions f, for instance. (This uses the integrability of the singularity in $P(x,y)$.)

Let us compute $\overline{\partial}^2 \circ Q_\epsilon$. We have that

$$\overline{\partial}_x^2 P_\epsilon(x,z) = \overline{\partial}_z^2 \Big(\frac{1}{\psi(x) - \psi(z)} \, \{\mu(\frac{x-z}{t}) - \mu(\frac{x-z}{\epsilon})\} \Big) \, \theta(\psi(z)) \, |\psi'(z)|^2 \tag{3.118}$$

on $\Lambda \times \Lambda$. Notice that we are converting x derivatives into z derivatives. We have used here the fact that $\frac{1}{\psi(x)-\psi(z)}$ is holomorphic away from the singularity. Therefore

$$\begin{aligned} &\overline{\partial}_x^2 Q_\epsilon f(x) = \\ &\quad \int_{\mathbf{C}} \frac{1}{\psi(x) - \psi(z)} \, \{\mu(\frac{x-z}{t}) - \mu(\frac{x-z}{\epsilon})\} \, \overline{\partial}_z^2 \{\theta(\psi(z)) \, |\psi'(z)|^2 \, f(\psi(z))\} \, dz \end{aligned} \tag{3.119}$$

for $x \in \Lambda$ and $f \in \mathcal{B}(E)$. Let us be careful about what this last $\overline{\partial}_z^2$ expression really means. Define a transfer operator $\mathcal{T}$ by

$$\mathcal{T} f(z) = \theta(\psi(z))\, |\psi'(z)|^2\, f(\psi(z)). \tag{3.120}$$

More precisely, we view $\theta(\psi(z))$ here as a smooth function defined on all of $\mathbf{C}$, and not just the domain Λ of ψ, by setting it to be 0 on $\mathbf{C}\backslash\Lambda$. This is reasonable, because $\operatorname{supp}\theta \subseteq M_2$ is a compact subset of $\psi(\Lambda)$, so that $\theta(\psi(z))$ vanishes off of the compact subset H_2 of Λ. With this interpretation, and similar remarks for $|\psi'(z)|^2$, $\mathcal{T}$ is a transfer operator in the sense of (3.13). Thus $\mathcal{T}$ maps $\mathcal{B}(E)$ boundedly into $\mathcal{B}(H_2)$, by Lemma 3.14. In particular, $\overline{\partial}^2 \mathcal{T} f$ is defined as a finite measure when $f \in \mathcal{B}(E)$, and so we can rewrite (3.119) as

$$\overline{\partial}_x^2 Q_\epsilon f(x) = \int_{\mathbf{C}} \frac{1}{\psi(x) - \psi(z)} \, \{\mu(\frac{x-z}{t}) - \mu(\frac{x-z}{\epsilon})\}\, \overline{\partial}_z^2 (\mathcal{T} f)(z)\, dz. \tag{3.121}$$

Here $\overline{\partial}_z^2 (\mathcal{T} f)(z)\, dz$ denotes the measure $\overline{\partial}^2 \mathcal{T} f$, which may or may not be absolutely continuous. Note that this measure is supported in H_2.

Before we take the limit as $\epsilon \to 0$ we need to record an estimate.

Sublemma 3.122. *Suppose that σ is a nonnegative measure on H_2, and consider the function of x defined on Λ by*

$$\int \frac{1}{|\psi(x) - \psi(z)|} \, |\mu(\frac{x-z}{t})|\, d\sigma(z). \tag{3.123}$$

If $t < s$ (as usual), then this is an integrable function supported on H_3 whose L^1 norm is bounded by a constant times $t\,\sigma(H_2)$.

The condition on the support comes from (3.99) and (3.74), as usual. For the L^1 bound we observe first that there is a constant $C > 0$ such that

$$|\psi(x) - \psi(z)| \geq C^{-1}\, |x - z| \quad \text{when } x \in H_3,\ z \in H_2. \tag{3.124}$$

Once we have this inequality the L^1 bound follows from Fubini's theorem and the fact that

$$\int \frac{1}{|w|}\, |\mu(\frac{w}{t})|\, dw = \text{ constant} \cdot t. \tag{3.125}$$

(Remember (3.99).) This proves Sublemma 3.122.

Let us check now that

$$\overline{\partial}_x^2 Q f(x) = \int_{\mathbf{C}} \frac{1}{\psi(x) - \psi(z)} \, \mu(\frac{x-z}{t})\, \overline{\partial}_z^2 (\mathcal{T} f)(z)\, dz \tag{3.126}$$

on Λ when $f \in \mathcal{B}(E)$. The point is simply to send $\epsilon \to 0$ in (3.121). Sublemma 3.122 (applied with t replaced with ϵ) implies that the right side of (3.121) converges to the right side of (3.126) in $L^1(\Lambda)$. On the other hand (3.117) implies that the left side of (3.121) converges to the left side of (3.126) on Λ in the sense of distributions. This proves (3.126).

Next we want to check that

$$\int_{\mathbf{C}} |\overline{\partial}_x^2 Qf(x)|\, dx \le C\, t\, \|f\|_{\mathcal{B}} \tag{3.127}$$

for some constant C when $f \in \mathcal{B}(E)$. We already know that $\operatorname{supp} Qf \subseteq H_3$, and that H_3 is a compact subset of Λ, and so it is enough to consider the integral over Λ. Thus we can use (3.126). Sublemma 3.122 implies that the left side of (3.127) is bounded by a constant times t times the total variation of the measure $\overline{\partial}_z^2(\mathcal{T}f)$. The total variation of $\overline{\partial}_z^2(\mathcal{T}f)$ is bounded by a constant times $\|f\|_{\mathcal{B}}$, since the transfer operator $\mathcal{T}$ maps $\mathcal{B}(E)$ boundedly into $\mathcal{B}(H_2)$, by Lemma 3.14. This yields (3.127).

This proves (3.111), which implies in turn that J_{12} maps $\mathcal{B}(E)$ into $\mathcal{B}(W)$ with norm $= O(t)$. This was the last step in the proof of Lemma 3.60, and so the proof of Lemma 3.60 is now complete.

Lemma 3.128. *Let E, H, and W be compact subsets of $\mathbf{C}$, and let g and h be functions on $\mathbf{C}$ such that g satisfies (3.10), h satisfies (3.46),* $\operatorname{supp} g \subseteq W$, *and* $\operatorname{supp} h \subseteq H$. *Then the operator $gS_thS_t : L^\infty(E) \to \mathcal{B}(W)$ is trace class for each $t \in (0,1]$, where we identify g and h with their corresponding multiplication operators, and*

$$\sup_{0<t\le 1} \text{ trace norm } gS_thS_t < \infty. \tag{3.129}$$

To prove this we begin with some preliminary reductions. We may as well assume that g is smooth, since otherwise we can view gS_thS_t as the composition of the operator of multiplication by g and $g_0S_thS_t$, where g_0 is some smooth function such that $g_0 \equiv 1$ on $\operatorname{supp} g$ and $\operatorname{supp} g_0 \subseteq W_0$ for some compact set $W_0 \supseteq W$. If we can prove the lemma for smooth g's, then we can prove it for $g_0S_thS_t$ (with W replaced by W_0), and then we can use the fact that multiplication by g defines a bounded operator from $\mathcal{B}(W_0)$ into $\mathcal{B}(W)$ (by Lemma 3.9) to get back to gS_thS_t. Thus we may assume that g is smooth.

Set $F_i = \{x \in \mathbf{C} : \operatorname{dist}(x, E \cup W) \le i\}$ $i = 1, 2$. We may assume that h is a smooth function which satisfies

$$h \equiv 1 \quad \text{ on } F_2. \tag{3.130}$$

Indeed, if not, let H_0 be a compact subset of $\mathbf{C}$ which contains F_2 and $\operatorname{supp} h$ in its interior, and let h_0 be a smooth function which satisfies $h_0 \equiv 1$ on $F_2 \cup \operatorname{supp} h$ and $\operatorname{supp} h_0 \subseteq H_0$. Then

$$gS_thS_t = gS_th\, h_0S_t = g\,[S_t, h]h_0S_t + h\, gS_th_0S_t. \tag{3.131}$$

The operator $g\,[S_t, h]h_0S_t$ is trace class, with

$$\sup_{0<t\le 1} \text{ trace norm } g\,[S_t, h]h_0S_t < \infty, \tag{3.132}$$

because of Lemmas 3.45 and 3.21. If we can prove the lemma for $gS_th_0S_t$ (with H replaced by H_0), then we shall know that $h\,gS_th_0S_t$ is trace class with bounded norm, since multiplication by h defines a bounded operator on $\mathcal{B}(W)$. This would then imply our original gS_thS_t is trace class with bounded norm.

Thus we may assume that h satisfies (3.130), and that g and h are both smooth.

Next, let us check that $g(S - S_t)hS_t : L^\infty(E) \to \mathcal{B}(W)$ is trace class with bounded norm. We know from Lemma 3.33 that $hS_t : L^\infty(E) \to \mathcal{B}(H)$ is trace class with norm $= O(t^{-1})$. We also know from Lemma 3.21 that $g(S - S_t) : \mathcal{B}(H) \to \mathcal{B}(W)$ is a bounded operator with norm $= O(t)$. Therefore $g(S - S_t)hS_t$ is trace class with bounded norm.

Thus we are reduced to showing that $gShS_t : L^\infty(E) \to \mathcal{B}(W)$ is a trace class operator with bounded norm.

We need to put S in a more convenient form. Define a convolution operator T by

$$Tf(x) = \int_{\mathbf{C}} \frac{1}{\pi} \frac{\overline{x} - \overline{y}}{x - y} f(y)\, dy. \tag{3.133}$$

This makes sense for integrable functions with compact support, for instance. We have that

$$S = T \circ \overline{\partial} \tag{3.134}$$

as operators acting on, say, smooth functions with compact support. This is not hard to check, using the fact that

$$\overline{\partial}(\frac{\overline{x}}{x}) = \frac{1}{x} \tag{3.135}$$

in the sense of distributions.

Since hS_t maps $L^\infty(E)$ into smooth functions with compact support, we have that

$$gShS_t(f) = gT\,\overline{\partial}(hS_t(f)) = gT(\overline{\partial}h)S_t(f) + gTh(\overline{\partial} \circ S_t)(f) \tag{3.136}$$

when $f \in L^\infty(E)$. Here $\overline{\partial}h$ means multiplication by the function $\overline{\partial}h$.

Sublemma 3.137. *The operator $gT(\overline{\partial}h)S_t : L^\infty(E) \to \mathcal{B}(W)$ is trace class with bounded trace norm.*

It suffices to show that $(\overline{\partial}h)S_t : L^\infty(E) \to \mathcal{B}(H)$ is trace class with bounded norm, since $gT : \mathcal{B}(H) \to \mathcal{B}(W)$ is bounded, by Lemma 3.12.

We can write $(\overline{\partial}h)S_t$ as

$$((\overline{\partial}h)S_t)(f)(x) = \int_E J(x, y)\, f(y)\, dy, \tag{3.138}$$

where

$$J(x, y) = \overline{\partial}h(x)\, \frac{1}{\pi}\, \frac{1}{x - y}\, \nu(\frac{x - y}{t}). \tag{3.139}$$

(See (3.19).) This is a smooth function of x and y, and $J(x, y) = 0$ when $x \notin H$, since $\operatorname{supp} h \subseteq H$. Thus $y \mapsto J(\cdot, y)$ defines a continuous mapping from E into $\mathcal{B}(H)$, and

Lemma 3.23 implies that $(\overline{\partial}h)S_t : L^\infty(E) \to \mathcal{B}(H)$ is trace class and that its trace norm is bounded by

$$|E| \cdot \sup_{y \in E} \|J(\cdot, y)\|_{\mathcal{B}} = |E| \cdot \sup_{y \in E} \int_{\mathbf{C}} |\overline{\partial}_x^2 J(x,y)|\, dx < \infty. \tag{3.140}$$

Let us check that

$$J(x,y) = \overline{\partial}h(x)\, \frac{1}{\pi}\, \frac{1}{x-y} \tag{3.141}$$

when $t \le 1$ and $y \in E$, i.e., that we can drop the ν. This is trivial when $x \notin \operatorname{supp} \overline{\partial}h$, since both sides of (3.141) then vanish. If $y \in E$ and $x \in \operatorname{supp} \overline{\partial}h$, then $|x-y| \ge 2$ because of (3.130). This implies that $|x-y| \ge t$ and hence that $\nu(\frac{x-y}{t}) = 1$ (by (3.18)), which gives (3.141) in this case too. Thus (3.141) holds when $t \le 1$ and $y \in E$.

Using (3.141) we see that the (3.140) does not depend on t when $t \in (0,1]$. It is finite, since $J(x,y)$ is smooth, and so we conclude that it is bounded for t in this range. This proves Sublemma 3.137.

Next we consider $gTh(\overline{\partial} \circ S_t)$.

Sublemma 3.142. *$gTh(\overline{\partial} \circ S_t)(f) = gT(\overline{\partial} \circ S_t)(f)$ when $f \in L^\infty(E)$ and $t \in (0,1]$.*

Set $E_t = \{x \in \mathbf{C} : \operatorname{dist}(x, E) \le t\}$. Let us check that

$$\operatorname{supp}\, (\overline{\partial} \circ S_t)(f) \subseteq E_t \quad \text{ when } f \in L^\infty(E). \tag{3.143}$$

Indeed,

$$(\overline{\partial} \circ S_t) f(x) = \int_{\mathbf{C}} \frac{1}{\pi}\, \frac{1}{x-y}\, (\overline{\partial}\nu)(\frac{x-y}{t})\; t^{-1}\, f(y)\, dy, \tag{3.144}$$

by (3.19) and calculus. We also have that

$$\overline{\partial}\nu(u) = 0 \qquad \text{when } |u| \ge 1 \text{ or } |u| \le \frac{1}{2}, \tag{3.145}$$

because of (3.18). This and (3.144) imply (3.143).

On the other hand, $h \equiv 1$ on E_t when $t \in (0,1]$, because of (3.130). This implies Sublemma 3.142.

To finish the proof of Lemma 3.128 it remains to show that $gT(\overline{\partial} \circ S_t) : L^\infty(E) \to \mathcal{B}(W)$ is trace class with bounded trace norm.

Define functions $k(x)$ and $b_t(x)$ on $\mathbf{C}$ by

$$k(x) = \frac{1}{\pi}\, \frac{\overline{x}}{x} \quad \text{ and } \quad b_t(x) = \frac{1}{\pi}\, t^{-1}\, \frac{1}{x}\, (\overline{\partial}\nu)(\frac{x}{t}). \tag{3.146}$$

Thus T and $\overline{\partial} \circ S_t$ are given by $Tf = k * f$ and $(\overline{\partial} \circ S_t) f = b_t * f$. We also have that

$$b_t(x) = 0 \quad \text{ when } |x| \ge t \qquad \text{and} \qquad \|b\|_\infty \le C\, t^{-2} \tag{3.147}$$

for some constant C. These facts follow from (3.145) and the observation that $\overline{\partial}\nu$ is bounded.

Given $f \in L^\infty(E)$ we have that $T(\overline{\partial} \circ S_t)(f) = k * b_t * f$, so that

$$(gT(\overline{\partial} \circ S_t))(f)(x) = \int_E g(x)\,(k * b_t)(x-y)\,f(y)\,dy. \tag{3.148}$$

Note that b_t and hence $k * b_t$ are smooth. Remember also that $\operatorname{supp} g \subseteq W$. We can use Lemma 3.23 to conclude that $gT(\overline{\partial} \circ S_t) : L^\infty(E) \to \mathcal{B}(W)$ is trace class with

$$\begin{aligned} \text{trace norm } gT(\overline{\partial} \circ S_t) &\le |E| \cdot \sup_{y \in E} \|g(\cdot)\,(k * b_t)(\cdot - y)\|_{\mathcal{B}} \\ &\le |E| \cdot \sup_{y \in E} \int_{\mathbf{C}} |\overline{\partial}_x^2 \{g(x)\,(k * b_t)(x-y)\}|\,dx \\ &= |E| \cdot \sup_{y \in E} \int_W |\overline{\partial}_x^2 \{g(x)\,(k * b_t)(x-y)\}|\,dx. \end{aligned} \tag{3.149}$$

In this last equality we have used the fact that g is supported in W.

To estimate the integral in (3.149) we use the formula

$$\begin{aligned} \overline{\partial}_x^2 \{g(x)\,(k * b_t)(x-y)\} &= (\overline{\partial}^2 g)(x)\,(k * b_t)(x-y) + 2(\overline{\partial} g)(x)\,\overline{\partial}_x (k * b_t(x-y)) \\ &\qquad + g(x)\,\overline{\partial}_x^2 (k * b_t(x-y)) \\ &= (\overline{\partial}^2 g)(x)\,(k * b_t)(x-y) + 2(\overline{\partial} g)(x)\,((\overline{\partial} k) * b_t(x-y)) \\ &\qquad + g(x)\,((\overline{\partial}^2 k) * b_t(x-y)). \end{aligned} \tag{3.150}$$

These derivatives of k should be taken in the sense of distributions, and then the convolution with b_t makes everything smooth again, since b_t is smooth.

Inserting the right side of (3.150) into the integral in (3.149) we get three terms to estimate. For the first we have that

$$\begin{aligned} \int_W |(\overline{\partial}^2 g)(x)\,(k * b_t)(x-y)|\,dx &\le \|\overline{\partial}^2 g\|_\infty\,\|k * b_t\|_\infty\,|W| \\ &\le \|\overline{\partial}^2 g\|_\infty\,\|k\|_\infty\,\|b_t\|_1\,|W|. \end{aligned} \tag{3.151}$$

Of course $\overline{\partial}^2 g$ is bounded (since g is smooth) and k is bounded (by inspection). We also have that $\|b_t\|_1$ is bounded, uniformly in t, as one can check from (3.147). Thus the contribution of this term to the integral in (3.149) is bounded.

For the second term on the right side of (3.150) we begin with the observation that

$$\overline{\partial} k(x) = \frac{1}{\pi}\,\frac{1}{x}. \tag{3.152}$$

This is easy to check. For the relevant integral we have that

$$\begin{aligned} \int_W |(\overline{\partial} g)(x)\,((\overline{\partial} k) * b_t(x-y))|\,dx &\le \|\overline{\partial} g\|_\infty \int_W |(\overline{\partial} k) * b_t(x-y)|\,dx \\ &\le \|\overline{\partial} g\|_\infty \int_W \int_{\mathbf{C}} \frac{1}{\pi}\,\frac{1}{|u|}\,|b_t(x-y-u)|\,du dx. \end{aligned} \tag{3.153}$$

Remember that $|x-y-u| \le t$ when $|b_t(x-y-u)| \neq 0$, because of (3.147). Let Z be a compact set which is large enough so that $u \in Z$ whenever $|x-y-u| \le 1$ for some $x \in W$ and $y \in E$. Using Fubini's theorem we get that

$$\int_W \int_{\mathbf{C}} \frac{1}{|u|} \, |b_t(x-y-u)| \, du dx \le \|b_t\|_1 \int_Z \frac{1}{|u|} \, du. \tag{3.154}$$

We have already seen that $\|b_t\|_1$ is bounded, uniformly in t, and this last integral over Z is also finite. Thus we conclude that the contribution of the second term on the right side of (3.150) to the right side of (3.149) is also bounded, uniformly in t.

The last term on the right side of (3.150) is $g(x)\,((\overline{\partial}^2 k) * b_t(x-y))$. For this we use the fact that $\overline{\partial}^2 k$ is the Dirac delta function at the origin, as in (3.4). Thus

$$g(x)\,((\overline{\partial}^2 k) * b_t(x-y)) = g(x)\, b_t(x-y). \tag{3.155}$$

The relevant integral for (3.149) reduces to

$$\int_W |g(x)\, b_t(x-y)| \, dx \le \|g\|_\infty \|b_t\|_1. \tag{3.156}$$

Again this is bounded, uniformly in t.

Thus we conclude from (3.149) and these estimates that

$$\sup_{0<t\le 1} \text{trace norm } gT(\overline{\partial} \circ S_t) < \infty. \tag{3.157}$$

This completes the proof of Lemma 3.128, because of (3.136) and Sublemmas 3.142 and 3.137.

Now we come to the main result of this section. For this we need to first set some notation and assumptions.

Let $\mathcal{M}$ be a transfer operator as in (3.13). We assume that g_ω, ψ_ω, and Λ_ω satisfy the same conditions as in the paragraph just after (3.13). Fix a compact set $B \subseteq \mathbf{C}$, and assume that $\Lambda_\omega, \psi_\omega(\Lambda_\omega) \subseteq B$ for each ω. We shall make the standing assumption that

$$g_\omega \text{ and } \overline{\partial} g_\omega \text{ satisfy (3.46) for each } \omega \in \Omega. \tag{3.158}$$

This ensures that $\mathcal{M}$ defines a bounded operator on $\mathcal{B}(B)$, by Lemma 3.14.

Fix a compact set K contained in the interior of B such that

$$K \supseteq \bigcup_{\omega \in \Omega} \{\text{supp } g_\omega \cup \psi_\omega(\text{supp } g_\omega)\}. \tag{3.159}$$

Let K_1 be another compact subset of the interior of B which contains K in its interior, and fix a smooth cut-off function η on $\mathbf{C}$ such that

$$\text{supp } \eta \subseteq K_1 \qquad \text{and} \qquad \eta \equiv 1 \quad \text{on a neighborhood of } K. \tag{3.160}$$

Note that

$$\mathcal{M}(f) \equiv 0 \qquad \text{on } B \backslash K \tag{3.161}$$

$$\mathcal{M}(\eta f) = \eta \mathcal{M}(f) = \mathcal{M}(f) \tag{3.162}$$

for all functions f. Notice also that multiplication by η defines a bounded operator on $\mathcal{B}(B)$, by Lemma 3.9.

Fix a complex number z, our spectral parameter. From now on we shall assume that

$$I - z\mathcal{M} \qquad \text{is invertible on } \mathcal{B}(B). \tag{3.163}$$

We can automatically extend $(I - z\mathcal{M})^{-1}$ to the larger space

$$\{f \in L^\infty(\mathbf{C}) : \eta f \in \mathcal{B}(B)\} \tag{3.164}$$

by the formula

$$(I - z\mathcal{M})^{-1} f = (1 - \eta) f + (I - z\mathcal{M})^{-1}(\eta f). \tag{3.165}$$

This is consistent with the original definition of $(I - z\mathcal{M})^{-1}$ on $\mathcal{B}(B)$ because of (3.162). Note that the image of $\mathcal{B}(B)$ under S or an S_t is always contained in (3.164), because of Lemma 3.21.

Define the transfer operator $\mathcal{N}$ by

$$\mathcal{N}\Phi(x) = \sum_{\omega \in \Omega} (\overline{\partial} g_\omega)(x)\, \Phi \circ \psi_\omega(x). \tag{3.166}$$

This enjoys the same sort of properties as $\mathcal{M}$ does – like (3.161) and (3.162) – since $\operatorname{supp} \overline{\partial} g_\omega \subseteq \operatorname{supp} g_\omega$. Our assumption (3.158) implies that $\mathcal{N}$ is bounded on $\mathcal{B}(B)$, because of Lemma 3.14.

Given integers k and l define a new transfer operator $\mathcal{M}_{k,l}$ by

$$\mathcal{M}_{k,l}\Phi(x) = \sum_{\omega \in \Omega} g_\omega(x)\, (\psi'_\omega(x))^k\, (\overline{\psi}'_\omega(x))^l\, \Phi \circ \psi_\omega(x). \tag{3.167}$$

Again these transfer operators satisfy the analogues of (3.161) and (3.162). We define $\mathcal{N}_{k,l}$ in the same manner, replacing g_ω with $\overline{\partial} g_\omega$ as in (3.166). Notice that

$$g_\omega\, (\psi'_\omega)^k\, (\overline{\psi}'_\omega)^l \text{ and } \overline{\partial} g_\omega\, (\psi'_\omega)^k\, (\overline{\psi}'_\omega)^l \text{ satisfy (3.46) for each } \omega \in \Omega. \tag{3.168}$$

This is easy to check, using (3.158) and the holomorphicity and hence smoothness of ψ_ω (and the fact that $\operatorname{supp} g_\omega$ is a compact subset of the domain of ψ_ω). A useful consequence of this observation is that

$$\text{the } \mathcal{M}_{k,l}\text{'s and } \mathcal{N}_{k,l}\text{'s are all bounded operators on } \mathcal{B}(B). \tag{3.169}$$

This follows from Lemma 3.14.

We are going to need to make the technical assumption that

$$I - z\mathcal{M}_{0,l} \quad \text{is invertible on } \mathcal{B}(B) \text{ for at least one of } l = 1, -1. \tag{3.170}$$

Once we know that this operator is invertible on $\mathcal{B}(B)$ we can extend it to (3.164) as before, using a formula like (3.165).

Define the kneading operators $\mathcal{D}_t = \mathcal{D}_t(z)$ by

$$\mathcal{D}_t = \mathcal{N}(I - z\mathcal{M})^{-1} S_t. \tag{3.171}$$

We can think of this initially as mapping elements of $\mathcal{B}(B)$ to some functions on $\mathbf{C}$, i.e., S_t maps $\mathcal{B}(B)$ into the space (3.164), and then $(I - z\mathcal{M})^{-1}$ and $\mathcal{N}$ act on this space. However, $\mathcal{D}_t$ is actually a bounded operator on $\mathcal{B}(B)$. To see this note that

$$\begin{aligned} \mathcal{D}_t &= \mathcal{N}(I - z\mathcal{M})^{-1} S_t \\ &= \mathcal{N}\,\eta\,(I - z\mathcal{M})^{-1} S_t = \mathcal{N}(I - z\mathcal{M})^{-1}\,\eta\, S_t, \end{aligned} \tag{3.172}$$

where we are identifying η with the associated multiplication operator. This identity follows from the analogue of (3.162) for $\mathcal{N}$ and the fact that $(I - z\mathcal{M})^{-1}$ commutes with multiplication by η (because of (3.162)). Once we have this identity we may conclude that $\mathcal{D}_t$ is a bounded operator on $\mathcal{B}(B)$, and even a trace class operator, because $\eta\, S_t$ is a trace class operator on $\mathcal{B}(B)$, by Lemma 3.33.

Theorem 3.173. *Let $\mathcal{M}, g_\omega, \Lambda_\omega$, etc., be as above. Assume in particular that (3.158), (3.163), and (3.170) hold. Then $\mathcal{D}_t$ is a trace class operator on $\mathcal{B}(B)$ for each $t \in (0, 1]$, and we have that*

$$\sup_{0<t\le 1} \text{ trace norm } \mathcal{D}_t^2 < \infty. \tag{3.174}$$

It would be much nicer if the trace norms of the $\mathcal{D}_t$'s themselves were bounded, but this does not seem to work.

The proof of the theorem is fairly simple given all of our previous lemmas. We already noted above that the $\mathcal{D}_t$'s are trace class, and so the only issue is the bound (3.174). Using (3.172) we can write out $\mathcal{D}_t^2$ as

$$\mathcal{D}_t^2 = \mathcal{N}(I - z\mathcal{M})^{-1}\,\eta\, S_t\, \mathcal{N}(I - z\mathcal{M})^{-1}\,\eta\, S_t. \tag{3.175}$$

To prove (3.174) it suffices to show that

$$\sup_{0<t\le 1} \text{ trace norm } \;\eta\, S_t\, \mathcal{N}(I - z\mathcal{M})^{-1}\,\eta\, S_t < \infty, \tag{3.176}$$

since $\mathcal{N}(I - z\mathcal{M})^{-1}$ is a bounded operator.

The main point is that we have two $\eta\, S_t$'s, so that we can try to use Lemma 3.128. We have to commute an $\eta\, S_t$ around other operators, but we have a lot of lemmas for doing precisely that.

Lemma 3.177. *$(\eta\, S_t\, \mathcal{N} - \mathcal{N}_{0,-1}\, \eta\, S_t)\,(I - z\mathcal{M})^{-1}\, \eta\, S_t$ is a trace class operator on $\mathcal{B}(B)$ for $t \in (0,1]$, and it has bounded trace norm.*

To see this we apply Lemma 3.60, with $\mathcal{M}$ replaced with $\mathcal{N}_{0,-1}$, so that $\mathcal{M}_{0,1}$ should be replaced with $\mathcal{N}$. We also take $E = W = B$ and $\chi = \eta$ in Lemma 3.60. Note that the coefficient functions for $\mathcal{N}_{0,-1}$ satisfy the requirements of Lemma 3.60, because of (3.168). From Lemma 3.60 we get a decomposition of $\eta(\mathcal{N}_{0,-1} S_t - S_t \mathcal{N}) : \mathcal{B}(B) \to \mathcal{B}(B)$ into $T_t + U_t$, where T_t is trace class with bounded norm and U_t has operator norm $= O(t)$. Notice that

$$\mathcal{N}_{0,-1}\, \eta\, S_t - \eta\, S_t \mathcal{N} = \eta(\mathcal{N}_{0,-1} S_t - S_t \mathcal{N}) = T_t + U_t, \tag{3.178}$$

because η commutes with $\mathcal{N}_{0,-1}$ (as in (3.162)). Hence

$$(\eta\, S_t\, \mathcal{N} - \mathcal{N}_{0,-1}\, \eta\, S_t)\,(I - z\mathcal{M})^{-1}\, \eta\, S_t = -(T_t + U_t)\,(I - z\mathcal{M})^{-1}\, \eta\, S_t. \tag{3.179}$$

The term with the T_t is trace class with bounded trace norm because T_t is and the other operators are uniformly bounded. The term with the U_t is also trace class with bounded trace norm because $\eta\, S_t$ is trace class with trace norm $O(t^{-1})$ (Lemma 3.33), because U_t has operator norm $O(t)$, and because $(I - z\mathcal{M})^{-1}$ is a bounded operator which does not depend on t. This proves Lemma 3.177.

To prove (3.176) it now suffices to show that

$$\sup_{0<t\le 1} \text{ trace norm } \; \eta\, S_t\, (I - z\mathcal{M})^{-1}\, \eta\, S_t < \infty, \tag{3.180}$$

because of Lemma 3.177 and because $\mathcal{N}_{0,-1}$ is a bounded operator on $\mathcal{B}(B)$, as in (3.169).

We want to commute an $\eta\, S_t$ around $(I - z\mathcal{M})^{-1}$. We can do this either from the left or from the right, and this corresponds to the ambiguity in (3.170). For the sake of definiteness we commute it from the left.

Assume that we can take $l = -1$ in (3.170). We want to analyze the operator

$$\Gamma = \eta\, S_t\, (I - z\mathcal{M})^{-1} - (I - z\mathcal{M}_{0,-1})^{-1}\, \eta\, S_t. \tag{3.181}$$

We can write this as

$$\Gamma = (I - z\mathcal{M}_{0,-1})^{-1}\, \Gamma_0\, (I - z\mathcal{M})^{-1}, \tag{3.182}$$

where

$$\Gamma_0 = (I - z\mathcal{M}_{0,-1})\, \eta\, S_t - \eta\, S_t\, (I - z\mathcal{M}). \tag{3.183}$$

We can rewrite Γ_0 as

$$\Gamma_0 = -z\{\mathcal{M}_{0,-1}\, \eta\, S_t - \eta\, S_t\, \mathcal{M}\}. \tag{3.184}$$

We apply Lemma 3.60 again, with the $\mathcal{M}$ in Lemma 3.60 replaced with $\mathcal{M}_{0,-1}$, so that the $\mathcal{M}_{0,1}$ in Lemma 3.60 corresponds to $\mathcal{M}$ here. As before, we are using (3.168) to know that the hypotheses in Lemma 3.60 on the coefficient functions are satisfied, and we take $E = W = B$ and $\chi = \eta$ in Lemma 3.60. We conclude that

$$\eta\, (\mathcal{M}_{0,-1}\, S_t - S_t\, \mathcal{M}) = T_t' + U_t', \tag{3.185}$$

where T'_t is trace class on $\mathcal{B}(B)$ with uniformly bounded trace norm, and U'_t is a bounded operator on $\mathcal{B}(B)$ with norm $= O(t)$. We also get

$$\Gamma_0 = -z(T'_t + U'_t), \tag{3.186}$$

because η commutes with $\mathcal{M}_{0,-1}$, as in (3.162). Using this and (3.182) we get a decomposition

$$\Gamma = T''_t + U''_t, \tag{3.187}$$

where T''_t is trace class on $\mathcal{B}(B)$ with uniformly bounded trace norm, and U''_t is a bounded operator on $\mathcal{B}(B)$ with norm $= O(t)$.

Let us use this to prove (3.180). We have that

$$\eta\, S_t\, (I - z\mathcal{M})^{-1}\, \eta\, S_t = \Gamma\, \eta\, S_t + (I - z\mathcal{M}_{0,-1})^{-1}\, (\eta\, S_t)^2, \tag{3.188}$$

by algebra. The last term is trace class with bounded trace norm, because of Lemma 3.128 and the boundedness of $(I - z\mathcal{M}_{0,-1})^{-1}$. The first term on the right splits into $T''_t\, \eta\, S_t + U''_t\, \eta\, S_t$. As usual, $T''_t\, \eta\, S_t$ is trace class with bounded norm because T''_t is, and $U''_t\, \eta\, S_t$ is trace class with bounded norm because $\eta\, S_t$ is trace class with norm $= O(t^{-1})$ and because U''_t has operator norm $= O(t)$. This takes care of the right side of (3.188), and so we conclude that the left side does have bounded trace norm, i.e., (3.180) holds.

If $l = 1$ in (3.170), then we perform a similar analysis of

$$(I - z\mathcal{M})^{-1}\, \eta\, S_t - \eta\, S_t\, (I - z\mathcal{M}_{0,1})^{-1}. \tag{3.189}$$

That is, we use Lemma 3.60 to prove that this operator admits a decomposition like (3.187), and then we use this decomposition to prove (3.180) in practically the same manner as before, using also Lemma 3.128.

This completes the proof of Theorem 3.173.

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