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Connection between Wightman Functions and Green Functions in p-Space.

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Resumé. — Dans le présent travail, après avoir repris l'étude des propriétés d'analyticité de la fonction $\mathscr{W}$ de Wightman dans le cas où le temps seul est variable complexe, nous en déduisons la fonction de Green G, étendant ainsi par une nouvelle méthode les résultats de O. Steinmann relatifs à la fonction à 4 points. La fonction G a pour valeur frontière la transformée de Fourier de la valeur moyenne du vide du produit T des champs et prolonge analytiquement la fonction retardée de L.S.Z. dans l'espace des impulsions. Finalement on établit un ensemble de propriétés qui caractérisent G en ce sens que si G possède ces propriétés, il existe une et une seule fonction $\tilde{\mathscr{W}}$ possédant les propriétés habituelles et telle que G en dérive.

1. – Introduction. x- and p-spaces.

The Wightman function $\mathscr{W}(z)$, $z = x + iy$ is defined as an analytic continuation of the vacuum expectation value

$$\mathscr{W}(x) = \langle A^{(0)}(x_0) A^{(1)}(x_1) \dots A^{(n)}(x_n) \rangle_0$$

of the local scalar fields $A^{(i)}(x_i)$ [1,2].

The Green function $G(k)$, $k = p + iq$ will be defined as an analytic continuation of the Fourier transform of the vacuum expectation value of a T-pro-

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(1) A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).
(2) D. HALL and A. WIGHTMAN: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **31**, no. 5 (1957).

duct [3].

$$G(p) = \mathscr{F}\langle TA^{(0)}A^{(1)} \dots A^{(n)}\rangle_0(p_1, \dots, p_n) .$$

It is convenient to introduce immediately the spaces of variables that will be used in the sequel.

Consider the space R^{n+1} of the independent real variables $t_0, t_1, \dots, t_n$. The quotient of this space by the equivalence relation

$$(1.1) \qquad t_0 - t_1 = t'_0 - t'_1 , \quad t_1 - t_2 = t'_1 - t'_2 , \quad \dots , \quad t_{n-1} - t_n = t'_{n-1} - t'_n$$

is a space R^n which will be called (t).

Similarly if $x_0, x_1, \dots, x_n$ or $y_0, y_1, \dots, y_n$ (resp. $z_0, z_1, \dots, z_n$) are sets of independent real (resp. complex) vector variables, the corresponding equivalence relations will yield spaces R^{4n} (resp. C^{4n}) which will be called (x) or (y) (resp. (z)).

If one considers vectors $(z_0^0, \boldsymbol{x}_0)$, $(z_1^0, \boldsymbol{x}_1)$, ..., $(z_n^0, \boldsymbol{x}_n)$ where the first (time) component is allowed to be complex, the other (space) components being real, one obtains in the same way a space $C_n \times R^{3n}$ called $(z^0, \boldsymbol{x})$.

Consider now the space R^{n+1} of the independent real variables $s_0, s_1, \dots, s_n$ The subspace of this space defined by

$$(1.2) \qquad s_0 + s_1 + \dots + s_n = 0$$

is a space R^n which will be called (s).

Similarly if $p_0, p_1, \dots p_n$ or $q_0, q_1, \dots, q_n$ (resp. $k_0, k_1, \dots, k_n$) are sets of independent real (resp. complex) vector variables, the corresponding subspaces are space R^{4n} (resp. C^{4n}) which will be called (p) or (q) (resp. (k)).

If one considers vectors $(k_0^0, \boldsymbol{p}_0)$, $(k_1^0, \boldsymbol{p}_1)$, ..., $(k_n^0, \boldsymbol{p}_n)$ where the first component is allowed to be complex, the other components being real, one obtains in the same way a space $C^n \times R^{3n}$ called $(k^0, \boldsymbol{p})$.

We can now introduce a bilinear form on (t) (s), namely

$$(1.3) \qquad s\cdot t = \sum_{i=0}^{n} s_i \cdot t_i = \sum_{i=1}^{n} s_i(t_i - t_0) .$$

Similarly

$$(1.3') \qquad p\cdot x = \sum_{i=0}^{n} p_i \cdot x_i = \sum_{i=1}^{n} p_i(x_i - x_0) ,$$

and so on.

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(3) J. SCHWINGER: *Annual International Conference on High Energy Physics at CERN* (1958).

For integrations over (t), (s); (x), (p), etc., we will use

$$\mathrm{d}t = \mathrm{d}(t_1 - t_0) \ \dots\ \mathrm{d}(t_n - t_0) \qquad \mathrm{d}s = \mathrm{d}s_1 \ \dots\ \mathrm{d}s_n\,, \tag{1.4}$$

$$\mathrm{d}x = \mathrm{d}(x_1 - x_0) \ \dots\ \mathrm{d}(x_n - x_0) \qquad \mathrm{d}p = \mathrm{d}p_1 \ \dots\ \mathrm{d}p_n\,. \tag{1.4'}$$

In the spaces just introduced, we have of course redundant variables. It is useful to keep them in general for reasons of symmetry. In particular cases however, other variables may be more suitable.

We have introduced above *four*-dimensional vectors for obvious physical reasons. All that will be said can however be easily generalized to $N+1$ dimensional vectors $(x^0, x^1, \dots, x^n)$ with metric $(x^0)^2-(x^1)^2-\dots-(x^N)^2$, $N \geqslant 1$

2. – The decomposition of (t) and (s) into cones.

Let us consider the set of all planes

$$t_{i'} - t_i = 0 \qquad\qquad i,\, i' = 0, 1, \dots, n;\ i \neq i' \tag{2.1}$$

in (t). They decompose (t) into a set $\mathscr{T}$ of open convex cones.

In such a cone T, every difference $t_{i'} - t_i$ has a well-defined sign so that we can order the t_i by increasing values into a characteristic sequence $t_{i_0} < t_{i_1} < \dots < t_{i_n}$.

On the other hand to every sequence $(i_0, i_1, \dots, i_n)$ there corresponds a permutation π of $\{0, 1, \dots, n\}$ such that $\pi\ (0, 1, \dots, n) = (i_0, i_1, \dots, i_n)$ (and to every permutation corresponds a sequence). There is thus a one-to-one correspondence between $\mathscr{T}$ and the symmetric group γ_{n+1} of permutations of $n+1$ objects, and $\mathscr{T}$ therefore contains $(n+1)!$ cones, each being defined by n relations:

$$t_{i_1} - t_{i_0} > 0 \quad \dots \quad t_{i_n} - t_{i_{n-1}} > 0\,.$$

We will often write for convenience

$$T(\pi) = (i_0, i_1, \dots, i_n)\,. \tag{2.2}$$

The n faces of $T(\pi)$:

$$t_{i_0} = t_{i_1} < t_{i_2} < \dots < t_{i_n}\,, \quad t_{i_0} < t_{i_1} = t_{i_2} < \dots < t_{i_n}\,, \quad \dots, \quad t_{i_0} < t_{i_1} < \dots < t_{i_{n-1}} = t_{i_n}\,.$$

can then be represented by

$$(i_0 \sim i_1, i_2, \dots, i_n)\,, \quad (i_0, i_1 \sim i_2, \dots, i_n)\,, \quad \dots, \quad (i_0, i_1, \dots, i_{n-1} \sim i_n)\,, \tag{2.3}$$

where the $\sim$ symbol allows for transposition of the adjacent indices.

The faces of lower dimension are represented with more $\sim$ symbols up to the vertices:

$$(2.4)\quad (i_0,\, i_1\sim i_2\sim \ldots \sim i_n)\,,\quad (i_0\sim i_1,\, i_2\sim \ldots\sim i_n)\,,\quad \ldots,\quad (i_0\sim i_1\sim \ldots \sim i_{n-1},\, i_n)\,.$$

Each cone (2) can thus be viewed as a formal $n-1$-simplex with n faces (3) and n vertices (4) (this simplex can be realized by cutting T with an appropriate affine plane). The set of all these simplexes together with their faces builds up a « simplicial complex » which is a « triangulation » of a $n-1$-sphere (*e.g.* the sphere $\sum_{i=1}^{n}(t_0-t_i)^2=1$ in (t)).

We go now over to (s) space and consider the set of all planes

$$(2.5)\qquad \sum_{i\in X} s_i = 0\,,\qquad X\subset\{0, 1, \ldots, n\}\,,\qquad X\neq\emptyset\,,\qquad X\neq\{0, 1, \ldots, n\}\,.$$

If X_1 and X_2 are such that $X_2=\complement X_1$, *i.e.* if they are complementary subsets of $\{0, 1, \ldots, n\}$, the planes $\sum_{i\in X_1} s_i=0$ and $\sum_{i\in X_2} s_i=0$ are of course identical.

We will call $\mathscr{S}$ the set of open convex cones into which (s) is decomposed by the planes (5). This set has a less simple geometrical structure than $\mathscr{T}$. For instance a cone $S\in\mathscr{S}$ is not always limited by n planes when $n\geqslant 4$. The situation is illustrated by the case of the cone S for $n=4$ which is defined by

$$(2.6)\qquad \begin{cases} s_0+s_2>0 & s_0+s_3>0 & s_0+s_4>0\,,\\ s_1+s_2>0 & s_1+s_3>0 & s_1+s_4>0\,. \end{cases}$$

It may be remarked that the subspace of (s) orthogonal to an intersection (different from 0) of planes (1) of (t) is an intersection of planes (5) the converse however is generally not true.

3. – Analyticity of the Wightman function.

The following axioms: Lorentz invariance, existence and uniqueness of the vacuum, stability of the vacuum and local commutativity imply that the Wightman function $\mathscr{W}(z)$ exists, is analytic in $U\pi\mathscr{R}_n'$ (the union of the permuted extended tubes) (*) and is invariant there under the homogeneous complex Lorentz group.

(*) Let us recall that the tube $\mathscr{R}_n$ is defined by

$$\eta_1^I = y_1-y_0\in V_+\,,\qquad \eta_2^I=y_2-y_1\in V_+, \ldots, \eta_n^I=y_n-y_{n-1}\in V_+\,,\quad x \text{ arbitrary}.$$

The extended tube is defined by $\mathscr{R}_n' = U_{\Lambda\in L_+(C)}\Lambda\mathscr{R}_n$.

$\mathcal{W}(z)$ has $(n+1)!$ distribution boundary values $\mathcal{W}^{\pi}(x)$ (which are assumed to be tempered) when the imaginary part of z tends to zero, z remaining inside some $\pi\mathcal{R}_n$, $\pi\in\gamma_{n+1}$.

Let

$$(3.1)\qquad \zeta_j^{\pi} = z_{i_j} - z_{i_{j-1}}\,, \qquad (\zeta_j^{\pi} = \xi_j^{\pi} + i\eta_j^{\pi})\,, \qquad P_j^{\pi} = \sum_{k=j}^{n} p_{i_k}\,,$$

with

$$\pi(0, 1, \ldots, n) = (i_0, i_1, \ldots, i_n)\,,$$

then

$$p\cdot x = \sum_{i=0}^{n} p_i\cdot x_i = \sum_{j=1}^{n} P_j^{\pi}\cdot\xi_j^{\pi}\,.$$

If we write

$$(3.2)\qquad \mathcal{W}^{\pi}(x) = \overline{\mathcal{F}}G^{\pi}(p) = (2\pi)^{-2n}\int dp\, \exp\,[ip\cdot x]\, G^{\pi}(p)\,,$$

the stability of the vacuum expresses itself by the support condition

$$(3.3)\qquad G^{\pi}(p) = 0 \quad \text{unless} \quad P_j^{\pi}\in\overline{V}_+\,, \qquad j = 1, \ldots, n$$

This allows the Fourier transform (2) to be extended to a Laplace transform analytic in $\pi\mathcal{R}_n$ and gives conditions on its behaviour at infinity. We will not however formulate these conditions since it is easier in practical cases to use directly (3.3).

Regarding Lorentz invariance, if space reflections are allowed, $\mathcal{W}$ is invariant under $L(C)$, if they are rejected, $\mathcal{W}$ is only invariant under $L_+(C)$, where the complex rotations in $L_+(C)$ have determinant $+1$.

Having introduced the basic properties of the Wightman functions that do not connect several of them (as positive-definiteness of the metric and the asymptotic condition would do) we proceed by studying $\mathcal{W}$ in the space $(z^0, \boldsymbol{x})$.

The decomposition of the space (y^0) into cones $T(\pi)$ induces a decomposition of $(z^0, \boldsymbol{x})$ into domains $R^{4n} + iT(\pi)$.

If $\pi\in\gamma_{n+1}$, the corresponding permuted tube and permuted extended tube are

$$\pi\mathcal{R}_n = \{z : \pi^{-1}z\in\mathcal{R}_n\}\,, \qquad \pi\mathcal{R}'_n = \{z : \pi^{-1}z\in\mathcal{R}'_n\}\,.$$

The real point in $\mathcal{R}'_n$ are the Jost points [4], those in $\pi\mathcal{R}'_n$ the « permuted » Jost points. We do not assume that $\mathcal{W}(z)$ is uniform in $U\pi\mathcal{R}'_n$ but if x is a Jost point or a permuted Jost point, $\mathcal{W}(z)$ is holomorphic and uniform in a neighbourhood of x [1,5].

(4) R. Jost: *Helv. Phys. Acta*, **30**, 409 (1957).

(5) D. Kleitman: *Bull. Am. Phys. Soc.*, **5**, 82, 79 (1960).

Since $R^{4n}+iT(\pi)$ is the trace on $(z^0, \boldsymbol{x})$ of $\pi\mathscr{R}_n$, $\mathscr{W}(z^0, \boldsymbol{x})$ is analytic in every $R^{4n}+iT(\pi)$.

Suppose now that $T(\pi')$ and $T(\pi'')$ have a face $(i_0, \ldots, i_{k-1} \sim i_k, \ldots, i_n)$ in common. The piece of plane F defined by

$$y^0_{i_0} < \ldots < y^0_{i_{k-1}} = y^0_{i_k} < \ldots < y^0_{i_n} \tag{3.4}$$

is then a common boundary of $R^{4n}+iT(\pi')$ and $R^{4n}+iT(\pi'')$.

Let x be a real point in $\pi'\mathscr{R}'_n$ (a permuted Jost point [4]), $\mathscr{W}(z^0, \boldsymbol{x})$ is analytic in a neighbourhood of x, and therefore at some point of F for which $(x_k - x_{k-1})^2 < 0$.

Let now z be any point of F for which $(x_k - x_{k-1}) < 0$. By a very small complex Lorentz transformation, z can be brought either in $\pi'\mathscr{R}_n$ or in $\pi''\mathscr{R}_n$, so that the restrictions of $\mathscr{W}$ to $R^{4n}+iT(\pi')$ and to $R^{4n}+iT(\pi'')$ can both be continued over F at z. These continuations coincide because the set of points of F for which $(x_k - x_{k-1})^2 < 0$ is connected and they coincide at some points of this set.

It is now easy to conclude that $\mathscr{W}$ is analytic at those points of the plane $y^0_k - y^0_{k-1} = 0$ for which $(x_k - x_{k-1})^2 < 0$ and which do not belong to other planes of singularities. The case of these intersections is dealt with by use of the Kantensatz [6,7] and we get the following.

THEOREM 1. – *$\mathscr{W}(z^0, \boldsymbol{x})$ can have singularities only if two of its arguments $(z^0_i, \boldsymbol{x}_i)$ and $(z^0_{i'}, \boldsymbol{x}_{i'})$, $i \neq i'$ are such that $y^0_{i'} - y^0_i = 0$ and $x_{i'} - x_i$ is not space-like.*

Let now the function $F(z)$ be invariant under $L^\uparrow_+$ and such that $F(z^0, \boldsymbol{x})$ is analytic in the domain $R^{4n}+iT(\pi)$.

We introduce the variables $\zeta^\pi_1, \ldots, \zeta^\pi_n$ of eq. (4.1) as co-ordinates in (z), writing

$$z = (\zeta^\pi_1, \ldots, \zeta^\pi_n) = (\zeta^\pi_1, 0, \ldots, 0) + (0, \zeta^\pi_2, \ldots, 0) + \ldots + (0, 0, \ldots, \zeta^\pi_n)\,.$$

If $z \in \pi\mathscr{R}_n$, each term in the right-hand side is of the form

$$(0, \ldots, \zeta_i, \ldots, 0) = \Lambda^{(i)}(0, \ldots, \chi_i, \ldots, 0)\,,$$

where $\Lambda^{(i)} \in L^\uparrow_+$ and $(0, \ldots, \chi_i, \ldots, 0)$ belongs to the boundary of $R^{4n}+iT(\pi)$ in $(z^0, \boldsymbol{x})$, $(0, \ldots, \zeta_i, \ldots, 0)$ is thus a limiting point of analyticity points of $\mathscr{W}$ and, by virtue of the tube theorem (*), the same is true for z. Since $\pi\mathscr{R}_n$ is

(*) We need here an unusual form of the tube theorem. That this holds is seen by referring to a proof of it based on the use of the continuity theorem.

(6) H. BEHNKE and P. THULLEN: *Ergeb. d. Math.*, **3**, no. 3 (Berlin, 1934).

(7) D. RUELLE: *Helv. Phys. Acta*, **32**, 135 (1959).

an open set, z is a point of analyticity of F and we obtain easily (²) the following

THEOREM 2. – *If the domain of analyticity of* $\mathcal{W}(z^0, \boldsymbol{x})$ *is as given in Theorem* 1 *and if* $\mathcal{W}(z)$ *is invariant under* $L_+^\uparrow$ *(resp.* $L^\uparrow$*)* $\mathcal{W}(z)$ *is analytic in* $U\pi R_n'$ *and invariant there under* $L_+(C)$ *(resp.* $L(C)$*).*

4. – The boundary values of the Wightman function.

We have introduced in the last paragraph the real boundary values $\mathcal{W}^\pi(x)$ of $\mathcal{W}(z)$. They may also be defined by

$$\mathcal{W}^\pi(x) = \lim_{y^0 \to 0} \mathcal{W}(z^0, \boldsymbol{x}) \,, \qquad y^0 \in T(\pi) \,. \tag{4.1}$$

This defines $(n+1)!$ « sheets » along (x) corresponding to the cones $T(\pi) \in \mathcal{T}$.

We divide now each sheet into 2^n oriented « intervals »

$$(\pi, \sigma) = (i_0 \gtrless i_1 \gtrless \dots \gtrless i_n) \tag{4.2}$$

by giving the differences $x^0_{i_k} - x^0_{i_{k-1}}$ a definite sign so that either $x_{i_{k-1}} < x_{i_k}$ or $x_{i_{k-1}} > x_{i_k}$. σ is thus an arbitrary family of n $>$ or $<$ signs.

Just as the $T(\pi)$, the $(n+1)!\,2^n$ intervals (π, σ) may be viewed as the $(n-1)$-simplexes of a formal complex with faces

$$(i_0 = i_1 \gtrless i_2 \gtrless \dots \gtrless i_n) \,, \quad (i_0 \gtrless i_1 = i_2 \gtrless \dots \gtrless i_n) \,, \quad \dots, \quad (i_0 \gtrless i_1 \gtrless \dots \gtrless i_{n-1} = i_n) \tag{4.3}$$

and vertices

$$(i_0 \gtrless i_1 = i_2 = \dots = i_n) \,, \ (i_0 = i_1 \gtrless i_2 = \dots = i_n) \,, \ \dots, \ (i_0 = i_1 = \dots = i_{n-1} \gtrless i_n), \tag{4.4}$$

where the $=$ sign allows for transposition of the adjacent indices.

A sum of intervals is called a $(n-1)$-cycle when its boundary is zero. We will mostly consider cycles made up of $(n+1)!$ intervals corresponding to the permutations $\pi \in \gamma_{n+1}$.

Let $s(\sigma)$ be the number of $<$ signs in σ, then a necessary and sufficient condition for

$$C = \sum_{\pi \in \gamma_{n+1}} (-)^{s(\sigma_\pi)} (\pi, \sigma_\pi)$$

to be a cycle is that whenever $T(\pi_1)$ and $T(\pi_2)$ differ only by a transposition of consecutive indices, σ_{π_1} and σ_{π_2} may differ only by the sign between these two indices.

Consider now a sequence (*)

$$(4.5) \qquad \varphi_r(x) \in \mathscr{S}_{4n}\,, \qquad r\text{: positive integer}$$

chosen once for all and such that $\varphi_r(x) \to \delta(x)$ in $\mathscr{S}^*_{4n}$ when $r \to \infty$.

We define $(\pi, \sigma)\mathscr{W} * \varphi_r(x) \in \mathscr{S}^*$ to be equal to $\mathscr{W}^\pi * \varphi_r(x)$ when the inequalities σ hold, and zero otherwise.

We make then the following assumptions on $\mathscr{W}$ and the sequence φ_r.

ASSUMPTIONS A. – *If* $C = \sum_\pi (-)^{s(\sigma_\pi)}(\pi, \sigma_\pi)$ *is a cycle:*

1) $C\mathscr{W} * \varphi_r(x)$ *converges towards a distribution in* $\mathscr{S}^*$ *when* $r \to \infty$.

2) $C\mathscr{W}(x) = \lim_{r\to\infty} C\mathscr{W} * \varphi_r(x)$ *is invariant under* $L_+^\uparrow$.

The first assumption is less stringent than it would be to require each $(\pi, \sigma)\mathscr{W} * \varphi_r(x)$ to have a limit.

The second one is not unnatural since it can be *proved* for points such that $x_{i'} - x_i \neq 0$ whenever $i' \neq i$. One has just to use Theorem 1 and to notice that the derivatives of $C\mathscr{W}$ with respect to the parameters of the Lorentz group vanish when all x_i are different.

If the assumptions A are fulfilled, we define the vacuum expectation value of the time-ordered product by $\breve{G}(x) = T\mathscr{W}(x)$, where the cycle T is defined by $T = \sum_\pi (\pi, >)$. This definition may of course depend upon the sequence φ_r.

5. – Shifting of integral paths and analyticity of the Green function.

Using eq. (3.1), we can write

$$(5.1) \qquad \mathscr{F}[(\pi, \sigma)\mathscr{W} * \varphi_r](p) = (2\pi)^{-2n}\int \mathrm{d}x \exp\left(-i \sum P_j^\pi \cdot \xi_j^\pi\right)(\pi, \sigma)\mathscr{W} * \varphi_r(x)\,,$$

π being held fixed and such that $\pi(0, 1, \ldots, n) = (i_0, i_1, \ldots, i_n)$, let σ' and σ'' differ only by the sign between i_{k-1} and i_k. Clearly then

$$\int \mathrm{d}\xi_k^\pi \exp\left[-iP_k^\pi \cdot \xi_k^\pi\right]\left[(\pi, \sigma')\mathscr{W} * \varphi_r(x) + (\pi, \sigma'')\mathscr{W} * \varphi_r(x)\right],$$

has the support property $P_k^\pi \in \overline{V}_+$.

(*) For a definition of the convolution product ($*$), of the functionnal space $\mathscr{S}$ and of its dual $\mathscr{S}^*$ (space of tempered distributions), see SCHWARTZ (8).

(8) L. SCHWARTZ: *Théorie des distributions*, t. 2 (Paris, 1951); t. 1, 2ème éd. (Paris, 1957).

Otherwise stated, when P_k^{π} is restricted to the complementary of $\overline{V}_+$, in particular when $P_k^{\pi 0} < 0$, the formula

$$\mathscr{F}[(\pi, \sigma')\mathscr{W} * \varphi_r](p) = - \mathscr{F}[(\pi, \sigma'')\mathscr{W} * \varphi_r](p) \tag{5.2}$$

allows the path of integration of ξ_k to be « shifted ».

We will apply this result to

$$G_r(p) = (2\pi)^{-2n}\int \mathrm{d}x \exp[-ip\cdot x]\, T\mathscr{W} * \varphi_r(x) = \sum_\pi \mathscr{F}[(\pi, >)\mathscr{W} * \varphi_r](p). \tag{5.3}$$

Let p^0 belong to some cone S of the family $\mathscr{S}$ into which (p^0) can be divided (see Section **2**), the $P_j^{\pi 0}$ then have a definite sign when π and j are fixed.

Shifting the path of integration whenever it is possible in $\mathscr{F}[(\pi, >)\mathscr{W} * \varphi_r](p)$ we get

$$\mathscr{F}[(\pi, >)\mathscr{W} * \varphi_r](p) = (-)^{s(\sigma)}\mathscr{F}[(\pi, \sigma_\pi^S)\mathscr{W} * \varphi_r](p), \qquad p^0 \in S, \tag{5.4}$$

where the support of $(\pi, \sigma_\pi^S)\mathscr{W} * \varphi_r(x)$ is

$$\xi_j^{\pi 0} \leqslant 0 \quad \text{if} \quad P_j^{\pi 0} > 0, \qquad \xi_j^{\pi 0} \geqslant 0 \quad \text{if} \quad P_j^{\pi 0} < 0. \tag{5.5}$$

So, if q is such that $q^0 \in S$ and $\boldsymbol{q} = 0$

$$\exp[q\cdot x](\pi, \sigma_\pi^S)\mathscr{W} * \varphi_r(x) \in \mathscr{S}^*. \tag{5.6}$$

On the other hand, $C^S = \sum_\pi (-)^{s(\pi)}(\pi, \sigma_\pi^S)$ is easily seen to be a cycle and $C^S\mathscr{W}$ is therefore Lorentz-invariant. (5.6) gives then

$$\exp[(\Lambda q)\cdot x]\, C^S\mathscr{W} \in \mathscr{S}^* \qquad \text{for } \Lambda \in L_+^\uparrow. \tag{5.7}$$

We will now use the following properties [9]:

1) If A is a distribution, the set Γ of all real points q such that $\exp[q\cdot x]A(x) \in \mathscr{S}^*$ is convex.

2) $\mathscr{F}\{\exp[q\cdot x]A\}(p)$ defines a Laplace transform, analytic when the variable $k = p + iq$ is in the tube $R^{4n} + i\,\overset{0}{\Gamma}$ ($\overset{0}{\Gamma}$: the interior of Γ).

We will call $\Gamma(S)$ the set Γ corresponding to the distribution $C^S\mathscr{W}$. Using 1) and the invariance of $\Gamma(S)$ under homotheties $s \to \alpha s$ ($\alpha > 0$), we see that any finite sum of vectors belonging to $\Gamma(S)$ belongs to $\Gamma(S)$.

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[9] J. L. Lions: *Suppl. Nuovo Cimento*, **14**, 9 (1959).

Let now $u^\alpha \in V_+$, $\alpha = 1, \ldots, 4$ and $\sum_{\alpha=1}^{4} u^\alpha = u$, where u is the unit vector along thet ime axis. If $s^\alpha \in S$, the point $q = (\sum_{\alpha=1}^{4} s_0^\alpha u^\alpha, \sum_{\alpha=1}^{4} s_1^\alpha u^\alpha, \ldots, \sum_{\alpha=1}^{4} s_n^\alpha u^\alpha)$ belongs to $\Gamma(S)$, because we can write $u^\alpha = r^\alpha \Lambda^\alpha u$, $r^\alpha > 0$, $\Lambda^\alpha \in L_+^\uparrow$.

If $s^1 = \ldots = s^4 = s$, $q = (s, \boldsymbol{o})$. If $s^1, \ldots, s^4$ are varied over neighbourhoods of s in S, q varies over a neighbourhood of $(s, \boldsymbol{o})$. Since this neighbourhood belongs to $\Gamma(S)$, we have proved that $s \in S$ implies $(s, \boldsymbol{o}) \in \overset{0}{\Gamma}(S)$.

Extending the Fourier transform $G^S(p) = \mathscr{F}C^S\mathscr{W}(p)$ to a Laplace transform $G^S(p+iq)$ analytic in the tube $R^{4n} + i\overset{0}{\Gamma}(S)$ and then restricting to the space $(k^0, \boldsymbol{p})$, we have proved:

1) $G^S(k^0, \boldsymbol{p})$ is analytic in $R^{4n} + iS$ (*i.e.* when $q^0 \in S$).

2) $\lim_{q^0 \to 0, q^0 \in S} G^S(k^0, \boldsymbol{p}) = G(p)$ if $p^0 \in S$.

Let $S_0 \in \mathscr{S}$ be defined by the equations $s_i > 0$, $1 \leqslant i \leqslant n$.

We introduce the retarded cycle R by $R = C^{S_0}$, $R\mathscr{W}$ then reduces to the well-known r function [10] (*) and $\mathscr{F}r(p)$ is analytic if p is a Jost point, *i.e.* if

$$(5.8) \qquad \Big(\sum_{i=1}^{n} \lambda_i p_i\Big)^2 < 0 \quad \text{whenever} \quad \lambda_i \geqslant 0\,, \quad \sum_{i=1}^{n} \lambda_i = 1\,.$$

A set $F \subset (s)$, $F \neq \emptyset$, will be called a face of $S \in \mathscr{S}$ if it is the interior in a subspace $\sum_{i \in X} s_i = 0$ of the intersection of this subspace and the closure $\bar{S}$ of S.

Let now S' and S'' have a face F in common. The piece of plane $R^{4n} + iF$ is then a common boundary of $R^{4n} + iS'$ and $R^{4n} + iS''$.

If a point $k \in R^{4n} + iF$ is such that $\big(\sum_{i \in X} p_i\big)^2 < 0$, it is possible to bring it either in $R^{4n} + i\overset{0}{\Gamma}(S')$ or in $R^{4n} + i\overset{0}{\Gamma}(S'')$ by a very small complex Lorentz transformation. This means that $G^{S'}(k^0, \boldsymbol{p})$ and $G^{S''}(k^0, \boldsymbol{p})$ can both be analytically continued through $R^{4n} + iF$ at k. In order to prove that these continuations agree in the connected set of all points of $R^{4n} + iF$ for which $\big(\sum_{i \in X} p_i\big)^2 < 0$, we will show that they agree at some point of this set.

Let p_J be a Jost point (eq. (5.8)) such that $p_J^0 \in F$ in (p^0). In any real neighbourhood of p_J there exist boundary values of $G^{S'}(k)$ and $G^{S''}(k)$ which coincide with $\mathscr{F}T\mathscr{W} = \mathscr{F}r$ which is analytic in a neighbourhood of p_J. There are thus points of F with $\big(\sum_{i \in X} p_i\big)^2 < 0$ such that $G^{S'}(k)$ and $G^{S''}(k)$ coincide, which proves the announced property.

(*) The connexion with the usual definition of r is given by ref. [11], eq. (3.11).

[10] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 319 (1957).

[11] N. NISHIJIMA: *Phys. Rev.*, **111**, 995 (1958).

The intersections of several surfaces $\sum_{i\in X} q_i^0 = 0$ are easily dealt with by use of the Kantensatz [6] and we get for the function $G(k)$, whose restriction to $R^{4n} + i\overset{0}{\Gamma}(S)$ is $G^S(k)$, the following

THEOREM 3. – *The function $G(k^0, \mathbf{p})$ can have singularities only if $\sum_{i\in X} q_i^0$ vanishes for some $X \subset \{0, 1, ..., n\}$, $X \neq \emptyset$, $X \neq \{0, 1, ..., n\}$ and $\sum_{i\in X} p_i$ is not space-like.*

6. – The multiple commutators.

We will now try to get more information about the connexion between the cones $S \in \mathscr{S}$ in (s) and the corresponding cycles C^S.

Let $\mathscr{B}_n$ be the abelian group generated by the cones $S \in \mathscr{S}$, and let $\mathscr{C}_n$ be the subgroup generated by the cycles C^S in the abelian group of all cycles defined in Section **4**. The mapping $S \to C^S$ extends by linearity to a homomorphism $\mathscr{B}_n \to \mathscr{C}_n$. This homomorphism is obviously onto, but it is not in general an isomorphism. Its kernel $\mathscr{A}_n$ is the set of linear combinations of $S \in \mathscr{S}$ such that the corresponding linear combinations of boundary values of $G(z)$ vanish identically.

We call Steinmann relations the resulting linear relations between the $G^S(p)$.

Let $X = \{i_0, i_1, ..., i_k\} \subset \{0, 1, ..., n\}$. We will call $(s)_X$ the subspace of (s) defined by the relation $\sum_{i\in X} s_i = 0$ in the subspace generated by $s_{i_0}, s_{i_1}, ..., s_{i_k}$. The decomposition of $(s)_X$ into a set $\mathscr{S}_X$ of cones is effected just as in (s) and to these cones we associate cycles C^S for the variables $z_{i_0}, z_{i_1}, ..., z_{i_k}$. We will call $\mathscr{C}_X$ the abelian group generated by the cycles thus formed.

Consider now two cones $S', S'' \in \mathscr{S}$ such that S' and S'' have in common the face F belonging to the plane $\sum_{i\in X} s_i = 0$ so that $\sum_{i\in X} s_i < 0$ in S' and $\sum_{i\in X} s_i > 0$ in S''.

If we write

$$C^{S''} - C^{S'} = \sum_{\pi} [(-)^{s(\sigma'')}(\pi, \sigma_\pi^{S''}) - (-)^{s(\sigma')}(\pi, \sigma_\pi^{S'})] , \tag{6.1}$$

where we have set for simplicity $\sigma' = \sigma_\pi^{S'}$, $\sigma'' = \sigma_\pi^{S''}$, the sum in the right-hand side extends only over those permutations $\pi \in \gamma_{n+1}$ for which

$$\pi(0, 1, ..., n) = (i_0, i_1, ..., i_n) \text{ and } X = \{i_0, ..., i_k\} \quad \text{or} \quad X = \{i_{n-k}, ..., i_n\}. \tag{6.2}$$

Let $\pi_1 \in \gamma_{k+1}$ and $\pi_2 \in \gamma_{n-k}$ be permutations of X and CX respectively.
We may then represent the permutations π of eq. (2) by $\pi_1\pi_2$ or $\pi_2\pi_1$, the

corresponding σ being of the form $\sigma_1 \gtrless \sigma_2$ or $\sigma_2 \gtrless \sigma_1$ respectively, so that

$$(6.3)\qquad \begin{cases} (\pi_1\pi_2, \sigma') = (\pi_1\pi_2, \sigma_1 > \sigma_2) & (\pi_1\pi_2, \sigma'') = (\pi_1\pi_2, \sigma_1 < \sigma_2)\,, \\ (\pi_2\pi_1, \sigma') = (\pi_2\pi_1, \sigma_2 < \sigma_1) & (\pi_2\pi_1, \sigma'') = (\pi_2\pi_1, \sigma_2 > \sigma_1)\,, \end{cases}$$

σ_1 (resp. σ_2) is determined by the signs of the $\sum_{i\in X_1} s_i$, $X_1 \subset X$, $X_1 \neq \emptyset$, $X_1 \neq X$ (resp. of the $\sum_{i\in X_2} s_i$, $X_2 \subset CX$. $X_2 \neq \emptyset$, $X_2 \neq CX$) in F, which are the same as in S' and S''. Since the sign conditions on the $\sum_{i\in X_1} s_i$ (resp. $\sum_{i\in X_2} s_i$) are compatible, they determine a cone $S_1 \in \mathscr{S}_X$ (resp. $S_2 \in \mathscr{S}_{CX}$) and one may write $\sigma_1 = \sigma_{\pi_1}^{S_1}$ (resp. $\sigma_2 = \sigma_{\pi_2}^{S_2}$) independently of whether $\pi = \pi_1\pi_2$ or $\pi = \pi_2\pi_1$.

Using (6.3), (6.1) can be written

$$(6.4)\qquad \begin{cases} C^{S''} - C^{S'} = \sum_{\pi_1}\sum_{\pi_2}[(-)^{s(\sigma_1>\sigma_2)}(\pi_1\pi_2, \sigma_1 > \sigma_2) - (-)^{s(\sigma_1<\sigma_2)}(\pi_1\pi_2, \sigma_1 < \sigma_2) + \\ \qquad + (-)^{s(\sigma_2<\sigma_1)}(\pi_2\pi_1, \sigma_2 < \sigma_1) - (-)^{s(\sigma_2>\sigma_1)}(\pi_2\pi_1, \sigma_2 > \sigma_1)] \\ \quad = \sum_{\pi_1}(-)^{s(\sigma_1)}\sum_{\pi_2}(-)^{s(\sigma_2)}[(\pi_1\pi_2, \sigma_1 > \sigma_2) + \\ \qquad + (\pi_1\pi_2, \sigma_1 < \sigma_2) - (\pi_2\pi_1, \sigma_2 > \sigma_1) - (\pi_2\pi_1, \sigma_2 < \sigma_1)]\,. \end{cases}$$

We introduce now the product and the commutator of two cycles C_1 and C_2 when these cycles have no variable in common

$$(6.5)\qquad \begin{cases} C_1 \cdot C_2 = \sum_{\pi_1}(-)^{s(\sigma_1)}(\pi_1, \sigma_1)\cdot\sum_{\pi_2}(-)^{s(\sigma_2)}(\pi_2, \sigma_2) \\ \quad = \sum_{\pi_1}(-)^{s(\sigma_1)}\sum_{\pi_2}(-)^{s(\sigma_2)}[(\pi_1\pi_2, \sigma_1 > \sigma_2) + (\pi_1\pi_2, \sigma_1 < \sigma_2)]\,, \\ [C_1, C_2] = C_1\cdot C_2 - C_2\cdot C_1\,. \end{cases}$$

The commutator of two cycles will be defined to be zero if they have at least one variable in common. We have thus proved the formula

$$(6.6)\qquad C^{S''} - C^{S'} = [C^{S_1}, C^{S_2}]\,.$$

Conversely, if S_1 and S_2 are arbitrary cones of $\mathscr{S}_X$ and $\mathscr{S}_{CX}$ respectively, it is easily seen that there exists at least one couple of cones S', $S'' \in \mathscr{S}$ such that eq. (6.6) holds. This means that the direct sum $\mathscr{C} = \sum_X \mathscr{C}_X$ has the structure of a Lie algebra for commutation.

From the above, it results that the cycle C^S corresponding to any cone $S \in \mathscr{S}$ is equal to the cycle C^{S_0} corresponding to a fixed cone S_0 plus a sum of commutators of cycles with a smaller number of variables.

One may thus reconstruct every cycle in $\mathscr{C}_n$ if one knows one cycle C^S, $S \in \mathscr{S}_X$ for each X. For instance one may take the retarded cycles, obtaining the following

THEOREM 4. – *The abelian group $\mathscr{C}$ generated by the cycles C^S is also generated as a Lie algebra by the retarded cycles R.*

If we restrict to $\mathscr{C}_n$, this means that the abelian group of all linear combinations with integral coefficients of the boundary values $G^S(p)$ of $G(k)$ is identical to the abelian group of the Fourier transforms of all linear combinations with integral coefficients of the vacuum expectation values of multiple commutators of retarded products and fields (each field $A^{(i)}(x_i)$ being used eventually in a R-product, once and only once in each multiple commutator).

7. – Introduction of masses.

When we introduced cycles along the real boundary values of $\mathscr{W}(z)$, we had to cut singularities (at the top of light cones). This difficulty was solved by a regularization process and assumptions about $\mathscr{W}$.

The purpose of this paragraph is to avoid similar troubles with $G(k)$ by introducing a non-zero minimum mass in the theory.

It will also be necessary for the following to introduce the truncated Wightman functions $\tilde{\mathscr{W}}$ [12]. No proof will be given here of the properties stated.

Let ϱ_k be the family of all partitions of the set $\{0, 1, ..., n\}$ into $k+1$ subsets: $X_0, X_1, ..., X_k$ and let $\mathscr{W}(z)_{X_j}$ be the Wightman function of the variables z_i such that $i \in X_j$.

We write then the reduction formula

$$\mathscr{W}(z) = \sum_{k=0}^{n} \sum_{\varrho_k} \prod_{j=0}^{k} \tilde{\mathscr{W}}(z)_{X_j} \tag{7.1}$$

and use it to define the truncated functions $\tilde{\mathscr{W}}$ recursively on the number of variables.

The function $\tilde{\mathscr{W}}(z)$ has all mathematical properties described above for $\mathscr{W}(z)$. A Green function can be deduced of it, which also has all the mathematical properties of $G(k)$. *It can be seen that it coincides in fact with $G(k)$.*

We know that the mass operator in Hilbert space has an eigenvalue equal to zero and corresponding to the vacuum. We shall assume that the rest of its spectrum is $\geqslant \mu$, $\mu > 0$.

([12]) R. HAAG: *Phys. Rev.*, **112**, 668 (1958) and *Suppl. Nuovo Cimento*, **14**, 131 (1959).

Let then $V^{\mu}_{+} = x: x \in V_{+},\ x^2 > \mu^2\}$, x: a vector in Minkowski space. We have the following (see [13])

THEOREM 5. – *Let* $\widetilde{\mathscr{W}}^{\pi}(x) = \overline{\mathscr{F}}\, \widetilde{G}^{\pi}(p) = (2\pi)^{-2n}\int \mathrm{d}p \exp[ipx]\, \widetilde{G}^{\pi}(p)$; $\widetilde{G}^{\pi}(p)$ then *satisfies the support condition* $\widetilde{G}^{\pi}_{j}(p) = 0$ *unless* $P^{\pi}_{j} \in \overline{V}^{\mu}_{+}$, $j = 1, \ldots, n$.

THEOREM 6. – *The function* $G(k^0, \boldsymbol{p})$ *can have singularities only if* $\sum_{i \in X} q^0_i$ *vanishes for some* $X \subset \{0, 1, \ldots, n\}$, $X \neq \emptyset$, $X \neq \{0, 1, \ldots, n\}$ and $\left(\sum_{i \in X} p_i\right)^2 \geqslant \mu^2$.

We gather now the information we have about the functions $\widetilde{\mathscr{W}}$ and G.

I. Properties of $\widetilde{\mathscr{W}}(z)$.

1) $\widetilde{\mathscr{W}}(z)$ *is invariant under* $L_{+}(C)$ *or* $L(C)$ *according to whether the theory is invariant under* $L^{\uparrow}_{+}$ *or* $L^{\uparrow}$.

2) *The singularities of* $\widetilde{\mathscr{W}}(z^0, \boldsymbol{x})$ *are given by the Theorem* 1.

3) $\widetilde{\mathscr{W}}(z)$ *has boundary values*

$$\widetilde{\mathscr{W}}^{\pi}(x) = \lim_{y^0 \to 0,\ y^0 \in (\pi)} \widetilde{\mathscr{W}}(z^0, \boldsymbol{y})$$

which are tempered distributions.

4) There are conditions on the behaviour of $\widetilde{\mathscr{W}}(z)$ at infinity which we replace by the *support conditions of Theorem* 5 *on* $\mathscr{F}\widetilde{\mathscr{W}}^{\pi}(p)$.

From these properties of $\widetilde{\mathscr{W}}$, we have derived the following

II. Properties of $G(p)$.

1) $G(k)$ *is invariant under* $L_{+}(C)$ *or* $L(C)$ *according to wheter the theory is invariant under* $L^{\uparrow}_{+}$ *or* $L^{\uparrow}$.

2) *The singularities of* $G(k^0, \boldsymbol{p})$ *are given by the Theorem* 6.

3) $G(k)$ *has boundary values*

$$G^{S}(p) = \lim_{q^0 \to 0,\ q^0 \in S} G(k^0, \boldsymbol{p})$$

which are tempered distributions.

4) *These boundary values are subjected to linear conditions: the Steinmann relations* (Section **6**).

5) There are conditions on the behaviour of $G(k)$ at infinity which we replace by the *support conditions on* $\overline{\mathscr{F}}G^{S}(x)$ *which follow from eq.* (5.5).

[13] D. RUELLE: *Thèse* (Bruxelles, 1959).

8. – Products.

In order to show that the information contained in the properties of the $\tilde{\mathscr{W}}$ function has been completely translated into terms of the properties of the G function, we will reconstruct $\tilde{\mathscr{W}}$ (not $\mathscr{W}$!) from G. This paragraph is devoted to an intermediate step in this reconstruction, namely the definition of products.

Since the distributions $G^S(p)$ are subjected to the Steinmann relations, they generate an abelian group which is isomorphic to the group $\mathscr{C}_n$ of cycles introduced in Section **6**. For facility, we will in fact identify the two groups and represent the $G^S(p)$ by the corresponding cycles. We will also introduce the commutators by the formula (6.6) and we will be allowed, in computations with multiple commutators, to use the relations

$$(8.1)\qquad [C_1, C_2] = -[C_2, C_1], \qquad [C_1' + C_1'', C_2] = [C_1', C_2] + [C_1'', C_2],$$
$$[C_1, [C_2, C_3]] + [C_2, [C_3, C_1]] + [C_3, [C_1, C_2]] = 0 .$$

Consider now a partition of $\{0, 1, ..., n\}$ into $k+1$ subsets $X_0, X_1, ..., X_k$. If $X_j = \{i_0, i_1, ..., i_{r(j)}\}$, we will write $x_{jj'} = x_{i_{j'}}$ and $p_{jj'} = p_{i_{j'}}$.

We define then

$$\mathrm{d}p_j = \mathrm{d}p_{j1} \dots \mathrm{d}p_{jr(j)} , \qquad P_j = \sum_{l=j}^{k} \sum_{i\in X_l} p_i .$$

Now,

$$\mathrm{d}p = \mathrm{d}p_0 \cdot \mathrm{d}P_1 \cdot \mathrm{d}p_1 \dots \mathrm{d}P_k \cdot \mathrm{d}p_k$$

and

$$\sum_{i=0}^{n} p_i \cdot x_i = \sum_{j=1}^{k} P_j (x_{j0} - x_{(j-1)0}) + \sum_{j=0}^{k} \sum_{j'=1}^{r(j)} p_{jj'} (x_{jj'} - x_{j0})$$

so that if

$$\bar{\bar{\mathscr{F}}}_{X_j} = (2\pi)^{-2r(j)} \int \mathrm{d}p_j \exp\,[i \sum_{j'=1}^{r(j)} p_{jj'} (x_{jj'} - x_{j0})],$$

we have

$$\bar{\bar{\mathscr{F}}} = \bar{\bar{\mathscr{F}}}_{X_0} \bar{\bar{\mathscr{F}}}_{X_1} \dots \bar{\bar{\mathscr{F}}}_{X_k} (2\pi)^{-2k} \int \mathrm{d}P_1 \dots \mathrm{d}P_k \exp\,[i \sum_{j=1}^{k} P_j (x_{j0} - x_{(j-1)0}) .$$

Let $C_j \in \mathscr{C}_{X_l}$, we will define products $C_0 \cdot C_1 \dots C_k$ with the following properties.

III. Properties of the products.

1) *The products are Lorentz-invariant tempered distributions.*

2) *Distributivity*: $C_0 \dots (C_i' + C_i'') \dots C_k = C_0 \dots C_i' \dots C_k + C_0 \dots C_i'' \dots C_k$.

3) $[\overline{\mathscr{F}}_{X_0}\overline{\mathscr{F}}_{X_1}\dots\overline{\mathscr{F}}_{X_k}\, C_0\cdot C_1\dots C_k]\cdot\exp\sum_{j=0}^{k}\sum_{j'=0}^{r(j)} s_{jj'}x^0_{jj'}\in\mathscr{S}^*$
when $C_j = C^{S_j}$, $S_j\in\mathscr{S}_{X_j}$ *and* $\sum_{j'=0}^{r(j)} s_{jj'}=0$, $(s_{j0}, s_{j1}, \dots s_{jr(j)})\in\overline{S}_j$.

4) *Support properties in* (p)*:*

$$C_0\cdot C_1\dots C_k \text{ vanishes unless } P_j\in\overline{V}^{\mu}_{+},\ 1\leqslant j\leqslant k.$$

5) *The following identity holds:*

$$C_0\dots C_{l-1}\cdot C_l\dots C_k - C_0\dots C_l\cdot C_{l-1}\dots C_k = C_0\dots[C_{l-1}, C_l]\dots C_k\,.$$

This last property allows the definition of multiple commutators of products of cycles, the number of dots ($\cdot$) plus the number of square brackets ([ ]) being equal to k. This will justify *a posteriori* our use of the symbols $[C_0, C_1\dots C_k]$ and $[C_0\dots C_{k-1}, C_k]$.

Properties III are trivial consequences of Properties II for one single cycle belonging to $\mathscr{C}_n$. We will now define the products recursively on the number of factors and show that III-1)-5) hold at each step.

By definition, let

$$[C_0, C_1\dots C_k] = \sum_{j=1}^{k} C_1\dots C_{j-1}\cdot[C_0, C_j]\cdot C_{j+1}\dots C_k\,. \tag{8.3}$$

First, we show that this expression vanishes unless $P_1^2\geqslant\mu^2$.

By the induction Assumption III-4), $C_1\dots C_{j-1}\cdot[C_0, C_j]\cdot C_{j+1}\dots C_k$ vanishes unless

$$P_2-P_1\in\overline{V}^{\mu}_{+},\ \dots,\ P_j-P_1\in\overline{V}^{\mu}_{+},\quad P_{j+1}\in\overline{V}^{\mu}_{+},\ \dots,\ P_k\in\overline{V}^{\mu}_{+}\,. \tag{8.4}$$

If $P^0_k<\varepsilon$, $0<\varepsilon<\mu$, the sum in the right-hand side of (8.3) reduces to one term with the support property $P_k-P_1\in\overline{V}^{\mu}_{+}$, so that $[C_0, C_1, \dots, C_k]$ vanishes unless $P^0_1\leqslant-\mu+\varepsilon$.

Similarly, if $P^0_k-P^0_1<\varepsilon$, $[C_0, C_1, \dots, C_k]$ vanishes unless $P^0_1\geqslant\mu-\varepsilon$. Let now $P^0_k>0$, $P^0_k-P^0_1>0$, using the induction Assumption III-4), we have

$$C_k\cdot C_1\dots C_{j-1}\cdot[C_0, C_j]\cdot C_{j+1}\dots C_{k-1}=0\,,\quad [C_0, C_k]\cdot C_1\dots C_{k-1}=0\,.$$

Therefore, using III-2)-5),

$$\begin{aligned}[C_0, C_1\dots C_k] &= \sum_{j=1}^{k-1} C_1\dots[C_0, C_j]\dots C_k + C_1\dots C_{k-1}\cdot[C_0, C_k]\\ &= \sum_{j=1}^{k-1}[C_1\dots[C_0, C_j]\dots C_{k-1}, C_k] + [C_1\dots C_{k-1}, [C_0, C_k]]\end{aligned}$$

$$= \sum_{j=1}^{k-1} \Big(\sum_{l=1}^{j-1} C_1 \dots [C_l, C_k] \dots [C_0, C_j] \dots C_{k-1} + C_1 \dots [[C_0, C_j], C_k] \dots C_{k-1} +$$

$$+ \sum_{l=j+1}^{k-1} C_1 \dots [C_0, C_j] \dots [C_l, C_k] \dots C_{k-1} + C_1 \dots [C_j, [C_0, C_k] \dots C_{k-1},$$

$$= \sum_{j=1}^{k-1} \Big(\sum_{l=1}^{j-1} C_1 \dots [C_l, C_k] \dots [C_0, C_j] \dots C_{k-1} + C_1 \dots [C_0, [C_j, C_k]] \dots C_{k-1} +$$

$$+ \sum_{l=1}^{k-1} C_1 \dots [C_0, C_j] \dots [C_l, C_k] \dots C_{k-1} \Big)$$

$$= \sum_{l=1}^{k-1} \Big(\sum_{j=1}^{l-1} C_1 \dots [C_0, C_j] \dots [C_l, C_k] \dots C_{k-1} + C_1 \dots [C_0, [C_l, C_k]] \dots C_{k-1} +$$

$$+ \sum_{j=l+1}^{k-1} C_1 \dots [C_l, C_k] \dots [C_0, C_j] \dots C_{k-1} \Big)$$

$$= \sum_{l=1}^{k-1} [C_0, C_1 \dots [C_l, C_k] \dots C_{k-1}] .$$

We have thus proved that $[C_0, C_1, \dots, C_k]$ vanishes unless $P_1^2 \geqslant \mu^2$ when $P_k^0 > 0$, $P_k^0 - P_1^0 > 0$.

As a whole, using the « principe de recollement des morceaux » [8], we have proved that $[C_0, C_1 \dots C_k]$ vanishes when $|P_1^0| < \mu - \varepsilon$, $0 < \varepsilon < \mu$.

Letting ε go to zero and using Lorentz invariance, we see that $[C_0, C_1 \dots C_k]$ vanishes unless $P_1^2 \geqslant \mu^2$.

We define the product $C_0 \cdot C_1 \dots C_k$ to be equal to $[C_0, C_1 \dots C_k]$ when $P_1^0 > -\mu$, and to zero when $P_1^0 < \mu$.

The product may also be defined to be equal to $[C_0 \dots C_{k-1}, C_k]$ when $P_k^0 > -\mu$ and to zero when $P_1^0 < \mu$. The proof of the equivalence of the two definitions is straightforward if one uses the same kind of developments as above.

It remains now to show that the properties III.1-5) III.1), hold. 2), 3), are direct consequences of the corresponding induction assumptions and of the definition of the product.

III.4). The product has been defined so that $P_1 \in \overline{V}_+^\mu$, introducing this into (8.4) one finds $P_j \in \overline{V}_+^\mu$ for $2 \leqslant j \leqslant k$.

III.5). We have

$$[C_0, C_1 \dots C_{l-1} \cdot C_l \dots C_k] = \sum_{j=1}^{l-2} C_1 \dots [C_0, C_j] \dots C_{l-1} \cdot C_l \dots C_k +$$

$$+ C_1 \dots [C_0, C_{l-1}] \cdot C_l \dots C_k + C_1 \dots C_{l-1} \cdot [C_0, C_l] \dots C_k +$$

$$+ \sum_{j=l+1}^{k} C_1 \dots C_{l-1} \cdot C_l \dots [C_0, C_j] \dots C_k ,$$

so that

$$[C_0, C_1 \dots C_{l-1} \cdot C_l \dots C_k] - [C_0, C_1 \dots C_l \cdot C_{l-1} \dots C_k]$$

$$= \sum_{j=1}^{l-2} C_1 \dots [C_0, C_j] \dots [C_{l-1}, C_l] \dots C_k + C_1 \dots [[C_0, C_{l-1}], C_l] \dots C_k + \\ + C_1 \dots [C_{l-1}, [C_0, C_l]] \dots C_k + \sum_{j=l+1}^{k} C_1 \dots [C_{l-1}, C_l] \dots [C_0, C_j] \dots C_k$$

$$= \sum_{j=1}^{l-2} C_1 \dots [C_0, C_j] \dots [C_{l-1}, C_l] \dots C_k + C_1 \dots [C_0, [C_{l-1}, C_l]] \dots C_k + \\ + \sum_{j=l+1}^{k} C_1 \dots [C_{l-1}, C_l] \dots [C_0, C_j] \dots C_k$$

$$= [C_0, C_1 \dots [C_{l-1}, C_l] \dots C_k] .$$

From this equation, the property results, except for

$$C_0 \cdot C_1 \dots C_k - C_1 \cdot C_0 \dots C_k = [C_0, C_1] \dots C_k$$

which is proved by considering the difference

$$[C_0 \cdot C_1 \dots C_{k-1}, C_k] - [C_1 \cdot C_0 \dots C_{k-1}, C_k] .$$

9. – Determination of the Wightman function from the Green function.

We will now prove the following

THEOREM 7. – *If a function $G(k)$ satisfies the conditions* II.1)-5) *of Section* **7**, *it determines uniquely a Wightman function $\tilde{\mathscr{W}}(z)$ from which it derives. $\tilde{\mathscr{W}}(z)$ satisfies the conditions* I.1)-4).

A restriction will be brought to this formulation at the end of the proof.

From $G(k)$ we have already derived the products $C_0 \cdot C_1 \dots C_n$ and proved the properties III.1)-5) starting from II.1)-5).

Let us first consider condition III.3).

Corresponding to the cone $S_j \in \mathscr{S}_{X_j}$ we define the convex closed cone $U(S_j)$:

$$U(S_j) = \{(x^0_{j0}, x^0_{1j}, \dots, x^0_{jr(j)}) : \sum_{j'=0}^{r(j)} s_{jj'} x_{jj'} \leqslant 0 \text{ whenever } (s_{j0}, s_{j1}, \dots, s_{jr(j)}) \in \bar{S}_j\}. \tag{9.1}$$

III.3) implies that $\bar{\mathscr{F}}_{X_0} \bar{\mathscr{F}}_{X_1} \dots \bar{\mathscr{F}}_{X_k} C_0 \cdot C_1 \dots C_n$ decreases faster than exponentially at infinity in the complementary of $U(S_j)$.

But then, it can be shown, using Lorentz invariance, that $\mathscr{F}_{X_0}\mathscr{F}_{X_1}\ldots\mathscr{F}_{X_k}C_0\cdot C_1\ldots C_k$ must actually vanish outside of $U(S_j)$.

So we replace III.3) by

III.3′). *Support properties in* (x): $\mathscr{F}_{X_0}\mathscr{F}_{X_1}\ldots\mathscr{F}_{X_k}C_0\cdot C_1\ldots C_k$ *where* $C_j = C^{S_j}$, $S_j\in\mathscr{S}_{X_j}$, *vanishes unless*

$$(x^0_{j0}, x^0_{j1}, \ldots, x^0_{jr(j)})\in U(S_j)\,, \qquad 0\leqslant j\leqslant k.$$

Consider now the case when $G(k)$ derives from a Wightman function $\tilde{\mathscr{W}}(z)$ in the way described in Section **5**.

Up to Fourier transformation, the products defined in the last paragraph are then identical with those defined by eq. (6.5) (*). The proof of this statement is easy, it proceeds by induction on the number of factors of the product and is based on the support property $P^\pi_k\in\overline{V}{}^\mu_+$ of

$$\int\mathrm{d}\xi^\pi_k \exp[-iP^\pi_k\cdot\xi^\pi_k](\pi,\sigma')\tilde{\mathscr{W}}*\varphi_r(x)+(\pi,\sigma'')\tilde{\mathscr{W}}*\varphi_r(x)]\,,$$

when $\pi(0, 1, \ldots, n) = (i_0, i_1, \ldots, i_n)$ and σ', σ'' differ only by the sign between i_{k-1} and i_k.

In particular, the boundary values of $\tilde{\mathscr{W}}$ are given by

$$\tilde{\mathscr{W}}^\pi(x) = \overline{\mathscr{F}}[A^{(i_0)}\cdot A^{(i_1)}\ldots A^{(i_n)}](x)\,, \tag{9.2}$$

where the « cycle » $A^{(i)}$ is C^{S_i}, S_i being the unique « cone » in $\mathscr{S}_{\{i\}}$ ($\mathscr{S}_{\{i\}}$ as well as S_i is of course reduced to a point).

Now, if $G(k)$ satisfies the conditions II.1)-5) but has not been derived from a function $\tilde{\mathscr{W}}(z)$, we define a function $\tilde{\mathscr{W}}(z)$ from $G(k)$ by equation (9.2).

Properties I.1), 3), 4) are then immediate consequences of III.1), 4). In order to show that I.2) holds too, we refer to the proof of Theorem 1 and see that this holds if

$$\overline{\mathscr{F}}[A^{(i_0)}\ldots A^{(i_{k-1})}\cdot A^{(i_k)}\ldots A^{(i_n)}](x)-\overline{\mathscr{F}}[A^{(i_0)}\ldots A^{(i_k)}\cdot A^{(i_{k-1})}\ldots A^{(i_n)}](x)$$

vanishes when $(x_k - x_{k-1})^2 < 0$.

To prove this we use first III.5) to write $A^{(i_{k-1})}\cdot A^{(i_k)} - A^{(i_k)}\cdot A^{(i_{k-1})} = [A^{(i_{k-1})}, A^{(i_k)}]$ then (6.6) and III.2) to see that $[A^{(i_{k-1})}, A^{(i_k)}]$ is the difference of two cycles in two variables with supports $x_k - x_{k-1}\in\overline{V}_+$ and $x_{k-1}-x_k\in\overline{V}_+$ respectively, and finally III.3′), 1) to show that these support properties remain in the full products.

(*) The proper extension of eq. (6.5) to products of $k+1$ cycles is immediate. These products are then applied to $\tilde{\mathscr{W}}$.

Now that we have constructed a function $\tilde{\mathscr{W}}(z)$ with all the desired properties, it remains just to show that the function $G(k)$ derived from $\tilde{\mathscr{W}}(z)$ by the method of Section **5** is the same as the one from which we have started.

To do this, we will simply show that the products of cycles C^{S_j}, $S_j \in \mathscr{S}_{X_j}$, $\{X_0, X_1, ..., X_k\}$: a partition of $\{0, 1, ..., n\}$, applied to $\tilde{\mathscr{W}}$ are identical, up to Fourier transformation, to the products defined directly from G. This is true indeed for $k = n$ because of our definition of $\tilde{\mathscr{W}}$, and what we want to prove is just that it is also true for $k = 0$.

We use thus recursion on k, k decreasing.

Let

$$\Pi_{1,2} = C^{S_0} \cdot C^{S_1} \dots C^{S_k}$$

be a product according to either definition.

Π_1 and Π_2 satisfy both the same support properties III.3′) in (x).

Now, if $k < n$, there is at least one cycle, say C^{S_j}, in more than one variable. Using (6.6) and III.5), it is possible to transform Π_1 or Π_2 into

$$\tilde{\Pi}_{1,2} = C^{S_0} \dots C^{\tilde{S}_j} \dots C^{S_k}, \qquad \tilde{S}_j \in \mathscr{S}_{X_j}$$

by adding to them a sum of products of $k+2$ factors.

So

$$\Pi_2 - \Pi_1 = \tilde{\Pi}_2 - \tilde{\Pi}_1 .$$

But if $\tilde{S}_j$ is the antipodal of S_j in $(s)_{X_j}$, $U(\tilde{S}_j)$ will also be the antipodal of $U(S_j)$. $\Pi_2 - \Pi_1$ vanishes thus *if the variables* x_{j0}, x_{j1}, ..., $x_{jr(j)}$ *are not all equal* and Theorem 7 is proved.

It remains to discuss the restriction brought to Theorem 7 by the fact that

$$\mathscr{F} G^S(x) = C^S \tilde{\mathscr{W}}(x)$$

only when all vectors x_i are different.

This reflects obviously the ambiguity of the definition of $C^S\tilde{\mathscr{W}}(x)$ which involves « cutting singularities ».

Any definition of the $C\tilde{\mathscr{W}}(x)$ involves a choice when some difference $x_{i'} - x_i$ vanishes, but should be such that

1) the identical linear relations between the general cycles C give rise to the same relations between the $C\tilde{\mathscr{W}}(x)$;

2) when C reduces to $A^{(i_0)} \cdot \sigma^{(i_1)} \dots A^{(i_n)}$, $C\tilde{\mathscr{W}}(x)$ should reduce to a boundary value of $\tilde{\mathscr{W}}(z)$.

1) and 2) were achieved here by the trick of regularizing $\tilde{\mathscr{W}}$ before cutting the singularities.

In conclusion and in order to avoid ambiguities it seems preferable to give oneself a function $G(k)$ with properties II.1)-5) rather than a function $\tilde{\mathscr{W}}(k)$ with properties I.1)-4).

* * *

The idea of the present work originated from the Thesis of O. STEINMANN (14) who treated the problem of the connexion between Wightman and Green functions in full details for the four-point function. A later paper of STEINMANN (15) on the same subject treats the general problem of the n-point function with methods and results rather different from what has been done here. A more related treatment is due to H. ARAKI (16). Most of this work (essentially up to Section **7** incl.) was done in summer 1959 while the author stayed at the E.T.H. as a « chercheur agréé de l'Institut Interuniversitaire des Sciences Nucléaires » (Belgium).

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(14) O. STEINMANN: *Helv. Phys. Acta*, **33**, 257 (1960).
(15) O. STEINMANN: *Helv. Phys. Acta*, **33**, 347 (1960).
(16) H. ARAKI: to be published.

RIASSUNTO (*)

Nel presente lavoro, dopo aver ripreso lo studio delle proprietà di analiticità della funzione $\mathscr{W}$ di Wightman nel caso in cui il solo tempo sia una variabile complessa, ne deduciamo la funzione G di Green, estendendo altresì con un nuovo metodo i risultati di O. Steinmann relativi alla funzione a 4 punti. La funzione G ha per valore al contorno la trasformata di Fourier del valore medio del vuoto del prodotto T dei campi e prolunga analiticamente la funzione ritardata di L.S.Z. nello spazio degli impulsi. Infine si stabilisce un assieme di proprietà che caratterizzano G nel senso che se G possiede tali proprietà, esiste una ed una sola funzione $\tilde{\mathscr{W}}$ che ha le proprietà solite e tale che ne derivi G.

(*) *Traduzione a cura della Redazione.*