ANALYTICITY OF GREEN'S FUNCTIONS OF DILUTE QUANTUM GASES

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In this note we point out that Ginibre's results on the reduced density matrices of quantum gases[1] have immediate implications for the existence and analyticity of Green's functions. If H_Λ is the Hamiltonian in the bounded region Λ, we define Green's functions by

$$G_\Lambda(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m;\ \zeta_1,\ldots,\zeta_m)$$
$$= Z^{-1}\operatorname{Tr}(A_1(\underset{\sim}{x}_1)e^{-(\zeta_2-\zeta_1)H_\Lambda}A_2(\underset{\sim}{x}_2)\ldots e^{-(\zeta_m-\zeta_{m-1})H_\Lambda}A_m(\underset{\sim}{x}_m)e^{-(\beta+\zeta_1-\zeta_m)H_\Lambda})$$

where $Z = \operatorname{Tr} e^{-\beta H_\Lambda}$ and $A_k(\underset{\sim}{x}_k)=a^*(x'_{k1})\ldots a^*(x'_{kp(k)})a(x''_{k1})\ldots a(x''_{kq(k)})$.
Let $\varphi_k \in L^2((\mathbb{R}^\nu)^{p(k)+q(k)})$ for Fermi statistics, or $\varphi_k=\varphi'_k\,\varphi''_k$ with $\varphi'_k \in L^2(\mathbb{R}^{\nu p(k)}), \varphi''_k \in L^2(\mathbb{R}^{\nu q(k)})$ for Bose statistics, we write

$$G^\varphi_\Lambda(\zeta_1,\ldots,\zeta_m)$$
$$= \int_{\Lambda^{p(1)+q(1)}} d\underset{\sim}{x}_1\ldots\int_{\Lambda^{p(m)+q(m)}} d\underset{\sim}{x}_m\,\varphi_1(\underset{\sim}{x}_1)\ldots\varphi(\underset{\sim}{x}_m)G_\Lambda(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m;\zeta_1,\ldots,\zeta_m)$$

In the case of a system of particles interacting through a suitable pair potential Φ and for small activity the operator $e^{-\lambda H_\Lambda}$, with $\lambda > 0$, may be defined in terms of Wiener integrals and is of trace class. The operators $e^{-\lambda H_\Lambda}A_k(\underset{\sim}{x}_k)\,e^{-\lambda H_\Lambda}$ can also be expressed in terms of Wiener integrals and are of trace class.

When λ is complex and $\operatorname{Re}\lambda > 0$, $e^{-\lambda H_\Lambda}$ is defined and analytic, therefore G_Λ is an analytic function of the complex variables $\zeta_k = \beta_k - it_k$ in the domain

$$\mathcal{D} = \{(\zeta_1,\ldots,\zeta_m) : \beta_1 < \ldots < \beta_m < \beta_1 + \beta\}$$

If $t_1 = \ldots = t_m$, and $\beta_1 < \ldots < \beta_m < \beta_1 + \beta$, G_Λ can be expressed in terms of Wiener integrals and it follows from Ginibre's analysis[2] that when $\Lambda \to \infty$ (e.g. Λ is a sphere centered at the origin and with radius tending to infinity),

$$G_\Lambda(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m;\ \beta_1, \ldots, \beta_m) \to G(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m;\ \beta_1, \ldots, \beta_m) \tag{1}$$

uniformly on compacts with respect to $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m$, and

$$G^{\varphi}_{\Lambda}(\beta_1, \ldots, \beta_m) \to G^{\varphi}(\beta_1, \ldots, \beta_m)$$

$$= \int d\underset{\sim}{x}_1 \ldots d\underset{\sim}{x}_m\ \varphi_1(\underset{\sim}{x}_1) \ldots \varphi_m(\underset{\sim}{x}_m)\ G(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m;\ \beta_1, \ldots, \beta_m) \tag{2}$$

1. Proposition. There exists a function $G(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m; \zeta_1, \ldots, \zeta_m)$ analytic with respect to $(\zeta_1, \ldots, \zeta_m) \in \mathfrak{D}$ and such that

$$\lim_{\Lambda \to \infty} G_\Lambda(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m; \zeta_1, \ldots, \zeta_m) = G(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m;\ \zeta_1, \ldots, \zeta_m)$$

uniformly on compacts with respect to $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m, \ \zeta_1, \ldots, \zeta_m$. Furthermore, if $(\zeta_1, \ldots, \zeta_m) \in \mathfrak{D}$,

$$\lim_{\Lambda \to \infty} G^{\varphi}_{\Lambda}(\zeta_1, \ldots, \zeta_m) = G^{\varphi}(\zeta_1, \ldots, \zeta_m)$$

$$= \int d\underset{\sim}{x}_1 \ldots d\underset{\sim}{x}_m\ \varphi_1(\underset{\sim}{x}_1) \ldots \varphi_m(\underset{\sim}{x}_m)\ G(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m;\ \zeta_1, \ldots, \zeta_m)$$

We notice first that $\mathfrak{D}$ is the union (over $n > 0$) of the sets

$$K_n = \{(\zeta_1, \ldots, \zeta_m) \in \mathfrak{D}: \beta_2 - \beta_1 \geq \frac{\beta}{2n}, \ldots, \beta_m - \beta_{m-1} \geq \frac{\beta}{2n}, \beta + \beta_1 - \beta_m \geq \frac{\beta}{2n}\ \}$$

If $(\zeta_1,\dots,\zeta_m) \in K_n$ we may express G_Λ in terms of operators $e^{(it_k - \frac{1}{4n}\beta)H_\Lambda} A_k(x_k)\, e^{-(it_k+\frac{1}{4n}\beta)H_\Lambda}$, and $e^{-\lambda H_\Lambda}$ with $0<\lambda<\beta$. Using Hölder's inequality[3)] we find an upper bound for $|G_\Lambda|$ in terms of the expressions

$$Z^{-1}\mathrm{Tr}\,[(e^{-\frac{1}{4n}\beta H_\Lambda} A_k(x_k)^* e^{-\frac{1}{2n}\beta H_\Lambda} A_k(x_k)\, e^{-\frac{1}{4n}\beta H_\Lambda})^n]$$

which are known by (1) to have a limit when $\Lambda \to \infty$. We may thus assume that G_Λ is bounded on each K_n uniformly with respect to Λ and $x_1,\dots,x_m$ in a compact, and the convergence of G_Λ on the real points of $\mathcal{D}$ implies its uniform convergence on the compacts of $\mathcal{D}$. The uniformity of the convergence with respect to $x_1,\dots,x_m$ on compacts follows from the uniformity of (1).

The proof of the convergence of G_Λ^φ proceeds like the proof of the convergence of G_Λ and shows in particular that the limit of G_Λ^φ is a bounded multilinear functional of $\varphi_1,\dots,\varphi_m$ (Fermi) or $\varphi_1',\dots,\varphi_m''$ (Bose) on the product of the relevant L^2 spaces, identification of the limit follows from (2), taking $\varphi_1,\dots,\varphi_m$ with compact supports.

2. Proposition. Let $m=2$ and let the pair potential $\Phi \in L^1(\mathbb{R}^\nu) \cap L^2(\mathbb{R}^\nu)$. Then G^φ extends to a bounded continuous function on $\bar{\mathcal{D}}$ such that

$$\lim_{\Lambda\to\infty} G_\Lambda^\varphi(\zeta_1,\zeta_2) = G^\varphi(\zeta_1,\zeta_2)$$

<u>uniformly on the compacts of the closure</u> $\overline{\mathcal{D}}$ <u>of</u> $\mathcal{D}$.

The operators $A_k(\varphi_k)\, e^{-\lambda H_\Lambda}$ are of trace class [consider $e^{-\lambda H_\Lambda} A_k(\varphi_k)^* A_k(\varphi_k)\, e^{-\lambda H_\Lambda}$] and, if $\beta_2 = \beta_1$ or $\beta_2 = \beta_1 + \beta$, we have

$$|G^{\varphi}_{\Lambda}(\zeta_1, \zeta_2)| \le [G_{1\Lambda}\, G_{2\Lambda}]^{1/2} \tag{3}$$

$$G_{k\Lambda} = Z^{-1}\mathrm{Tr}[(A_k(\varphi_k)^* A_k(\varphi_k) + A_k(\varphi_k)\, A_k(\varphi_k)^*)e^{-\beta H_\Lambda}] \tag{4}$$

We assume first Bose statistics. The reduced density matrices are integral kernels of bounded operators in L^2 . When (Λ_n) tends to infinity these bounded operators form a bounded sequence converging in the strong operator topology[1). Therefore there exists $C > 0$ such that

$$G_{k\,\Lambda_n} \le C \int d\underset{\sim}{x}_k\, |\varphi_k(\underset{\sim}{x}_k)|^2 \tag{5}$$

for all n . Since G^{φ}_{Λ} is analytic and bounded, (3) holds for all $(\zeta_1, \ldots, \zeta_2)$ in $\overline{\mathcal{D}}$ and, using (5), this gives

$$|G^{\varphi}_{\Lambda_n}(\zeta_1, \zeta_2)| \le C\, \|\varphi_1\|_2\, \|\varphi_2\|_2 \tag{6}$$

for all n and $(\zeta_1, \zeta_2) \in \overline{\mathcal{D}}$. In the case of Fermi statistics, (6) holds again (with $C=1$) because

$$\|A_k(\varphi_k)\| \le \|\varphi_k\|_2$$

In view of (6) it suffices to prove the proposition when $A_k(\varphi_k)$ is of the form

$$A_k(\varphi_k) = a^*(\psi'_{k'})\dots a^*(\psi'_{kp(k)})a(\psi''_{k'})\dots a(\psi''_{kq(k)})$$

where $\psi'_{k'}\dots\psi''_{kq(k)}$ are of class C^2 with compact support.

We have

$$\frac{d}{d\zeta_2} G^{\varphi}_{\Lambda} = Z^{-1}\mathrm{Tr}\{A_1(\varphi_1)e^{-(\zeta_2-\zeta_1)H_\Lambda}[A_2(\varphi_2),H_\Lambda]e^{-(\beta+\zeta_1-\zeta_2)H_\Lambda}\}$$

and therefore

$$\left| \frac{d}{d\zeta_2} G^{\varphi}_{\Lambda} \right| \le [G_{1\Lambda}\, G'_{2\Lambda}]^{1/2}$$

where $G'_{2\Lambda}$ is given by (4) with $A_k(\varphi_k)$ replaced by $[A_2(\varphi_2),H_\Lambda]$. Since $\psi'_{21},\dots,\psi''_{2q(2)}$ are of class C^2 with compact support, the commutator of $A_2(\varphi_2)$ with the kinetic energy part of H_Λ is again of the form $A(\varphi)$. In view of this $G'_{2\Lambda}$ is a sum of integrals of reduced density matrices $G'_\Lambda(x_1,\dots,x_r)$ multiplied by continuous functions $\psi(x_i)$ with compact support, and the pair potential $\Phi(x_k-x_j)$. The pair potential appears as factor 0, 1 or 2 times; if $\Phi(x_k-x_j)$ appears there also appears a factor $\psi(x_j)$ or $\psi(x_k)$; for each variable x_i in $G'_\Lambda(x_1,\dots,x_r)$ which does not appear in a factor $\Phi(x_j-x_i)$ there is a factor $\psi(x_i)$. Using the condition $\Phi \in L^1(\mathbb{R}^\nu) \cap L^2(\mathbb{R}^\nu)$ and the fact that the reduced density matrices G'_Λ are bounded functions, uniformly in Λ [1)] we obtain a bound on $G'_{2\Lambda}$ which is independent of Λ .

Therefore $\left|\frac{d}{d\zeta_2} G^{\varphi}_{\Lambda}\right| = \left|\frac{d}{d\zeta_1} G^{\varphi}_{\Lambda}\right|$ is bounded on $\mathfrak{D}$ uniformly in Λ. The convergence of G^{φ}_{Λ} in $\mathfrak{D}$ implies then its uniform convergence on the compacts of $\overline{\mathfrak{D}}$.

3. Remark. Let $m=3$, $\Phi \in L^1(\mathbb{R}^{\nu}) \cap L^2(\mathbb{R}^{\nu})$. In the case of Fermi statistics, G^{φ} extends to a bounded continuous function on $\overline{\mathfrak{D}}$ such that

$$\lim_{\Lambda \to \infty} G^{\varphi}_{\Lambda}(\zeta_1, \zeta_2, \zeta_3) = G^{\varphi}(\zeta_1, \zeta_2, \zeta_3)$$

uniformly on the compacts of the closure $\overline{\mathfrak{D}}$ of $\mathfrak{D}$.

To estimate $\left|\frac{d}{d\zeta_i} G^{\varphi}_{\Lambda}\right|$ it suffices to consider the expression

$$Z^{-1} \operatorname{Tr} \{e^{(-\beta+it)H_{\Lambda}} [A_i, H_{\Lambda}] e^{it'H_{\Lambda}} A_j e^{it''H_{\Lambda}} A_k\}$$

and similar ones where $[A_i, H_{\Lambda}]$ and A_j, A_k are circularly permuted. If $[A_i, H_{\Lambda}]$ occupies the middle position we rewrite the expression in terms of $[A_j, H_{\Lambda}]$, $[A_k, H_{\Lambda}]$. The rest of the argument goes as for $m=2$ (using the boundedness of A_j).

4. Proposition. In the case of Fermi statistics, introduce the operators

$$A_k(\varphi_k, f_k) = \int dt \ f(t) \, e^{it H_{\Lambda}} A_k(\varphi_k) \, e^{-it H_{\Lambda}}$$

where $f_k \in L^1(\mathbb{R})$, then the limit

$$\lim_{\Lambda \to \infty} Z^{-1} \operatorname{Tr}(A_1(\varphi_1, f_1) \ldots A_m(\varphi_m, f_m)\, e^{-\beta H_\Lambda})$$

exists.

It is sufficient to prove this for f_k of class C^1 with compact support. We construct Green's functions with the operators $A_k(\varphi_k, f_k)$ instead of $A_k(\varphi_k)$ and use the fact that the derivatives of these functions are bounded in $\mathcal{D}$ uniformly with respect to Λ.

5. Remark. Proposition 4 has obvious implications for the description of time evolution of a dilute Fermi gas. It does not, however, exhibit this time evolution as a group of automorphisms of the C^*-algebra of the anticommutation relations. Streater[4)] and Hepp[5)] have shown that such a group of automorphisms exists for some non local interactions.

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Footnotes.

1) See J. Ginibre. J. Math. Phys. 6, 238-251 (1965); 6, 252-262 (1965); 6; 1432-1446 (1965); J. Ginibre p. 148 in Statistical Mechanics (proceedings of the I.U.P.A.P. meeting, Copenhagen, 1966) edited by T. Bak, Benjamin, New York, 1967.

2) This result is contained in C. Gruber's thesis [Princeton, 1968, unpublished], see also J. Ginibre and C. Gruber, Commun. Math. Phys. 11, 198-213 (1969).

3) See N. Dunford and J. Schwartz, Linear Operators , Interscience, New York, 1963, Lemma XI. 9-20, p 1105.

4) R.F. Streater. Commun. Math. Phys. 7, 93-98 (1968)

5) K. Hepp. Unpublished.