by explicit argument.

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Extension of the Lee-Yang Circle Theorem

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We study the zeros of the partition function of a classical spin system and compute a region of the complex fugacity plane where they necessarily lie. We recover the Lee-Yang circle theorem as a special example; we also find that a spin system with finite-range interaction has no phase transition at high temperature.

Asano's recent results[1] have revived interest in the celebrated Lee-Yang circle theorem,[2] by giving it a more conceptual proof. Here an extension of the circle theorem to noncircular regions is proved and applied to problems in statistical mechanics.

(1) Statement of results.— Let P be a complex polynomial in several variables, which is of degree 1 with respect to each, i.e., Λ is a finite set and

$$P(z_\Lambda) = \sum_{X \subset \Lambda} c_X z^X,$$

where $z_\Lambda = (z_x)_{x\in\Lambda}$, and $z^X = \prod_{x\in X} z_x$.

Theorem: Let (Λ_α) be a finite covering of Λ, and for every $x \in \Lambda_\alpha$ let $M_{\alpha x}$ be a closed subset of the complex plane C such that $0 \notin M_{\alpha x}$. For each α we assume that the polynomial

$$P_\alpha(z_{\Lambda_\alpha}) = \sum_{X \subset \Lambda_\alpha} c_{\alpha X} z^X$$

does not vanish when $z_x \notin -\prod_\alpha(-M_{\alpha x})$, all $x \in \Lambda$. Then the polynomial

$$P(z_\Lambda) = \sum_{X \subset \Lambda} z^X \prod_\alpha c_{\alpha(\Lambda_\alpha \cap X)}$$

does not vanish when[3] $z_x \notin -\prod_\alpha(-M_{\alpha x})$, all $x \in \Lambda$.

The proof is given in Section 2. This result extends a theorem by Lee and Yang,[2] and can be used in the same way to obtain regions free of zeros for polynomials in one variable. More precisely, let the Λ_α be the two-point subsets of Λ: $\Lambda_\alpha = \{x, y\}$ and $c_{\alpha X} = a_{xy}$ when $X = \{x\}$ or $X = \{y\}$, $c_{\alpha X} = 1$ when $X = \emptyset$ or $X = \{x, y\}$. For real a_{xy} and $-1 \leqslant a_{xy} \leqslant 1$ we may take $M_{\alpha x} = \{z \in C : |z| \geqslant 1\}$; hence[4]

$$Q(\xi) = \sum_{X \subset \Lambda} \xi^{|X|} \prod_{x\in X} \prod_{y \in X} a_{xy}$$

does not vanish when $|\xi| < 1$. By symmetry $Q(\xi)$ does not vanish when $|\xi| > 1$, hence the zeros of Q have absolute value 1; this is the Lee-Yang circle theorem.

Let Φ be a real function on $(Z_m)^\nu$ (ν-tuples of integers mod m, "periodic lattice") with $\Phi(x) = \Phi(-x)$, and take $\Lambda = (Z_m)^\nu$. Let again the Λ_α be the two-point subsets of Λ: $\Lambda_\alpha = \{x, y\}$, and write $c_{\alpha X} = \exp[-\beta\Phi(x-y)]$ when $X = \{x, y\}$ and $c_{\alpha X} = 1$ when $X = \emptyset$, $\{x\}$, or $\{y\}$. We may then take

$$M_{\alpha x} = \Delta_{xy}{}^\beta,$$

where

$$\Delta_{xy}{}^\beta = \{z \in C : |z+1| \leqslant (1 - e^{\beta\Phi(x-y)})^{1/2}\} \quad \text{for } \Phi(x-y) \leqslant 0,$$

$$\Delta_{xy}{}^\beta = \{z \in C : |z e^{-\beta\Phi(x-y)} + 1| \leqslant (1 - e^{-\beta\Phi(x-y)})^{1/2}\} \quad \text{for } \Phi(x-y) \geqslant 0,$$

and we find that

$$Q(\xi) = \sum_{X \subset \Lambda} \xi^{|X|} \exp[-\beta \sum_{\{x,y\} \subset X} \Phi(x-y)]$$

can vanish only when

$$\xi \in \Gamma^\beta = -\prod_{y \in Z_m{}^\nu} (-\Delta_{0y}{}^\beta).$$

The region Γ^β is sketched for small and large values of β in Fig. 1. For small β, Γ^β does not

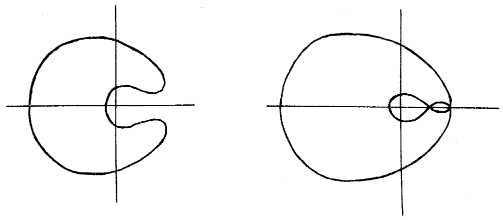

FIG. 1. The region Γ^β for small β (left) and large β (right).

intersect the positive real axis. Therefore a lattice gas with finite-range interaction has no phase transition at high temperature. This result was known[5] but is obtained here with minimum technicality.

The Lee-Yang circle theorem implies that an Ising ferromagnet can have at most one phase transition (at $\xi = 1$). The above proof implies that the zeros of $Q(\xi)$ remain close to the unit circle when a small perturbation (possibly many-body) is added to the original ferromagnetic pair interaction. From this one can deduce the following: An infinite Ising ferromagnet has only one equilibrium state at $\xi \neq 1$; in particular, the thermodynamic limit of the correlation functions is independent of boundary conditions.[6]

(2) Proof of theorem. – When the Λ_α are disjoint, P is just the product of the P_α (with disjoint sets of variables) and the theorem is trivial. To prove the theorem in general we first form the product of the P_α with disjoint sets of variables and then obtain P by successive "contractions." These contractions (introduced by Asano[1]) are described in the following proposition, from which the theorem is immediately obtained.

Lemma: Let A, B be closed subsets of C which do not contain 0. Suppose that the complex polynomial

$$a + bz_1 + cz_2 + dz_1z_2$$

can vanish only when $z_1 \in A$ or $z_2 \in B$. Then

$$a + dz$$

can vanish only when $z \in -AB$.

Since $0 \notin A$, $0 \notin B$, we have $a \neq 0$. If $d = 0$ there is nothing to prove. If $d \neq 0$, $ad - bc = 0$, we have

$$a + bz_1 + cz_2 + dz_1z_2 = d(z_1 + c/d)(z_2 + a/c)$$

therefore $-c/d \in B$, and $a/c \in B$, and $a + dz$ vanishes only when $z = -a/d \in -AB$.

Let now $d \neq 0$, $ad - bc \neq 0$, and write[7]

$$\varphi(z) = -(a + bz)/(c + dz), \quad \psi(z) = a/dz,$$

where φ, ψ are now considered as mappings of the Riemann sphere (add a point at infinity to C, A, B). If we write $\omega = \varphi\psi^{-1}$, $z_2 = \omega z_1$ is equivalent to

$$ab + adz_1 + adz_2 + cdz_1z_2 = 0$$

showing that $\omega^2 = 1$: ω is an involution. Since B is a proper closed set, $\omega(B)$ cannot be interior to B [otherwise also $\omega^2(B)$ would be interior to B]. Thus

$$\omega(B) \cap \sim\bar{B} \neq \emptyset$$

where $\sim\bar{B}$ is the closure of the complement of B. By assumption $\sim B \subset \varphi(A)$ and since $\varphi(A)$ is closed, $\sim\bar{B} \subset \varphi(A)$. Hence

$$\omega(B) \cap \varphi(A) \neq \emptyset,$$

or

$$\varphi\psi^{-1}(B) \cap \varphi(A) \neq \emptyset,$$

or

$$B \cap \psi(A) \neq \emptyset$$

which proves the lemma.

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[4] We let $|X|$ be the number of points in X.

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[6] A proof of this result will be published elsewhere.

[7] I am indebted to F. J. Dyson for communicating this argument to me, and for his kind permission to reproduce it. Dyson's elegant proof extends to the case where $\psi(z) = -(e + fz)/(g + hz)$ and $ah + de = bg + cf$.