CURRENTS, FLOWS, and DIFFEOMORPHISMS

by

David Ruelle and Dennis Sullivan

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 BURES-sur-YVETTE - FRANCE

[43]

CURRENTS, FLOWS, and DIFFEOMORPHISMS

by David Ruelle and Dennis Sullivan

Here we describe a geometrical object in a manifold which determines by integration of differential forms a <u>closed</u> current in the sense of de Rham. In certain examples from dynamical systems, for example Anosov diffeomorphisms, these geometrical objects will lead to non-trivial real homology classes.

Our "geometrical currents" share some of the features of incompressible flows in the manifold. A geometrical current[1] is made up of three things - a <u>partial foliation</u> in the manifold which is <u>oriented</u> and provided with a <u>transversal measure</u> .

To go into more detail, to give a <u>partial foliation</u> (the streamlines of the current) we give a closed subset S of our smooth manifold W divided into connected subsets L_α . There is a collection of closed disks of the form

$$D^k \times D^{n-k} = (\text{horizontal disk} \times \text{vertical disk})$$

called "flow boxes" whose interiors cover W and which meet the L_α nicely. Namely, each L_α intersects one of the flow boxes $D^k \times D^{n-k}$ in a collection of "horizontal disks" $\{D^k \times y\}$. We assume each horizontal disk is smoothly embedded in W with tangent planes varying continuously over the entire flow box.

---

[1] This terminology is not meant to exclude other geometrical objects defining currents. The examples we discuss have a marked laminar quality related to the incompressibility needed to obtain a <u>closed</u> current.

Now we describe the transversal measure on our streamlines $\{L_\alpha\}$ which is the analogue of incompressibility. By a transversal T we mean a smooth (n-k) dimensional submanifold which is transversal to each L_α . T is small if it can be surrounded by a single flow box. Now the part of a small transversal in the support S of the current can be slid horizontally to one of the standard transversals $x \times D^{n-k}$. Maps between parts of small transversals constructed by iterations of this operation and its inverse are called canonical isomorphisms .

A transversal measure μ for our geometric current provides each small transversal with a measure of finite mass. We assume each such measure is supported on the part of the transversal intersecting the support of the current and that the canonical isomorphisms are measure preserving .

Finally, to complete our notion of geometric current we need the analogue of direction of flow. This is the orientation ν which continuously assigns to each point x of the support S of the current, an orientation of the manifold L_α passing through x .

The three objects (L_α , μ , ν) define a geometric current with support S .

Remark. In what follows it will be clear that we can dispense with introducing those horizontal disks $\{D^k \times y\}$ of the flow box $D^k \times D^{n-k}$ which are not contained in the support S of our geometric current. Therefore it suffices to define partial flow boxes $\{D^k \times K^{n-k}\}$ where K^{n-k} is a closed subset of D^{n-k} such that $S \cap D^{n-k} \subset K^{n-k}$ (transversal measures and orientation remain defined as above). This fact will prove useful in the study of Axiom A diffeomorphisms below, where partial flow boxes are available but it is not known if they can be completed to flow boxes $D^k \times D^{n-k}$. For simplicity we shall however continue to consider the flow boxes $D^k \times D^{n-k}$, and leave to the reader the obvious extension of the results to partial flow boxes.

We can speak of two geometric currents (L_α, μ, ν) and (L'_α, μ', ν') <u>intersecting transversally</u> in a third current $(L''_\alpha, \mu'', \nu'')$ if <u>W is oriented</u>. For this we assume the layers L_α and L'_α intersect transversally in the layers L''_α . We assume there is one system of flow boxes working simultaneously for all three currents,

$$(D^k \times D^c)_i = (D^{k'} \times D^{c'})_i = (D^{k''} \times D^{c''})_i$$

where c, c' , and c'' are the codimensions. Then $D^{k''} = D^k \cap D^{k'}$ and $D^{c''} = D^c \times D^{c'}$ and we have the usual relation

$$\begin{aligned}\text{codimension } (L_\alpha, \mu, \nu) &+ \text{codimension } (L'_\alpha, \mu', \nu') \\ &= \text{codimension } (L''_\alpha, \mu'', \nu'') \ .\end{aligned}$$

The orientations of these currents combine with the orientation of W to orient the transversals D^c , $D^{c'}$, and $D^{c''}$.

We assume

$$\nu''_t = \nu_t \times \nu'_t$$

for these transversal orientations. Finally we assume the relation

$$\mu'' = \mu \times \mu' \quad \text{on} \quad D^{c''} = D^c \times D^{c'}$$

holds for the transversal measures.

The de Rham current associated to a geometric current.

Fix a geometric current (L_α, μ, ν) of dimension k and support S contained in W. We assume W is covered by a locally finite system of flow boxes $(D^k \times D^{n-k})_i$ provided with a partition of unity. Then each compactly supported k-form w on W becomes decomposed into a finite sum $w = \Sigma w_i$ where w_i is supported in the i^{th} flow box.
Now we can integrate w_i over each horizontal $(D^k \times y)_i$ and obtain a continuous function f_i on $(D^{n-k})_i$. We can then average this function using the transversal measure μ to obtain a real number c_i. Note that c_i only depends on the horizontal integrals over the layers of $(D^k \times D^{n-k})_i \cap S$ because the support of μ is contained in $(D^{n-k})_i \cap S$. There is also a sign question which is fixed using the orientation ν in case the flow box really intersects S. So we can define a current in the sense of de Rham from the geometric current (L_α, μ, ν) by the formula

$$<(L_\alpha, \mu, \nu), w> = \sum_i c_i = \sum_i \int_{(D^{n-k})_i} \mu(dy) \left(\int_{(D^k \times y)_i} w_i \right) ,$$

where $w = \Sigma_i w_i$.

It is easy to see that the de Rham current only depends on the partial foliation L_α, the transverse measure μ, and the orientation ν, and not on the system of flow boxes and the partition of unity. If another choice of flow boxes and partition of unity were made, one could find a common refinement. So it would suffice for this point to consider the case where we have a system of flow boxes $(D^k \times D^{n-k})_{ij}$ inside $(D^k \times D^{n-k})_i$, covering the support of w_i. If w_i were thus further broken down as a sum $w_i = \sum_j w_{ij}$, let $\widetilde{w}_{ij}$ be defined on all of $D^k_{ij} \times D^{n-k}_j$ by extending by zero. (We leave consideration of the point about compatibility of horizontal layers outside S to the reader.) Then by the invariance of μ under canonical isomorphism we have

$$c_{ij} = \int_{(D^{n-k})_{ij}} \mu(dy) \left(\int_{(D^k \times y)_{ij}} w_{ij} \right) = \int_{(D^{n-k})_i} \mu(dy) \left(\int_{(D^k \times y)_{ij}} \tilde{w}_{ij} \right) .$$

$$\text{So} \quad \sum_j c_{ij} = \int_{(D^{n-k})_i} \mu(dy) \left(\sum_j \int_{(D^k \times y)_{ij}} \tilde{w}_{ij} \right)$$

$$= \int_{(D^{n-k})_i} \mu(dy) \left(\int_{(D^k \times y)_i} w_i \right)$$

$$= c_i .$$

This demonstrates the independence of choice.

A second remark is that our de Rham current is closed. That is, if $w = d\varphi$ where φ also has compact support then $< (L_\alpha, \mu, \nu), d\varphi > = 0$. This follows by writing a finite sum $\varphi = \sum_i \varphi_i$ and observing the inside integrals in the definition, $\int_{(D^k \times y)_i} d\varphi_i$, vanish.

We also note that if the support of w does not meet S then $< (L_\alpha, \mu, \nu) , w > = 0$.

So if c is the codimension of our geometric current (L_α, μ, ν) and $U \subset W$ is the open complement of its support S then we have the

Proposition 1: A geometrical current defines a canonical cohomology class

$$[L_\alpha, \mu, \nu] \in H^c(W, U) ,$$

if W is oriented.

Now we prove a precise geometrical form of the cohomological formula

$$[L_\alpha, \mu, \nu] \cup [L'_\alpha, \mu', \nu'] = [L''_\alpha , \mu'', \nu'']$$

when (L_α, μ, ν) and (L'_α, μ', ν') intersect transversally in $(L''_\alpha, \mu'', \nu'')$.

First, recall that any n-k form ν on W determines a (k dimensional) current by the fomula

$$< \nu, w > = \int_W w \wedge \nu ,$$

after fixing an orientation of W .

Such currents can be multiplied using the wedge products of forms,

$$< \nu, \cdot > \wedge < \nu', \cdot > = < \nu \wedge \nu', \cdot > .$$

One can try to extend this multiplication to more general currents by continuity since these form (or smooth) currents are dense in all currents.

Currents can be smoothed by convolution [5].

This approximation (or regularization) procedure for currents is linear, commutes with the boundary operator, approximately preserves the support, and (can) proceed by smoothing the current in one coordinate patch at a time [5].

In the case of two currents defined by integration along two submanifolds intersecting transversally the multiplication procedure works in the limit and there is the formula [5]

$$\lim_{\delta \to 0} R_\delta(M) \wedge R_\delta(M') = \int_{M \cap M'} (\cdot)$$

where $C \to R_\delta C$ is the regularization which depends on choices and a degree of approximation parameter δ . This convergence is uniform for evaluation on uniformly bounded forms supported in a given compact set of W .

In these terms we can state the

Proposition 2: If two geometrical currents (L_α, μ, ν) and (L'_j, μ', ν') intersect transversally in $(L''_\alpha, \mu'', \nu'')$, then for each compactly supported form w we have

$$\lim_{\delta \to 0} (R_\delta(L_\alpha, \mu, \nu) \wedge R_\delta(L'_\alpha, \mu', \nu'))(w)$$

$$= \langle (L''_\alpha, \mu'', \nu''), w \rangle \; .$$

Proof: It suffices to discuss forms supported in a single flow box

$$D = D^k \times D^c = D^{k'} \times D^{c'} = D^{k''} \times D^{c''}$$

where $D^{k''} = D^k \cap D^{k'}$ and $D^{c''} = D^c \times D^{c'}$.

Regularizing uses a specific n-form φ_δ (involving choices) on $D \times D$ which converges to the diagonal current as $\delta \to 0$. The n-form φ_δ combines with a k-current on D to yield an n-k form on D , $R_\delta C$. For example, each $D^k \times y \subset D$ defines a current (by integration) and $R_\delta(D^k \times y)$ is the n-k form $\int_{z \in D^k \times y} \varphi_\delta(z, \cdot)$. Also $R_\delta(L_\alpha, \mu, \nu)$ is given by the expression $\int_{D^c} \mu(dy) \left(\int_{D^k \times y} \varphi_\delta(z, \cdot) \right)$, in this flow box.

Thus we can write

$$R_\delta(L_\alpha, \mu, \nu) = \int_{D^c} \mu(dy)\, (R_\delta(D^k \times y)) \quad ,$$

expressing the basic linearity property of regularizing. We can also write

$$R_\delta\, (L'_\alpha, \mu', \nu') = \int_{D^{c'}} \mu'(dy)\, (R_\delta(D^{k'} \times y')) \quad .$$

If we approximate these integrals by finite sums

$$R_\delta(L_\alpha, \mu, \nu) \sim \Sigma \mu_i R_\delta(D^k \times y_i)$$

$$R_\delta(L'_\alpha, \mu', \nu') \sim \Sigma \mu'_i R_\delta(D^{k'} \times y'_i)$$

we see that

$$R_\delta(L_\alpha, \mu, \nu) \wedge R_\delta(L'_\alpha, \mu', \nu') \sim \Sigma \mu_i \mu'_j R_\delta(D^k \times y_i) \wedge R_\delta(D^{k'} \times y'_j) .$$

As $\delta \to 0$ $R_\delta(D^k \times y_i) \wedge R_\delta(D^{k'} \times y'_j)$ approaches the current defined by $D^k \cap D^{k'} \times (y_i, y'_j)$ i.e. $D^{k''} \times y''_{ij}$ where $y''_{ij} = (y_i, y'_j)$ in $D^{c''} = D^c \times D^{c'}$. [5]. Since this convergence is uniform in y''_{ij} , and $\mu'' = \mu \times \mu'$ on $D^{c''} \cap S''$,

$$R_\delta(L_\alpha, \mu, \nu) \wedge R_\delta(L'_\alpha, \mu', \nu') \to \int_{D^{c''} \cap S''} \mu(dy'')(\int_{D^{k''} \times y} (\cdot))$$

as currents. This completes the proof of proposition 2.

As a corollary we can write the equation of currents

$$(L_\alpha, \mu, \nu) \wedge (L'_\alpha, \mu', \nu') = (L''_\alpha, \mu'', \nu'')$$

in an oriented manifold. We also obtain the cohomological equation

$$[L_\alpha, \mu, \nu] \cup [L'_\alpha, \mu', \nu'] = [L''_\alpha, \mu'', \nu'']$$

where

$$[L_\alpha, \mu, \nu] \in H^c(W, U)$$

$$[L'_\alpha, \mu', \nu'] \in H^{c'}(W, U')$$

$$[L''_\alpha, \mu'', \nu''] \in H^{c''}(W, U'') ,$$

$c'' = c + c'$ are the codimensions, and $U'' = U \cup U'$ are the open sets complementary to the respective supports of the currents.

Examples of geometric currents

i) We have already seen that a single oriented submanifold of W defines a geometric current. The existence of flow boxes is clear and the transverse measure is the point mass along the submanifold.

ii) Consider a linear foliation of the plane by parallel lines. We have a measure on transversals defined by projecting onto orthogonals and we can choose an orientation. If we pass to the standard quotient torus we obtain a geometric current representing the real homology class $X\cos\theta + Y\sin\theta$ where θ is the angle between the linear foliation and the x-axis and X and Y are the standard generators of H_1 (torus, R) .

iii) Let S denote the standard solid torus $S^1 \times D^2$ in R^3 . Consider a diffeomorphism $S \to$ interior S which squeezes in and stretches around twice. Then $\bigcap_{u=0}^{\infty} f^n S$ is the dyadic solenoid which is locally homeomorphic to $C \times R$ where C is the Cantor set of bi-infinite sequences of 0's and 1's . The transversal Cantor sets are canonically endowed with the $(\frac{1}{2}, \frac{1}{2})$ measure because the ambiguity in the identification is the interchange of zero and one. Choosing an orientation yields a geometric current supported on the solenoid and representing the generator of H_1 (solid torus). The theoretical discussion behind this example is given below for general Axiom A diffeomorphisms.

iv) Let φ_t be any volume preserving flow on W . Suppose w is an invariant volume form and X is the vector field generating the flow. If X is never zero, then we obtain a geometric current using the flow lines of φ_t , oriented by X , and the transversal measure defined by the (n-1) form $\eta(\cdot,\ldots,\cdot) = w(X,\cdot,\ldots,\cdot)$. The de Rham current associated to this geometric current, described above by two integrations, is also given by the formula

$$C_{w,X}(\nu) = \int_W \nu(X) \wedge w \quad .$$

This formula defines a current for any pair (w, X) but it only yields a <u>closed</u> current when w is invariant along the flow determined by X .

Using the ergodic theorem when W is compact and the pair (φ_t, w) is ergodic*) , one can show that the homology class of this current can be approximated as follows - start at (w-almost) any point x and travel along the flow for a long time T until a point x' near x is reached. The $1/T$ times the cycle made up of the traveled path plus a small path connecting x and x' gives the homological approximation, see [6].

These incompressible flow examples include the torus example above, as well as any Hamiltonian dynamical system. However, if the zeroes of the vector field X are not "laminar in nature" one finds new "geometrical currents" outside the class discussed above.

v) Another way to generalize the torus example is to consider higher dimensional foliations. If $\{L_\alpha\}$ is a k-dimensional foliation of W which can be oriented by ν and provided with a transversal measure μ , then of course we have a geometric current (L_α, μ, ν) .

In general, a non trivial transversal measure may not exist for a foliation, and if it exists it is only unique in certain cases. If the manifold is compact such a measure generalizes the notion of a compact leaf. For each compact leaf determines the transversal measure which is just the point mass along this leaf and zero elsewhere. In the Reeb foliation of S^3

*) That is, φ_t leaves no set invariant whose w measure is positive and less than the total mass of W .

the compact leaf determines the only transversal measure.

In another example, the manifold W is the unit tangent bundle of an oriented negatively curved compact surface M^2 of higher genus The geodesic flow on W, which defines a geometric current, is the transversal intersection of two codimension one foliations. These foliations are transversal to the foliation of W defined by circles of unit tangent vectors. This latter foliation also defines a geometric current by pulling up the volume form of the surface to give a transverse measure.

Since the Euler characteristic of M^2 is non-zero this current of circles defines the zero homology class. It follows that neither of the transversal codimension one foliations admit transverse measures. If one did, the intersection with the current of circles would be non-zero homologically, which is a contradiction.

Axiom A diffeomorphisms

Now we turn to the discussion for which the introduction was intended.

Let f be a diffeomorphism of the compact manifold M. Recall $x \in M$ is called wandering if x has a neighbourhood U which moves away under f, i.e. $f^n U \cap U = \emptyset$ for $n \geq 1$, and non-wandering if it is not wandering. The set of non-wandering points form a closed invariant set $\Omega = \Omega(f)$. Axiom A assumes the density of periodic points in Ω and the existence of a df invariant splitting of the tangent bundle of M along Ω, $\tau = E^u \oplus E^s$, where E^u is exponentially expanded by $(df)^n$ and E^s similarly contracted (relative to some Riemannian metric).

Our constructions here are based on assembling several of the striking geometrical consequences of this axiom established by several workers.

Let Λ be a closed invariant subset of Ω which is open in Ω and contains a dense orbit (basic set). The Axiom A splitting has a constant dimension, $(n-k, k)$ along Ω and we assume that the bundles E^u and E^s admit orientations. Let

$$W^s_\Lambda = \{x \in M : f^n x \to \Lambda \text{ as } n \to +\infty\}$$

$$W^u_\Lambda = \{x \in M : f^{-n} x \to \Lambda \text{ as } n \to +\infty\} .$$

Now W^s_Λ can be filled up by k-dimensional submanifolds of M, the "stable manifolds" of Λ. Similarly W^u_Λ can be filled up by the "unstable manifolds of Λ." These two systems of submanifolds are oriented, laminar (they admit partial flow boxes) and can be transversely measured. Thus we can obtain two geometric currents of dimensions k and $n-k$ respectively, supported by W^s_Λ and W^u_Λ, which intersect transversally in Λ.

To begin the construction we choose an adapted Riemann metric on M and for sufficiently small $\epsilon > 0$ and $x \in \Lambda$ define the local stable and unstable manifolds

$$W^s_x(\epsilon) = \{y \in M : d(f^n x, f^n y) < \epsilon \text{ for all } n \geq 0\}$$

$$W^u_x(\epsilon) = \{y \in M : d(f^{-n} x, f^{-n} y) < \epsilon \text{ for all } n \geq 0\} .$$

Thus $f\, W^s_x(\epsilon) \subset W^s_{fx}(\epsilon)$ and $f^{-1} W^u_x(\epsilon) \subset W^u_{f^{-1}(x)}(\epsilon)$.

By stable manifold theory [4], if $x \in \Lambda$, there are partial flow boxes

$$D^k \times K^{n-k}$$

where K^{n-k} is a closed subset of $W^u_x(\epsilon)$ so that, for x' near x, $W^s_{x'}(\epsilon)$ intersects $D^k \times K^{n-k}$ in exactly one horizontal factor. There are also partial flow boxes

$$K^k \times D^{n-k}$$

where K^k is a closed subset of $W^s_x(\epsilon)$ so that, for n' near x, $W^u_{x'}(\epsilon)$ intersects $K^k \times D^{n-k}$ in exactly one vertical factor. Furthermore, near x, Λ agrees with the cartesian product $K^k \times K^{n-k}$. (Figure 1)

This picture has many consequences, for example if

$$W^s_\Lambda(\epsilon) = \cup\{W^s_x(\epsilon) : x \in \Lambda\} \ , \ W^u_\Lambda(\epsilon) = \cup\{W^u_x(\epsilon) : x \in \Lambda\}$$

then $W^s_\Lambda(\epsilon) \cap W^u_\Lambda(\epsilon) = \Lambda$. Also there are canonical homeomorphisms between parts of $W^s_x(\epsilon)$ and $W^s_{x'}(\epsilon)$ and also $W^u_x(\epsilon)$ and $W^u_{x'}(\epsilon)$, denoted $p^s_{x,x'}$ and $p^u_{x,x'}$, defined by projection along the appropriate factor. The projections are denoted p^s_x and p^u_x. We suppose α is chosen so that, for x, x' in Λ and $d(x, x') < \alpha$, p^s_x is defined on $W^s_{x'}(\alpha) \cap \Lambda$ and imbeds this set in $W^s_x(\epsilon) \cap \Lambda$. Similarly, for p^u_x.

Writing $B_x(\beta) = \{y \in M : d(x, y) < \beta\}$ we can choose $\beta < \alpha$ such that, if $x \in \Lambda$,

$$B_x(\beta) \cap W^u_\Lambda(\epsilon) \subset \cup \{W^u_y(\epsilon) : y \in W^s_x(\epsilon) \cap \Lambda\}$$

and

$$p^s_x(B_x(\beta) \cap W^u_\Lambda(\epsilon)) \subset W^s_x(\alpha)$$

and similarly when s and u are interchanged.

1. Theorem. Let $\log \lambda$ be the topological entropy of $f|\Lambda$. For each $x \in \Lambda$ there is a measure $\mu^s_x \geq 0$ on $W^s_x(\epsilon)$ and a measure $\mu^u_x \geq 0$ on $W^u_x(\epsilon)$ such that:

(a) supp $\mu^s_x \subset W^s_x(\epsilon) \cap \Lambda$, supp $\mu^u_x \subset W^u_x(\epsilon) \cap \Lambda$

(b) <u>If</u> $d(x, x') < \alpha$, <u>the image by</u> $p^s_{x,x'}$ <u>of the restriction of</u> $\mu^s_{x'}$ <u>to</u> $W^s_{x'}(\alpha) \cap \Lambda$ <u>is the restriction of</u> μ^s_x <u>to</u> ${}^{s}_{x,x'}(W^s_{x'}(\alpha) \cap \Lambda)$:

$$p^s_{x,x'}(\mu^s_{x'} | W^s_{x'}(\alpha) \cap \Lambda) = \mu^s_x | p^s_{x,x'}(W^s_{x'}(\alpha) \cap \Lambda)$$

<u>Similarly</u>

$$p^u_{x,x'}(\mu^u_{x'} | W^u_{x'}(\alpha) \cap \Lambda) = \mu^u_x | p^u_{x,x'}(W^u_{x'}(\alpha) \cap \Lambda)$$

(c) $(f^{-1} \mu^s_x - \lambda^{-1} \mu^s_{f^{-1}x}) | W^s_{f^{-1}x}(\epsilon) = 0$

$(f \mu^u_x - \lambda^{-1} \mu^u_{fx}) | W^u_{fx}(\epsilon) = 0$

(d) <u>Let</u> $[.,.] : (W^u_x(\epsilon) \cap \Lambda) \times (W^s_x(\epsilon) \cap \Lambda) \longrightarrow \Lambda$ <u>be defined by</u> $\{[y, z]\} = W^s_y(2\epsilon) \cap W^u_z(2\epsilon)$. <u>Then</u>

$$[\cdot, \cdot] (\mu^u_x \times \mu^s_x) | B_x(\beta) \cap \Lambda = \rho | B_x(\beta) \cap \Lambda$$

<u>where</u> ρ <u>is Bowen's measure on</u> Λ , i.e., <u>the unique</u> f-<u>invariant probability measure on</u> Λ <u>such that its measure-theoretical entropy is</u> $\log \lambda$.

This theorem is proved by Sinai ([7] Theorem 5.1) for Anosov diffeomorphisms. The general case follows simply by using Bowen's construction of Markov partitions for basic sets [1]. For completeness we give the argument in the Appendix.

As a corollary we see that the unstable manifolds near Λ form a geometric current C^s_ϵ . The stable bundle E^s is assumed oriented so we have a continuous orientation of the $W^s_x(\epsilon)$. The $p^u_{x,x'}$ generate the canonical isomorphisms among the standard transversals $W^u_{x'}(\epsilon) \cap (D^k \times K^{n-k})$ of the partial flow box $D^k \times K^{n-k}$. The theorem, parts (a) and (b), then provides us with a <u>transversal measure</u> μ^u generated by canonical isomorphism from μ^u_x .

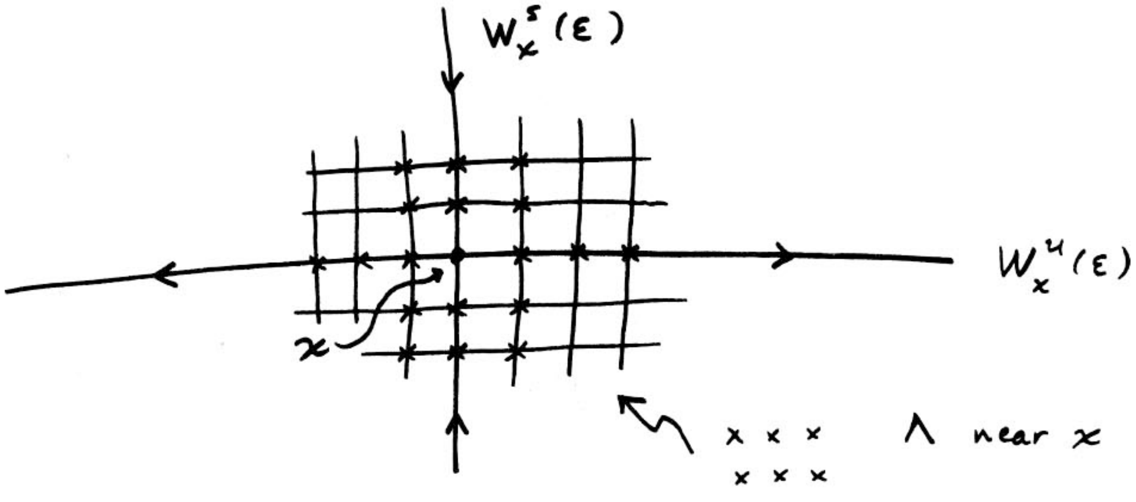

figure 1

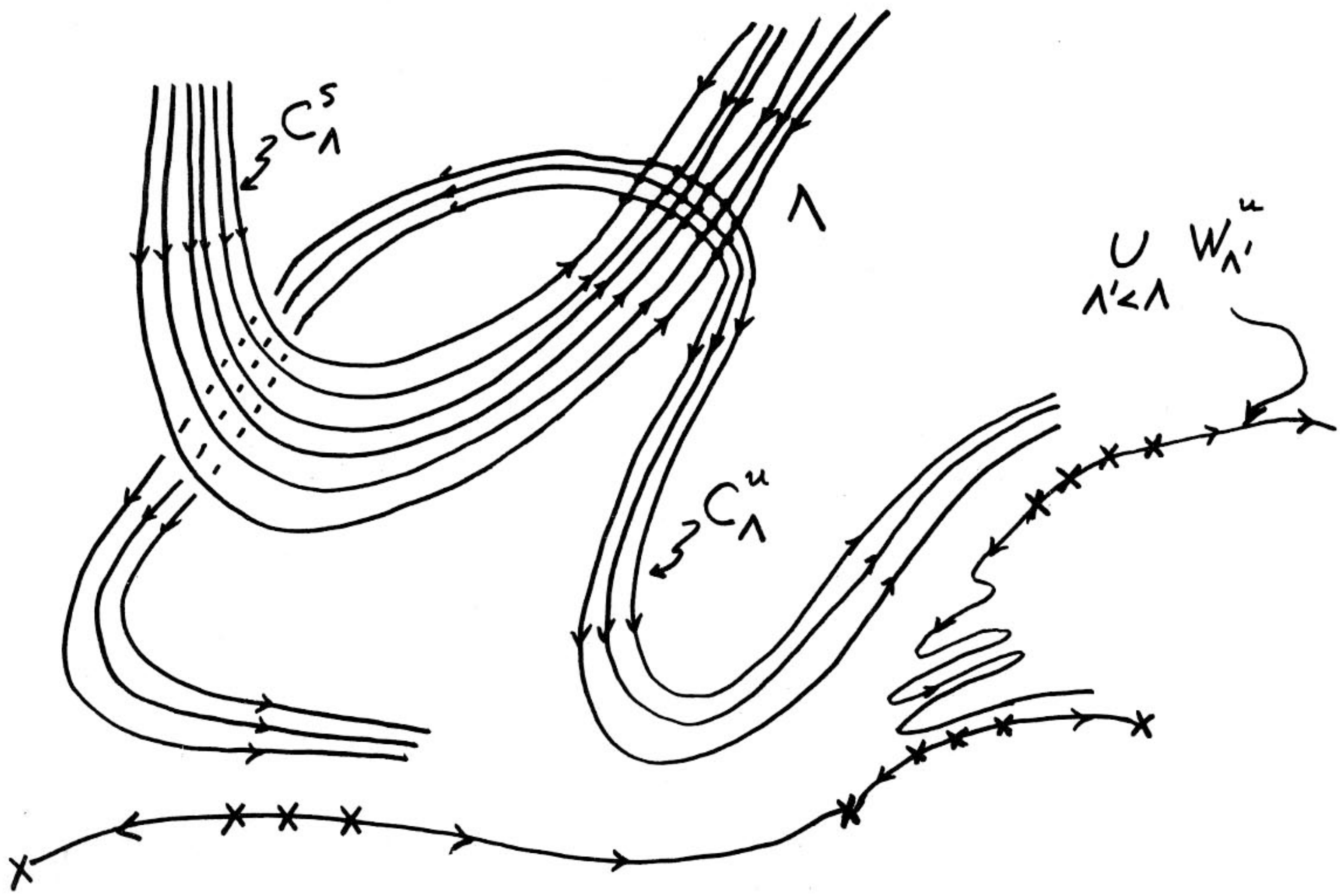

figure 2

To go further we must make an orientation assumption about f. Namely that f preserves (or reverses) the orientation on E^s.

Then by part (c) of the theorem, f maps the geometric current C^s_ϵ to $\pm \lambda^{-1} C^s_\epsilon$ (when restricted to a small neighbourhood of Λ). Using this fact we can construct an extension C^s of C^s_ϵ to the complement in M of the set

$$X^s_\Lambda = \bigcap_{n \geq 0} \text{clos.}\ (W^s_\Lambda \setminus f^{-n} W^s_\Lambda(\epsilon))$$

It is known [4] that $X^s_\Lambda \cap W^s_\Lambda = \emptyset$. Therefore every $x \in X^s_\Lambda$ belongs to $W^s_{\Lambda'}$ for a basic set $\Lambda' \neq \Lambda$. Furthermore if the Axiom-A diffeomorphism f satisfies the No-Cycle condition, then

$$(X^s_\Lambda \cap W^s_{\Lambda'} \neq \emptyset) \Rightarrow (\Lambda' > \Lambda) \qquad (*)$$

where $\Lambda' > \Lambda$ means that there is a sequence $\Lambda' = \Lambda_1, \Lambda_2, \ldots, \Lambda_{n+1} = \Lambda$ such that

$$W^u_{\Lambda_i} \cap W^s_{\Lambda_{i+1}} = \emptyset$$

for $i=1,\ldots,n$. How to prove (*), using [4], was shown to us by J. Palis. Altogether we see that if f satisfies Axiom-A and the No-Cycle condition, C^s is defined on

$$M \setminus \bigcup_{\Lambda' : \Lambda' > \Lambda} W^s_{\Lambda'} \qquad \text{(Figure 2)}$$

Similarly we construct a geometric current C^u using f, the $W^u_x(\epsilon)$, the measures μ^s_x, and the orientation of E^u which we suppose is preserved or reversed by f.

We also denote by C^s and C^u the de Rham currents determined by these geometric currents.

We summarize all this in the following theorem which makes use of the further fact [4], [2] that

$$W^s_\Lambda \cap W^u_\Lambda = \Lambda$$

Theorem 2. There are naturally defined geometric currents C^s of dimension k on $M \setminus \bigcup_{\Lambda':\Lambda'>\Lambda} W^s_{\Lambda'}$ and C^u of dimension $n-k$ on $M \setminus \bigcup_{\Lambda':\Lambda'<\Lambda} W^u_{\Lambda'}$. C^s is supported by the stable manifolds of Λ and C^s by the unstable manifolds of Λ. C^s and C^u intersect transversally in Λ which becomes a 0-current via the Bowen measure ρ. On the level of de Rham currents we can say

a) C^s and C^u are closed and thus define cohomology classes $[C^s]$ and $[C^u]$ in $H^k(M \setminus \bigcup_{\Lambda':\Lambda'>\Lambda} W^s_{\Lambda'}, M \setminus \bigcup_{\Lambda':\Lambda'\geq\Lambda} W^u_{\Lambda'})$ and $H^{n-k}(M \setminus \bigcup_{\Lambda':\Lambda'<\Lambda} W^u_{\Lambda'}, M \setminus \bigcup_{\Lambda':\Lambda'\leq\Lambda} W^u_{\Lambda'})$ respectively.

b) $C^s \wedge C^u$ is defined as a current and coincides with the 0-dimensional current (Λ, Bowen measure). It follows that

$$[C^s] \cup [C^u] = (\Lambda \text{ , Bowen measure})^{\text{dual}}$$

in $H^n(M, M \setminus \Lambda)$.

c) Finally $f\,C^s = \pm\, 1/\lambda\, C^s$ and $f\,C^u = \pm\,\lambda\, C^u$ on the level of currents where $\log \lambda$ is the topological entropy of f/Λ.

Remark: The orientation assumption on f cannot always be satisfied for example in the horseshoe example where Λ is totally disconnected.

The orientation assumption is fulfilled when Λ is connected for example when Λ is an attractor and $f|\Lambda$ is topologically mixing. In fact, Theorem 2 was in part motivated by the homological study of expanding attractors in [8] where the classes $[C^s]$ and $[C^u]$ are studied in this special case.

(See for example the solenoid case iii) above.)

Finally, we note a consequence of Theorem 2 and Proposition 2. Let f be an Anosov diffeomorphism of a smooth manifold M. In our terms this means the non-wandering set is all of M and f satisfies Axiom A. The classification of these diffeomorphisms whose study has inspired much of the work on Axiom A diffeomorphisms can hopefully be reduced to a study of the algebraic topology of M. As a small step in this direction we have the following

Corollary: If $f : M \to M$ is an Anosov diffeomorphism with orientable stable and unstable foliations then there are canonical homology classes (defined by these foliations) $S \in H_k(M, R)$ and $U \in H_{n-k}(M, R)$ so that $S \cap U = 1$ in $H_o(M, R)$ and $fU = \pm \lambda U$ and $fS = \pm 1/\lambda S$ where $\log \lambda$ is the topological entropy of f.

Appendix: proof of Theorem 1.

Let $\mathcal{C}$ be a Markov partition of Λ (see [1]) and, if $E, F \in \mathcal{C}$, write

$$t_{EF} = \begin{cases} 1 & \text{if } f(\text{int } E) \cap \text{int } F \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Define

$$\Sigma = \{(E_i) \in \mathcal{C}^{\mathbb{Z}} : t_{E_i E_{i+1}} = 1 \quad \text{for all } i \in \mathbb{Z}\}$$

and $\pi : \Sigma \to \Lambda$ by

$$\{\pi(E_i)\} = \bigcap_{i \in \mathbb{Z}} f^{-1} E_i$$

Then it is known that π is a continuous surjective map and $f \circ \pi = \pi \circ \sigma$ where σ is the shift on Σ. Let ν be the unique σ-invariant probability measure on Σ such that its entropy $h_\nu(\sigma)$ is maximum (Parry), then $\rho = \pi \nu$

is the unique f-invariant probability measure on Λ such that its entropy is maximum and the abstract dynamical systems (σ, ν) and (f, ρ) are isomorphic.

Let t be the matrix indexed by $\mathcal{C} \times \mathcal{C}$ with elements t_{EF}. From the Perron-Frobenius theorem [3] it follows that t and its adjoint t^* have eigenvectors a, b with components $a_E > 0$, $b_E > 0$ correponding to the same eigenvalue $\lambda > 0$, and we may assume that $\Sigma_{E \in \mathcal{C}} b_E a_E = 1$. If $k \leq \ell$, the measure ν satisfies

$$\nu\{(E_i) \in \Sigma : E_k = F_k, \dots, E_\ell = F_\ell\}$$

$$= \lambda^{-(\ell-k)} b_{F_k} t_{F_k F_{k+1}} \cdots t_{F_{\ell-1} F_\ell} a_{F_\ell}$$

which expresses that (σ, ν) is a Markov chain. We also have $\log \lambda = h_\nu(\sigma) = h_f(\rho)$.

Let Σ^+ resp. Σ^- be defined like Σ with $\mathbb{Z}$ replaced by $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$ resp. $\mathbb{Z}^- = \{n \in \mathbb{Z} : n \leq 0\}$. We define $\sigma : \Sigma^+ \to \Sigma^+$ and $\sigma^{-1} : \Sigma^- \to \Sigma^-$ in the obvious manner. There are measures $\nu^\pm \geq 0$ on $\Sigma^\pm$ such that, for $k \geq 0$,

$$\nu^+\{(E_i) \in \Sigma^+ : E_o = F_o, \dots, E_k = F_k\} = \lambda^{-k} t_{F_o F_1} \cdots t_{F_{k-1} F_k} a_{F_k}$$

$$\nu^-\{(E_i) \in \Sigma^- : E_{-k} = F_{-k}, \dots, E_o = F_o\} = \lambda^{-k} b_{F_{-k}} t_{F_{-k} F_{-k+1}} \cdots t_{F_{-1} F_o}$$

We may assume that the sets in the Markov partition $\mathcal{C}$ have diameter less then $\epsilon/3$. If $y = \pi(F_i)$, let $\pi^+_{y,F_o} : \{(E_i) \in Z^+ : E_o = F_o\} \to W^u_x(\epsilon) \cap F_o$ be defined by

$$\pi^+_{y,F_o}(E_i) = \pi(\ldots,F_{-2}, F_{-1}, F_o, E_1, E_2,\ldots)$$

so that

$$\{\pi^+_{y,F_o}(E_i)\} = W^u_y(\varepsilon) \cap \bigcap_{i\in\mathbb{Z}^+} f^{-i} E_i$$

The image of $\nu^+|\{(E_i) \in \Sigma^+ : E_o = F_o\}$ by π^+_{y,F_o} is a measure μ^u_{y,F_o} on $W^u_y(\varepsilon) \cap F_o$. The measure μ^s_{y,F_o} is defined similarly. We have

$$f\ \pi^+_{y,F_o}(E_i) = \pi^+_{y^*(E_1),E_1}(\sigma(E_i))$$

for any $y^*(E_1) \in E_1 \cap f(W^u_x(\varepsilon) \cap F_o)$. The image of $\nu^+|\{(E_1) \in \Sigma^+ : E_o = F_o$ and $E_1 = F_1\}$ by σ is $\lambda^{-1}\ \nu^+|\{(E_i) \in \Sigma^+ : E_o = F_1\}$. Therefore

$$f\ \mu^u_{y,F_o} = \lambda^{-1} \sum_{F_1\in\mathcal{C}} t_{F_oF_1}\ \mu^u_{y^*(F_1),\ F_1} \tag{*}$$

If $x \in \Lambda$ we define the measure μ^u_x on $W^u_x(\varepsilon)$ as follows. For each $E \in \mathcal{C}$ such that $W^u_x(\varepsilon) \cap E \neq \emptyset$, choose $y \in W^u_x(\varepsilon) \cap E$. Let supp $\mu^u_x \subset W^u_x(\varepsilon) \cap \Lambda$ and μ^u_x restricted to E be $\mu^u_{y,E}$ restricted to $W^u_x(\varepsilon) \cap E$. We define μ^s_x similarly. For the definition of μ^u_x to make sense we have to check that if $W^u_x(\varepsilon) \cap E \cap E' \neq \emptyset$ the definition of μ^u_x on E agrees with that on E' . We also want to verify that μ^u_x , $\mu^u_{x'}$ correspond by $p^u_{x,x'}$ as asserted in (b). To prove both facts it suffices to show that $\mu^u_{y',E'}$ and $\mu^u_{y,E}$ correspond by the projection $p^u_{y'}$ along stable manifolds. More precisely we want to show that the image of $\mu^u_{y,E}$ by $p^u_{y'}$ agrees with $\mu^u_{y',E'}$ on $(W^u_{y'}(\varepsilon) \cap E') \cap p^u_{y'}(W^u_y(\varepsilon) \cap E)$. If $E = E'$ this is the case because of

$$\pi^+_{y',E'} = p^u_{y'}\ \pi^+_{y,E}$$

and the definition of $\mu^u_{y,E}$, $\mu^u_{y',E'}$. If $E \neq E'$ we use (*) repeatedly to compare $f^k\ \mu^u_{y,E}$ and $f^k\ \mu^u_{y',E'}$. Let $\mathcal{I}^+$ be the set of points

$(E_i) \in \Sigma^+$ such that $E_o = E$ and there exists $(E_i') \in \Sigma^+$ such that

(i) $E_o' = E'$

(ii) $p^u_{y',E'}\ \pi^+_{y,E}(E_i) = \pi^+_{y',E'}(E_i')$

(iii) $E_i \neq E_i'$ for all $i \in \mathbb{Z}^+$.

We have to prove that $\nu^+(\Upsilon^+) = 0$.

Let Υ be the set of points $(E_i) \in \Sigma$ such that $E_o = E$ and there exists $(E_i') \in \Sigma^+$ such that

(i) $E_o' = E'$

(ii) $pr^u_{y',E'}\ \pi(E_i) = \pi^+_{y',E'}(E_i')$

(iii) $E_i \neq E_i'$ for all $i \in \mathbb{Z}^+$

The claim that $\nu^+(\Upsilon^+) = 0$ will follow from $\nu(\Upsilon) = 0$.

If $z \in \pi\Upsilon$, then for all $k \in \mathbb{Z}^+$ there exist E_k , E_k' distinct such that $f^k z \in E_k$ and

$$\lim_{k\to\infty} d(f^k z, E_k') = 0$$

uniformly in z . The ρ-measure of the boundary of each set in the Markov partition $\mathcal{C}$ vanishes. Therefore

$$\nu(\Upsilon) = \rho(\pi\Upsilon) = \lim_{k\to\infty} \rho(f^k\ \pi\Upsilon) = 0$$

We have thus verified (a) and (b); (c) follows from (*). Consider now the map

$$\{(E_i) \in \Sigma^+ : E_o = E\} \times \{(E_i) \in \Sigma^- : E_o = E\} \to \{(E_i) \in \Sigma : E_o = E\}$$

such that

$$(E_o, E_1,\ldots) \times (\ldots,E_{-1}, E_o) \to (\ldots,E_{-1}, E_o, E_1,\ldots)$$

The image of

$$(\sigma^+ | \{(E_i) \in \Sigma^+ : E_o = E\}) \times (\sigma - | \{(E_i) \in \Sigma^- : E_o = E\})$$

by this map is $\sigma | \{(E_i) \in \Sigma : E_o = E\}$, and (d) follows readily.

References

[1] R. Bowen. Markov partitions for Axiom A diffeomorphisms. Amer. J. Math. 92, 725-747 (1970)

[2] R. Bowen. w-limit sets for Axiom A diffeomorphisms. Preprint.

[3] F.R. Gantmacher. The theory of matrices. 2 vol. Chelsea, New York, 1964.

[4] M. Hirsch, J. Palis, C. Pugh, and M. Shub. Inventiones math. 9, 121-134 (1970).

[5] G. de Rham. Variétés différentiables. 2nd ed. Hermann, Paris, 1960.

[6] S. Schwartzmann. Asymptotic cycles. Annals of Math. 66, 270-284 (1957).

[7] Ia. G. Sinai. Markov partitions and C-diffeomorphisms. Funkts. Anal. Prilozh. 2 n° 1, 64-89 (1968).

[8] D. Sullivan and R. Williams. Homological properties of attractors (in preparation).

IHES/74/M/084
Juin 1974