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Zeta-Functions for Expanding Maps and Anosov Flows

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Abstract. Given a real-analytic expanding endomorphism of a compact manifold M, a meromorphic zeta function is defined on the complex-valued real-analytic functions on M. A zeta function for Anosov flows is shown to be meromorphic if the flow and its stable-unstable foliations are real-analytic.

Introduction

In a previous note [12] we have defined generalized zeta-functions for diffeomorphisms and flows. These zeta-functions have as their argument a function on the manifold rather than a number (cf. [1, 15]). We have indicated certain analyticity properties of these zeta-functions in the case of Axiom-A diffeomorphisms or flows.

More detailed properties of meromorphy are obtained in the present paper, on the basis of stronger assumptions. These new assumptions combine hyperbolicity ("Axiom A") with real-analyticity requirements. We discuss now some of the results.

I. *Let M be a real-analytic connected compact manifold, $f\colon M \mapsto M$ a real analytic expanding map, and φ a complex-valued real-analytic function on M. Then the series*

$$\zeta = \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} \varphi(f^k x)$$

converges for small $|u|$ and extends to a meromorphic function of u in the entire complex plane.

Actually, ζ is also a meromorphic function of φ. The hyperbolicity assumption is here that f is expanding, i.e. $\|(Tf)v\| \geqq \theta \|v\|$ with $\theta > 1$ for some Riemann metric.

II. *Let (f^t) be a real-analytic Anosov flow on a real-analytic manifold such that the stable and unstable manifolds form real-analytic foliations. If $\lambda(\gamma)$ denotes the* *prime period of a periodic orbit γ, the product*

$$\zeta(s)=\prod_{\gamma\,\text{periodic}}(1-e^{-s\lambda(\gamma)})$$

converges for large $\operatorname{Re}s$ *and extends to a meromorphic function of* s *in the entire complex plane.*

Information is also obtained on the growth properties of ζ. II applies in particular to the geodesic flow of a compact manifold N with constant negative sectional curvature[1]. In particular

$$s\mapsto\prod_{\gamma\,\text{periodic}}\prod_{k=0}^{\infty}(1-e^{-(s+k)\lambda(\gamma)})$$

is meromorphic, in agreement with the results of Selberg for surfaces of constant negative curvature [13].

The proof of I and II (more precisely of Theorems 2 and 3 below) is based on Theorem 1 below which is of a more technical nature. We shall go rather fast over the proof of Theorem 2 because *P.* Cartier has now a better proof of a more precise result [4].

Theorem 1. *Let I be a finite set and $D=\sum_{i\in I}D_i$ the disjoint sum of connected open sets $D_i\subset\mathbb{C}^N$. Let t be a $I\times I$ matrix with entries $t_{ij}=0$ or 1 and, if $t_{ij}=1$, let $\psi_{ij}\colon D_j\mapsto D_i$ be holomorphic and such that clos $\psi_{ij}D_j$ is compact in D_i. We write*

$$J_n=\{(i,\ldots,i_n)\in I^n\colon t_{i_1i_2}=\cdots=t_{i_{n-1}i_n}=t_{i_ni_1}=1\}.$$

(a) *If $(i_1,\ldots,i_n)\in J_n$, then $\psi_{i_ni_1}\circ\psi_{i_1i_2}\circ\cdots\circ\psi_{i_{n-1}i_n}$ has one and only one fixed point $z_{(i_1,\ldots,i_n)}$.*

(b) *If φ is complex-valued analytic on D, let*

$$a_n=\sum_{(i_1,\ldots,i_n)\in J_n}\prod_{k=1}^{n}\varphi(z_{(i_{k+1}\ldots i_n i_1\ldots i_k)}).$$

Then the series $\sum_{n=1}^{\infty}\frac{1}{n}a_nu^n$ converges in a neighborhood of the origin, and

$$\zeta(u\varphi)=\exp\sum_{n=1}^{\infty}\frac{1}{n}a_nu^n$$

extends to a meromorphic function of u:

$$\zeta(u\varphi)=\frac{d_1(u\varphi)}{d_2(u\varphi)}$$

where $u\mapsto d_1(u\varphi)$, $d_2(u\varphi)$ are entire functions.

There is an open set D' such that $\operatorname{clos}D'\subset D$ *and if $\mathscr{B}$ is the Banach space of complex continuous functions on* $\operatorname{clos}D'$ *which are analytic in D', with the uniform*

[1] The conditions of II are satisfied because, if $\hat{N}$ is the universal covering space of N, then the Lie group of isometries of $\hat{N}$ acts transitively on the unit tangent bundle of $\hat{N}$ (see for instance [5] p. 207)

norm, then d_1, d_2 *extend to entire functions on* $\mathscr{B}$ *satisfying majorizations*

$$|d_i(\varphi)-1| \leqq \|\varphi\| \exp[C(\log(\|\varphi\|+e))^2].$$

In particular $u \mapsto d_1(u\varphi), d_2(u\varphi)$ *are entire functions of order* 0 *of* u; *their zeroes (arranged by increasing modulus and repeated according to multiplicity) tend to infinity exponentially fast.*

Notice first that we can, without changing the problem, replace each D_i by a bounded set D_i' such that $\operatorname{clos} D_i' \subset D_i$. We may thus assume that the D_i are bounded and that each ψ_{ij} is holomorphic from a neighborhood of $\operatorname{clos} D_j$ to D_i. Also, φ is continuous on $\operatorname{clos} D$, and we shall prove that d_1, d_2 are entire on $\mathscr{B}$ where $\mathscr{B}$ is the Banach space of functions continuous on $\operatorname{clos} D$, and analytic in D.

Part (a) of Theorem 1 results from the following lemma applied to $D = D_{i_n}$ and $\psi = \psi_{i_n i_1} \circ \psi_{i_1 i_2} \circ \cdots \circ \psi_{i_{n-1} i_n}$.

Lemma 1. *Let* D *be a bounded connected open subset of* $\mathbb{C}^N$ *and* ψ *be holomorphic from a neighborhood of* $\operatorname{clos} D$ *to* D. *Then*

$$\bigcap_{l=1}^{\infty} \psi^k \operatorname{clos} D$$

consists of a single point $\tilde{z}$. *The eigenvalues of the derivative of* ψ *at* $\tilde{z}$ *are strictly less than* 1 *in absolute value.*

Let $\varphi: D \mapsto \mathbb{C}$ be analytic, and define

$$\varphi_l = \varphi \circ \psi^l \qquad \text{for } l \geqq 1.$$

Then φ and the φ_l are bounded uniformly on a neighborhood of ψ ($\operatorname{clos} D$). Therefore the φ_l and their derivatives are bounded uniformly on $\operatorname{clos} D$. Let $(\tilde{\varphi}_l)$ be a subsequence converging uniformly on $\operatorname{clos} D$, and let φ be its limit. We have

$$\max_{z \in \operatorname{clos} D} |\tilde{\varphi}_l(z)| \geqq \max_{z \in \psi \operatorname{clos} D} |\tilde{\varphi}_l(z)| \geqq \max_{z \in \operatorname{clos} D} |\tilde{\varphi}_{l+1}(z)|$$

so that

$$\max_{z \in \operatorname{clos} D} |\tilde{\varphi}(z)| = \max_{z \in \psi \operatorname{clos} D} |\tilde{\varphi}(z)|.$$

Since D is connected, $\tilde{\varphi}$ is constant. This shows that φ is constant on $\bigcap_{l=1}^{\infty} \psi^l \operatorname{clos} D$. Since this is true for all φ, this intersection consists of a single point $\tilde{z}$.[2] As a consequence, the derivative of ψ^l tends to 0 when $l \to \infty$, so that the eigenvalues of the derivative of ψ at $\tilde{z}$ must be strictly less than 1 in absolute value.

Fredholm Theory

We assume that D is a bounded open subset of $\mathbb{C}^N$ and that ψ is holomorphic from a neighborhood of $\operatorname{clos} D$ to D. Let $\mathscr{H}_k$ be the Fréchet space of holomorphic

[2] As pointed out to me by D. Mayer, the above proof is essentially to be found in Hervé [8] p. 83. Another proof that ψ has only one fixed point, using the index of the map, was indicated to me by N. Kuiper

exterior forms of order k on D, $0 \leqq k \leqq N$. We identify $\mathscr{H}_0$ with the space of analytic functions on D. Consider those $\omega \in \mathscr{H}_k$ such that their coefficients have a continuous extension to $\operatorname{clos} D$; those ω form a Banach space $\mathscr{B}_k(D)$ with the uniform norm. If $\varphi \in \mathscr{B}_0(D)$, a linear map $L_k: \mathscr{B}_k(D) \mapsto \mathscr{B}_k(D)$ is defined by

$$(L_k \omega)(z) = \varphi(z) \cdot [(\wedge^k \psi'_z)(\omega \circ \psi(z))] \tag{1}$$

where ψ'_z is the derivative of ψ at z.

This formula also defines a map $\mathscr{H}_k \mapsto \mathscr{B}_k(D)$ which is bounded. Since $\mathscr{H}_k$ is a nuclear space ([6], II, Corollaire p. 56), this map is nuclear of order 0 ([6], II, Corollaire 4, p. 39 and Corollaire 2, p. 61). Composing with the continuous injection $\mathscr{B}_k(D) \mapsto \mathscr{H}_k$, we see that L_k is nuclear of order 0 ([6], I, p. 84 and II, p. 9). Therefore ([6], II, Corollaire 4, p. 18) L_k corresponds to a unique Fredholm kernel, of order 0. In particular

$$\operatorname{Tr} L_k = \sum_i u_i, \qquad \det(1 - uL_k) = \prod_i (1 - u u_i)$$

where the u_i are the eigenvalues of L_k repeated according to multiplicity. The Fredholm determinant $\det(1 - uL_k)$ is an entire function of order 0 of u ([6], II, Théorème 4, p. 16).

Lemma 2. *With the assumptions of Lemma* 1 *and definition* (1) *we have*

$$\sum_{k=0}^{N} (-1)^k \operatorname{Tr} L_k = \varphi(\tilde{z}).$$

If D' is a non-empty open connected subset of D such that $\psi \operatorname{clos} D' \subset D'$, L_k extends in an obvious manner to a nuclear operator L'_k of order 0 on $\mathscr{B}_k(D')$ with the same eigenvalues and therefore the same trace. We choose for D' a polydisk centered at $\tilde{z}$. By a suitable linear change of coordinates in $\mathbb{C}^N$ we can assume that ψ'_z is in Jordan normal form consisting of blocks

$$\begin{pmatrix} \lambda & \varepsilon & 0 & \dots \\ 0 & \lambda & \varepsilon & \dots \\ 0 & 0 & \lambda & \dots \\ \dots & & & \end{pmatrix}$$

with $|\lambda| < \theta < 1$ and $\varepsilon \neq 0$ arbitrary. It is clear that for sufficiently small ε, $v > 0$, the polydisk

$$D(v) = \{z \in \mathbb{C}^N : |z_k - \tilde{z}_k| < v \text{ for } k = 1, \dots, N\}$$

is mapped by ψ into $D(\theta v)$. With $D' = D(v)$ we have

$$\begin{aligned}(L'_k \omega)(z) &= \varphi(z) \cdot [(\wedge^k \psi'_z)(\omega \circ \psi(z))] \\ &= \varphi(z) \cdot \left[(\wedge^k \psi'_z) \left(\prod_{k=1}^{N} \int \frac{\rho(\zeta_k)\, d\zeta_k \wedge d\bar{\zeta}_k}{\tilde{z}_k + \zeta_k - [\psi(z)]_k}\right) \omega(\tilde{z} + \zeta)\right]\end{aligned}$$

where ρ is a suitable C^∞ function with support in $\{u \in \mathbb{C} : \theta v < |u| < v\}$. This formula also defines a nuclear operator L''_k of order 0 on the Banach space of exterior forms of order k with continuous coefficients on $\operatorname{clos} D(v)$ (L''_k can be obtained by composing a continuous operator from the forms with continuous coefficients on $\operatorname{clos} D(v)$ to the forms with analytic coefficients on $D(\theta' v)$, $\theta < \theta' < 1$, and a nuclear operator from the forms with analytic coefficients on $D(\theta' v)$ to the forms with continuous coefficients on $\operatorname{clos} D(v)$). Since L'_k and L''_k have the same eigenvalues, we have $\operatorname{Tr} L'_k = \operatorname{Tr} L''_k$ and $\operatorname{Tr} L''_k$ can be computed by standard Fredholm theory (see [7]):

$$\operatorname{Tr} L''_k = \left(\prod_{k=1}^{N} \int \frac{\rho(\zeta_k)\, d\zeta_k \wedge d\bar{\zeta}_k}{\tilde{z}_k + \zeta_k - [\psi(\tilde{z}+\zeta)]_k} \right) \varphi(\tilde{z}+\zeta) \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}+\zeta}).$$

According to the above, we can replace $D(v)$ by $D(v/l)$ for $l > 1$. This amounts to replacing $\rho(\zeta_k)$ by $l\rho(l\zeta_k)$, hence

$$\operatorname{Tr} L''_k = \left(\prod_{k=1}^{N} \int \frac{\rho(\zeta_k)\, d\zeta_k \wedge d\bar{\zeta}_k}{\zeta_k - l[\psi(\tilde{z}+\zeta/l) - \tilde{z}]_k} \right) \varphi(\tilde{z}+\zeta/l) \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}+\zeta/l})$$

and in the limit $l \to \infty$

$$\operatorname{Tr} L''_k = \left(\prod_{k=1}^{N} \int \frac{\rho(\zeta_k)\, d\zeta_k \wedge d\bar{\zeta}_k}{\zeta_k - [\psi'_{\tilde{z}}(\zeta)]_k} \right) \varphi(\tilde{z}) \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}}).$$

Letting now $\varepsilon \to 0$ we an integrate, finding

$$\operatorname{Tr} L''_k = \varphi(\tilde{z}) \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}}) \prod_{k=1}^{N} (1-\lambda_k)^{-1}$$

where the λ_k are the eigenvalues of ψ'_z. Therefore

$$\operatorname{Tr} L_k = \frac{\varphi(\tilde{z}) \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}})}{\det(1-\psi'_{\tilde{z}})}$$

hence

$$\sum_{k=0}^{N} (-1)^k \operatorname{Tr} L_k = \varphi(\tilde{z}) [\det(1-\psi'_{\tilde{z}})]^{-1} \sum_{k=0}^{N} (-1)^k \operatorname{Tr}(\Lambda^k \psi'_{\tilde{z}}) = \varphi(\tilde{z}).$$

Proof of Part (b) of Theorem 1. We revert to the notation of the theorem, and define an operator $\mathscr{L}_k$ on $\mathscr{B}_k(D)$ by

$$(\mathscr{L}_k \omega)(z) = \varphi(z) \sum_{i:t_{ij}=1} (\Lambda^k(\psi_{ij})'_z)(\omega \circ \psi_{ij}(z)) \qquad \text{if } z \in D_j.$$

If $t_{ij}=1$, let also $\mathscr{L}_{ji}\colon \mathscr{B}_k(D_i) \to \mathscr{B}_k(D_j)$ be defined by

$$(\mathscr{L}_{ji}\omega)(z) = \varphi(z)(\Lambda^k(\psi_{ij})'_z)(\omega \circ \psi_{ij}(z))$$

for fixed k. The operator $\mathscr{L}_k$ is nuclear of order 0 and

$$\operatorname{Tr}(\mathscr{L}_k^n) = \sum_{(i_1,\ldots,i_n)\in J_n} \operatorname{Tr}(\mathscr{L}_{i_n i_{n-1}} \cdots \mathscr{L}_{i_2 i_1} \mathscr{L}_{i_1 i_n})$$

where

$$(\mathscr{L}_{i_n i_{n-1}} \cdots \mathscr{L}_{i_2 i_1} \mathscr{L}_{i_1 i_n} \omega)(z) = \varphi(z) \cdot \varphi(\psi_{i_{n-1} i_n} z) \cdot \cdots \cdot \varphi(\psi_{i_1 i_2} \cdots \psi_{i_{n-1} i_n} z),$$

$$(\Lambda^k(\psi_{i_n i_1})'_{\psi_{i_1 i_2} \cdots \psi_{i_{n-1} i_n} z}) \cdots (\Lambda^k(\psi_{i_{n-1} i_n})'_z),$$

$$\omega \circ \psi_{i_n i_1} \circ \psi_{i_1 i_2} \circ \cdots \circ \psi_{i_{n-1} i_n}(z) = \left[\prod_{k=1}^{n} \varphi(\psi_{i_k i_{k+1}} \cdots \psi_{i_{n-1} i_n} z)\right] (\Lambda^k \psi'_z)(\omega \circ \psi(z))$$

and we have put $\psi = \psi_{i_n i_1} \circ \psi_{i_1 i_2} \circ \cdots \circ \psi_{i_{n-1} i_n}$. Comparing with (1) and using Lemma 2, we find

$$\sum_{k=0}^{N} (-1)^k \operatorname{Tr}(\mathscr{L}_k^n) = \sum_{(i_1, \ldots, i_n) \in J_n} \prod_{k=1}^{n} \varphi(\psi_{i_k i_{k+1}} \cdots \psi_{i_{n-1} i_n} z_{(i_1, \ldots, i_n)}) = a_n .$$

By Fredholm theory ([7] p. 350), the following series in u:

$$\sum_{n=1}^{\infty} \frac{u^n}{n} \operatorname{Tr}(\mathscr{L}_k^n)$$

has non-vanishing convergence radius, and

$$\exp\left[-\sum_{n=1}^{\infty} \frac{u^n}{n} \operatorname{Tr}(\mathscr{L}_k^n)\right] = \det(1 - u\mathscr{L}_k).$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n} a_n u^n$ has non-vanishing convergence radius and

$$\exp \sum_{n=1}^{\infty} \frac{1}{n} a_n u^n = \prod_{k=0}^{N} [\det(1 - u\mathscr{L}_k)]^{(-1)^{k+1}}. \tag{2}$$

The Fredholm determinants in the right-hand side are entire functions of u of order 0. It is also known ([7], Proposition 1, p. 346–347) that these determinants are entire functions of $u\mathscr{L}_k$ on the Banach space $\mathscr{B}_k(D)' \hat{\otimes} \mathscr{B}_k(D)$ of Fredholm kernels on $\mathscr{B}(D)$. Since the map $u\varphi \mapsto u\mathscr{L}_k$ is $\mathbb{C}$-linear and continuous from $\mathscr{B}(D) = \mathscr{B}_0(D)$ to $\mathscr{B}_k(D)' \hat{\otimes} \mathscr{B}_k(D)$, the Fredholm determinants are entire functions of $u\varphi \in \mathscr{B}(D)$. More can however be said because $\mathscr{H}_k$ is a space of analytic functions. We have indeed

$$\mathscr{L}_k = \sum_i \lambda_i \psi'_i \otimes (\varphi \cdot \varphi_i)$$

where (λ_i) is a sequence of complex numbers with exponential decrease, (ψ'_i) and (φ_i) are bounded sequences in $\mathscr{B}_k(D)'$ and $\mathscr{B}_k(D)$ respectively ([6], II, Remark 9, c. p. 63). One can thus estimate the coefficients of $\det(1 - u\mathscr{L}_k)$, finding a bound $A \|\varphi\|^n n^{n/2} e^{-Bn^2}$ for the absolute value of the coefficient of u^n. Therefore

$$|\det(1 - \mathscr{L}_k) - 1| \leqq \|\varphi\| \exp[C(\log(\|\varphi\| + e))^2] \tag{3}$$

(where $\log e = 1$). It is then well-known that the zeros of $u \mapsto \det(1 - u\mathscr{L}_k)$ tend to infinity exponentially fast. From (2) and (3) the estimates of the theorem are easily obtained.

Theorem 2. *Let M be a real-analytic connected compact manifold, and f a real analytic expanding endomorphism of M. This means that for some Riemann metric*[3] *there exists $\theta > 1$ such that*

$$\|(Tf)v\| \geqq \theta \|v\|$$

for all $v \in TM$.

If φ is a complex-valued real-analytic function on M, let

$$a_n = \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} \varphi(f^k x)$$

where $\mathrm{Fix} f^n = \{x \in M : f^n x = x\}$. *Then the power series*

$$\sum_{n=1}^{\infty} \frac{1}{n} a_n u^n$$

converges in a neighborhood of 0, *and*

$$\zeta(u\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} a_n u^n$$

extends to a meromorphic function of u:

$$\zeta(u\varphi) = \frac{d_1(u\varphi)}{d_2(u\varphi)}.$$

If D is an open neighborhood of M in a complexification of M, and $\mathscr{B}$ the Banach space of continuous functions on $\mathrm{clos} D$ *which are analytic in D, with the uniform norm, then d_1, d_2 extend to entire functions on $\mathscr{B}$ satisfying majorizations*

$$|d_i(\varphi) - 1| \leqq \|\varphi\| \exp[C(\log(\|\varphi\| + e))^2].$$

For general results on expanding endomorphisms of compact manifolds, see Shub [14] and Hirsch [9].

In view of published results it is convenient to introduce the space

$$\bar{M} = \{(x_n)_{n \geqq 0} : f x_{n+1} = x_n \text{ for all } n \geqq 0\}.$$

With respect to the metric

$$d((x_n), (y_n)) = \sum_n \frac{1}{\theta^n} d(x_n, y_n)$$

$\bar{M}$ is a compact metric space. Defining

$$\bar{f}(x_n) = (f x_n), \qquad \pi(x_n) = x_0$$

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[3] More generally one can assume that there exist $c > 0$, $\theta > 1$ such that

$$\|(Tf^n)v\| \geqq c\theta^n \|v\|$$

for all $n \geqq 0$, $v \in TM$. One can then find a metric for which $c = 1$ by Mather's argument (see [10]) we have $\pi \circ \bar{f} = f \circ \pi$. The map $\bar{f}$ is a homeomorphism, with inverse

$$\bar{f}^{-1}(x_n) = (y_n), \quad y_n = x_{n+1}.$$

Let $X = (x_n) \in \bar{M}$, $\delta > 0$, and define

$$W_X^s(\delta) = \{Y \in \bar{M} : d(\bar{f}^n X, \bar{f}^n Y) \leqq \delta \text{ for all } n \geqq 0\},$$

$$W_X^u(\delta) = \{Y \in \bar{M} : d(\bar{f}^{-n} X, \bar{f}^{-n} Y) \leqq \delta \text{ for all } n \geqq 0\}.$$

Then, for sufficiently small δ,

$$W_X^s(\delta) \subset \pi^{-1}(\pi X).$$

If $Y, Z \in W_X^s(\delta)$, with $Y = (y_n)$, $Z = (z_n)$, we find thus

$$\begin{aligned}
d(\bar{f}^n Y, \bar{f}^n Z) &= \sum_{k=0}^{\infty} \frac{1}{\theta^k} d(f^n y_k, f^n z_k) \\
&= \sum_{k=0}^{n-1} \frac{1}{\theta^k} d(f^{n-k} y_0, f^{n-k} z_0) \\
&\quad + \sum_{k=n}^{\infty} \frac{1}{\theta^k} d(y_{k-n}, z_{k-n}) \\
&= \frac{1}{\theta^n} \sum_{l=0}^{\infty} \frac{1}{\theta^l} d(y_l, z_l) = \frac{1}{\theta^n} d(Y, Z).
\end{aligned}$$

This is part of the following easily proved statement.

Fact 1. *There are positive numbers $\lambda < 1$, ε and γ such that the following is true.*
For $n \geqq 0$,

$$d(\bar{f}^n Y, \bar{f}^n Z) \leqq \lambda^n d(Y, Z) \quad \text{if } Y, Z \in W_X^s(\gamma),$$

$$d(\bar{f}^{-n} Y, \bar{f}^{-n} Z) \leqq \lambda^n d(Y, Z) \quad \text{if } Y, Z \in W_X^u(\gamma).$$

If $d(X, Y) \leqq \varepsilon$, then $W_X^s(\gamma) \cap W_Y^u(\gamma)$ consists of a single point, which we denote by $[X, Y]$. The map

$$[\cdot, \cdot] : \{(x, y) \in \bar{M} \times \bar{M} : d(X, Y) \leqq \varepsilon\} \to \bar{M}$$

is continuous.

This is the same Fact 1 from which Bowen [2] derives the existence of a Markov partition. For completeness we also mention that periodic points are dense in $\bar{M}$ and that $\bar{f}$ is topologically transitive. Given $N > 0$ and $(x_n) \in \bar{M}$, we can find arbitrarily close to x_N a point $y \in M$ periodic with respect to f (Shub [14], Theorem 1, g). Then $(f^N y, f^{N-1} y, \ldots, y, f^{p-1} y, \ldots)$ is arbitrarily close to (x_n) and periodic. Topological transitivity follows from the fact that there is a point $x \in M$ with dense orbit for f (Shub [14], Theorem 1, f), then any $X \in \pi^{-1} x$ has dense orbit for $\bar{f}$.

Notice now that π is a bijection of the periodic points of period p for $\bar{f}$ to the periodic points of period p for f. We shall use a method for counting periodic points of Axiom-A basic sets, which is also applicable to $\bar{M}$, and which is due to

Manning [11]. Given a Markov partition of $\bar{M}$, Manning's formula expresses $\operatorname{card}\operatorname{Fix}\bar{f}^n$ as a sum of terms $\pm\operatorname{card}\operatorname{Fix}\tau_\alpha^n$, where the τ_α are subshifts of finite type. In fact to each periodic point of τ_α is associated a periodic point of $\bar{M}$, and the formula

$$\operatorname{card}\operatorname{Fix}\bar{f}^n=\sum_\alpha \pm\operatorname{card}\operatorname{Fix}\tau_\alpha^n$$

holds over each periodic point X of $\bar{M}$. This gives the possibility of computing a_n as a sum of contributions $\pm a_{\alpha n}$ of the subshifts τ_α. We consider one subshift τ_α, with symbol set $\{S_i\}$ and transition matrix (t_{ij}). To each symbol S_i is associated a finite set $\{R_{(i)l}\}$ of rectangles of the Markov partition, such that $R(S_i)=\bigcap_l R_{(i)l}\neq\emptyset$. If $(S_{i(k)})_{k\in\mathbb{Z}}$ is a point of $\operatorname{Fix}\tau_\alpha^n$, i.e., a periodic sequence of symbols, the point $X\in\operatorname{Fix}\bar{f}^n$ associated with $(S_{i(k)})$ satisfies

$$X\in R(S_{i(0)}),\ \dots,\bar{f}^{n-1}X\in R(S_{i(n-1)}).$$

This determines X uniquely by expansiveness of $\bar{f}$. The point $x=\pi X\in\operatorname{Fix}f^n$ is then uniquely determined by

$$x\in\pi R(S_{i(0)}),\ \dots,f^{n-1}x\in\pi R(S_{i(n-1)}).$$

If the Markov partition has been chosen sufficiently fine, the expanding character of f permits the construction of connected open sets D_i in a complexification of M, such that $D_i\supset\pi R(S_i)$ and $fD_i\supset\operatorname{clos}D_j$ when $t_{ij}=1$. Let $\psi_{ij}:D_j\mapsto D_i$ be the map such that $f\circ\psi_{ij}$ is the identity on D_j. We find that Theorem 1 applies in the present situation (up to replacement of the D_i by subsets of $\mathbb{C}^N$) and gives information on the $a_{\alpha n}$. Since $a_n=\sum_\alpha\pm a_{\alpha n}$, Theorem 2 follows from Theorem 1.

Theorem 3. *Let (f^t) be an Anosov flow on the compact manifold M. We suppose that M, (f^t) are real-analytic, and that the (strong) stable and unstable manifolds form real-analytic foliations of M. If $\lambda(\gamma)$ denotes the prime period of a periodic orbit γ, the function*

$$\zeta(s)=\prod_{\gamma\,\text{periodic}}(1-e^{-s\lambda(\gamma)})$$

defined for large $\operatorname{Re}s$ *extends to a meromorphic function, which is the quotient $d_1(s)/d_2(s)$ of two entire functions of order $\leqq 2$ satisfying estimates*

$$|d_i(s)-1|\leqq e^{-\alpha\operatorname{Re}s}\quad\text{when }\operatorname{Re}s\geqq R$$

for some α, $R>0$.

For the proof we rely heavily on Bowen [3].

By definition of an Anosov flow, the vector field defining (f^t) does not vanish on M. The tangent bundle has a (f^t) invariant continuous splitting

$$TM=E^s+E^u+E^0.$$

Here E^0 is 1-dimensional tangent to the flow; furthermore

$$\|(Tf^t)v\| \leqq ce^{-\kappa t}\|v\| \qquad \text{for } v \in E^s,\ t \geqq 0,$$

$$\|(Tf^{-t})v\| \leqq ce^{-\kappa t}\|v\| \qquad \text{for } v \in E^u,\ t \geqq 0$$

for some Riemann metric, and some c, $\kappa > 0$. Given $t_0 > 0$ we can choose a new metric such that the above conditions hold with $c = 1$ (and suitable $\kappa > 0$) when $t \geqq t_0$. The (strong) stable and unstable manifolds are respectively tangent to E^s and E^u. The center-stable and center-unstable manifolds are respectively tangent to $E^s + E^0$ and $E^u + E^0$.

A symbolic dynamics can be constructed from local cross sections by a finite number of small open disks in hypersurfaces transverse to the flow (see Bowen [3] Section 2). We can here choose each A_i to be a real-analytic manifold fibered by disks contained in stable manifolds (contracting fibers).

Let $x \in A_i$, and suppose that $y = f^t x \in A_j$ with $\varepsilon_1 < t < \varepsilon_2$ (some fixed small ε_1, $\varepsilon_2 > 0$). Then t is a real-analytic function of $x: t = \lambda_{A_i A_j}(x)$ defined on an open subset of A_i.

Let S be a finite symbol set, $(t_{\xi\xi'})$ a "transition" matrix indexed by $S \times S$ with entries 0 and 1, and τ the shift on the space

$$\Lambda = \{(\xi_m)_{m \in \mathbb{Z}} \in S^{\mathbb{Z}} : t_{\xi_m \xi_{m+1}} = 1 \text{ for all } m\}.$$

Suppose that to each $\xi \in S$ a disk $A(\xi) \in \{A_i\}$ is associated. Then, under appropriate conditions, for each $\underline{\xi} = (\xi_m)_{m \in \mathbb{Z}} \in \Lambda$ there is exactly one point $\pi\underline{\xi} \in A(\xi_0)$ and a sequence $(t_m)_{m \in \mathbb{Z}}$ of real numbers such that $\varepsilon_1 < t_m - t_{m-1} < \varepsilon_2$ and $f^{t_m} x \in A(\xi_m)$. [In particular, symbolic dynamics corresponds to a bijection $S \to \{A_i\}$, but we need to consider a more general situation.] If $\underline{\xi} \in \operatorname{Fix} \tau^n$, the (f^t) orbit γ through $\pi(\underline{\xi})$ is periodic, and the quantity

$$\lambda(\underline{\xi}) = \sum_{k=0}^{n-1} \lambda_{A(\xi_k) A(\xi_{k+1})}(\pi \tau^k \underline{\xi}) \tag{4}$$

is a multiple of the prime period $\lambda(\gamma)$ of this orbit.

Bowen ([3], Section 5) constructs a finite number of shifts τ_α with properties as above, and integers $l(\alpha)$ such that

$$\sum_\alpha (-1)^{l(\alpha)+1} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{\xi} \in K_{\alpha n}} e^{-s\lambda_\alpha(\underline{\xi})} = \sum_{\gamma \text{ periodic}} e^{-s\lambda(\gamma)}. \tag{5}$$

Here $K_{\alpha n}$ is the set of points $\underline{\xi}$ of prime period n for the shift τ_α, and $\lambda_\alpha(\underline{\xi})$ the expression (4) for this shift. The series in (5) converge for sufficiently large $\operatorname{Re} s$. We have thus (for large $\operatorname{Re} s$)

$$\zeta(s) = \prod_{\gamma \text{ periodic}} (1 - e^{-s\lambda(\gamma)})$$

$$= \exp\left[-\sum_{\gamma \text{ periodic}} \sum_{m=1}^{\infty} \frac{1}{m} e^{-ms\lambda(\gamma)}\right]$$

$$=\exp\left[-\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\alpha}(-1)^{l(\alpha)+1}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\xi\in K_{\alpha n}}e^{-ms\lambda_\alpha(\xi)}\right]$$

$$=\exp\sum_{\alpha}(-1)^{l(\alpha)}\sum_{r=1}^{\infty}\frac{1}{r}\sum_{\xi\in\mathrm{Fix}\,\tau_\alpha^r}\exp\left[-s\sum_{k=0}^{r-1}\lambda_{A(\xi_k)A(\xi_{k+1})}(\pi_\alpha\tau_\alpha^k\underline{\xi})\right]. \tag{6}$$

The set on which the shift τ_α acts is

$$\Lambda_\alpha=\{(\xi_m)_{m\in\mathbb{Z}}\in(S_\alpha)^{\mathbb{Z}}: t_{\alpha\xi_m\xi_{m+1}}=1 \text{ for all } m\}.$$

Given an integer $N\geqq 0$ there is a set $\Lambda_{\alpha,N}$ consisting of sequences $(\eta_m)_{m\in\mathbb{Z}}$ of elements of

$$S_{\alpha,N}=\{(\xi_n)_{-N\leqq n\leqq N}\in(S_\alpha)^{2N+1}: t_{\alpha\xi_n\xi_{n+1}}=1 \text{ for } -N\leqq n\leqq N-1\}$$

and a homeomorphism $h_{\alpha,N}$ of Λ_α onto $\Lambda_{\alpha,N}$ so that

$$(h_{\alpha,N}\underline{\xi})_m=(\xi_{m-N},\xi_{m-N+1},\dots,\xi_{m+N}).$$

The transition matrix $t_{\alpha,N}$ of $\Lambda_{\alpha,N}$ is easily constructed. We have $h_{\alpha,N}\circ\tau_\alpha=\tau_{\alpha,N}\circ h_{\alpha,N}$ where $\tau_{\alpha,N}$ is the shift on $\Lambda_{\alpha,N}$. For sufficiently large N, the sets

$$R_\zeta=\{\pi_\alpha\circ h_{\alpha,N}^{-1}\underline{\eta}: \underline{\eta}=(\eta_m)_{m\in\mathbb{Z}}\in\Lambda_{\alpha,N} \text{ and } \eta_0=\zeta\}$$

with $\zeta\in S_{\alpha,N}$ have arbitrarily small diameters. If $\zeta=(\xi_n)_{-N\leqq n\leqq N}$ we write $A(\zeta)=A(\xi_0)$, and

$$\lambda_\zeta(y)=\lambda_{A(\zeta_0)A(\xi_1)}(y).$$

Then (6) becomes

$$\zeta(s)=\exp\sum_{\alpha}(-1)^{l(\alpha)}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\underline{\eta}\in\mathrm{Fix}(\tau_{\alpha,N})^n}\exp\left[-s\sum_{k=0}^{n-1}\lambda_{\zeta_k}(\pi_\alpha\circ h_{\alpha,N}^{-1}\circ(\tau_{\alpha,N})^k\underline{\eta})\right]. \tag{7}$$

For each $\zeta\in S_{\alpha,N}$ choose $x_\zeta\in R_\zeta$; then $x_\zeta\in A(\zeta)$ and the intersection of $A(\zeta)$ with the (local) center-unstable manifold through x_ζ is a real-analytic manifold $C(\zeta)$.

If the (ζ_0,ζ_1) matrix element of $t_{\alpha,N}$ is equal to 1, a real-analytic map $f_{\zeta_0\zeta_1}$ of a neighborhood of x_{ζ_0} in $C(\zeta_0)$ to $C(\zeta_1)$ is defined by

$$y\mapsto f^t y\mapsto z$$

where $\varepsilon_1<t<\varepsilon_2$, and $f^t y\in A(\zeta_1)$, and $f^t y$, z belong to the same contracting fiber of $A(\zeta_1)$. We can assume that the metric on M is such that

$$\|(Tf^{-t})v\|\leqq e^{-\kappa t}\|v\| \qquad \text{for } v\in E^u,\ t\geqq\varepsilon_1.$$

Use now on $T_x C(\zeta)$ the metric obtained by projection along E_x^0 of the metric in E_x^u. It is then clear that for sufficiently large N, $Tf_{\zeta_0\zeta_1}^{-1}$ is a contraction.

From this contracting property and compactness arguments we find that, if N has been chosen sufficiently large, *for each* $\zeta\in S_{\alpha,N}$ *there is an open disk* D_ζ *in a complexification of* $C(\zeta)$ *such that*

(a) The projection of R_ζ on $C(\zeta)$ along contracting fibers is contained in D_ζ.

(b) λ_ζ is defined on $D_\zeta\cap C(\zeta)$ and extends to an analytic function $\tilde{\lambda}_\zeta$ on D_ζ.

(c) If the (ζ', ζ) matrix element of $t_{\alpha,N}$ is equal to 1, $f_{\zeta'\zeta}^{-1}$ is defined on $D_\zeta \cap C(\zeta)$ and extends to a holomorphic map $\psi_{\zeta'\zeta}: D_\zeta \mapsto D_{\zeta'}$ such that $\operatorname{clos}(\psi_{\zeta'\zeta} D_\zeta)$ is compact in $D_{\zeta'}$.

Theorem 1 applies thus in the present situation with $I = S_{\alpha,N}$, $t = t_{\alpha,N}$ and $\varphi = \exp(-s\tilde{\lambda}_\zeta)$ on D_ζ.

On the other hand

$$\sum_{\underline{\eta} \in \operatorname{Fix}(\tau_{\alpha,N})^n} \exp\left[-s \sum_{k=0}^{n-1} \lambda_{\zeta_k}(\pi_\alpha \circ h_{\alpha,N}^{-1} \circ (\tau_{\alpha,N})^k \underline{\eta})\right]$$

$$= \sum_{(\zeta_1, \dots, \zeta_n) \in J_n} \prod_{k=1}^{n} \varphi(\psi_{i_k i_{k+1}} \dots \psi_{i_{n-1} i_n} z_{(\zeta_1, \dots, \zeta_n)})$$

where we have used the fact that λ_ζ is constant on a contracting fiber. Comparing with (7) we see that Theorem 3 follows from Theorem 1.

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