$$G^{0L'}_{1J'll'1S'} = -\frac{2M}{\sqrt{4\pi}}\sqrt{\frac{2l'+1}{2L'+1}}L_{L'}({}^{2S'+1}l_{J'}{}'), \quad G^{1jj}_{1J'll'1S'} = 2M\sqrt{\frac{2l'+1}{2j+1}}M_j({}^{2S'+1}l_{J'}{}'), \quad F^{j1}_{1J'll'1S'} = 2M\sqrt{\frac{2l'+1}{2j+1}}E_j({}^{2S'+1}l_{J'}{}').$$

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ON MANIFOLDS OF PHASE COEXISTENCE*

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Using a theorem on convex functions due to Israel, it is shown that a point of coexistence of n + 1 phases cannot be isolated in the space of interactions, but lies on some "infinite dimensional manifold".

1. Introduction

A remarkably effective technique has recently been introduced by R. B. Israel [1] to prove the existence of phase transitions in lattice systems. As a corollary to Israel's work we show in this note that a point of coexistence of at least n + 1 phases cannot be isolated in the space of interactions, but lies in some infinite dimensional set. On physical grounds (Gibbs phase rule) one would expect that this set is a "manifold of codimension n". In particular for n = 2 one would think that it separates locally the space of potentials into two regions, at least for an appropriate choice of this space of potentials. We do not prove such strong results, but what we prove goes in the right direction.†

From a technical point of view we do little else than repeat Israel's arguments with some modifications. We shall present the results in the framework of classical statistical mechanics, which we describe in Sec. 2. Israel's technique is sketched in Sec. 3. Our new results are in Secs. 4 and 5.

* This article appeared in the Russian original in a translation by Yu. Sukhov. For the English edition, Professor Ruelle has kindly provided the original text. The Russian displayed equations and some in-text equations have been used.

† Physically related results, using totally different mathematical techniques, have been obtained by Pirogov and Sinai [2-4]; see also [5].

Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France. Published in Teoreticheskaya i Matematicheskaya Fizika, Vol. 30, No. 1, pp. 40-47, January, 1977. Original article submitted June 4, 1976.

The most striking result (Sec. 5) is about lattice gases with pair interactions. If a pair potential is prescribed at a finite number of sites, one can find an extension and a chemical potential for which there are two phases with different densities. If a pair potential φ_0 and a chemical potential μ_0 are given for which there are two phases with different densities, and if one prescribes a small modification of φ_0 at a finite number of sites one can, by a small modification at the remaining sites, and a small change of the chemical potential, obtain a pair interaction with again two phases with different densities.

2. Equilibrium States in Classical Lattice Statistical Mechanics*

Let F be a finite set, with the discrete topology. If $\xi=(\xi_x)\in F^{Z^\nu}$ and $a\in\mathbf{Z}^\nu$, we let the translations of $\mathbf{Z}^\nu$ act on F^{Z^ν} by $(\tau^a\xi)_x=\xi_{x+a}$. The space F^{Z^ν} with the product topology is compact. Let Ω be a nonempty closed subset of F^{Z^ν}, invariant under the τ^a. The τ^a (restricted to Ω) are homeomorphisms. For finite $\Lambda\subset\mathbf{Z}^\nu$, let Ω_Λ be the image in F^Λ of the projection of Ω. We call *interaction* a real function Φ on the union of the Ω_Λ such that Φ vanishes on $\Omega_\varnothing$, Φ is invariant under translations of $\mathbf{Z}^\nu$ and

$$|\Phi|=\sum_{X\ni 0}\frac{1}{\operatorname{card} X}\sup_{\xi_X\in\Omega_X}|\Phi(\xi_X)|<\infty.$$

The interactions form a Banach space $\mathfrak{A}$ with norm $|\cdot|$.

Let $\mathscr{C}$ be the space of real continuous functions on Ω. It is a Banach space with the uniform norm $\|\cdot\|$. The dual $\mathscr{C}^*$ of $\mathscr{C}$ consists of the measures on Ω; we let I in $\mathscr{C}^*$ be the set of (translation) invariant probability measures, i.e., of measures σ such that $\sigma(A)=\sigma(A\circ\tau^x)$ for all $A\in\mathscr{C}$ and $x\in\mathbf{Z}^\nu$. The set I is convex, and compact with respect to the w^* topology on $\mathscr{C}$ (vague topology).

If $\Phi\in\mathfrak{A}$, we define $A_\Phi\in\mathscr{C}$ by

$$A_\Phi(\xi)=\sum_X{}^*\Phi(\xi\upharpoonright X),$$

where the sum extends over those X such that 0 is the "middle" element of X [ordering $\mathbf{Z}^\nu$ lexicographically, 0 is the element at the position $\operatorname{ent}[\frac{1}{2}(\operatorname{card} X+1)]$ in X]. Other definitions are possible, giving the same value for $\sigma(A_\Phi)$ when $\sigma\in I$. The definition chosen here has the virtue that the image of $\mathfrak{A}$ by $\Phi\to A_\Phi$ is $\mathscr{C}$, and that $\|A\|=\inf\{|\Phi|: A=A_\Phi\}$. In particular, if A depends only on the restriction of its argument to some finite $\Delta\subset\mathbf{Z}^\nu$, we write $A\in\mathscr{C}_\Delta$, and there exists Φ such that $A=A_\Phi$, $\Phi(\xi\upharpoonright X)=0$ unless X is a translate of some finite Λ, and $|\Phi|=\|A\|$. It will be a matter of convenience whether one prefers to use interactions in $\mathfrak{A}$, or continuous functions in $\mathscr{C}$.

The *pressure* P^Φ of $\Phi\in\mathfrak{A}$, or the pressure $P(A)$ of $A\in\mathscr{C}$, is defined in a standard way, and $P^\Phi=P(A_\Phi)$. The function $P:\mathscr{C}\to\mathbf{R}$ is convex continuous. The (mean) entropy of $\sigma\in I$ is denoted by $s(\sigma)$. The function $s: I\to\mathbf{R}$ is affine upper semi-continuous ≥ 0. One has a *variational principle*

$$P(A)=\max_{\sigma\in I}(s(\sigma)+\sigma(A)), \tag{1}$$

and conversely

$$s(\sigma)=\inf_{A\in\mathscr{C}}(P(A)-\sigma(A)).$$

The set I_A of those $\sigma\in I$ which make the right-hand side of (1) maximum, is the set of *equilibrium states* for A.

Let V be a Banach space, V^* is dual and $f: V\to\mathbf{R}$ a continuous convex function. We say that $\sigma\in V^*$ is *f-bounded* if there exists $c\in\mathbf{R}$ such that $\sigma\leq f+c$. We say that $\sigma\in V^*$ is *tangent* to f at $x\in V$ if $f(x+y)\geq f(x)+\sigma(y)$ for all $y\in V$.

The P-bounded elements of $\mathscr{C}^*$ are precisely the invariant probability measures $\sigma\in I$. The elements $\sigma\in\mathscr{C}^*$ tangent to P at A constitute the set I_A of equilibrium states for A. If f is the function $\Phi\to P^\Phi$ on $\mathfrak{A}$, the f-bounded elements of $\mathfrak{A}^*$ are of the form $\Psi\to\sigma(A_\Psi)$ with $\sigma\in I$ and the elements of $\mathfrak{A}$ tangent to f at Φ are of the form $\Psi\to\sigma(A_\Psi)$ with $\sigma\in I_{A_\Phi}$.

The convex compact set I is a Choquet simplex. This means that every $\rho\in I$ is the barycenter of a

* The theory sketched here generalizes the well-known results for lattice gases (see [6]).

unique measure m_ρ on I, carried by the extremal points of I. The extremal points of I are called ergodic states, and m_ρ gives the ergodic decomposition of ρ. Given $A \in \mathscr{E}$, let $\hat{A}: I \to \mathbf{R}$ be defined by $\hat{A}(\sigma) = \sigma(A)$; the measure m_ρ is then determined by

$$m_\rho\Big(\sum_{i=1}^{l} \hat{A}_i\Big) = \lim_{\Lambda \uparrow \infty} \rho\Big[\prod_{i=1}^{l}\Big(\frac{1}{\operatorname{card}\Lambda}\sum_{x\in\Lambda} A_i \circ \tau^x\Big)\Big], \tag{2}$$

where $\Lambda \nearrow \infty$ means "limit in the sense of Van Hove", for instance Λ is a cube with side going to infinity.

The set I_A of equilibrium states for A is convex, compact, and a Choquet simplex. Its extremal points are ergodic states, which implies that the unique decomposition of $\rho \in I_A$ into extremal points of I_A is the same as the ergodic decomposition of ρ given by m_ρ.

3. Israel's Theory

Israel's technique is to approximate invariant states by equilibrium states, using the following general results on convex functions.

THEOREM 1. Let V be a Banach space, $f: V \to \mathbf{R}$ be convex continuous, and C be a closed convex cone with apex 0 in V. If $\sigma_0 \in V^*$ is f-bounded, $x_0 \in V$ and $\varepsilon > 0$, there is $\sigma_0 \in V^*$ tangent to f at x with $x \in x_0 + C$,

$$\|x-x_0\| \leqslant \frac{1}{\varepsilon}[f(x_0)-\sigma_0(x_0)-s(\sigma_0)],$$

and $\sigma(y) \geqslant \sigma_0(y) - \varepsilon\|y\|$ $(\forall y \in C)$, where we have written $s(\sigma_0) = \inf\{f(y)-\sigma_0(y):\ y\in V\}$.

For applications to classical lattice statistical mechanics, we take $V = \mathfrak{A}$ and $f: \Phi \to P^\Phi$. One then obtains the following lemma and theorem.

LEMMA. Let $A_1, A_2 \in \mathscr{E}$ and $S \subset \mathbf{Z}^\nu$. We define a convex cone

$$\mathscr{L}_S = \Big\{a_1A_1 + a_2A_2 + \frac{1}{2}\sum_{x\in\mathbf{Z}^\nu} b_x A_1(A_2\circ\tau^x) + b_{-x}(A_1\circ\tau^x)A_2 : a_1, a_2, b_x \in \mathbf{R},\ b_x \geqslant 0,\ b_x = 0, \text{ if } x \notin S \text{ and } \sum_{x\in S} b_x < \infty\Big\}.$$

We assume that $A_1, A_2 \in \mathscr{E}_\Delta$ for some finite $\Delta \subset \mathbf{Z}^\nu$. Given $\sigma_0 \in I$, $B \in \mathscr{E}$ and $\varepsilon > 0$ there exist $B \in B_0 + \mathscr{L}_S$ and $\sigma \in I_B$ such that

$$\|B - B_0\| \leqslant \frac{1}{\varepsilon}[P(B_0) - \sigma_0(B_0) - s(\sigma_0)] \tag{3}$$

and

$$\sigma(A_1(A_2\circ\tau^x)) - \sigma(A_1)\sigma(A_2) \geqslant \sigma_0(A_1(A_2\circ\tau^x)) - \sigma_0(A_1)\sigma_0(A_2) - 3\varepsilon\|A_1\|\cdot\|A_2\|$$

for all $x \in S$.

THEOREM 2. Let $A \in \mathscr{E}_\Delta$ for some finite $\Delta \subset \mathbf{Z}^\nu$, and define a convex cone

$$\mathscr{L} = \Big\{aA + \sum_{x\in\mathbf{Z}^\nu} b_x A(A\circ\tau^x) : a, b_x \in \mathbf{R},\ b_x \geqslant 0, \sum b_x < \infty\Big\}.$$

A. Let $\sigma_0', \sigma_0'' \in I$ be such that $\sigma_0'(A) \neq \sigma_0''(A)$. Given $C \in \mathscr{E}$ there exist $B \in C + \mathscr{L}$ and two equilibrium states $\sigma', \sigma'' \in I_B$ such that $\sigma'(A) \neq \sigma''(A)$.

B. Let σ_0', σ_0'' be equilibrium states for $C \in \mathscr{E}$ such that $\sigma_0'(A) \neq \sigma_0''(A)$. Given $\varepsilon > 0$, one can choose $\delta > 0$ such that if $C' \in \mathscr{E}$ and $\|C' - C\| < \delta$, there exist $B \in C' + \mathscr{L}$ with $\|B - C'\| < \varepsilon$, and two equilibrium states $\sigma', \sigma'' \in I_B$ with $\sigma'(A) \neq \sigma''(A)$.

We indicate, for later use, how Theorem 2 is obtained from the lemma.

Write $\sigma_0 = \frac{1}{2}(\sigma_0' + \sigma_0'')$. The assumption $\sigma_0'(A) \neq \sigma_0''(A)$ implies that

$$m_{\sigma_0}(\hat{A}^2) = \lim_{\Lambda\nearrow\infty} \sigma_0\Big[\Big(|\Lambda|^{-1}\sum_{x\in\Lambda} A\circ\tau^x\Big)^2\Big] > \sigma_0(A)^2.$$

Choose $\varepsilon > 0$ such that

$$\lim_{\Lambda\nearrow\infty} \sigma_0\Big[\Big(|\Lambda|^{-1}\sum_{x\in\Lambda} A\circ\tau^x\Big)^2\Big] \geqslant \sigma_0(A)^2 + 4\varepsilon\|A\|^2.$$

We apply the lemma with $A_1 = A_2 = A$ and $S=\mathbf{Z}^\nu$ (B_0 will be chosen later), obtaining $B\in B_0+\mathscr{L}$ and $\sigma \in I_B$ such that

$$\|B-B_0\| \leqslant \frac{1}{\varepsilon}[P(B_0)-\sigma_0(B_0)-s(\sigma_0)]$$

and

$$\sigma\left[\left(|\Lambda|^{-1}\sum_{x\in\Lambda} A\cdot\tau^x\right)^2\right]-\sigma(A)^2 \leqslant \sigma_0\left[\left(|\Lambda|^{-1}\sum_{x\in\Lambda} A\cdot\tau^x\right)^2\right]-\sigma_0(A)^2-3\varepsilon\|A\|^2.$$

Therefore

$$m_\sigma(\hat{A}^2) = \lim_{\Lambda\nearrow\infty} \sigma\left[\left(|\Lambda|^{-1}\sum_{x\in\Lambda} A\circ\tau^x\right)^2\right]-\sigma(A)^2 \geqslant \varepsilon\|A\|^2.$$

From this follows that there exist σ' and σ'' in the support of m_σ with $\sigma'(A)\neq\sigma''(A)$. Taking $B_0 = C$, we obtain Assertion A.

Suppose now that $\sigma_0', \sigma_0''\in I_C$. Choose $\delta > 0$ such that if $C'\in\mathscr{E}$ and $\|C'-C\|<\delta$ we have $P(C')-\sigma_0(C')-s(\sigma_0)<\varepsilon^2$. Taking $B_0 = C'$, we have $B\in C'+\mathscr{L}$, and (3) gives $\|B-C'\|<\varepsilon$, proving Assertion B.

4. Coexistence of Phases

In the above lemma and theorem we could restrict our attention to interactions Φ such that $\Phi(\xi\restriction_X)=0$ when card $X > 2$ card Δ (or to corresponding elements of $\mathscr{E}$). Theorem 2 deals with the situation when there are at least two different equilibrium states. This corresponds physically to the coexistence of at least two phases. Part B of Theorem 2 shows that an interaction Ψ_0 (or a function C) for which several phases coexist cannot be isolated: it lies in an "infinite dimensional manifold" of such interactions. One should check that these interactions are not all "physically equivalent" [Φ, Ψ are physically equivalent if there exists $c \in \mathbf{R}$ such that $\sigma(A_\Phi)=\sigma(A_\Psi)+c$ for all $\sigma \in I$], and that the "manifold" is not dense. This will be done in the special case treated in the next section.

The coexistence of at least n + 1 phases can be treated in a similar manner. Let $A_1,\ldots,A_n\in\mathscr{E}_\Delta$, and let $A=\sum a_iA_i$, with $\sum a_i^2=1$. We assume that $\sigma_0^{(0)},\sigma_0^{(1)},\ldots,\sigma_0^{(n)}\in I$ are such that $\sigma_0^{(0)}(A)=\sigma_0^{(1)}(A)=\ldots=\sigma_0^{(n)}(A)$ holds for no choice of $a_1, \ldots, a_n$. Defining

$$\sigma_0 = \frac{1}{n+1}\sum_{i=0}^{n}\sigma_0^{(i)},$$

we have $m_{\sigma_0}(\hat{A}^2)-\sigma_0(A)^2\geqslant 4\varepsilon\|A\|^2$ with some $\varepsilon > 0$ independent of $a_1, \ldots, a_n$. Let $\mathscr{L}$ be the linear space generated by the A_i and $A_i(A_j\circ\tau^x)$, and let $B_0\in\mathscr{E}$. By an easy extension of the lemma, there exists $B\in B_0+\mathscr{L}$ such that

$$\|B-B_0\| \leqslant \frac{1}{\varepsilon}[P(B_0)-\sigma_0(B_0)-s(\sigma_0)],$$

and $\sigma \in I_B$ such that

$$|\sigma(A\cdot(A\circ\tau^x))-\sigma(A)^2-[\sigma_0(A(A\circ\tau^x))-\sigma_0(A)^2]|\leqslant 3\varepsilon\|A\|^2$$

for all $a_1, \ldots, a_n$ and all $x \in \mathbf{Z}^\nu$. Therefore $m_\sigma(\hat{A}^2)-\sigma(A)^2\geqslant\varepsilon\|A\|^2$, proving that the dimension of I_B is at least n: at least n + 1 phases coexist. Again an interaction for which at least n + 1 phases coexist cannot be isolated.

5. Lattice Gases with Pair Interactions

We consider a system with $F = \{0, 1\}$ and $\Omega=\{0, 1\}^{\mathbf{Z}^\nu}$. We define $A\in\mathscr{E}_{\{0\}}$ by $A(\xi) = \xi_0$ (A takes therefore the values 0 and 1). We shall use "pair" interactions Φ, such that $\Phi(\xi\restriction_X)=0$ if $|X| > 2$ and $\Phi(\xi\restriction_{\{0\}})=-\mu A(\xi)$, $\Phi(\xi\restriction_{\{0,x\}})=\varphi(x)A(\xi)A(\tau^x\xi)$ for $x \neq 0$. Here $\mu \in \mathbf{R}$ and $\varphi(x)=\varphi(-x)\in\mathbf{R}$ is defined for $x \neq 0$. Notice that

$$|\Phi| = \mu+\frac{1}{2}\sum_{x\neq 0}|\varphi(x)|.$$

I. Let $0\in M\subset\mathbf{Z}^\nu$, where M is finite and $M = -M$. Suppose that a function $\bar{\varphi}$: $M\setminus\{0\}\to\mathbf{R}$ is given such that $\bar{\varphi}(x)=\bar{\varphi}(-x)$. Then one can extend $\bar{\varphi}$ to φ: $\mathbf{Z}^\nu\setminus\{0\}\to\mathbf{R}$ such that

$$\sum_{x\neq 0} |\varphi(x)| < \infty,$$

and find μ such that there are two equilibrium states σ' and σ'' for Φ satisfying $\sigma'(A) = \sigma''(A)$.

II. Let μ_0 and φ_0 correspond to a pair interaction Φ_0. We assume that σ_0' and σ_0'' are equilibrium states for Φ_0, and that $\sigma_0'(A)\neq\sigma_0''(A)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $0\in M\subset \mathbf{Z}^\nu$, where M is finite, M = –M. Suppose that $\tilde{\varphi}: M\setminus\{0\}\to\mathbf{R}$ satisfies $\tilde{\varphi}(x)=\tilde{\varphi}(-x)$ and

$$\frac{1}{2}\sum_{x\in M\setminus\{0\}} |\tilde{\varphi}(x)-\varphi_0(x)| < \delta.$$

One can then extend $\tilde{\varphi}$ to $\varphi: \mathbf{Z}^\nu\setminus\{0\}\to\mathbf{R}$ and find μ such that $\varphi(x)=\varphi(-x)$, $\varphi(x)\leqslant\varphi_0(x)$ if $x\notin M$,

$$|\mu-\mu_0| + \frac{1}{2}\sum_{x\notin M} |\varphi(x)-\varphi_0(x)| < \varepsilon,$$

and there are two equilibrium states σ' and σ'' for the interaction Φ corresponding to μ and φ satisfying $\sigma'(A)\neq\sigma''(A)$.

To prove I and II it suffices to imitate the proof of Theorem 2, using the lemma with $S=\mathbf{Z}^\nu\setminus M$. Notice that

$$m_\sigma(\hat{A})^2 = \lim_{\Lambda\nearrow\infty} \rho\left[|\Lambda|^{-2} \sum_{\substack{x,\,y\in\Lambda,\\ x-y\notin M}} (A\circ\tau^x)(A\circ\tau^y)\right].$$

6. Discussion

The Assertion I above shows that if a pair interaction Φ_0 has two equilibrium states with different densities, then close to Φ_0 there is an infinite dimensional set of pair interactions Φ which have two equilibrium states with different densities.

As pointed out in Sec. 4, we should check that these interactions Φ are not physically equivalent. This follows from [7]: pair interactions Φ_0 and Φ are physically equivalent if and only if $\mu_0 = \mu$ and $\varphi_0 = \varphi$.

We should also check that the pair interactions which have two equilibrium states with different densities do not form a dense set. This results from the convergence of low activity expansions (see, for instance, [6], Sec. 4.2.6).

7. A Heuristic Theory of Phase Transitions

A theory of phase transitions would assert that if exactly n + 1 phases coexist for the interaction Φ_0, then there passes through Φ_0 a manifold of codimension n of coexistence of n + 1 phases (in an appropriate space V of interactions). The n + 1 phases at Φ_0 correspond to elements $\alpha_0, \alpha_1, \ldots, \alpha_n$ of V*, which are all equal to a linear functional w on a subspace X of codimension n of V. The restriction $P\restriction_{(\Phi_0+X)}$ of the pressure to Φ_0 + X has a unique tangent at Φ_0 (Hahn–Banach theorem).

Let Y be a subspace of dimension n of V transversal to X, and $\beta_0, \beta_1, \ldots, \beta_n\in Y^*$ be the restrictions Y of $\alpha_0, \alpha_1, \ldots, \alpha_n$. Following the ideas of Sec. 4 one can show that there exist $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_n\in Y^*$ arbitrarily close to $\beta_0, \beta_1, \ldots, \beta_n$, and $\delta > 0$ and $\varkappa > 0$ such that the following is true:

<u>Assertion.</u> For each $\Phi \in$ X there exists $\psi(\Phi) \in$ V such that

$$\|\psi(\Phi)\|\leqslant\varkappa[P(\Phi_0+\Phi)-P(\Phi_0)-w(\Phi)], \tag{4}$$

and n + 1 phases coexist for the interaction $\Phi_0 + \Phi + \psi(\Phi)$; more precisely, if $\|\Phi\| < \delta$

$$P\restriction_{(\Phi+\psi(\Phi)+Y)}\geqslant P(\Phi+\psi(\Phi))+\bar{\beta}_i, \qquad i=0, 1, \ldots, n. \tag{5}$$

For each $\Phi \in$ X we have

$$\lim_{\lambda\to 0}\frac{1}{\lambda}[P(\Phi_0+\lambda\Phi)-P(\Phi_0)-w(\lambda\Phi)]=0, \tag{6}$$

because $P\restriction_{(\Phi_0+X)}$ has a unique tangent at Φ_0. It is therefore tempting to assume that

$$\frac{P(\Phi_0+\Phi)-P(\Phi_0)-w(\Phi)}{\|\Phi\|}\to 0,\ \text{when}\ \Phi\in X,\ \|\Phi\|\to 0.$$

If that is the case one can choose δ and the above function ψ such that $\psi(\Phi)\in Y$ if $\|\Phi\|<\delta$. Furthermore ψ is then unique such that (4), (5) hold, and continuous (this is a sort of implicit function theorem). The image $\psi(\{\Phi\in X:\|\Phi\|<\delta\})$ is the desired manifold of phase coexistence. One can show that it is tangent to X, and intersection of manifolds of coexistence of less than n + 1 phases in the expected simplicial configuration. The details will be given elsewhere.

Unfortunately, (6) cannot be true in general. In fact for a one-dimensional lattice gas, if two phases with different densities coexist for a pair potential φ_0, there are finite-range pair potentials arbitrarily close to φ_0, and for those there is no phase transition. It is not clear at this point if (6) will hold in cases of some generality, or if the above discussion has only heuristic value.

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SOLUTIONS OF THE BBGKY HIERARCHY. CLASSICAL STATISTICS

A.K. Vidybida

The BBGKY hierarchy of classical kinetic equations is regarded as a single abstract evolution equation in the space of sequences of functions that are integrable with respect to the momenta and translationally invariant with respect to the coordinates. An expression is obtained for solving the equations in the form of a number of nonlinear operators applied to the initial data.

The BBGKY hierarchy of kinetic equations [1] describes the dynamics of infinite statistical systems and is a chain of coupled integrodifferential equations for the distribution functions. Earlier, in [2], an expression has been obtained for the solution of the Cauchy problem for the BBGKY hierarchy in the Banach space of sequences of functions that are integrable with respect to all arguments. A shortcoming of the expression is that its application to distribution functions that describe a real system, i.e., not more than bounded with respect to the configuration coordinates, leads to volume divergences in each order in $1/v$. The aim of the present paper is to obtain an expression free of this shortcoming.

In Sec. 1, we introduce the space b of sequences of functions that are translationally invariant with respect to the coordinates [4] and integrable with respect to the momenta, and auxiliary constructions are performed in it. In Sec. 2, we derive an expression for the solution of the Cauchy problem for the case when the initial condition lies in b. In Secs. 3 and 4, this expression is transformed to an "pseudononlinear" form (see Eq. (13)). This means that the evolution operator, which is linear, is represented in a form in which each of its orders in $1/v$ is a nonlinear operator. In Sec. 4 arguments in favor of such a representation are adduced.

Institute of Theoretical Physics, Academy of Sciences of the Ukrainian SSR. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 30, No. 1, pp. 48-56, January, 1977. Original article submitted March 2, 1976.