SENSITIVE DEPENDENCE ON INITIAL CONDITION AND TURBULENT BEHAVIOR

OF DYNAMICAL SYSTEMS*

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Abstract. The asymptotic behavior of differentiable dynamical systems is ana-lyzed. We discuss its description by asymptotic measures and the "turbulent behavior" associated with sensitive dependence on initial condition.

1. Generalities

The purpose of this talk is to discuss some qualitative features of the time evolution of natural systems. The time evolution is described by an equation

$$x_{t+1} = f(x_t) \qquad \text{(discrete time)} \qquad (1)$$

or

$$\frac{dx_t}{dt} = X(x_t) \qquad \text{(continuous time)} \qquad (2)$$

The qualitative features that we want to discuss are those associated with sensitive dependence on initial condition. We shall try to analyse sensitive dependence on initial condition, see how it manifests itself as turbulent be-havior, and find the simplest examples in which it occurs.

In applications x_t represents the state (at time t) of the natural system under consideration, and may vary in some infinite dimensional space (as for instance in hydrodynamics). For the convenience of the mathematical dis-

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cussion we shall however suppose that the state space M of our system is finite dimensional.

The time evolution is defined by maps $f^t : M \mapsto M$ (t discrete or continuous). We shall assume that either M is $\mathbb{R}^m$ and there is some bounded open $U \subset \mathbb{R}^m$ such that closure $f^t U \subset U$ for $t > 0$, or that M is a compact differentiable manifold[*]. We assume that the map f in (1) or the vector field X in (2) are C^r , i.e. r times continuously differentiable with $r \geq 1$.

In what follows I shall try to be mathematically correct, or at least not misleading. However it has to be realized that many of the questions to be discussed are poorly understood mathematically. I shall definitely not limit myself to those topics which are completely elucidated. I shall try to give an idea of what lies beyond, at the cost of some conjectures and heuristic considerations.

2. Sensitive dependence on initial conditions.

It will often be convenient to distinguish the following three cases

I. <u>Maps</u> : discrete time, f not necessarily invertible.

II. <u>Diffeomorphisms</u> : discrete time, f has a differentiable inverse, so that f^t is defined for $t = -1, -2,...$

III. <u>Flows</u> : continuous time.

Sensitive dependence on initial condition means that if there is a small change δx_o in the initial condition x_o , the corresponding change $\delta x_t = f^t(x_o + \delta x_o) - f^t(x_o)$ of $x_t = f^t(x_o)$ grows and becomes large when t becomes large. More precisely we require δx_t to <u>grow exponentially with</u> t . Why this requirement is reasonable will appear in the next section.

The difference $f^t(x_o + \delta x_o) - f^t(x_o) = \delta x_t$ is meaningful only if our manifold M is $\mathbb{R}^m$. In general, it is preferable to interpret δx_t as a vec-

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[*] A compact differentiable manifold may always be thought of as imbedded in $\mathbb{R}^N$ for suitably large N . The "tangent spaces" to M may then be identified with subspaces of $\mathbb{R}^N$ as intuition dictates.

tor tangent to the manifold; we have then

$$\delta x_t = (T_{x_o} f^t)\, \delta x_o$$

where $T_{x_o} f^t$ is a linear operator mapping the tangent space $T_{x_o} M$ to M at x_o to the tangent space $T_{x_t} M$ at x_t . If M is $\mathbb{R}^m$, then the tangent spaces can be identified with $\mathbb{R}^m$ and $T_x f^t$ is just the $m \times m$ matrix of partial derivatives of $f^t(x)$ with respect to x . The discussion of sensitive dependence on initial condition translates thus into the study of $T_x f^t$ for large t . We shall not try to make statements valid for all $x \in M$ (all initial conditions), but rather for almost all x with respect to some probability measure ρ (on M) invariant under f^t (i.e. invariant under time evolution). We shall try to argue later why an ensemble average (with respect to some ρ) corresponds to time average for "most" initial conditions. For the moment we accept as a fact of life the fact that x is distributed according to the f^t-invariant probability measure ρ . Then <u>non commutative ergodic theorem</u> of Oseledec describes then the behavior of $T_x f^t$ for large t .

3. The non commutative ergodic theorem.

We first give the version of the theorem which is appropriate for the study of maps (case I of Section 2).

<u>Theorem 1.</u> <u>Let</u> (M, Σ, ρ) <u>be a probability space and</u> $\tau : M \to M$ <u>a measurable map preserving</u> ρ . <u>Let also</u> $T : M \to M_n(\mathbb{R})$ <u>be a measurable map into the</u> $m \times m$ <u>matrices, such that</u> [*]

$$\log^+ \|T(\cdot)\| \in L^1(M, \rho)$$

<u>and write</u> $T_x^n = T(\tau^{n-1}x) \ldots T(\tau x) T(x)$.

<u>There is</u> $\Omega \subset M$ <u>such that</u> $\rho(\Omega) = 1$ <u>and for all</u> $x \in \Omega$

$$\lim_{n \to \infty} (T_x^{n*} T_x^n)^{\frac{1}{2}n} = \Lambda_x$$

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[*] We write $\log^+ x = \max\{0, \log x\}$.

exists [* denotes matrix transposition].

Let $\exp \lambda_x^{(1)} < \cdots < \exp \lambda_x^{(s(x))}$ be the eigenvalues of Λ_x [with possibly $\lambda_x^{(1)} = -\infty$], and $U_x^{(1)}, \ldots, U_x^{(s(x))}$ the corresponding eigenspaces. If $V_x^{(r)} = U_x^{(1)} + \cdots + U_x^{(r)}$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log \|T_x^n u\| = \lambda_x^{(r)} \quad \text{when} \quad u \in V_x^{(r)} \backslash V_x^{(r-1)}$$

for $r = 1, \ldots, s(x)$.

The theorem published by Oseledec [6] assumes τ and T invertible. Its proof has been simplified by Raghunathan [8]. The above result can be obtained by modifying Raghunathan's argument.

Let $m_x^{(r)} = \dim U_x^{(r)} = \dim V_x^{(r)} - \dim V_x^{(r-1)}$. The numbers $\lambda_x^{(1)}, \ldots, \lambda_x^{(s(x))}$, with multiplicities $m_x^{(1)}, \ldots, m_x^{(s(x))}$ constitute the spectrum of (ρ, τ, T) at x . The $\lambda_x^{(r)}$ are also called characteristic exponents. When n tends to ∞ , $\frac{1}{n} \log \|T_x^n\|$ tends to the maximum characteristic exponent $\lambda_x^{(s(x))}$. The spectrum is τ-invariant; if ρ is τ-ergodic the spectrum is almost everywhere constant.

To apply theorem 1 to differentiable maps of M it suffices, if $M = \mathbb{R}^m$, to take $\tau = f$, $T(x) = T_x f$. If M is a compact manifold, it may be necessary to cut it into a finite number of measurable pieces, each of which is diffeomorphic[*] to a subset of $\mathbb{R}^m$, so that $T_x M$ is identified to $\mathbb{R}^m$ for all x . We see thus that the asymptotic behavior of $\delta x_t = (T_x f^t) \delta x$ is exponential with t for large t . We have sensitive dependence on initial condition if the maximum characteristic exponent is strictly positive. Of course, other characteristic exponents are allowed to be negative.

If f is a diffeomorphism (case II of Section 2), the following version of the non commutative ergodic theorem gives extra information.

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[*] We require thus the existence of a differentiable map with differentiable inverse, from an open neighborhood of the closure of the piece of M considered, to an open set in $\mathbb{R}^m$.

Theorem 2. Keeping the notation and assumptions of theorem 1, let τ have a measurable inverse, let T^{-1} exist such that

$$\log^+ \|T(\cdot)^{-1}\| \in L^1(M, \rho)$$

and write $T_x^{-n} = T(\tau^{-n}x)^{-1} \ldots T(\tau^{-2}x)^{-1} T(\tau^{-1}x)^{-1}$

We can assume that for $x \in \Omega$ there is a splitting $\mathbb{R}^m = W_x^{(1)} \oplus W_x^{(2)} \oplus \ldots$ such that the following limits exists

$$\lim_{k \to \pm \infty} \frac{1}{k} \log \|T_x^k n\| = \lambda_x^{(r)} \quad \text{if} \quad u \in W_x^{(r)} \setminus \{0\}$$

Obviously the $\lambda_x^{(r)}$ are the characteristic exponents, and the $m_x^{(r)} = \dim W_x^{(r)}$ their multiplicities.

The splitting $\mathbb{R}^m = W_x^{(1)} \oplus W_x^{(2)} \oplus \ldots$ depends measurably on x. If M is a manifold and $\tau = f$ a diffeomorphism, the splitting is in general not continuous. One may assume that Ω is a Borel set of measure 1 with respect to every f-invariant measure, and that the dependence of the splitting and the spectrum on $x \in \Omega$ is Borel.

Suppose that one can take for Ω a closed f-invariant set, that there is $\varepsilon > 0$ such that the spectrum is disjoint from the interval $(-\varepsilon, +\varepsilon)$, and that the spaces

$$W_x^+ = \sum_{r: \lambda_x^{(r)} > 0} W_x^{(r)} \quad , \quad W_x^- = \sum_{r: \lambda_x^{(r)} < 0} W_x^{(r)}$$

depend continuously on $x \in \Omega$. One says then that Ω is hyperbolic. This is the main ingredient in the Axiom A of Smale. There exists a detailed theory of Axiom A diffeomorphisms, to which we shall refer to test various ideas. By contrast the ergodic theory of non-axiom A diffeomorphisms is in a state close to non-existence.

We shall make no special discussion of the non commutative ergodic theorem for flows (case III). The results are those expected.

4. Asymptotic measures.

The use of the non commutative ergodic theorem assumes that x is distributed according to some f-invariant measure ρ (for simplicity we discuss the discrete time case). We have to examine this assumption, and also try to restrict the choice of ρ , the invariant measures being often quite numerous [*].

A natural idea is to define ρ as a time average

$$\rho = \lim_{n\to\infty} \frac{1}{n} \sum_{k=o}^{n-1} \delta_{f^k x} \qquad (3)$$

where δ_x is the Dirac measure at x and the limit is in the sense that the integrals of continuous functions converge (vague limit). This procedure may not work : one can find a diffeomorphism f and an open set (non empty) of x such that (3) fails to exist (R. Bowen, private communication). Nevertheless one can hope that the limit exists in many cases. For C^2 axiom A diffeomorphisms [**] one can show that there is a set M' such that M\M' has zero Lebesgue measure [***] and the limit (3) exists for x ∈ M , taking a finite number of values. The asymptotic measures determined in this manner for Axiom A diffeomorphisms are precisely those ergodic measures which make maximum the expression

$$p(\rho) = h(\rho) - \int \rho(dx) \sum_{r:\lambda_x^{(r)} > 0} m_x^{(r)} \lambda_x^{(r)} \qquad (4)$$

the maximum being in fact zero. In (4), h(ρ) is the entropy (or Kolmogorov - Sinai invariant) of ρ with respect to f .

One can verify [11] that for every differentiable map f of a com-

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*) An axiom A diffeomorphism may have either only finitely many ergodic measures (carried by periodic orbits) or continuously many ergodic measures.

**) There is a similar result for axiom A flows. See [14], [9], [2].

***) By "Lebesgue measure" on a compact manifold M we mean the measure associated with any Riemann metric (any two such measures are equivalent).

pact manifold M into itself, and every f-invariant measure ρ , $p(\rho) \le 0$,
where $p(\rho)$ is defined by (4). This suggests that one should look for the
limits (3) among the measures satisfying (4) <u>provided one discards a set of</u>
<u>x with zero Lebesgue measure</u>. Here is a heuristic argument in favor of that
idea.

Let ν be the Lebesgue measure on M , normalized to 1. Suppose the
measures

$$\frac{1}{n} \sum_{k=0}^{n-1} f^k \nu$$

tend to a limit ρ when $n \to \infty$, then one can expect that the $\frac{1}{n} \sum_{0}^{n-1} \delta_{f^k x}$
tends for ν - almost all x to ρ or to one of its ergodic components
$\left[\text{If}\ \frac{1}{n} \sum_{0}^{n-1} \delta_{f^k x} \right.$ tends to a limit ρ_x for ν-almost all x , then
$\rho = \int \nu(dx)\ \rho_x \left.\right]$. We assume that for ρ-almost all $\xi \in M$, there is an
"unstable manifold" $\mathcal{V}_{\xi}^{-}$ tangent to $W_{\xi}^{-} = \sum\limits_{r:\lambda_{\xi}^{(r)} < 0} W_{\xi}^{(r)}$, such that the family
of the $\mathcal{V}_{\xi}^{-}$ is invariant under f . $\left[\text{Such a result is known in} \right.$
certains cases, see Pesin $\lceil 7 \rceil$ $\left.\right]$. The manifolds $\mathcal{V}_{\xi}^{-}$ are expanded by f^n ex-
ponentially fast for large n . Due to this stretching along the manifolds
$\mathcal{V}_{\xi}^{-}$, the measures $f^n \nu$ for large n will tend to remain smooth in the di-
rection parallel to the V_{ξ}^{-} (while their density may acquire a large transverse
derivative). Therefore we expect ρ to have conditional measures along the
$\mathcal{V}_{\xi}^{-}$ which have continuous density with respect to Lebesgue measure on the $\mathcal{V}_{\xi}^{-}$.
If that is the case, one can estimate the entropy of ρ to be at least the
integral of the logarithm of the expansion coefficient along the unstable mani-
folds, i.e.

$$h(\rho) \ge \int \rho(dx) \sum_{r:\lambda_x^{(r)} > 0} m_x^{(r)} \lambda_x^{(r)}$$

Since the reverse inequality also holds, we have $p(\rho) = 0$.

To the above heuristic argument there should correspond a theorem.
Its precise formulation and conditions of applicability are not yet known but
should be more general than the rather restricted axiom A class. The important
question of the stability of the asymptotic measures under small stochastic
perturbations is discussed below in the appendix.

We can try to make use of the equality

$$h(\rho) = \sum_{r:\lambda_x^{(r)} > 0} m_x^{(r)} \lambda_x^{(r)} \tag{5}$$

for asymptotic measures ρ (we assume ρ ergodic and the right-hand side is given its ρ-almost everywhere constant value). In particular, the right-hand side is quite accessible numerically, while $h(\rho)$ is not. One can thus estimate $h(\rho)$ by taking some initial condition ξ, computing $x = f^N\xi$ for some large N, and then computing $\sum_{r:\lambda_x^{(r)} > 0} m_x^{(r)} \lambda_x^{(r)}$ by Theorem 1 of Section 3. Notice that if one somehow knows that the topological entropy

$$\sup \{h(\rho) : \rho \text{ invariant}\}$$

vanishes, then (5) implies that there is no sensitive dependence on initial condition. The converse is a theorem : if all characteristic exponents of all invariant measures are ≤ 0, then (because $p(\rho) \le 0$) the topological entropy is 0.

Since the measures which satisfy (5) are those which maximize $p(\rho)$ (as given by (4)), one can try to approximate them by a variational procedure. It remains to be seen if one can use this method practically to get an idea of the asymptotic measures for a problem such as that of fluid turbulence.

Finally, let us remark (after Benettin et al [1]) that the largest characteristic exponent $\chi = \lambda_x^{(s(x))}$, which characterizes the sensitive dependence on initial condition, satisfies

$$\frac{1}{m} h(\rho) \le \chi \le h(\rho)$$

if (5) holds.

5. Turbulent behavior.

It is generally impractical to change the initial condition of a dynamical system occuring in nature, and to observe the behavior of δx_t as t increases. There are however indirect ways in which the sensitive dependence on initial condition manifests itself, giving rise to what we shall call turbulent behavior [*]. In particular, x_t will be neither asymptotically constant

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[*] The connection with the theory of fluid turbulence is not discussed here. See Lorenz [4], Ruelle and Takens [12].

nor periodic, but will have an apparently erratic appearance.

If one can measure the position x_t at each time with x high but only finite precision as is the case for natural phenomena - it is found that systems with sensitive dependence on initial condition loose information at the rate $\sum_{r:\lambda_x^{(r)} \geq 0} m_x^{(r)} \lambda_x^{(r)}$ (i.e. $h(\rho)$ according to the relation (5)). One would like to deduce from this some decay properties of the time correlation functions

$$F_{\varphi\psi}(t) = \int \rho(dx) \; \varphi(x) \; \psi(f_x^t) - \left[\int \rho\varphi \right] \left[\int \rho\psi \right]$$

when $|t| \to \infty$ (φ and ψ are assumed differentiable). At this moment a theorem is known only for axiom A diffeomorphisms, where it has been proved that $F_{\varphi\psi}$ decreases exponentially at infinity. In particular the positive measure

$$\omega \to \sum_{t \in \mathbb{Z}} e^{-i\omega t} \; F_{\varphi\varphi}(t)$$

has continuous density, a property known as continuous (frequency) spectrum. One expects continuous spectrum to occur under much more general conditions than axiom A . Actually continuous spectrum is observed experimentally in fluid turbulence (see particularly [3]).

6. Simplest examples of turbulent behavior.

The simplest examples of sensitive dependence on initial condition have been reviewed recently in [10]. We recall that the lowest dimension for which sensitive dependence occurs is respectively 1,2,3 in cases I,II,III of Section 2 (maps, diffeomorphisms, flows).

On the 3-torus, a flow with sensitive dependence on initial condition can be obtained by an arbitrarily C^2-small perturbation of a quasiperiodic flow. For a m-torus $m > 3$, C^2 can be replaced by C^∞ (see [5]) . This means that if a small coupling is introduced between 3 or more oscillators, turbulent behavior may result. Using oscillating electric circuits, it should be possible to visualize the transition to continuous spectrum when a suitable coupling is introduced between the oscillators. Alternatively if the frequencies are in the audible range, the transition to continuous spectrum should corre-spond to a change in the musical nature of the corresponding sound. These expe-

riments - contrasting the coupling of 3 oscillators with the coupling of 2 oscillators - have not been performed as far as I know. Since they are easy I strongly suggest that they should be attempted[*].

7. Conclusions.

The domain of research which has been reviewed here is one where progress is slow, due to great mathematical difficulties. The potential applications are however very important, both from the purely theoretical viewpoint (understanding of turbulence), and from a very practical viewpoint (discussion of individual systems with sensitive dependence on initial condition). It seems to me that even in the present very imperfect state of the theory it has started to be possible to interpret in a meaningful way some of the typical aspects of "turbulent" differentiable dynamical systems.

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[*] A computer study of three coupled oscillators has been made by Sherman and Mc Laughlin [13]. Unfortunately their paper does not make clear exactly what mathematical system is treated.

It has to be remarked that, although the flows on T^2 do not have sensitive dependence on intial condition, they may have only the trivial eigenfunction 1 [see A.N. Kolmogorov "On dynamical systems with integral invariants on the torus", Dokl. Akad. Nauk SSSR 93 N° 5, 763-766 (1953)]. This situation appears to be exceptional (M. Herman, private opinion), and it is not clear what the Fourier transform of $F_{\varphi\varphi}$ then looks like.

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Appendix

In section 4 we have justified the consideration of certain <u>asymptotic</u> <u>measures</u> by the fact that they describe the ergodic averages for almost all initial conditions (with respect to Lebesgue measure). Another point of view is that the time evolution (1) or (2) is in practice always perturbed by some noise, being thus replaced by a stochastic process. One can argue heuristically that the same class of measures is obtained in this manner as before, namely invariant measures which are "continuous along the unstable direction". This is because the smearing corresponding to the noise term preserves the "continuity along the unstable direction". In the axiom A case this argument has been made rigorous (see below the references to Sinai and Kifer).

One more remark about the asymptotic measures. The conditional measure on an unstable manifold has a density with respect to the "Lebesgue measure" on that leaf and, by f-invariance, these densities satisfy proportionality relations involving the Jacobian of f in the unstable direction. These conditions correspond to the "Gibbs state" condition just as the variational principle of Section 4 corresponds to the definition of equilibrium states in statistical mechanics. The asymptotic measures or "ensembles" representing turbulence thus bear, technically, a remarkable resemblance with the ensembles of equilibrium statistical mechanics.

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