What Are the Measures Describing Turbulence?

David RUELLE

IHES, 91440 Bures-sur-Yvette

It is believed that the average properties of a dissipative dynamical system, in particular a turbulent flow, are described by a measure invariant under time evolution. We discuss here the problem of determining such measures (asymptotic measures).

§ 1. Introduction

The history of ideas on turbulence is a very confused one, showing an astonishing coexistence of incoherent and often contradicting ideas. One very reasonable view is that turbulence should be described by some probability measure μ on the space S of states (velocity fields) of the viscous fluid under consideration. The Hopf equation expresses the invariance under time evolution of the measure μ (or its Fourier transform). Starting with the invariance equation, approximation schemes for determining μ have been set up, assuming gaussianity, making truncations, etc. I think it is fair to say that these schemes have not been very successful.

To understand this failure, it is illuminating to examine the set of measures invariant under time evolution for simple differential equations of the type

$$\frac{dx}{dt}=X(x), \quad x\in \boldsymbol{R}^m \tag{1·1}$$

for instance systems obtained by truncation of the full hydrodynamic equations. This is done in § 2.

In § 3 and the rest of this report we discuss what measures are appropriate for the description of the asymptotic behavior of (1·1) or the corresponding discrete time evolution

$$x_{t+1}=f(x_t). \tag{1·2}$$

We follow a recent proposal[10] and indicate new results.

§ 2. Invariant measures for simple differentiable dynamical systems

Lorenz[6] studied numerically the following system, obtained by truncating the hydrodynamic equations describing convection

$$\dot{x} = -10x + 10y\,,$$
$$\dot{y} = -xz + 28x - y\,,$$
$$\dot{z} = xy - \frac{8}{3}z\,.$$

His conclusions and those of later studies agree with the results obtained in studying a variety of other systems. They can be summarized as follows:

1) The solutions of the equations tend asymptotically to a set with complicated structure, called a *strange attractor*. This set looks locally like a manifold times a Cantor set; in particular it has Lebesgue measure zero.

2) The solutions exhibit *sensitive dependence on initial condition*: A small error on initial condition in general grows exponentially with time.

3) There are many ergodic invariant measures. [For Axiom A systems there are uncountably many, this is probably the case also for the Lorenz system].

The sensitive dependence on initial condition is the more important fact; we shall come back to it later.

Property 1) shows that nothing like a Gaussian measure can occur: invariant measures are necessarily singular with respect to the Lebesgue measure. Furthermore, Property 3) shows that the Hopf equation expressing invariance of μ is very far from having a unique solution. If specific results are obtained from gaussianity assumptions and truncation, it is not clear wha: their significance is.

A surprising fact, in view of Property 3), is that ergodic averages produced with the computer tend to *one* and the same ergodic measure μ, whatever the initial condition:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\,\varphi(n(t)) = \int \varphi(y)\,\mu(dy) \tag{2·1}$$

for every continuous function φ. This is true for the Lorenz system. In other systems a finite number of these *asymptotic* measures do occur.*) In hydrodynamical experiments also one measure describes the behavior of the system in general, although an infinity of ergodic measures undoubtedly exist in the turbulent regime. We shall now try to understand why this is the case.

§ 3. Asymptotic measures

An idea as to what measures μ are selected to represent the ergodic

*) In some systems an infinite number of measures will occur, due for instance to group invariance.

averages (2·1) is provided by the study of Axiom A dynamical systems.*) For such systems, if one discards a set of Lebesgue measure zero of initial conditions, only a finite number of measures are produced by the ergodic averages (2·1). These measures are further characterized by the fact that they are stable under small stochastic perturbations.

In the discussion of hydrodynamic experiments or experiments with other dissipative physical systems, discarding a set of Lebesgue measure zero of initial conditions is reasonable.**) In the case of computer studies of solutions of differential equations, small stochastic perturbations provided by round-off errors, are certainly present. In both cases, the result is the same: Finitely many *asymptotic measures* are selected *in the case of Axiom A systems.*

Instead of solutions of differential equations we may consider discrete time dynamical systems (see Eq. (1·2)). The ergodic averages become then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\varphi(f^k x)=\int\varphi(y)\,\mu(dy). \tag{3·1}$$

The discrete time dynamical systems have been better studied than their continuous time analogues. Usually theorems are first proved for discrete time, the more difficult continuous time case is solved somewhat later. For this reason we shall from here on shift to the discrete time language, and speak of *diffeomorphisms* instead of differential equations.

It is natural to look for the asymptotic measures for general diffeomorphisms (or flows) among those which satisfy some conditions which characterize them in the Axiom A case. Before doing this we have to introduce the notion of characteristic exponents.

§ 4. Characteristic exponents

Let f be a diffeomorphism and μ an ergodic measure (for f) with compact support. Denote by $T_x f$ the tangent map to f (resp. the matrix of partial derivatives if f is a diffeomorphism of $\boldsymbol{R}^m$). It follows from the multiplicative ergodic theorem of Oseledec that μ-almost everywhere one can decompose the tangent space (resp. $\boldsymbol{R}^m$) into a direct sum $W_n^{(1)}\oplus\cdots\oplus W_n^{(s)}$ such that

*) It will not be necessary for our purposes to define these systems. They are a reasonably large class of differentiable dynamical systems, which are fairly well understood theoretically. For their study, the reader may consult S. Smale[13] and R. Bowen.[2] For the results on asymptotic measures see Sinai,[12] Ruelle,[8] Bowen and Ruelle[3] and the monograph of Bowen.[1] See also the papers of Kifer[5] on small stochastic perturbations.

**) We treat the phase space of the dissipative system as finite dimensional. We shall not discuss here the problems connected with the fact that in most cases it has actually infinite dimension. Let us say however that in many cases of interest it appears that the motions lie asymptotically in finite dimension, see Mallet-Paret.[7]

$$\lim_{k\to\pm\infty}\frac{1}{k}\log\|T_x f^k u\|=\lambda^{(r)}\quad\text{if}\quad u\in W_x^{(r)}.$$

The numbers $\lambda^{(r)}$ are called characteristic exponents. In particular, the largest characteristic exponent $\lambda^{(s)}$ satisfies

$$\lim_{n\to+\infty}\frac{1}{n}\log\|T_x f^n\|=\lambda^{(s)}$$

since we assumed μ ergodic, the characteristic exponents are constant (a.e.). If the asymptotic behavior of the orbit $(f^n x)$ is described by μ, and if the largest characteristic exponent $\lambda^{(s)}$ is strictly positive, then we have *sensitive dependence on initial condition.*

If the largest characteristic exponent $\lambda^{(s)}$ is strictly negative, then the support of μ is an attracting periodic orbit. Although this last result is intuitively natural, the proof of it which I know is not elementary.[11]

§ 5. Characterization of asymptotic measures

In the Axiom A case, the asymptotic measures are precisely those ergodic measures μ which satisfy the following equivalent conditions.

(a) *Variational principle.* The quantity $h(\mu)$-$\Lambda(\mu)$ is maximum (this maximum is zero). Here $\Lambda(\mu)$ denotes the sum of the positive characteristic exponents (with multiplicity); $h(\mu)$ denotes the entropy (Kolmogorov-Sinai invariant).

(b) *Smoothness of conditional measures.* The measure μ is smooth along unstable directions. More precisely, the conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure on these unstable manifolds. Unstable manifolds will be defined below.

The equivalence between the above two conditions is proved by methods of statistical mechanics. The smoothness of conditional measures means that μ is a Gibbs state (for some interaction), and the variational principle means that μ is an equilibrium state. One uses then the equivalence between Gibbs states and equilibrium states.

Also for diffeomorphisms which do not satisfy Axiom A, it is natural to look for asymptotic measures among those satisfying either the variational principle, or the smoothness of conditional measures. The equivalence between these conditions is not known in general. We are here in the realm of conjectures, but two results are known.

(a) *An inequality related to the variational principle* (Ruelle,[9] Katok[4]). If f is a C^1 map of a compact manifold, and μ any f-ergodic measure, then

$$h(\mu)\leq\Lambda(\mu).$$

(b) *Existence of unstable manifolds* (Ruelle[11]).

Let f be a $C^{1+\varepsilon}$ diffeomorphism of a compact manifold. Define the *unstable manifold* $\mathcal{P}_x^{(-)}$ through x by

$$\mathcal{P}_x^{(-)}=\left\{y\in M:\limsup_{n\to\infty}\frac{1}{n}\log d(f_x^{-n},f_y^{-n})<0\right\}.$$

Let also $U_x^{(-)}=W_x^{(p)}+\cdots+W_x^{(s)}$ be the sum of the spaces corresponding to strictly positive characteristic exponents. Then for almost all x (with respect to any f-invariant measure μ) $\mathcal{P}_x^{(-)}$ is a smooth manifold tangent to $U_x^{(-)}$ at x. $\mathcal{P}_x^{(-)}$ may actually be dense in parts of M, and therefore a more careful formulation is: $\mathcal{P}_x^{(-)}$ is the image of $U_x^{(-)}$ by an injective imbedding tangent to the identity at x. Similar results had been proved earlier by Pesin in a special case.

§ 6. Conclusions

The programme of studying asymptotic measures for general diffeomorphisms is only at its beginning. If one is optimistic, one may hope that the asymptotic measures will play for dissipative systems the sort of role which the Gibbs ensembles have played for statistical mechanics. Even if that is the case, the difficulties encountered in statistical mechanics in going from Gibbs ensembles to a theory of phase transitions may serve as a warning that we are, for dissipative systems, not yet close to a real theory of turbulence.

References

1) R. Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms", *Lecture Notes in Mathematics, No. 470* (Springer, Berlin, 1975).
2) R. Bowen (to appear).
3) R. Bowen and D. Ruelle, Inventiones Math. **29** (1975), 181.
4) S. Katok, The Estimation from above for the Topological Entropy of a Diffeomorphism (to appear).
5) Yu. I. Kifer, Doklady Akad. Nauk SSSR **216** (1974), 979 [English translation]; Soviet Math. Doklady **15** (1974), 918; Izv. Akad. Nauk SSSR; Ser. Mat. **38** (1974), 1091 [English translation]; Math. USSR Izvestija **8** (1974), 1083.
6) E. N. Lorenz, J. Atmos. Sci. **20** (1963), 130.
7) J. Mallet-Paret, J. Diff. Equations. **22** (1976), 331.
8) D. Ruelle, Am. J. Math. **98** (1976), 619.
9) D. Ruelle, Boll. Soc. Bras. Mat. **9** (1978), 83.
10) D. Ruelle, "Sensitive Dependence on Initial Condition and Turbulent Behavior of Dynamical Systems", Ann. N. Y. Acad. Sci. (to appear).
11) D. Ruelle, "Ergodic Theory of Differentiable Dynamical Systems" (to appear).
12) Ya. G. Sinai, Uspekhi Mat. Nauk **27** (1972), 21 [English translation]; Russian Math. Surveys **27** (1972), 21.
13) S. Smale, Bull. Am. Math. Soc. **73** (1967), 747.

Discussion

H. Haken: It is known from the work of May and others that the variety of solutions of $x_{t+1}=f(x_t)$ is richer than that of solutions of the corresponding differential equation $dx/dt=g(x)$. Could you therefore comment on the relation between these two types of equations?

D. Ruelle: Solutions of differential equations in some fixed dimension n are less rich than discrete time evolutions in the same dimension. They rather correspond to discrete time evolution in dimension $n-1$.

J. L. Lebowitz: Am I correct in thinking that the hope is still very very far away, it is true, but nevertheless the hope is that there is an invariant measure of the general type you discussed for the Navier-Stokes equation at high Reynolds numbers which will contain such things as the Kolmogorov law, etc.?

D. Ruelle: That is the remote hope.

M. S. Green: What does your maximum principle mean for an ergodic Hamiltonian system?

D. Ruelle: For systems with a smooth invariant measure, the variational principle becomes an identity derived by Pesin and Margulis.

M. S. Green: Could you expand on the remark in the beginning of your talk that random elements in the numerical calculations such as round-off errors have an important role to play even though such elements may be very small?

D. Ruelle: If the initial point x is arbitrary and no randomness is present, every ergodic measure can appear as

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}.$$

M. S. Green: I speak from the prejudice that the origin of the dissipative systems in Hamiltonian systems of so many degrees of freedom must have an important bearing on the physical interpretation of strange attractors, in spite of the fact that fluctuations from this source are extremely small. Your reply clarifies, at least for me, the natural molecular fluctuations and their associate phase space measures have the effect of picking out the particular ergodic measure on the strange attractor which satisfies your maximum principle as the physically significant one.

N. G. van Kampen: The aim is to understand turbulence as a consequence of the hydrodynamic equations, *as they stand*. No molecular aspects need be taken into consideration, unless you adopt the unusual view that turbulence is not a purely hydrodynamical phenomenon.

D. Ruelle: It is an interesting idea that the microscopic fluctuations may provide the element which fixes the choice of the asymptotic measure describing the macroscopic behavior of the system. Otherwise, as Prof. van Kampen indicates, turbulence is completely described by the macroscopic hydrodynamic equations. For instance, I believe that multiplying the amplitude of microscopic fluctuations by a factor of 2 would have no visible effect.

M. S. Green: I think I may differ from Prof. van Kampen about the task of theoretical physics in explaining turbulence. Statistical mechanics delivers to the hydrodynamicist the Navier-Stokes equations together with very small but nonzero random stresses and heat currents which represent the molecular fluctuations. I do not see any principle which requires us to explain turbulence using the deterministic equations alone. If the natural fluctuations have a role to play in picking out the physically relevant measure among the many permitted by the deterministic equations. I do not think we have committed a methodological solecism. On the contrary, we have thereby diminished the number of explanatory principles something I find very satisfying.

J. D. Gunton: The mathematical models which you study can apparently be obtained in some cases from truncation of the hydrodynamic equations. Since Dr. Yahata has shown at this meeting that a partial understanding of the instabilities in Couette flow can be obtained by a truncation of the Navier-Stokes equations, I would like to know if some of your ideas can be applied to his work.

D. Ruelle: I expect that the ideas I described apply to the truncations of hydrodynamic equations studied by Dr. Yahata and others.

H. Haken: It might be possible to introduce a classification of different kinds of turbulence by means of your measures. Has anything been done in this direction?

D. Ruelle: A classification of the type you suggest would be obtained by counting the number of positive or zero characteristic exponents.

P. C. Martin: I like to make the analogy between the Ruelle-Takens picture for turbulence and the statistical dynamic picture of a crystal. It is a useful concept to understand that a crystal is a state with broken translational symmetry and that a state with broken spatial translational symmetry has Bragg peaks. Likewise it is a useful concept to understand that dissipative dynamical systems can have attractors more complicated than limit cycles and fixed points and that the time dependent correlations of such states have continuous frequency spectra. This classification theory helps us to recognize that we do not need external noise for turbulence just as we do not need external periodic potentials to create a crystal. However, crystals come in many variation and have many proportions and a solid state physicist wishes to understand a great deal more than that a crystal differs from a liquid because it gives rise to Bragg peaks. In particular, the fact that in some asymptotic region, e.g., for second order structural phase transitions, there is a universal behavior for the long range correlation functions of substances has little to do with the definition of a crystal. Indeed the behavior of the correlations in this limit can be understood by a different sort of ideas (via the renormalization group and fixed points) that does not distinguish between structural phase transitions and other transitions in fluids with order parameters of the same symmetry. In the same way, the Kolmogorov spectrum and strong turbulence might well be independent of whether the noise is externally generated or not. At the very least, an understanding of strong turbulence involves showing that in some asymptotic limit, the behavior of the correlations on strange attractors associated with fluids has "universal" properties of a very specific type. It is a long way from Bragg peaks to critical exponents and it is a long way from continuous frequency spectra to Kolmogorov-like "universal" spatial correlations.