MEASURES DESCRIBING A TURBULENT FLOW

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Abstract

Recent attempts at understanding hydrodynamic turbulence have used the ideas of strange attractors, characteristic exponents and stable manifolds for differentiable dynamical systems in finite dimensional spaces. This use was somewhat metaphorical, because hydrodynamic evolution is defined in infinite dimensional functional spaces. A recent study indicates that many results on finite dimensional dynamical systems carry over to dynamical systems in infinite dimensional Hilbert spaces under certain compactness assumptions. This is the case in particular for the time evolution defined by the Navier-Stokes equations in a bounded region of $\mathbb{R}^2$ or $\mathbb{R}^3$.

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Corrections and additions

p. 1, bottom, add

This paper was presented at the conference on "Nonlinear Dynamics" organized by the New York Academy of Sciences, December 17 to 21, 1979.

p. 4, insert $^{**)}$ at the end of ℓ. 16 and add the following footnote at the end of the page

**) Actually, the analyticity is explicitly proved in Iooss [20] and already implicitly in Fujita and Kato [21], as G.Iooss kindly pointed out to me.

p. 5, replace ℓ. 4 by

(a) $\|u(T_0)\| \leqq R$

p. 5, replace ℓ. 6 by

In particular, if $d = 2$, the ball $\{u \in H_0^1 : \|u\| \leqq R_0\}$ is admissible for sufficiently large R_0 , T_0 .

p. 5, replace ℓ. 12 and beginning of ℓ. 13 by

The set $\Lambda = \cap_{t>T_0} f^t M$ is compact.

p. 6, replace "Proposition 5" in ℓ. 1 by

Proposition 1.5 .

p. 8, ℓ. 3, after $\Theta > 0$, insert

$\lambda > 0$.

p. 9, ℓ. 5, replace $\ldots \leqq \lambda(u)\}$ by

$\ldots \leqq -\lambda(u)\}$.

p. 13, bottom, add

[20] G.Iooss : "Sur la deuxième bifurcation d'une solution stationnaire de systèmes du type Navier-Stokes." Arch. Rat. Mech. Anal. 64, 339-369 (1977).

[21] H. Fujita and T. Kato. "On the Navier-Stokes initial value problem I". Arch. Rat. Mech. Anal. 16, 269-315 (1964).

0. Introduction

The motion of a fluid in a region Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$ is defined by a function $t \longrightarrow v(t)$, where $v(t)$ belongs to some functional space $\mathcal{H}$ of velocity fields in Ω . In a turbulent regime one expects $v(t)$ to be distributed according to some probability law. This probability law is defined by a measure ρ on $\mathcal{H}$, invariant under the deterministic time evolution of the system.

One has reached some understanding of the invariant measures for a time evolution in a finite-dimensional space $\mathcal{H}$. The notions of strange attractor, sensitive dependence on initial condition , and characteristic exponents have been useful in this respect. Also it is almost everywhere possible to define stable and unstable manifolds, and one can in a number of cases identify the measures which are stable under small stochastic perturbations. In this paper we discuss the extension of results obtained for finite-dimensional dynamical systems to the more realistic case of the time evolution defined in a Hilbert space by the Navier-Stokes equation.

I wish to thank C. Foias for a very useful conversation on the analyticity of solutions of the Navier-Stokes equation.

1. Navier-Stokes theory

We summarize here some results on the time evolution defined by the Navier-Stokes equation in a bounded domain Ω . The pioneering work of J. Leray on the Navier-Stokes equation was followed by contributions by E. Hopf, O.A. Ladyzhenskaya, J.L. Lions and others *) . We follow here Foias and Temam [2] ; their paper contains a convenient exposition as well as a good list of references.

Let thus Ω be bounded open in $\mathbb{R}^d$, d = 2 or 3. We assume that $\partial\Omega$ is of class C^2 , that Ω is locally on one side of $\partial\Omega$, and that $\partial\Omega$ consists of a finite number of connected components. The Navier-Stokes equation and incompressibility condition are

$$\frac{\partial\vec{v}}{\partial t} + \sum_i v_i \frac{\partial\vec{v}}{\partial x_i} = \nu\,\Delta\vec{v} - \vec{\nabla}p + \vec{g} \qquad (1)$$

$$\vec{\nabla}\cdot\vec{v} = 0 \qquad (2)$$

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*) See the monographs of Ladyzhenskaya [5] , Lions [6], and Temam [19] .

to be satisfied for $(x,t) \in \Omega \times [0,+\infty)$. The boundary conditions are

$$\vec{v} = \vec{\varphi} \quad \text{on} \quad \partial\Omega \times [0,+\infty) \tag{3}$$

$$\vec{v} = \vec{v}_0 \quad \text{on} \quad \Omega \times \{0\} \tag{4}$$

The constant ν is the kinematic viscosity. The external force $\vec{g}$ is assumed to be square-integrable *) in Ω . The velocity $\vec{\varphi}$ is tangent to the boundary and is assumed to extend to a divergence free vector field with square-integrable second derivatives in Ω . Writing $\vec{u} = \vec{v} - \vec{\varphi}$, we may replace (1), (3) by

$$\frac{\partial\vec{u}}{\partial t} + \sum_i \left(u_i \frac{\partial\vec{u}}{\partial x_i} + \varphi_i \frac{\partial\vec{u}}{\partial x_i} + u_i \frac{\partial\vec{\varphi}}{\partial x_i}\right)$$

$$= \nu\, \Delta\vec{u} - \vec{\nabla}p + \left(\vec{g} - \varphi_i \frac{\partial\vec{\varphi}}{\partial x_i}\right) \tag{5}$$

$$\vec{u} = 0 \quad \text{on} \quad \partial\Omega \tag{6}$$

One looks then for a solution $\vec{u}$ of (5) in the Hilbert space H_0^1 of divergence free vector fields in Ω vanishing on the boundary **) with the Dirichlet norm

$$\|v\| = \left[\int \sum_{i,j} (\partial_i v_j(x))^2\, dx\right]^{\frac{1}{2}} \tag{7}$$

The divergence-free condition takes care of (2). Projecting (1) on divergence-free vector fields eliminates the pressure term $\vec{\nabla}p$, and it is easy to make sense of the other terms. Altogether, one has to find $u(t) = \vec{u}(\cdot,t) \in H_0^1$ satisfying

$$\frac{du}{dt} = F(u) \tag{8}$$

$$u(0) = u_0 \tag{9}$$

We say that a solution of (8) is <u>regular</u> on an interval I if $u : I \longmapsto H_0^1$ is continuous.

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*) In this section, integrability is always with respect to Lebesgue measure.

**) Specifically, H_0^1 is the completion with respect to (9) of the space of C^∞ divergence-free vector fields with compact support in Ω .

1.1. Theorem : (Uniqueness). Given $T > 0$, there can be only one regular solution on $[0,T]$ with given initial condition u_0 .

We shall denote by H^2 the Hilbert space of divergence free vector fields on Ω with norm

$$|||\vec{u}||| = \left[\int_\Omega [\sum_i u_i^2 + \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \sum_{i,j,k} \left(\frac{\partial u_i}{\partial x_i \partial x_j}\right)^2] dx\right]^{\frac{1}{2}}$$

(in particular $\vec{\varphi} \in H^2$) . We also denote by $H^1_{0\mathbb{C}}$, $H^2_{\mathbb{C}}$ the complexifications of H^1_0 , H^2 .

1.2. Theorem (Existence and analyticity). There is a constant C (depending on Ω , ν , $\vec{g}$, $\vec{\varphi}$) such that if we write $t(A) = C(1+A^2)^{-2}$, then every $u_0 \in H^1_0$ determines a regular solution $u(t)$ with $\|u(t)\|^2 \le 1 + 2A^2$ in $[0,t(\|u_0\|)]$. The map $(u_0,t) \longmapsto u(t)$ extends to a holomorphic function *)

$$\{u_0 \in H^1_{0\mathbb{C}} : \|u_0\| < A\} \times \{s\, e^{i\theta} : 0 < s < t(A)\cdot\cos^3\theta\} \longmapsto H^1_{0\mathbb{C}} \cap H^2_{\mathbb{C}}$$

and $\{u(t) : \|u_0\| < A\}$ is bounded in $H^2_{\mathbb{C}}$ for fixed $t \in (0,t(A))$. The derivative of $u(t)$ with respect to u_0 is an injective linear map $H^1_{0\mathbb{C}} \longmapsto H^1_{0\mathbb{C}} \cap H^2_{\mathbb{C}}$.

These results are explicit or implicit in Foias and Temam [2] (see especially the proof of Lemma 3.1).

1.3. Theorem (Dimension 2) : If $d = 2$, every $u_0 \in V$ determines a regular solution $u(t)$ in $[0,+\infty)$. There are a constant $S > 0$ and a continuous function $s : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ such that

$$\limsup_{t\to\infty} \|u(t)\| \le S$$

and

$$\|u(t)\| \le s(\|u_0\|) \qquad \text{for all } t \ge 0$$

See Foias and Temam [2] Section 2 . If $d = 2$, we may write $u(t) = S^t u_0$, where $S^t : H^1_0 \longmapsto H^1_0 \cap H^2$ for all $t > 0$. If $d = 3$, we impose now a condition such that S^t with similar properties can be defined.

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*) To be precise, it is a holomorphic function with values in the Hilbert space $H^1_{0\mathbb{C}}$, and a holomorphic function with values in the Hilbert space $H^2_{\mathbb{C}}$. The space $H^1_0 \cap H^2$ is closed in H^2 and the H^2 norm restricted to it is equivalent to $\vec{u} \mapsto [\int (\Delta(\vec{u}))^2\, dx]^{\frac{1}{2}}$.

1.4. Definition : We say that an open set $M \subset H_0^1$ is an admissible set (of initial conditions) if every $u_0 \in M$ determines a regular solution u on $[0,+\infty)$, and there are R , $T_0 \geq 0$ such that

(a) $$\limsup_{t\to\infty} \|u(t)\| \leq R$$

(b) $$u(t) \in M \qquad \text{for all } t \geq T_0$$

In particular, if $d = 2$, $M = H_0^1$ is admissible with $T_0 = 0$.

1.5. Proposition : Let M be an admissible set of initial conditions, and S^t be defined by $S^t u_0 = u(t)$ for $u_0 \in M$, $t \geq T_0$. If $t \geq T_0$, S^t is injective and real analytic $M \longmapsto M$. The semigroup property $f^t \circ f^{t'} = f^{t+t'}$ holds. The maps $(u,t) \longmapsto f^t u$, $Df^t(u)$ are continuous from $M \times (T_0,+\infty)$ to M and the bounded operators on H_0^1 respectively.

There is an (S^t) invariant compact set Λ such that $S^t u \longrightarrow \Lambda$ for all $u \in M$, when $t \to \infty$. For $u \in \Lambda$, $t > T_0$, the operator $DS^t(u)$ is compact and injective.

Using the compactness of the map $H^1_{0\mathbb{C}} \cap H^2_{\mathbb{C}} \hookrightarrow H^1_{0\mathbb{C}}$ one obtains Proposition 5 from results stated earlier. Of course Proposition 5 could be strengthened on various points. The weak form which we have selected will however be sufficient for later purposes.

2. Characteristic exponents and invariant manifolds

Let a differentiable dynamical system on a finite dimensional manifold M be given, with an invariant probability measure ρ . Then, the multiplicative ergodic theorem of Oseledec [8] holds, defining the characteristic exponents. From the theory of Pesin [9] , [10] , [11] follows also the existence almost everywhere of stable and unstable manifolds *). We shall now indicate an infinite dimensional extension of these results to Hilbert spaces (this extension is not trivial, see Ruelle [17]). Related results for Banach spaces have been obtained by R. Mañé (private communication).

From now on M will be an open subset of a separable real Hilbert space

*) Pesin's formulation requires ρ to be smooth, but this assumption is unnecessary, see [16] .

$\mathfrak{H}$, and (S^t) a semiflow on M, with the properties stated in Proposition 5. In particular (S^t) may be the Navier-Stokes time evolution. Every (S^t)-invariant measure has its support in the compact set Λ.

2.1. Theorem (First multiplicative ergodic theorem) : Given an (S^t)-invariant Borel function $\mu : M \longmapsto \mathbb{R}$, there is a Borel set $\Gamma \subset M$ such that $S^t\Gamma \subset \Gamma$ for $t \geq T_0$, and $\rho(\Gamma) = 1$ for every (S^t)-invariant probability measure ρ. If $u \in \Gamma$, there are an integer $s \geq 0$, reals $\mu^{(1)} > \ldots > \mu^{(s)} > \mu$ and finite-codimensional spaces $\mathfrak{H} = V_u^{(1)} \supset \ldots \supset V_u^{(s)} \supset V_u^{(s+1)}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log\|DS^t(u)v\| = \mu^{(r)} \quad \text{if} \quad v \in V_u^{(r)} \setminus V_u^{(r+1)}$$

for $r = 1,\ldots,s$, and

$$\limsup_{t\to\infty} \frac{1}{t} \log\|DS^t(u)v\| \leq \mu \quad \text{if} \quad v \in V_u^{(s+1)}$$

The functions $u \longmapsto s, \mu^{(1)},\ldots,\mu^{(s)}, V_u^{(2)},\ldots,V_u^{(s+1)}$ are Borel and $u \longmapsto s, \mu^{(1)},\ldots,\mu^{(s)}$, codim $V_u^{(2)},\ldots,$codim $V_u^{(s+1)}$ are (S^t)-invariant. [Note : the codimension codim V is the dimension of the orthogonal complement $V^\perp$] .

2.2. Theorem (Second multiplicative ergodic theorem) : Given an (S^t)-invariant Borel function $\mu : M \longmapsto \mathbb{R}$, there is a Borel set $\tilde{\Gamma} \subset \Lambda$ such that $S^t\tilde{\Gamma} = \tilde{\Gamma}$ for $t \geq T_0$, and $\rho(\tilde{\Gamma}) = 1$ for every (S^t)-invariant probability measure ρ. If $u \in \tilde{\Gamma}$, there are an integer $s \geq 0$, reals $\mu^{(1)} > \ldots > \mu^{(s)} > \mu$ and finite-dimensional spaces $\{0\} = \tilde{V}_u^{(0)} \subset \tilde{V}_u^{(1)} \subset \ldots \subset \tilde{V}_u^{(s)}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log\|(DS^t(S^{-t}u))^{-1}v\| = -\mu^{(r)} \quad \text{if} \quad v \in \tilde{V}_u^{(r)} \setminus \tilde{V}_u^{(r-1)}$$

for $r = 1,\ldots,s$, and

$$\liminf_{t\to\infty} \frac{1}{t} \log\|(DS^t(S^{-t}u))^{-1}v\| \geq -\mu \quad \text{if} \quad v \notin \tilde{V}_u^{(s)}$$

The functions $u \longrightarrow s, \mu^{(1)},\ldots,\mu^{(s)}, \tilde{V}_u^{(1)},\ldots,\tilde{V}_u^{(s)}$ are Borel and $n \longmapsto s, \mu^{(1)},\ldots,\mu^{(s)}$, dim $\tilde{V}_u^{(1)},\ldots,$dim $\tilde{V}_u^{(s)}$ are (S^t) invariant.

2.3. Remarks : Assumptions of differentiability rather than real analyticity are sufficient for the above theorems to hold. Theorem 2.1 holds without injectivity assumption on S^t and $DS^t(u)$. Almost everywhere with respect to every (S^t)-

invariant probability measure, the quantities s, $\mu^{(r)}$, codim $V^{(r+1)}$ occuring in Theorem 2.1 are equal to s, $\mu^{(r)}$, dim $\tilde{V}^{(r)}$ in Theorem 2.2. This justifies the confusion in notation for s and $\mu^{(r)}$. We have also, for almost all u, $V_u^{(r+1)} \cap \tilde{V}_u^{(r)} = \{0\}$ and $V_u^{(r+1)} + \tilde{V}_u^{(r)} = \mathcal{H}$ (transversality). Furthermore if $\delta_r(u)$ is the minimum of the component orthogonal to $V_u^{(r+1)}$ of a vector $v \in \tilde{V}_u^{(r)}$ with $\|v\| = 1$, then

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \delta_r(f^t x) = 0$$

It is easy to let μ go to $-\infty$ in Theorems 2.1 and 2.2.

The $\mu^{(r)}$ are called <u>characteristic exponents</u>. The <u>multiplicity</u> $m^{(r)}$ of $\mu^{(r)}$ is codim $V^{(r+1)}$ - codim $V^{(r)}$ = dim $\tilde{V}^{(r)}$ - dim $\tilde{V}^{(r-1)}$.

2.4. <u>Theorem</u> (Local stable manifolds) : <u>Let</u> Θ, λ, r <u>be</u> (S^t)-<u>invariant Borel functions on</u> Γ <u>with</u> $\Theta > 0$, $\lambda < 0$, r <u>integer</u> $\in [0,s]$, <u>and</u>

$$\mu^{(r+1)} < \lambda < \mu^{(r)}$$

(<u>where we have written</u> $\mu^{(0)} = +\infty$, $\mu^{(s+1)} = \mu$).

<u>Replacing possibly</u> Γ <u>by a smaller set retaining the properties of Theorem 2.1 one may construct Borel functions</u> $\beta > \alpha > 0$ <u>and</u> $\gamma > 1$ <u>on</u> Γ <u>with the following properties</u>.

(a) <u>If</u> $u \in \Gamma$ <u>the set</u>

$$\mathcal{V}_u^\lambda = \{v \in M : \|u-v\| \le \alpha(u) \text{ and } \|S^t u - S^t v\| \le \beta(u) \exp t\lambda(u) \text{ for all } t \ge T_0\}$$

<u>is contained in</u> Γ <u>and is a finite codimensional real analytic submanifold of the ball</u> $\{v \in M : \|u-v\| \le \alpha(u)\}$. <u>For each</u> $v \in \mathcal{V}_u^\lambda$, <u>the tangent</u> $T_v \mathcal{V}_u^\lambda$ <u>is</u> $V_v^{(r+1)}$. <u>More generally</u>, <u>for every</u> $r' \in [0,s]$, <u>the function</u> $v \longrightarrow V_v^{(r'+1)}$ <u>is real analytic on</u> $\mathcal{V}_u^\lambda$.

(b) <u>If</u> $v, w \in \mathcal{V}_u^\lambda$, $t \ge T_0$, <u>then</u>

$$\|S^t v - S^t w\| \le \gamma(u) \|v - w\| \exp t\lambda(u)$$

(c) <u>If</u> $u \in \Gamma$, <u>then</u> $\alpha(S^t u)$, $\beta(S^t u)$, $\gamma(S^t u)^{-1}$

decrease less fast than the exponential $e^{-\Theta t}$ when $t \to \infty$.

2.5. Theorem (Local unstable manifolds) : Let Θ, λ, r be (S^t)-invariant Borel functions on $\tilde{\Gamma}$, with $\Theta > 0$, r integer $\in [0,s]$, and

$$\mu^{(r+1)} < \lambda < \mu^{(r)}$$

(where we have written $\mu^{(0)} = +\infty$, $\mu^{(s+1)} = \mu$).

Replacing possibly $\tilde{\Gamma}$ by a smaller set retaining the properties of Theorem 2.2., one may construct Borel functions $\tilde{\beta} > \tilde{\alpha} > 0$ and $\tilde{\gamma} > 1$ on $\tilde{\Gamma}$ with the following properties.

(a) If $u \in \tilde{\Gamma}$, the set

$$\tilde{\mathcal{V}}_u^\lambda = \{v \in M : \|u - v\| \le \tilde{\alpha}(u) \text{ and }$$
$$\|S^{-t}u - S^{-t}v\| \le \tilde{\beta}(u) \exp[-t\,\lambda(u)] \text{ for all } t \ge T_0\}$$

is contained in $\tilde{\Gamma}$ and is a finite dimensional real analytic submanifold of the ball $\{v \in M : \|u - v\| \le \alpha(u)\}$. For each $v \in \tilde{\mathcal{V}}_u^\lambda$, the tangent $T_v \tilde{\mathcal{V}}_u^\lambda$ is $\tilde{V}_v^{(r)}$. More generally, for every $r' \in [0,s]$, the function $v \longmapsto \tilde{V}_v^{(r')}$ is real analytic on $\tilde{\mathcal{V}}_u^\lambda$.

(b) If $v, w \in \tilde{\mathcal{V}}_u^\lambda$, $t \ge T_0$, then

$$\|S^{-t}v - S^{-t}w\| \le \tilde{\gamma}(u)\, \|v - w\| \exp[-t\,\lambda(u)]$$

(c) If $u \in \tilde{\Gamma}$, then $\tilde{\alpha}(S^{-t}u)$, $\tilde{\beta}(S^{-t}u)$, $\tilde{\gamma}(S^{-t}u)^{-1}$ decrease less fast than the exponential $e^{-\Theta t}$ when $t \to \infty$.

2.6. Remarks : Theorem 2.4. holds without injectivity assumptions on S^t and $DS^t(u)$. The manifolds $\mathcal{V}_u^\lambda$ and $\tilde{\mathcal{V}}_u^\lambda$ are local generalized stable and unstable manifolds respectively. They do not in general depend continuously on u, but the construction implies measurability properties.

If ρ is an (S^t)-invariant probability measure such that the characteristic exponents $\mu^{(r)}$ are almost everywhere non-zero, let

$$\mu^{(r(Q+1))} < \lambda < 0 < \tilde{\lambda} < \mu^{(r(Q))}$$

Then $\mathcal{V}_u^\lambda$ and $\tilde{\mathcal{V}}_u^{\tilde{\lambda}}$ are respectively local stable and local unstable manifolds in the strict sense. They intersect transversally at u for ρ-almost all u.

One can define global stable and unstable manifolds, we shall consider only the latter.

2.7. Theorem (Global unstable manifolds): With the notation of Theorems 2.2 and 2.5., one can choose $\widetilde{\Gamma}$ such that, if $u \in \widetilde{\Gamma}$, the set

$$\widetilde{\mathfrak{W}}_u^\lambda = \{v \in M : \limsup_{t\to\infty} \frac{1}{t} \log \|S^{-t}u - S^{-t}v\| \le \lambda(u)\}$$

is contained in $\widetilde{\Gamma}$ and is the image of $\widetilde{V}_u^{(r)}$ by an injective real analytic immersion tangent to the identity at u.

This applies in particular to the global unstable manifold in the strict sense :

$$\widetilde{\mathfrak{W}}_u = \{v \in M : \limsup_{t\to\infty} \frac{1}{t} \log \|S^{-t}u - S^{-t}v\| < 0 \}$$

3. What are the measures describing turbulence ?

We have assumed that there is a compact attracting set Λ such that $S^t u_0$ tends to Λ for $u_0 \in M$ and $t \to \infty$. If we choose the smallest Λ with this property, (S^t) extends to a group of homeomorphisms of Λ. In the case of Navier-Stokes time evolution, the work of Mallet-Paret [7] (and Foias and Temam [2]) shows that Λ has finite topological dimension. The stable and unstable manifold theorems give further geometric information on Λ and the flow near it. A large subset $\widetilde{\Gamma}$ of Λ (in the sense that $\rho(\widetilde{\Gamma}) = 1$ for every invariant probability measure ρ) consists of global unstable manifolds. The presence of strictly positive characteristic exponents corresponds to sensitive dependence on initial condition. If there is no zero characteristic exponent, the unstable manifolds are intersected transversally (almost everywhere) by (local) stable manifolds defined in a neighborhood of Λ.

If the stable (i.e. contracting) manifolds would form a continuous family covering a neighborhood of Λ, one would have a rather good geometric picture of the dynamics near Λ. At present one understands well only a limited class of dynamical systems (on finite dimensional manifolds) ; these are essentially the systems for which Axiom A holds. In the Axiom A case, one can decompose Λ into subsets (so-called "basic" sets), the most important ones being the "Axiom A attractors". If Λ is an Axiom A attractor, the local stable manifolds form a continuous family covering a neighborhood of Λ. There is then a distinguished invariant probability measure ρ on Λ (call it asymptotic measure) with the following properties *).

(a) For almost every u_0 with respect to Lebesgue measure in a neighborhood of Λ

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \varphi(S^t u_0) = \int \rho(du)\, \varphi(u)$$

for all the continuous real functions φ on a neighborhood of Λ.

(b) Stability under small stochastic perturbations : Let ρ_ε be the stationary probability measure for a small stochastic perturbation of our dynamical system (ε is a small parameter telling how small the perturbation is). Then

$$\lim_{\varepsilon\to\infty} \int \rho_\varepsilon(du)\, \varphi(u) = \int \rho(du)\, \varphi(u)$$

for all continuous real functions φ on a neighborhood of Λ. [We have not indicated what kind of small stochastic perturbation is allowed, for a precise formulation, see [4]].

The asymptotic measure ρ on an Axiom A attractor is ergodic, and characterized by either of the following properties.

(I) The conditional measures of ρ on unstable manifolds are absolutely continuous with respect to Lebesgue measure on these manifolds.

(II) The measure theoretic entropy $h(\rho)$ is equal to the sum of the positive characteristic exponents (counted with their multiplicity). [For an arbitrary invariant probability measure σ one has $h(\sigma) \le \int \sigma(dx) \sum_{r:\mu^{(r)}>0} m^{(r)}(x)\mu^{(r)}(x)$, see [13]].

It is natural to try to describe turbulence by an (S^t)-invariant probability measure ρ. Since there are in general uncountably many different ergodic measures, the question arises which one to choose. In [14] we suggested to look for the "asymptotic measures" describing turbulence among those which satisfy I or II. The following remarks are in order about that proposal.

(1) I implies II (in finite dimension) as shown by P. Walters and A. Katok (unpublished).

(2) An example (due to R. Bowen and A. Katok) shows that there is sometimes no

*) See Bowen and Ruelle [1] for (a) and Kifer [3] , [4] for (b). These are based on earlier work of Sinai [18] and Ruelle [12] .

measure satisfying I or II. It is not known how exceptional this situation is.

(3) Stability under small stochastic perturbations makes sense in infinite dimension and is a reasonable condition to impose on a measure describing turbulence. To see this one can estimate the time it takes for thermal fluctuations to be magnified by the sensitive dependence on initial condition, up to the point of becoming macroscopic. (Thermal fluctuations are a kind of stochastic perturbations always present in a fluid).One finds (see Ruelle [15]) that this time is relatively short.

Altogether, the use of I and II still appears promising for the research of measures describing turbulence.

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