CHARACTERISTIC EXPONENTS AND INVARIANT MANIFOLDS IN HILBERT SPACE

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Abstract :

The multiplicative ergodic theorem and the construction almost everywhere of stable and unstable manifolds (Pesin theory) are extended to differentiable dynamical systems on Hilbert manifolds under some compactness assumptions. The results apply to partial differential equations of evolution and also to non-invertible maps of compact manifolds.

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March 1980

IHES/P/80/11

This work was supported in part by the National Science Foundation under Grant M.C.S. 78-02 721.

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0. *Introduction* :

The classical theory of stable (or unstable) manifolds for a diffeomorphism or flow on a compact manifold holds under certain conditions of hyperbolicity *) . These conditions have been successively removed by Brin and Pesin [1], and by Pesin [10] , [11], [12] . In the latter work, the existence of a smooth invariant measure ρ is assumed, and the stable manifolds are shown to exist almost everywhere with respect to ρ . The existence of a smooth invariant measure is however not necessary, and it was shown in Ruelle [14] that there is always an invariant Borel set foliated by stable manifolds which has measure 1 with respect to every invariant probability measure **) .

An essential feature of the theory of stable manifolds "almost everywhere" is the use of the multiplicative ergodic theorem ***) . Under weak conditions this theorem implies the existence almost surely of the limit

$$\lim_{n\to\infty} (T^{n*} T^{n})^{1/2n} = \Lambda \tag{0.1}$$

for a random matrix product

$$T^n = T_n \cdot \ldots \cdot T_2 \cdot T_1 \quad .$$

It is shown in [14] that a small perturbation $T_k \longmapsto T_k + \delta T_k$, where δT_k decreases exponentially with k , simply replaces Λ in (0,1) by another matrix Λ' with the same eigenvalues. From this the stable manifold theorem can be obtained.

In the present paper we consider differentiable maps, rather than diffeomorphisms (or semiflows rather than flows). Furthermore we allow Hilbert manifolds rather than finite dimensional ones. If certain boundedness conditions hold, and if the tangent map Tf is compact (or more generally satisfies some

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*) See in particular the monograph by Hirsch, Pugh, and Shub [4], and references given there to earlier work.

**) A. Fathi, M. Herman and Ch. Yoccoz have obtained similar results in an unpublished seminar. Such results can also be obtained by Pesin's method (see Katok [5]).

***) Due to Oseledec [9] ; see also Millionščikov [8] Zaharevič [16], and Raghunathan [13] .

"discrete spectrum" condition) one can define local stable and unstable manifolds almost everywhere for f . To obtain global stable manifolds (resp. unstable manifolds) one has to require that Tf has dense range (resp. is injective).

Section 7 of this paper indicates how to obtain a large variety of results on local or global stable or unstable manifolds in finite or infinite dimensional manifolds under various differentiability and "spectrum" conditions. In particular, a proof of the claims made in Ruelle and Shub [15] for the finite dimensional case is obtained. Another application concerns semiflows on an open subset of a Hilbert space ; the results are described below *). Semiflows on Hilbert spaces occur as solutions of partial differential equations of evolution, and we have in mind applications to hydrodynamics turbulence.

The theorems of [15] on maps of finite dimensional manifolds, and those quoted below on semiflows on Hilbert spaces are typical of many more results obtainable from Section 7. It would take too much space to state all these results explicitly **) .

The present paper is largely parallel to the previous paper [14], obtaining analogous results by analogous methods. The differences are however nontrivial, and necessitate a careful reworking of the arguments.

0.1. *Semiflows on Hilbert space*

In this subsection, $\mathfrak{H}$ will be a separable real Hilbert space, and M an open subset of $\mathfrak{H}$. A $C^{\underset{\sim}{r}}$ semiflow (f^t) is defined on M , with $\underset{\sim}{r} > 1$, $\underset{\sim}{r} = \infty$, or $\underset{\sim}{r} = \omega$. By this we mean that a $C^{\underset{\sim}{r}}$ map $f^t : M \longmapsto M$ is defined for $t \geqq T_0$ (some $T_0 \geqq 0$) , that $f^{t+t'} = f^t \circ f^{t'}$, and that ***)

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*) Related results for Banach spaces have been obtained by R. Mañé (private communication).

**) Notice that, among other things, we could treat problems of the billiard type, where differentiability is not assumed everywhere.

***) D denotes the derivative. In view of the continuity requirements with respect to t, the study of the semiflow (f^t) may be replaced by the study of the map f^T for some suitable choice of $T > 0$.

$(x,t) \longmapsto f^t x$, $Df^t(x)$ are continuous from $M \times [T_0,+\infty)$ to M , and the bounded operators on $\mathfrak{h}$ respectively. We further assume that the set $\Lambda = \bigcap_{t > T_0} f^t M$ is compact, and that $Df^t(x)$ is a compact operator for $x \in \Lambda$, $t > T_0$.

If r is not integer, a map is C^r if its $[r]$-th derivative is Hölder continuous of exponent $r - [r]$. For integer r we deviate from standard use in this section, and require only that the $r-1$ st derivative be Lipschitz.

In I, II below we state multiplicative ergodic theorems and in IV, V, VII, stable and unstable manifold theorems.

I. Given an (f^t)-invariant Borel function $\mu : M \longrightarrow \mathbb{R}$, there is a Borel set $\Gamma \subset M$ such that $f^t\Gamma \subset \Gamma$ for $t \geqq T_0$, and $\rho(\Gamma) = 1$ for every (f^t)-invariant probability measure ρ . If $x \in \Gamma$, there are an integer $S \geqq 0$, reals $\mu^{(1)} > \ldots > \mu^{(S)} > \mu$ and finite-codimensional spaces $\mathfrak{h} = V_x^{(1)} \supset \ldots \supset V_x^{(S)} \supset V_x^{(S+1)}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log \|Df^t(x)\, u\| = \mu^{(r)} \quad \text{if} \quad u \in V_x^{(r)} \setminus V_x^{(r+1)}$$

for $r = 1,\ldots,S$, and

$$\lim_{t\to\infty} \frac{1}{t} \log \|Df^t(x)\, u\| \leqq \mu \qquad \text{if} \quad u \in V_x^{(S+1)}$$

The functions $x \longmapsto S$, $\mu^{(1)}, \ldots, \mu^{(S)}$, $V_x^{(2)}, \ldots, V_x^{(S+1)}$ are Borel and $x \longmapsto S$, $\mu^{(1)}, \ldots, \mu^{(S)}$, codim $V_x^{(2)}$, ... , codim $V_x^{(S+1)}$ are (f^t)-invariant. [Note : the codimension codim V is the dimension of the orthogonal complement $V^\perp$].

II. Let the maps f^t be injective and let the derivatives $Df^t(x)$ for $x \in \Lambda$ be injective $\mathfrak{h} \longmapsto \mathfrak{h}$. Given an (f^t)-invariant Borel function $\mu : M \longmapsto \mathbb{R}$, there is a Borel set $\tilde{\Gamma} \subset \Lambda$ such that $f^t\tilde{\Gamma} = \tilde{\Gamma}$ for $t \geqq T_0$, and $\rho(\tilde{\Gamma}) = 1$ for every (f^t)-invariant probability measure ρ . If $x \in \tilde{\Gamma}$, there are an integer $S \geqq 0$, reals $\mu^{(1)} > \ldots > \mu^{(S)} > \mu$ and finite-dimensional spaces

$\{0\} = \tilde{V}_x^{(0)} \subset \tilde{V}_x^{(1)} \subset \ldots \subset \tilde{V}_x^{(S)}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log \|(Df^t(f^{-t}x))^{-1} u\| = -\mu^{(r)} \text{ if } u \in \tilde{V}_x^{(r)} \setminus \tilde{V}_x^{(r-1)}$$

for $r = 1,\ldots,S$, and

$$\liminf_{t\to\infty} \frac{1}{t} \log \|(Df^t(f^{-t}x))^{-1} u\| \geqq -\mu \text{ if } u \notin \tilde{V}_x^{(S)}$$

The functions $x \longmapsto S$, $\mu^{(1)}, \ldots, \mu^{(S)}, \tilde{V}_x^{(1)}, \ldots, \tilde{V}_x^{(S)}$ are Borel and and $x \longmapsto S$, $\mu^{(1)}, \ldots, \mu^{(S)}$, $\dim \tilde{V}_x^{(1)}, \ldots, \dim \tilde{V}_x^{(S)}$ are (f^t)-invariant.

III. Almost everywhere with respect to every (f^t)-invariant probability measure, the quantities S , $\mu^{(r)}$, codim $V^{(r+1)}$ occuring in I are equal to S , $\mu^{(r)}$, $\dim \tilde{V}^{(r)}$ in II . This justifies the confusion in notation for S and $\mu^{(r)}$. We have also, for almost all x , $V_x^{(r+1)} \cap \tilde{V}_x^{(r)} = \{0\}$ and $V_x^{(r+1)} + \tilde{V}_x^{(r)} = \mathfrak{h}$ (transversality). Furthermore if $\delta_r(x)$ is the minimum of the component orthogonal to $V_x^{(r+1)}$ of a vector $u \in \tilde{V}_x^{(r)}$ with $\|u\| = 1$, then

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \delta_r(f^t x) = 0 .$$

It is easy to let μ go to $-\infty$ in I and II.

IV. <u>Local stable manifolds</u>

Let Θ, λ, r be (f^t)-invariant Borel functions on Γ with $\Theta > 0$, $\lambda < 0$, r integer $\in [0,S]$, and

$$\mu^{(r+1)} < \lambda < \mu^{(r)}$$

(where we have written $\mu^{(0)} = +\infty$, $\mu^{(S+1)} = \mu$) . Replacing possibly Γ by a smaller set retaining the properties of I one may construct Borel functions $\beta > \alpha > 0$ and $\gamma > 1$ on Γ with the following properties.

(a) If $x \in \Gamma$ the set

$$V_x^\lambda = \{y \in M : \|x-y\| \leqq \alpha(x) \text{ and } \|f^t x - f^t y\| \leqq \beta(x) \exp t\,\lambda(x) \text{ for all } t \geqq T_0\}$$

is contained in Γ and is a finite codimensional C^{r} submanifold*) of the ball $\{y \in M : \|x-y\| \leqq \alpha(x)\}$. For each $y \in V_x^\lambda$, the tangent $T_y V_x^\lambda$ is $V_y^{(r+1)}$. More generally, for every $r' \in [0,S]$, the function $y \mapsto V_y^{(r'+1)}$ is of class C^{r-1} on V_x^λ .

(b) If y , $z \in V_x^\lambda$, $t \geqq T_0$, then

$$\|f^t y - f^t z\| \leqq \gamma(x)\, \|y-z\| \exp t\ \lambda(x)$$

(c) If $x \in \Gamma$, then $\alpha(f^t x)$, $\beta(f^t x)$, $\gamma(f^t x)^{-1}$ decrase less fast than the exponential $e^{-\Theta t}$ when $t \to \infty$.

V. Local unstable manifolds

We retain the injectivity assumptions of II. Let Θ, λ , r be (f^t)-invariant Borel functions on $\tilde{\Gamma}$, with $\Theta > 0$, $\lambda > 0$, r integer $\in [0,S]$, and

$$\mu^{(r+1)} < \lambda < \mu^{(r)}$$

(where we have written $\mu^{(0)} = +\infty$, $\mu^{(S+1)} = \mu$) . Replacing possibly $\tilde{\Gamma}$ by a smaller set retaining the properties of II, one may construct Borel functions $\tilde{\beta} > \tilde{\alpha} > 0$ and $\tilde{\gamma} > 1$ on $\tilde{\Gamma}$ with the following properties.

(a) If $x \in \tilde{\Gamma}$, the set

$$\tilde{V}_x^\lambda = \{y \in M : \|x-y\| \leqq \tilde{\alpha}(x) \text{ and } \|f^{-t}x - f^{-t}y\| \leqq \tilde{\beta}(x) \exp[-t\lambda(x)] \text{ for all } t \geqq T_0\}$$

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*) Remember that in this section our definition of C^{r} deviates from standard use if r is integer.

is contained in $\tilde{\Gamma}$ and is a finite dimensional C^r submanifold of the ball $\{y \in M : \|x-y\| \leq \alpha(x)\}$. For each $y \in \tilde{\mathcal{V}}_x^\lambda$, the tangent $T_y \tilde{\mathcal{V}}_x^\lambda$ is $\tilde{V}_y^{(r)}$. More generally, for every $r' \in [0,S]$, the function $y \longmapsto \tilde{V}_y^{(r')}$ is of class C^{r-1} on $\tilde{\mathcal{V}}_x^\lambda$.

(b) If $y, z \in \tilde{\mathcal{V}}_x^\lambda$, $t \geq T_0$, then

$$\|f^{-t}y - f^{-t}z\| \leq \tilde{\gamma}(x)\,\|y-z\| \,\exp[-t\,\lambda(x)]$$

(c) If $x \in \tilde{\Gamma}$, then $\tilde{\alpha}(f^{-t}x)$, $\tilde{\beta}(f^{-t}x)$, $\tilde{\gamma}(f^{-t}x)^{-1}$ decrease less fast than the exponential $e^{-\Theta t}$ when $t \to \infty$.

VI. The manifolds $\mathcal{V}_x^\lambda$ and $\tilde{\mathcal{V}}_x^\lambda$ do not in general depend continuously on x, but the construction implies measurability properties on which we shall not elaborate.

If ρ is an (f^t)-invariant probability measure such that the characteristic exponents $\mu^{(r)}$ are almost everywhere nonzero, let

$$\mu^{(r(Q+1))} < \lambda < 0 < \tilde{\lambda} < \mu^{(r(Q))}$$

The local stable and unstable manifolds (in the strict sense) $\mathcal{V}_x^\lambda$, $\tilde{\mathcal{V}}_x^{\tilde{\lambda}}$ intersect transversally at x for ρ-almost all x.

Under suitable transversality conditions (for instance if $Df(x)$ has dense range for all $x \in M$) one can define global stable manifolds.

VII. <u>Global unstable manifolds</u>

With the notation and assumptions of II and V one can choose $\tilde{\Gamma}$ such that, if $x \in \tilde{\Gamma}$, the set

$$\tilde{\mathcal{W}}_x^\lambda = \{y \in M : \limsup_{t\to\infty} \frac{1}{t} \log \|f^{-t}x - f^{-t}y\| \leq -\lambda(x)\}$$

is contained in $\tilde{\Gamma}$ and is the image of $\tilde{V}_x^{(r)}$ by an injective C^r immersion tangent to the identity at x.

0.2 *General notation*

Throughout this paper, $\mathfrak{h}$ denotes a real or complex Hilbert space ; the distinction is made explicit when needed. T^* is the transpose or Hermitian conjugate of a bounded operator T in $\mathfrak{h}$. The space $\mathfrak{h}^{\wedge q}$ is the q-th exterior power of $\mathfrak{h}$; it consists of the completely antisymmetric elements of the Hilbert space tensor product of q copies of $\mathfrak{h}$. If T is a bounded operator on $\mathfrak{h}$, $T^{\wedge q}$ is the bounded operator on $\mathfrak{h}^{\wedge q}$, restriction of $T \otimes \ldots \otimes T$.

We denote by f^+ the positive part of a real function f ; for instance $\log^+ x = \max\{0, \log x\}$.

1. *A limit theorem for products of operators*

1.1. Theorem : Let $(T_n)_{n>0}$ be a sequence of bounded operators in $\mathcal{H}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T_n\| \leqq 0 \tag{1.1}$$

We write

$$T^n = T_n \cdot \ldots \cdot T_2 \cdot T_1$$

For some integer $Q \geqq 0$ we assume that

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\| = \ell_q \tag{1.2}$$

exists if $q \leqq Q$, and that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(T^n)^{\wedge (Q+1)}\| \leqq \ell_{Q+1} \tag{1.3}$$

where $\ell_1, \ldots, \ell_{Q+1}$ are finite. We define $\mu^{(1)} > \ldots > \mu^{(S+1)}$ and $r(q)$ for $q = 1, \ldots, Q+1$ so that

$$\sum_{k=1}^{q} \mu^{(r(k))} = \ell_q \quad \text{for} \quad q = 1, \ldots, Q+1 \tag{1.4}$$

If $Q > 0$ we assume that $r(Q) = S$, $r(Q+1) = S+1$, so that $\mu^{(r(Q))} > \mu^{(r(Q+1))}$.

If we keep the largest Q eigenvalues (counted with multiplicity) of $T^{n*} T^n$, and replace the others by 0 , we obtain an operator $[T^{n*} T^n]_Q$, well defined if n is large enough.

(a)

$$\lim_{n\to\infty} ([T^{n*} T^n]_Q)^{1/2n} = \Lambda_Q$$

exists. Its nonzero eigenvalues are $\exp \mu^{(1)}, \ldots, \exp \mu^{(S)}$. We denote by $U^{(1)}, \ldots, U^{(S)}$ the corresponding eigenspaces.

(b) Let $V^{(r)}$ be the orthogonal complement of $U^{(1)} + \dots + U^{(r-1)}$ for $r = 1, \dots, S+1$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n u\| = \mu^{(r)} \quad \text{if} \quad u \in V^{(r)} \setminus V^{(r+1)}$$

for $r = 1, \dots, S$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T^n u\| \leqq \mu^{(S+1)} \quad \text{if} \quad u \in V^{(S+1)}$$

Let $t_n^{(1)} \geqq t_n^{(2)} \geqq \dots$ be the eigenvalues of $(T^{n*}T^n)^{1/2}$ repeated according to multiplicity. If there is a continuous spectrum, its maximum is considered as an eigenvalue of infinite multiplicity. By (1.2), the limits

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{p=1}^{q} t_n^{(p)} = \ell_q$$

exist for $q = 1, \dots, Q$. Using (1.4) this gives

$$\lim_{n\to\infty} \frac{1}{n} \log t_n^{(q)} = \mu^{(r(q))} \quad , \qquad 1 \leqq q \leqq Q \tag{1.5}$$

Furthermore, by (1.3), (1.4)

$$\limsup_{n\to\infty} \frac{1}{n} \log t_n^{(Q+1)} \leqq \mu^{(r(Q+1))} \tag{1.6}$$

Let $U_n^{(r)}$, for $r = 1, \dots, S$, be the space spanned by the eigenvectors of $(T^{n*}T^n)^{1/2}$ corresponding to the eigenvalues $t_n^{(p)}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log t_n^{(p)} = \mu^{(r)}$$

$U_n^{(r)}$ is unambiguously defined for large n, and has finite dimension $m^{(r)}$.

To prove Theorem 1.1. we follow the approach of Raghunathan [13] (as we did in [14] for the finite dimensional case). We shall use the following result.

1.2. Lemma : Let $U_n^{(S+1)}$ denote the orthogonal complement of $U_n^{(1)} + \ldots + U_n^{(S)}$. Given $\delta > 0$ there is $K > 0$ such that, for all $n,k > 0$, and $1 \leqq r$, $r' \leqq S+1$,

$$\max\{|(u,u')| : u \in U_n^{(r)}, u' \in U_{n+k}^{(r')}, \|u\| = \|u'\| = 1\}$$

$$\leqq K \quad \exp[-n(|\mu^{(r')} - \mu^{(r)}| - \delta)] \tag{1.7}$$

We first prove (1.7) for $r' < r$. Equivalently, it suffices to prove that, if $v_{rr'}^k$ is the orthogonal projection of $u \in \sum_{i \geqq r} U_n^{(i)}$ in $\sum_{j \leqq r'} U_{n+k}^{(j)}$, then

$$\|v_{rr'}^k\| \leqq K\|u\| \exp[-n(\mu^{(r')} - \mu^{(r)} - \delta)] \tag{1.8}$$

It will be convenient to assume δ less than all $|\mu^{(i)} - \mu^{(j)}|$ for $i \neq j$, and to write $\delta^* = \delta/S$. In view of (1.1) there is $C > 0$ such that, for all n,

$$\log \|T_{n+1}\| \leqq C + n \frac{\delta^*}{4}$$

For large n we have thus

$$\|v_{rr'}^1\| \exp[(n+1)(\mu^{(r')} - \frac{\delta^*}{4})] \leqq \|T^{n+1}u\|$$

$$\leqq \|T_{n+1}\| \cdot \|T^n u\|$$

$$\leqq \exp[C + n\frac{\delta^*}{4}] \cdot \|u\| \exp[n(\mu^{(r)} + \frac{\delta^*}{4})]$$

If n is so large that $C - \mu^{(r')} + \frac{\delta^*}{4} \leqq n\frac{\delta^*}{4}$ this gives

$$\|v_{rr'}^1\| \leqq \|u\| \exp[-n(\mu^{(r')} - \mu^{(r)} - \delta^*)]$$

From this we deduce in particular

$$\|v^k_{r,r-1}\| \leqq \sum_{j=0}^{k-1} \|u\| \exp[-(n+j)(\mu^{(r-1)}-\mu^{(r)}-\delta^*)]$$

$$\leqq K_1\|u\| \exp[-n(\mu^{(r-1)}-\mu^{(r)}-\delta^*)]$$

with $K_1 = \{1 - \exp[-(\mu^{(r-1)}-\mu^{(r)}-\delta^*)]\}^{-1}$. Therefore also

$$\|v^k_{r,r-2}\| \leqq \sum_{j=0}^{k-1} \|u\| \exp[-(n+j)(\mu^{(r-2)}-\mu^{(r)}-\delta^*)]$$

$$+ \sum_{j=0}^{k-1} K_1\|u\| \exp[-n(\mu^{(r-1)}-\mu^{(r)}-\delta^*)] \exp[-(n+j)(\mu^{(r-2)}-\mu^{(r-1)}-\delta^*)]$$

$$\leqq K_2 \|u\| \exp[-n(\mu^{(r-2)}-\mu^{(r)}-2\delta^*)]$$

In general

$$\|v^k_{rr'}\| \leqq K_{r-r'} \|u\| \exp[-n(\mu^{(r')}-\mu^{(r)}-(r-r')\delta^*)]$$

Since $(r-r')\delta^* < \delta$, this proves (1.8).

It remains to prove (1.7) with $r' > r$. Choose unit vectors $u \in U_n^{(r)}$, $u' \in U_{n+k}^{(r')}$ and let F be the finite dimensional space spanned by all $U_n^{(i)}$, $U_{n+k}^{(j)}$ with $i, j \leqq S$, and u' if $r' = S+1$. We take an orthogonal basis (u_α) of F , containing u , and such that each u_α is in $U_n^{(r_\alpha)}$ for some r_α . Similarly let (u'_β) be an orthonormal basis of F containing u' , and such that each u'_β is in $U_{n+k}^{(r'_\beta)}$ for some r'_β . The matrix U with elements (u_α, u'_β) is unitary. We estimate the minors of U using the fact that

$$|(u_\alpha, u'_\beta)| \leqq K \exp[-n(\mu^{(r'_\beta)} - \mu^{(r_\alpha)} - \delta)] \quad \text{if } r'_\beta < r_\alpha$$

$$|(u_\alpha, u'_\beta)| \leqq 1 \qquad \text{in any case.}$$

We have $\dim F \leqq 2Q + 1$, and we may assume $K \geqq 1$. In view of $U^* = U^{-1}$, this gives

$$|(u_\alpha, u'_\beta)| \leqq (2Q)! \; K^{2Q}[\exp -n(\mu^{(r_\alpha)} - \mu^{(r'_\beta)} - 2Q\delta)]$$

if $r'_\beta > r_\alpha$. In particular

$$|(u,u')| \leqq K' \exp[-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta')]$$

with $K' = (2Q)! \, K^{2Q}$, $\delta' = 2Q\delta$. This is again of the form (1.7), and completes the proof of the lemma.

The lemma shows that $(U_n^{(r)})_{n>0}$ is a Cauchy sequence for every r . From this, and (1.5), part (a) of Theorem 1.1 follows.

We may write $U^{(r)} = \lim_{n\to\infty} U_n^{(r)}$ for $r = 1, \ldots , S+1$, and obtain from (1.7)

$$\max\{|(u,u')| : u \in U^{(r)}, u' \in U_n^{(r')}, \|u\| = \|u'\| = 1\}$$

$$\leqq K \exp[-n(|\mu^{(r')} - \mu^{(r)}| - \delta)]$$

Therefore, for $0 \neq u \in U^{(r)}$, and sufficiently large n ,

$$\frac{1}{n} \log \frac{\|T^n u\|}{\|u\|} \leqq \mu^{(r)} + 2\delta \qquad \text{if } r = 1, \ldots , S+1$$

$$\frac{1}{n} \log \frac{\|T^n u\|}{\|u\|} \geqq \mu^{(r)} - 2\delta \qquad \text{if } r = 1, \ldots , S \quad .$$

Thus

$$\lim_{n\to\infty} \frac{1}{n}\|\log T^n u\| = \mu^{(r)} \qquad \text{if } 0 \neq u \in U^{(r)} , \; r \leqq S$$

$$\limsup_{n\to\infty} \frac{1}{n}\|\log T^n u\| \leqq \mu^{(S+1)} \qquad \text{if } 0 \neq u \in U^{(S+1)}$$

and part (b) of Theorem 1.1 follows.

1.3. *Remark* :

Suppose that $\mathfrak{h}$ has infinite dimension. Instead of taking Q finite, let us assume that (1.2) exists for all integers $q > 0$. Define the $\mu^{(r)}$ and $r(q)$ by (1.4) for all $q > 0$. If $r(\cdot)$ takes only a finite number $S+1$ of values, one can apply Theorem 1.1. with that choice of S . Otherwise any finite choice of S is possible, and the spaces $U^{(r)}$ are independent of S for $r \leqq S$.

If $r(\cdot)$ takes infinitely many values, we have thus a natural definition of $U^{(r)}$ for all integers $r > 0$. Let again $V^{(r)}$ be the orthogonal complement of $U^{(1)} + \ldots + U^{(r-1)}$, and write $V^{(\infty)} = \bigcap_r V^{(r)}$. Let also $\mu^{(\infty)} = \inf_r \mu^{(r)}$ (finite or $-\infty$) . Then

$$\lim_{n\to\infty} \frac{1}{n} \|\log T^n u\| = \mu^{(r)} \qquad \text{if} \quad u \in V^{(r)} \setminus V^{(r+1)}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T^n u\| \leqq \mu^{(\infty)} \qquad \text{if} \quad u \in V^{(\infty)} .$$

Furthermore, if $\mu^{(\infty)} = -\infty$,

$$\lim_{n\to\infty} (T^{n*} T^n)^{1/2n} = \Lambda$$

exists in norm, and is a compact operator.

2. *Multiplicative ergodic theorems*

In this section the multiplicative ergodic theorem known for $Q \times Q$ matrices *) is extended to bounded operators in Hilbert space. In fact Proposition 2.1. below shows that the conditions (1.1) and (1.2) of Theorem 1.1. are satisfied almost everwhere in a measure-theoretic setting. The conjunction of Theorem 1.1. and Proposition 2.1. yields a multiplicative ergodic theorem. We do not state this theorem explicitly, but note its consequences for compact operators (Corollary 2.2.) and for unitary plus compact operators (Corollary 2.3).

2.1. Proposition : Let (M,Σ,ρ) be a probability space and $f : M \longmapsto M$ a measurable map preserving ρ. We assume $\mathfrak{h}$ separable, and let $T : M \longmapsto \mathcal{L}(\mathfrak{h})$ be measurable **) to the bounded operators, such that

$$\log^+ \| T(\cdot) \| \in L^1(M,\rho)$$

We write

$$T_x^n = T(f^{n-1}x) \cdot \ldots \cdot T(fx) \cdot T(x)$$

there is then $\Gamma^+ \subset M$ such that $f\,\Gamma^+ \subset \Gamma^+$, $\rho(\Gamma^+) = 1$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log \| T(f^{n-1}x) \| \leqq 0 \tag{2.1}$$

if $x \in \Gamma^+$. Furthermore there are f-invariant functions $\ell_q^+ : \Gamma^+ \longmapsto \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \| (T_x^n)^{\wedge q} \| = \ell_q^+(x) \tag{2.2}$$

if $x \in \Gamma^+$, for all integers $q > 0$.

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*) See Oseledec [9] , Raghunathan [13] .

**) i.e. the inverse images of open sets for the weak (or the strong) operator topology are measurable.

(2.1) follows from the integrability of $\log^+\|T(\cdot)\|$ and the ergodic theorem. It suffices to prove (2.2) for $q = 1$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n\| = \ell_1^+(x) \quad .$$

This is an extension of a theorem of Furstenberg and Kesten [3] ; it follows immediately from the subadditive ergodic theorem of Kingman [6], [7] . For convenience the latter theorem is reproduced in the Appendix.

2.2. Corollary (Multiplicative ergodic theorem for compact operators I) *).
We keep the notation and assumptions of Proposition 2.1., and suppose that $T(x)$ is compact for all x . There is then $\Gamma \subset M$ such that $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and the following properties hold if $x \in \Gamma$.

(a) $$\lim_{n\to\infty} (T_x^{n*}\, T_x^n)^{1/2n} = \Lambda_x$$

exists in norm and is a compact operator.

Let $\exp\mu_x^{(1)} > \exp\mu_x^{(2)} > \ldots$ be the nonzero eigenvalues of Λ_x . The $\mu_x^{(r)}$ are real, the sequence possibly terminates at $\mu_x^{(s)}$; otherwise we write $s = \infty$ (where $s = s_x$ may depend on x). Let $U_x^{(1)}$, $U_x^{(2)}$, ... be the corresponding eigenspaces and $m_x^{(r)} = \dim U_x^{(r)}$.

(b) The functions $x \longmapsto s_x$, $\mu_x^{(r)}$, $m_x^{(r)}$ are f-invariant. We let $V_x^{(r)}$ be the orthogonal complement of $U_x^{(1)} + \ldots + U_x^{(r-1)}$ for $r < s+1$. Let also $V_x^{(s+1)}$ be the nullspace of Λ_x . Then

$$\lim_{n\to\infty} \frac{1}{n}\|\log\, T_x^n u\| = \mu_x^{(r)} \qquad \text{if} \quad u \in V_x^{(r)} \setminus V_x^{(r+1)}$$

for $r = 1,2, \ldots$ $(r < s+1)$ and

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n u\| = -\infty \qquad \text{if} \quad u \in V_x^{(s+1)} \quad .$$

*) A complement to this multiplicative ergodic theorem is given in Corollary 3.4.

Let Γ^+ and the ℓ_q^+ be as in Proposition 2.1. Write

$$S_N = \{x \in \Gamma^+ : \lim_{q\to\infty} \frac{1}{q} \ell_q^+(x) \geqq -N\} \quad .$$

Then

$$-N\rho(S_N) \leqq \int_{S_N} \rho(dx) \frac{1}{q} \ell_q^+(x)$$

$$\leqq \int_{S_N} \rho(dx) \frac{1}{q} \log\|T(x)^{\wedge q}\| \quad .$$

When $q\to\infty$ we have

$$\frac{1}{q} \log\|T(x)^{\wedge q}\| \to -\infty$$

because $T(x)$ is compact. Since

$$\frac{1}{q} \log\|T(\cdot)^{\wedge q}\| \leqq \log^+\|T(\cdot)\| \in L^1(M, \rho)$$

we must have $\rho(S_N) = 0$ for all real N. Writing

$$\Gamma = \{x \in \Gamma^+ : \lim_{q\to\infty} \frac{1}{q} \ell_q^+(x) = -\infty\} \tag{2.3}$$

we have thus $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$. In view of (2.1), (2.2), (2.3), the corollary follows from Remark 1.3.

2.3. *Corollary* (Multiplicative ergodic theorem for unitary plus compact operators). We keep the notation and assumptions of Proposition 2.1. and let $\dim \mathfrak{H} = \infty$. For each x we suppose that $T(x)$ is the sum of a unitary and a compact operator. We assume that $T(x)$ is invertible and that

$$\log^+\| T(\cdot)^{-1}\| \in L^1(M,\rho)$$

There is then $\Gamma \subset M$ such that $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and the following properties hold if $x \in \Gamma$.

(a) $$\lim_{n\to\infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x$$

exists in norm, invertible, with $\Lambda_x - \mathbb{1}$ compact.

(b) Let $V_x^\mu \subset \mathfrak{h}$ be the spectral subspace corresponding to the part of the spectrum of $\operatorname{Log} \Lambda_x$ in $(-\infty,\mu]$, where μ belongs to the spectrum. Then

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n u\| = \mu \qquad \text{if} \quad u \in V_x^\mu \setminus \bigcup_{\mu'<\mu} V_x^{\mu'} .$$

Notice that the assumptions are symmetric with respect to interchange of $T(\cdot)$ and $T(\cdot)^{*-1}$. Let Γ^+ , ℓ_q^+ be as in Proposition 2.1., and Γ^- , ℓ_q^- be the corresponding quantities for $T(\cdot)^{*-1}$. Given $\varepsilon > 0$, let

$$S_\varepsilon^\pm = \{x \in \Gamma^\pm : \lim_{q\to\infty} \frac{1}{q} \ell_q^\pm(x) \geqq \varepsilon\} .$$

Then

$$\varepsilon\rho(S_\varepsilon^\pm) \leqq \int_{S_\varepsilon^\pm} \rho(dx) \frac{1}{q} \ell_q^\pm(x)$$

$$\leqq \int_{S_\varepsilon^\pm} \rho(dx) \frac{1}{q} \log\|(T(x)^{\pm 1})^{\wedge q}\| .$$

When $q\to\infty$ we have

$$\frac{1}{q} \log\|(T(x)^{\pm 1})^{\wedge q}\| \to 0$$

because $T(x)$ is the sum of a unitary and a compact operator. Since

$$\frac{1}{q} \log\|(T(\cdot)^{\pm 1})^{\wedge q}\| \leqq \log^+\|T(\cdot)^{\pm 1}\| \in L^1(M,\rho)$$

we must have $\rho(S_\varepsilon^\pm) = 0$. Writing

$$\Gamma = \{x \in \Gamma^+ \cap \Gamma^- : \lim_{q\to\infty} \frac{1}{q} \ell_q^+(x) = \lim_{q\to\infty} \frac{1}{q} \ell_q^-(x) = 0\}$$

we have thus $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$.

If $x \in \Gamma$, we may apply Remark 1.3. with $T_n = T(f^{n-1}x)$ and with $T_n = T(f^{n-1}x)^{*-1}$. Part (a) of the corollary results immediately.

Let $\mu^{(1)} > \ldots > \mu^{(S)} > \ldots > 0 > \ldots > \mu^{(-S')} > \ldots > \mu^{(-1)}$ be the eigenvalues of $\log \Lambda_x$. Let $U_n^{(r)}$ for $r = 1,\ldots, S$ be defined as in the proof of Theorem 1.1., when $T_n = T(f^{n-1}x)$. Let $U_n^{(-r)}$ for $r = 1, \ldots , S'$ be similarly defined when $T_n = T(f^{n-1}x)^{*-1}$. Let also $U_n^{(0)}$ denote the orthogonal complement of $U_n^{(1)} + \ldots + U_n^{(S)} + U_n^{(-1)} + \ldots + U_n^{(-S')}$. Given $\delta > \mu^{(S+1)} - \mu^{(-S'-1)}$ there is $K > 0$ such that, for all $n,k > 0$, and $-S' \leqq r$, $r' \leqq S$, the inequality (1.7) holds, where we have written $\mu^{(0)} = 0$. The proof is analogous to that of Lemma 1.2 and left to the reader. In particular we obtain

$$\max\{|(u,u')|: u \in U^{(r)} , u' \in U_n^{(r')}, \|u\| = \|u'\| = 1\}$$
$$\leqq K \exp[-n(|\mu^{(r')}-\mu^{(r)}| -\delta)]$$

where $U^{(r)} = \lim_{n\to\infty} U_n^{(r)}$. This permits an estimate of the growth of $\|T_x^n u\|$ for $0 \neq u \in U^{(r)}$ and the proof of part (b) of the corollary follows.

3. *A condition on sequences of operators*

We shall formulate in Section 4 a perturbation theorem for products of operators T_n . For the theorem to hold, the sequence (T_n) will have to verify stronger conditions than were imposed in Theorem 1.1. We shall refer to these conditions collectively as (S) . Here we state condition (S) and show that it holds almost everywhere in a measure-theoretic setting. As a consequence we obtain a second multiplicative ergodic theorem for compact operators (Corollary 3.4.).

3.1. *Condition* (S)

(S1) The assumptions and notation of Theorem 1.1. are retained

Define $T^{n,m} = T_n \cdot \ldots \cdot T_{m+1}$ for $n \geqq m+1$ ($T^{n,n}$ is the identity), and

$$V_m = \{u \in \mathfrak{h} : \limsup_{n\to\infty} \frac{1}{n-m} \log \|T^{n,m}u\| \leqq \mu^{(S+1)}\} \quad .$$

In particular $V_0 = V^{(S+1)}$. Let $\xi_0^{(1)}, \ldots, \xi_0^{(Q)}$, V_0 span $\mathfrak{h}$, where $\xi_0^{(k)}$ is a unit vector in $V^{(r(k))} \setminus V^{(r(k)+1)}$ for $k = 1, \ldots, Q$. Write

$$t_n^{(k)} = \| T_n \, \xi_{n-1}^{(k)} \| , \quad \xi_n^{(k)} = T_n \xi_{n-1}^{(k)} / t_n^{(k)} \quad .$$

(S2) For all n , the codimension of V_n is Q , or equivalently $\xi_n^{(1)}, \ldots, \xi_n^{(Q)}$, V_n span $\mathfrak{h}$ (See below) .

(S3) Let $\hat{T}^{n,m}$ denote the restriction of $T^{n,m}$ to V_m . Given $\varepsilon > 0$ there is $\kappa_\varepsilon > 0$ such that

$$\log \|\hat{T}^{n,m}\| \leqq (n-m)\, \mu^{(S+1)} + n\varepsilon + \kappa_\varepsilon \tag{3.1}$$

Since, by (S1) , we have

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{m=1}^{n} t_m^{(k)} = \lim_{n\to\infty} \frac{1}{n} \log \|T^n \xi_0^{(k)}\| = \mu^{(r(k))}$$

we may also assume that

$$\left|\log \prod_{m=M+1}^{N} t_m^{(k)} - (N-M)\,\mu^{(r(k))}\right| \leqq N\varepsilon + \kappa_\varepsilon \tag{3.2}$$

(S4) Given $\varepsilon > 0$, there is $D_\varepsilon \geqq 1$ such that if

$$u = \sum_{k=1}^{Q} u^{(k)}\,\xi_n^{(k)} + u^{(Q+1)} \quad , \quad u^{(Q+1)} \in V_n$$

then

$$|u^{(k)}| \leqq D_\varepsilon\, e^{n\varepsilon}\,\|u\| \quad \text{for } k = 1, \ldots , Q+1$$

(We write $\|u^{(Q+1)}\| = |u^{(Q+1)}|$) .

From Theorem 1.1 (a) we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n \xi_0^{(1)} \wedge \ldots \wedge T^n \xi_0^{(Q)}\| = \ell_Q$$

$$= \sum_{k=1}^{Q} \lim_{n\to\infty} \frac{1}{n} \log\| T^n \xi_0^{(k)}\| \quad .$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|T^n \xi_0^{(1)} \wedge \ldots \wedge T^n \xi_0^{(Q)}\|}{\|T^n \xi_0^{(1)}\| \ldots \|T^n \xi_0^{(Q)}\|} = 0 \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{1}{n-m} \log \frac{\|T^{n,m} \xi_m^{(1)} \wedge \ldots \wedge T^{n,m}\, \xi_m^{(Q)}\|}{\|T^{n,m} \xi_m^{(1)}\| \ldots \|T^{n,m}\, \xi_m^{(Q)}\|} = 0$$

If a linear combination of $\xi_m^{(1)}$, ... , $\xi_m^{(Q)}$ were in V_m , the limit would be < 0 . Therefore $\xi_m^{(1)} + V_m$, ... , $\xi_m^{(Q)} + V_m$ are linearly independent in $\mathfrak{h}/V_m$, and the two conditions in (S2) are equivalent.

3.2. *Proposition* : We keep the notation and assumptions of Proposition 2.1. Let a measurable f-invariant integer-valued function $Q(\cdot) \geqq 0$ be given on Γ^+ . We suppose that for almost all x either $Q(x) = 0$, or $\ell_{Q(x)}^+(x)$ is finite and

$$\ell_2^+(x) - \ell_1^+(x) < \ell_1^+(x) \qquad \text{if} \quad Q(x) = 1$$
$$\ell_{Q+1}^+(x) - \ell_Q^+(x) < \ell_Q^+(x) - \ell_{Q-1}^+(x) \quad \text{if} \quad Q = Q(x) > 1 \tag{3.4}$$

There is then $\Gamma \subset \Gamma^+$ such that $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and if $x \in \Gamma$, condition (S) holds with $T_n = T(f^{n-1}x)$, $Q = Q(x)$, $\ell_q = \ell_q^+(x)$ for $q = 1, \dots, Q(x)$, and finite $\ell_{Q+1} \geqq \ell_{Q+1}^+(x)$. The quantities $r(q)$, $\mu^{(r)}$, $V^{(r)}$ occuring in Theorem 1.1., the vectors $\xi_0^{(1)}, \dots, \xi_0^{(Q)}$, the constants κ_ε of (S3) and D_ε of (S4) can all be chosen to depend measurably on x .

Proposition 2.1. yields (2.1), (2.2). Therefore (1.1), (1.2), (1.3) are satisfied. We have taken $\ell_1, \dots, \ell_Q$ equal to $\ell_1^+(x), \dots, \ell_Q^+(x)$ so that $r(1), \dots, r(Q) = S$ and $\mu^{(1)}, \dots, \mu^{(S)}$ are determined by (1.4) . All these are f-invariant measurable functions of x . The functions $x \longmapsto V^{(r)}$ are measurable, and $x \longmapsto \xi_0^{(1)}, \dots, \xi_0^{(Q)}$ may be chosen measurable. It remains to choose ℓ_{Q+1} and $\mu^{(S+1)}$ finite f-invariant measurable such that $\mu^{(S+1)} < \mu^{(S)}$ if $Q > 0$, and

$$\mu^{(1)} \geqq \ell_1^+(x) \qquad \text{if} \quad Q = 0$$
$$\mu^{(S+1)} \geqq \ell_{Q+1}^+(x) - \ell_Q^+(x) \qquad \text{if} \quad Q > 0 \tag{3.5}$$

That this is possible follows from (3.4) . All the assumptions of Theorem 1.1. hold therefore for almost all x , and (S1) is thus verified.

Since $T^{n,m}$ at x is T^{n-m} at $f^m x$, the space V_m at x is V_0 at $f^m x$, i.e. , $V^{(S+1)}$ at $f^m x$. The codimension of this space is $Q(f^m x) = Q(x)$. This proves (S2).

For $n \geqq 1$, define

$$F_n(x) = \log \|\hat{T}^{n,0}(x)\| \quad .$$

Theorem A.1. applies and the limit

$$F(x) = \lim_{n\to\infty} \frac{1}{n} F_n(x)$$

exists for almost all x . By theorem 1.1.(b) and the theorem of Banach-Steinhaus we have, if $Q > 0$,

$$F(x) \leqq \ell^+_{Q+1}(x) - \ell^+_Q(x) \quad .$$

The converse inequality follows from (2.2) with $q = Q+1$. Thus

$$F(x) = \ell^+_{Q+1}(x) - \ell^+_Q(x) \qquad \text{if} \quad Q > 0$$

$$F(x) = \ell^+_1(x) \qquad \text{if} \quad Q = 0$$

In view of (3.5) we have then

$$\mu^{(S+1)} \geqq F(x)$$

so that we may apply Corollary A.2., proving (S3) . Clearly $x \longmapsto \kappa_\varepsilon$ satisfying (3.1), (3.2) can be chosen measurable.

Let $V^\perp_n(x)$ be the subspace (of dimension Q) of $\mathfrak{h}$ orthogonal to $V_n(x) = V_0(f^n x)$. A map $\text{Ұ}(x) : V^\perp_0(x) \longmapsto V^\perp_0(fx)$ is defined by $T(x)\eta = \text{Ұ}(x)\eta + \zeta$, $\zeta \in V_0(fx)$. The multiplicative ergodic theorem (for instance Corollary 2.2.) may be applied to $\text{Ұ}(x)$. It shows that, for almost all x , the limits

$$\lim_{n\to\infty} \frac{1}{n} \log\| \text{Ұ}(f^{n-1}x) \cdot \ldots \cdot \text{Ұ}(fx)\, \text{Ұ}(x)\eta\|$$

exist for all $\eta \in V^\perp_0(x)$.

Write

$$\xi = \eta_0 + \zeta_0 \quad , \quad T_x^n \xi = \eta_n + \zeta_n$$

where $x \longmapsto \xi$ is measurable to $\mathcal{A} \vee V_0(x)$, and $\eta_n \in V_n^{\perp}(x)$, $\zeta_n \in V_n(x)$.
From the above we know that

$$\lim_{n\to\infty} \frac{1}{n} \log \| \eta_n \|$$

exists almost everywhere. We have

$$\|\zeta_n\| \leqq \|\hat{T}^{n,0}\| \cdot \|\zeta_0\| + \sum_{m=1}^{n} \|\hat{T}^{n,m}\| \cdot \|T(f^{m-1}x)\| \cdot \|\eta_{m-1}\|$$

so that, using (S3) and (2.1), we obtain almost everywhere

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\zeta_n\| \leqq \max\{\mu^{(S+1)}, \lim_{n\to\infty} \frac{1}{n} \log\|\eta_n\| \}$$

This implies

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n \xi\| = \lim_{n\to\infty} \frac{1}{n} \log\|\eta_n\|$$

or

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|\eta_n\|}{\|T_x^n \xi\|} = 0 \tag{3.6}$$

We apply (3.6) with $T^{\wedge Q}$ replacing T and $\xi_0^{(1)} \wedge \ldots \wedge \xi_0^{(Q)}$ replacing ξ . If $\eta_n^{(k)}$ denotes the component of $\xi_n^{(k)}$ orthogonal to V_n , we find that

$$\lim_{n\to\infty} \frac{1}{n} \log|\det(\eta_n^{(1)} , \ldots , \eta_n^{(Q)})| = 0 \tag{3.7}$$

where we have used (3.3) . Given $\varepsilon > 0$ there is thus $D_\varepsilon > 1$ such that

$$|\det(\eta_n^{(1)}, \ldots, \eta_n^{(Q)})| \geqq \frac{Q}{D_\varepsilon - 1} e^{-n\varepsilon}$$

almost everywhere, with $x \longmapsto D_\varepsilon$ measurable. If $\mathfrak{h} \ni u = v + w$ with $v \in V_n^{\perp}(x)$, $w \in V_n(x)$, we can write

$$v = \sum_{k=1}^{Q} u^{(k)} \eta_n^{(k)}$$

with $|u^{(k)}| \leqq \frac{D_\varepsilon - 1}{Q} e^{n\varepsilon} \|u\|$. Hence the vector

$$u^{(Q+1)} = u - \sum_{k=1}^{Q} u^{(k)} \xi_n^{(k)}$$

is in $V_n(x)$, and $\|u^{(Q+1)}\| \leqq D_\varepsilon e^{n\varepsilon} \|u\|$, proving (S4).

3.3. *Proposition* : Let (S) be satisfied, and define $\widetilde{V}^{(r)} = U^{(1)} + \ldots + U^{(r)}$ for $r = 1, \ldots, S$, and $\widetilde{V}^{(0)} = \{0\}$. If $(u_n)_{n \geqq 0}$ satisfies $T_n^* u_n = u_{n-1}$ and

$$\liminf_{n\to\infty} \frac{1}{n} \log \|u_n\| < -\mu^{(S+1)} \tag{3.8}$$

then $u_0 \in \widetilde{V}^{(S)}$. Conversely for every $u_0 \in \widetilde{V}^{(S)}$ there is such a sequence $(u_n)_{n \geqq 0}$, it is unique and

$$\lim_{n\to\infty} \frac{1}{n} \log \| u_n \| = -\mu^{(r)} \tag{3.9}$$

if $u_0 \in \widetilde{V}^{(r)} \setminus \widetilde{V}^{(r-1)}$, for $r = 1, \ldots, S$.

If $u_m \notin V_m^{\perp}$, there is $u \in V_m$ with $(u, u_m) \neq 0$, hence

$$0 = \lim_{n\to\infty} \frac{1}{n} \log |(u, T^{n,m*} u_n)|$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \|T^{n,m} u\| + \liminf_{n\to\infty} \frac{1}{n} \log \| u_n \|$$

$$\leqq \mu^{(S+1)} + \liminf_{n\to\infty} \frac{1}{n} \log \|u_n\|$$

in contradiction with (3.8). Therefore (3.8) implies $u_0 \in \widetilde{V}^{(S)}$ and $u_m \in V_m^{\perp}$.

Define now $\check{T}_m : V_{m-1}^{\perp} \longrightarrow V_m^{\perp}$ so that $\check{T}_m u$ is the orthogonal projection of $T_m u$ in $V_m^{\perp}$. In view of (S1) and (S4),

$$\lim_{n\to\infty} \frac{1}{n} \log\| (\check{T}^n)^{\wedge q} \| = \ell_q \tag{3.10}$$

for $q = 1, \ldots, Q$. We have thus

$$\lim_{n\to\infty} \frac{1}{n} \log\|\check{T}_n\| = 0$$

$$\lim_{n\to\infty} (\check{T}^{n*}\check{T}^n)^{1/2n} = \check{\Lambda}$$

where $\check{\Lambda}$ is the restriction of Λ_Q to $\widetilde{V}^{(S)}$.

From (3.10) with $q = Q$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log\|(\check{T}_n)^{\wedge Q}\| = 0$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n} \log\|(\check{T}_n)^{*-1}\| = 0 .$$

Since we have also

$$\lim_{n\to\infty}((\check{T}^{n*-1})^*(\check{T}^{n*-1}))^{1/2n} = \check{\Lambda}^{-1}$$

Theorem 1.1. applies to the sequence $(\check{T}_n^{*-1})$, and (3.9) follows.

3.4. *Corollary* (Multiplicative ergodic theorem for compact operators II). <u>With the notation and assumptions of Proposition 2.1. and Corollary 2.2.</u>, Γ <u>may be chosen such that the following holds if</u> $x \in \Gamma$.

(c) <u>Let</u> $\widetilde{V}^{(r)} = U^{(1)} + \ldots + U^{(r)}$. <u>If</u> $(u_n)_{n\geq 0}$ <u>satisfies</u> $T_n^* u_n = u_{n-1}$, <u>and</u>

$$\liminf_{n\to\infty} \frac{1}{n} \log\|u_n\| \neq +\infty$$

Then $u_0 \in \bigcup_{r<s+1} \tilde{V}^{(r)}$. Conversely for every $u_0 \in \bigcup_{r<s+1} \tilde{V}^{(r)}$ there is such a sequence $(u_n)_{n\geqq 0}$, it is unique and

$$\lim_{n\to\infty} \frac{1}{n} \log \|u_n\| = -\mu^{(r)} \quad \text{if} \quad u_0 \in \tilde{V}^{(r)} \setminus \tilde{V}^{(r-1)} .$$

It suffices to take Γ as in Proposition 3.2., and apply Proposition 3.3.

3.5. *Multiplicative ergodic theorems with respect to* f^{-1} .

We place ourselves again in the situation of Proposition 3.1., but assume now that f has a measurable inverse. (If this condition is not satisfied, the dynamical system can be extended so that f is replaced by an invertible $\tilde{f}$). Since we have $\log^+ \|T^* \circ f^{-1}(\cdot)\| \in L^1$ we may formulate multiplicative ergodic theorems where f is replaced by f^{-1} and T by $T^* \circ f^{-1}$.

Let

$$\tilde{\ell}_q(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{T}^n_x)^{\wedge q}\|$$

where $\tilde{T}^n_x = T^*(f^{-n}x) \cdot \ldots \cdot T^*(f^{-1}x)$. Since $(T^n_x)^* = \tilde{T}^n_{f^n x}$, we have

$$\ell_q(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(T^n_x)^{\wedge q}\| = \lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{T}^n_{f^n x})^{\wedge q}\|$$

The functions ℓ_q and $\tilde{\ell}_q$ are almost everywhere f-invariant and therefore almost everywhere equal. Therefore the "characteristic exponents" $\mu^{(r)}$, and the "multiplicities" $\dim U^{(r)}$ are almost everywhere the same for f^{-1} , $T^* \circ f^{-1}$ as for f , T .

Denote by $U^{(-r)}_x$ the eigenspace corresponding to the eigenvalue $\exp \mu^{(r)}$ of the operator

$$\tilde{\Lambda}_x = \lim_{n\to\infty} ([\tilde{T}^{n*}_x \tilde{T}^n_x]_Q)^{1/2n}$$

where we assume $r \leqq r(Q) < r(Q+1)$. Define

$$V^{(r)} = (U^{(1)} + \ldots + U^{(r-1)})^{\perp}$$
$$V^{(-r)} = U^{(-1)} + \ldots + U^{(-r)}$$

for $r = 1,2, \ldots$, so that $\operatorname{codim} V^{(r+1)} = \dim V^{(-r)}$. We show now that, almost everywhere,

$$V^{(r+1)} \cap V^{(-r)} = \{0\} \tag{3.11}$$

$$V^{(r+1)} + V^{(-r)} = \mathfrak{H} \quad . \tag{3.12}$$

Let E be the set of x such that $V^{(r+1)} \cap V^{(-r)} \neq \{0\}$. Given a nonvanishing $u_0 \in V^{(r+1)} \cap V^{(-r)}$ there is, by Proposition 3.3., a unique sequence (u_n) such that $T(f^{-n}x)u_n = u_{n-1}$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \| u_n \| \leqq -\mu^{(r)} \quad .$$

Furthermore,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_x^n u_0\| \leqq \mu^{(r+1)} \quad .$$

Given $\delta > 0$, let E_n be the subset of E consisting of those x such that

$$\|u_n\| \leqq \|u_0\| \exp n(-\mu^{(r)} + \delta) \tag{3.13}$$

if $u_0 \in V_x^{(r+1)} \cap V_x^{(-r)}$, and

$$\|T_x^n u\| \leqq \|u\| \exp n(\mu^{(r+1)} + \delta) \tag{3.14}$$

if $u \in V_x^{(r+1)} \cap V_x^{(-r)}$. We have $T_x^n u \in V_{f^n x}^{(r+1)} \cap V_{f^n x}^{(-r)}$. If $x \in f^{-n} E_n$, (3.13) implies thus

$$\|u\| \leqq \|T_x^n u\| \exp n(-\mu^{(r)} + \delta) \tag{3.15}$$

Therefore (3.14), (3.15) yields $\mu^{(r)} - \mu^{(r+1)} \leqq 2\delta$ if $x \in E_n \cap f^{-n} E_n$.

Since $\rho(E_n \cap f^{-n} E_n) \to \rho(E)$ when $n \to \infty$, we have $\mu^{(r)} - \mu^{(r+1)} \leqq 2\delta$ almost everywhere on E. Since δ is arbitrary, this gives $\rho(E) = 0$. We have proved (3.11) ; (3.12) follows from $\operatorname{codim} V^{(r+1)} = \dim V^{(-r)}$.

Defining

$$\delta(x) = \min_{u \in V_x^{(-r)} : \|u\|=1} \|\text{component of } u \text{ orthogonal to } V_x^{(r+1)}\|$$

we have for almost all x

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \delta(f^k x) = 0 \qquad (3.16)$$

For $k \to +\infty$, this results from (S4). Using the invertible maps $\overset{\vee}{T}(x) : V_x^{(r+1)\perp} \longmapsto V_{fx}^{(r+1)\perp}$ one obtains (3.16) for $k \to \pm\infty$.

4. *A perturbation theorem* :

4.1. *Theorem* : Let $(T_n)_{n>0}$ be a sequence of bounded operators in $\mathfrak{h}$ such that condition (S) holds.

Let $\eta > 0$ be given, and for $T' = (T'_n)_{n>0}$, write

$$||T'-T|| = \sup_n ||T'_n - T_n||\, e^{3n\eta} \tag{4.1}$$

and $T'^n = T'_n \cdot \ldots \cdot T'_2 \cdot T'_1$. There are then constants δ , $A > 0$, $B_\varepsilon > 1$ (for any $\varepsilon > 0$) with the following properties.

If $||T'-T|| \leqq \delta$, then

$$\lim_{n\to\infty} \frac{1}{n} \log ||(T'^n)^{\wedge q}|| = \ell_q \qquad \text{for } q = 1, \ldots, Q \tag{4.2}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log ||(T'^n)^{\wedge(Q+1)}|| \leqq \ell_{Q+1} \tag{4.3}$$

and (S) holds for (T'_n) . When $Q > 0$ we write

$$\lim_{n\to\infty} ([T'^{n*}\, T'^n]_Q)^{1/2n} = \Lambda'_Q$$

and let $P^{(r)}(T')$ denote the orthogonal projection of Λ'_Q corresponding to the eigenvalue $\exp \mu^{(r)}$. Writing also $P^{(S+1)}(T') = 1 - \sum_1^S P^{(r)}(T')$ we have

$$||T'^n\, P^{(r)}(T')|| \leqq B_\varepsilon \exp n\, (\mu^{(r)} + \varepsilon) \tag{4.4.a}$$

for $r = 1, \ldots, S+1$, and

$$||(T'^n)^{*-1}\, P^{(r)}(T')|| \leqq B_\varepsilon \exp n\, (-\mu^{(r)} + \varepsilon) \tag{4.4.b}$$

for $r = 1, \ldots, S$. If (T''_n) is like (T'_n) , with $||T''-T|| \leqq \delta$, then

$$||P^{(r)}(T') - P^{(r)}(T'')|| \leqq A \, ||T'-T''|| \tag{4.5}$$

If $N \geqq 0$, the sequence $T_{(N)} = (T_{n+N})_{n>0}$ again satisfies the condition (S). For any prescribed $\Theta > 0$, the corresponding constants δ^{-1}, A, B_ε may be chosen to increase with N less fast than the exponential $e^{N\Theta}$.

Let $0 < \eta' < \eta$ and define

$$||T'^{\wedge q} - T^{\wedge q}|| = \sup_n ||T_n'^{\wedge q} - T_n^{\wedge q}|| \, e^{3n\eta'} \quad .$$

Assuming $||T'-T|| \leqq$ const., (S1) implies the existence of $E_q > 0$ such that

$$||T'^{\wedge q} - T^{\wedge q}|| \leqq E_q \, ||T'-T|| \quad . \tag{4.6}$$

Using this and the replacements $T_n, T_n' \longmapsto T_n^{\wedge q}, T_n'^{\wedge q}$ we see that the proof of (4.2) reduces to the case $q = 1$, and the proof of (4.3) to the case $Q = 0$. (The reader will check that condition (S) is verified by $T^{\wedge q}$ with $Q = 1$ or $Q = 0$).

We have thus to prove

$$\lim_{n\to\infty} \frac{1}{n} \log ||T'^n|| = \mu^{(1)} \tag{4.7}$$

if $Q > 0$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log ||T'^n|| \leqq \mu^{(1)} \tag{4.8}$$

if $Q = 0$.

Proof of (4.8) :

Taking $Q = 0$ in (S3) we see that, given $\varepsilon > 0$, there is M such that

$$||T^{n,m}|| \leqq \exp[(n-m)\, \mu^{(1)} + 2n\varepsilon] \tag{4.9}$$

if $n > M$. We define $T'^{n,m}$ like $T^{n,m}$. If we write $T'_n = T_n + (T'_n - T_n)$, expand $T'^{N,M}$ in 2^{N-M} terms, and use (4.1) and (4.9) we find

$$\|T'^{N,M}\| \leqq \exp 2N\varepsilon \cdot \prod_{n=M+1}^{N} (\exp \mu^{(1)} + \|T'-T\| e^{n(-3\eta+2\varepsilon)}) .$$

Hence, if $\varepsilon < \eta$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \|T'^N\|$$

$$\leqq \limsup_{N\to\infty} \frac{1}{N} \log \|T'^{N,M}\| \leqq \mu^{(1)} + 2\varepsilon$$

and from this (4.8) follows.

We shall use later the following consequence of our proof :

$$\limsup_{n\to\infty} \frac{1}{n-m} \log \|(T'^{n,m})^{\wedge q}\| \leqq \sum_{k=1}^{q} \mu^{(r(k))} \tag{4.10}$$

which holds for all $m \geqq 0$, and $1 \leqq q \leqq Q+1$.

Proof of (4.7) :

We let now $Q > 0$. If we find $u \in \mathfrak{h}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|T'^n u\| \geqq \mu^{(1)}$$

we have

$$\liminf_{n\to\infty} \frac{1}{n} \log\|T'^n\| \geqq \mu^{(1)} .$$

Together with (4.8) this yields (4.7) . Therefore (4.7) follows from Proposition 4.2. below (with $M = 0$, $K = 1$).

Using (4.6) we conclude that (4.2) and (4.3) hold provided

$$\|T'-T\| \leqq \min_{1\leqq q\leqq Q} \delta_q/E_q \tag{4.11}$$

where δ_q corresponds to δ_1 in the replacement of T by $T^{\wedge q}$ in Proposition 4.2.

4.2. *Proposition* : <u>There exists</u> $\delta_1 > 0$ <u>such that</u>

$$\lim_{n\to\infty} \frac{1}{n} \log \|T'^{n,M} u\| \geqq \mu^{(r(K))}$$

<u>provided</u> $M \geqq 0$, $1 \leqq K \leqq Q$, $\|T'-T\| \leqq \delta_1$ <u>and</u> $u \in \mathcal{U}_M^K$ <u>with</u>

$$\mathcal{U}_M^K = \{ \sum_{j=1}^{Q} u^{(j)} \xi_M^{(j)} + u^{(Q+1)}$$

$$: u^{(Q+1)} \in V_M \text{ and } \max_{j \leqq K} |u^{(j)}| > \max_{K<k\leqq Q+1} |u^{(k)}| \}.$$

We shall prove for later use a result somewhat stronger than the above proposition.

Given $u \in \mathfrak{h}$, $M \geqq 0$, we write

$$T'^{n,M} u = \sum_{j=1}^{Q} u_n^{(j)} \xi_n^{(j)} + u_n^{(Q+1)}$$

where $u_n^{(Q+1)} \in V_n$.

We have, using (S4) with $D_\eta = D$,

$$|u_n^{(k)}| \leqq t_n^{(k)} |u_{n-1}^{(k)}| + D\delta_1 e^{-2n\eta} \sum_{j=1}^{Q+1} |u_{n-1}^{(j)}| \tag{4.12}$$

$$|u_n^{(k)}| \geqq t_n^{(k)} |u_{n-1}^{(k)}| - D\delta_1 e^{-2n\eta} \sum_{j=1}^{Q+1} |u_{n-1}^{(j)}| \tag{4.13}$$

for $k = 1 , \dots , Q$, and

$$|u_n^{(Q+1)} - T_n u_{n-1}^{(Q+1)}| \leqq D\delta_1 e^{-2n\eta} \sum_{j=1}^{Q+1} |u_{n-1}^{(j)}| \tag{4.14}$$

We shall estimate the $|u_n^{(k)}|$ under the assumption that

$$(\forall n \geqq M) \quad \max_{j<J} |u_n^{(j)}| \leqq \max_{k\geqq J} |u_n^{(k)}| \tag{4.15}$$

for some fixed J . Notice in particular that

$$J \leqq K \qquad \text{if} \qquad u \in \mathcal{U}_M^K \tag{4.16}$$

Instead of using (3.1) one may express (S3) by

$$\log\|\hat{T}^{n,m}\| \leqq (n-m)\ \mu^{(S+1)} + n\ \varepsilon(n) \tag{4.17}$$

with $\varepsilon(n)$ decreasing towards 0 when $n \to \infty$.

If $k \geqq J$ we write

$$t_n^{(k)*} = t_n^{(k)} \ \exp(\mu^{(r(J))} - \mu^{(r(k))}) \qquad .$$

Taking $\varepsilon = \eta/2$ in (3.2) we find

$$\prod_{m=M+1}^{n-1} t_m^{(j)*} \ / \prod_{m=M+1}^{n} t_m^{(k)*} \leqq C\ e^{n\eta} \tag{4.18}$$

$$\exp[(n-1-M)\mu^{(r(J))}+(n-1)\varepsilon(n-1)] \ / \prod_{m=M+1}^{n} t_m^{(k)*} \leqq Ce^{n\eta} \tag{4.19}$$

$$\prod_{m=M+1}^{n-1} t_m^{(j)*} \ /\exp[(n-M)\mu^{(r(J))}] \leqq Ce^{n\eta} \tag{4.20}$$

$$\exp[(n-1-M)\mu^{(r(J))}+(n-1)\varepsilon(n-1)]/\exp[(n-M)\mu^{(r(J))}] \leqq Ce^{n\eta} \tag{4.21}$$

where C is independent of J,j,k,M,n (with $j,k \geqq J$, $0 \leqq M < n$).

From (4.12) we obtain

$$|u_n^{(k)}| \leqq t_n^{(k)*}|u_{n-1}^{(k)}| + D\delta_1 e^{-2n\eta}\ (Q+1) \ \max_j |u_{n-1}^{(j)}| \tag{4.22}$$

for $k = J , \dots , Q$. From (4.14) we get

$$|u_n^{(Q+1)} - T_n u_{n-1}^{(Q+1)}| \leqq D\delta_1 e^{-2n\eta}(Q+1) \ \max_j |u_{n-1}^{(j)}| \quad . \tag{4.23}$$

If $U_M = \max_j |u_M^{(j)}|$ and $n \geqq M$ we claim that

$$|u_n^{(k)}| \leqq \prod_{m=M+1}^{n} t_m^{(k)*} \cdot \prod_{m=M+1}^{n} (1+(Q+1)\ CD\delta_1 e^{-m\eta})\cdot U_M \tag{4.24}$$

for $k = J , \ldots , Q$, and

$$|u_n^{(Q+1)}| \leqq \exp[(n-M)\mu^{(r(J))}+n\varepsilon(n)]\cdot \prod_{n=M+1}^{n} (1+(Q+1)CD\delta_1 e^{-m\eta})\cdot U_M \tag{4.25}$$

Clearly (4.24) and (4.25) hold for $n = M$. Inserting (4.24),(4.25) and (4.15) in the right-hand side of (4.22) , and using (4.18), (4.19) we reproduce (4.24) . On the other hand, from (4.23) we get

$$|u_N^{(Q+1)}| \leqq \|\hat{T}^{N,M}\| U_M + \sum_{n=M+1}^{N} \|\hat{T}^{N,n}\| (Q+1) D\delta_1 e^{-2n\eta} \max_j |u_{n-1}^{(j)}| .$$

Using (4.17), (4.24), (4.25), (4.15), (4.20) and (4.21) yields

$$|u_N^{(Q+1)}| \leqq \exp((N-M)\ \mu^{(r(J))} + N\varepsilon(N)+\kappa) \times$$

$$\times \quad [U_M + \sum_{n=M+1}^{N} (Q+1)\ CD\delta_1 e^{-n\eta} \prod_{m=M+1}^{n-1} (1+(Q+1)CD\delta_1 e^{-m\eta})\cdot U_M]$$

which reproduces (4.25) . This proves (4.24) and (4.25).

We choose

$$\delta_1 = \frac{1}{(Q+1)CD} \prod_{m=1}^{\infty} (1-e^{-m\eta})^2 \tag{4.26}$$

In this way $(Q+1)CD\delta_1 < 1$, and

$$\begin{aligned} C_1 = \frac{\prod_{m=1}^{\infty}(1+(Q+1)CD\delta_1 e^{-m\eta})}{\prod_{m=1}^{\infty}(1-e^{-m\eta})} &\leqq \prod_{m=1}^{\infty} \frac{1+e^{-m\eta}}{1-e^{-m\eta}} . \\ &\leqq \prod_{n=1}^{\infty} (1-e^{-n\eta})^{-2} = \frac{1}{(Q+1)CD\delta_1} . \end{aligned} \tag{4.27}$$

Therefore (4.24) and (4.25) give

$$|u_n^{(k)}| \leqq C_1 \prod_{m=M+1}^{n} t_m^{(k)*} \cdot \prod_{m=M+1}^{n} (1-e^{-m\eta}) \cdot U_M \tag{4.28}$$

for $k = J , \ldots , Q$, and

$$|u_n^{(Q+1)}| \leqq C_1 \exp[(n-M)\mu^{(r(J))} + n\varepsilon(n)] \cdot \prod_{m=M+1}^{n} (1-e^{-m\eta}) \cdot U_M \tag{4.29}$$

We define now J to be the largest integer such that (4.15) holds. If $J \leqq Q$ we may choose $M' \geqq M$ such that $|u_{M'}^{(J)}| = \max_J |u_{M'}^{(j)}| = U_{M'}$. Inserting then (4.28), (4.29) with M replaced by M', and (4.18), (4.19) into (4.13) we get

$$|u_n^{(J)}| \geqq t_n^{(J)} \; |u_{n-1}^{(J)}|$$

$$- \quad (Q+1) C C_1 D \delta_1 e^{-n\eta} \prod_{m=M'+1}^{n} t_m^{(J)} \prod_{m=M'+1}^{n-1} (1-e^{-m\eta}) \cdot |u_{M'}^{(J)}|$$

for $n > M'$. Using (4.27) gives

$$|u_n^{(J)}| \geqq t_n^{(J)} \; [|u_{n-1}^{(J)}| - e^{-n\eta} \prod_{m=M'+1}^{n-1} t_m^{(J)} \cdot \prod_{m=M'+1}^{n-1} (1-e^{-m\eta}) \cdot |u_{M'}^{(J)}|]$$

which implies, by induction,

$$|u_n^{(J)}| \geqq \prod_{m=M'+1}^{n} t_m^{(J)} \cdot \prod_{m=M'+1}^{n} (1-e^{-m\eta}) \cdot |u_{M'}^{(J)}| \tag{4.30}$$

In view of (4.28), (4.29), we have

$$\limsup_{n\to\infty} \frac{1}{n} \log\|T'^{n,M}u\| \leqq \mu^{(r(J))}$$

for all J . On the other hand (4.30) and (S4) yield

$$\liminf_{n\to\infty} \frac{1}{n} \log\|T'^{n,M}u\| \geqq \mu^{(r(J))}$$

if $J \leqq Q$. We have thus

$$\lim_{n\to\infty} \frac{1}{n} \log\|T'^{n,M}u\| = \mu^{(r(J))} \quad \text{if} \quad J \leqq Q$$
$$\limsup_{n\to\infty} \frac{1}{n} \log\|T'^{n,M}u\| \leqq \mu^{(S+1)} \quad \text{if} \quad J = Q+1 \quad . \tag{4.31}$$

Proposition 4.2 follows from (4.16) and (4.31).

Proof of (4.4a) :

Tbeorem 1.1. now applies with (T'_n) replacing (T_n) . Therefore if u is in the range of $P^{(r)}(T')$, and $u \neq 0$, (4.31) shows that $r(J) = r$. In particular (4.28) and (4.29), with $M = 0$, prove (4.4.a).

Proof of (4.5) :

Let $K \leqq Q$, with $r(K+1) = r(K) + 1$, and define

$$V'_m = \{u \in \mathfrak{h} : \limsup_{n\to\infty} \frac{1}{n-m} \log\|T'^{n,m}u\| \leqq \mu^{(r(K)+1)}\} \quad .$$

Let $\hat{T}'^{n,m}$ denote the restriction of $T'^{n,m}$ to V'_m . If $u \in V'_M$, (4.31) implies that $J > K$. Therefore, by (4.28), (4.29) , (S4) , one can take $\varepsilon'(n)$ decreasing towards 0 when $n\to\infty$, such that

$$\log\|\hat{T}'^{n,m}\| \leqq (n-m)\ \mu^{(r(J))} + n\,\varepsilon'(n)$$
$$\leqq (n-m)\ \mu^{(r(K)+1)} + n\,\varepsilon'(n) \quad . \tag{4.32}$$

Proposition 4.2. shows that $\xi_M^{(1)}$, ... , $\xi_M^{(K)}$ are linearly independent modulo V'_M . We also know that Theorem 1.1. applies with $(T'_{n+M})_{n>0}$ replacing (T_n) ; the $\mu^{(r)}$ remain the same. In particular by taking $q = K+1$ in (4.10) we see that codim $V'_M = K$. Therefore there exist numbers λ_{kj} such that

$$\xi_M^{(k)} - \sum_{j=1}^{K} \lambda_{kj}\ \xi_M^{(j)} \in V'_M \qquad \text{for } k = K+1 \text{ , ... , } Q \text{ .} \tag{4.33}$$

Similarly, if $u^{(Q+1)} \in V_M$, there exist numbers λ_j such that

$$u^{(Q+1)} - \sum_{j=1}^{K} \lambda_j \, \xi_M^{(j)} \in V_M' \quad . \tag{4.34}$$

In view of Proposition 4.2. we have

$$|\lambda_{kj}| \leqq 1 \quad , \; |\lambda_j| \leqq |u^{(Q+1)}|$$

for $k = K+1 , \ldots , Q$, and $j = 1 , \ldots , K$. Given any $u \in \mathfrak{H}$ we may write $u = \sum_{k=1}^{Q} u^{(k)} \xi_n^{(k)} + u^{(Q+1)}$ with $u^{(Q+1)} \in V_n$ where $|u^{(k)}| \leqq D_\varepsilon e^{n\varepsilon} \|u\|$ by (S4) . Using (4.33), (4.34) one finds

$$u = \sum_{j=1}^{K} v^{(j)} \, \xi_n^{(j)} + v^{(K+1)} \quad , \quad v^{(K+1)} \in V_n' \tag{4.35}$$

with

$$|v^{(j)}| \leqq (Q-K+2) \, D_\varepsilon \, e^{n\varepsilon} \|u\| \qquad \text{for} \quad j = 1, \ldots , K$$

because $v^{(j)} = u^{(j)} + \sum_{k=K+1}^{Q} u^{(k)} \lambda_{kj} + \lambda_j$. Therefore also

$$|v^{(K+1)}| \leqq (Q-K+3) \, D_\varepsilon \, e^{n\varepsilon} \|u\|$$

so that (4.35) holds with

$$|v^{(j)}| \leqq D_\varepsilon' \, e^{n\varepsilon} \|u\| \qquad \text{for} \quad j = 1, \ldots , K+1 \; . \tag{4.36}$$

We may assume that we have chosen κ_ε' , and $D_\varepsilon' \geqq D_\varepsilon$, independent of K and T' (subject to $K \leqq Q$, $\|T'-T\| \leqq \delta$) .

Let now $0 < \alpha \leqq 1$, and define

$$\mathcal{V} = \{ \sum_{j=1}^{K} v^{(j)} \, \xi_0^{(j)} + \frac{1}{\alpha} v^{(K+1)}$$

$$: v^{(K+1)} \in V_0' \quad \text{and} \quad \max_{j \leqq K} |v^{(j)}| > |v^{(K+1)}| \} \; .$$

For $u \in \mathcal{V}$ we write

$$T''^n u = \sum_{j=1}^{K} v_n^{(j)} \xi_n^{(j)} + \frac{1}{\alpha} v_n^{(K+1)} .$$

The inequalities (4.12), (4.13), (4.14) are replaced by

$$|v_n^{(k)}| \leqq t_n^{(k)} |v_{n-1}^{(k)}| + D'\delta' e^{-2n\eta} \sum_{j=1}^{K+1} |v_{n-1}^{(j)}|$$

$$|v_n^{(k)}| \geqq t_n^{(k)} |v_{n-1}^{(k)}| - D'\delta' e^{-2n\eta} \sum_{j=1}^{K+1} |v_{n-1}^{(j)}|$$

$$|v_n^{(K+1)} - T_n' v_{n-1}^{(K+1)}| \leqq D'\delta' e^{-2n\eta} \sum_{j=1}^{K+1} |v_{n-1}^{(j)}|$$

where we have written $D' = D_\eta'$, and we assume

$$||T'-T''|| \leqq \alpha\delta' , ||T'-T|| \leqq \delta' , ||T''-T|| \leqq \delta' \tag{4.37}$$

(We shall fix $\delta' \leqq \delta_1$ later). We estimate the $|v_n^{(k)}|$ in much the same way as we estimated the $|u_n^{(k)}|$ in the proof of Proposition 4.2. We have mostly to replace Q by K , (S3) by (4.32), and (S4) by (4.35), (4.36) . We let here J be the largest integer such that

$$(\forall n) \qquad \max_{j \leqq J} |v_n^{(j)}| \leqq \max_{k \geqq J} |v_n^{(k)}|$$

(in particular $J \leqq K$) . We retain the definition of $t_n^{(k)*}$ and choose $C' \geqq C$ such that (4.18), (4.19), (4.20), (4.21) hold with κ , $\varepsilon(n)$ replaced by κ' , $\varepsilon'(n)$. Finally we take

$$\delta' = \frac{1}{(Q+1)C'D'} \prod_{m=1}^{\infty} (1-e^{-m\eta})^2 \tag{4.38}$$

and obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log ||T''^n u|| \geqq \mu^{(r(J))} \geqq \mu^{(r(K))}$$

when $u \in V$ and (4.37) holds.

Therefore if u is in the range of $P^{(r(K+1))}(T'') + \ldots + P^{(r(Q+1))}(T'')$ we have $u \notin V$, i.e.,

$$\max_{j \leqq K} |v^{(j)}| \leqq |v^{(K+1)}|$$

and since $(P^{(1)}(T') + \ldots + P^{(r(K))}(T'))v^{(K+1)} = 0$ we have

$$\|(P^{(1)}(T') + .. + P^{(r(K))}(T'))(P^{(r(K+1))}(T'') + .. + P^{(r(Q+1))}(T''))\| \leqq K\alpha D'$$

where we have used (4.36) with $n = 0$, $j = K+1$, $\varepsilon = \eta$. Let us write $P' = P^{(1)}(T') + \ldots + P^{(r(K))}(T')$, $P'' = P^{(1)}(T'') + \ldots + P^{(r(K))}(T'')$. In view of (4.37) we may write

$$\|P'(1-P'')\| \leqq K\alpha D' = \frac{KD'}{\delta'} \|T'-T''\| \quad .$$

Interchanging T' and T'' yields

$$\|P''(1-P')\| \leqq \frac{KD'}{\delta'}\|T'-T''\|$$

so that

$$\|P'-P''\| \leqq 2 \frac{KD'}{\delta'} \|T'-T''\| \quad .$$

Therefore

$$\|P^{(r)}(T')-P^{(r)}(T'')\| \leqq 4 \frac{QD'}{\delta'} \|T'-T''\| \tag{4.39}$$

for $r = 1, \ldots , r(Q) = S$. This proves (4.5) .

<u>Proof of</u> (4.4.b) :

Let P_M be the orthogonal projection on $V_M^{\perp}$ where V_M is defined in Section 3.1., and let $P_M(T')$ be the corresponding projection where T is replaced by T' .

An easy modification of Proposition 4.2. (see the proof of (4.5)) yields that, if $0 < \alpha \leqq 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|T'^{n,M}u\| \geqq \mu^{(S)}$$

provided $\|T'-T\| \leqq \alpha\delta_1$ and $u \in \mathcal{U}_M$, with

$$\mathcal{U}_M = \{ \sum_{j=1}^{Q} u^{(j)} \xi_M^{(j)} + \frac{1}{\alpha} u^{(Q+1)}$$

$$: u^{(Q+1)} \in V_M \text{ and } \max_{j\leqq Q} |u^{(j)}| > | u^{(Q+1)} |\} .$$

Therefore, if u is in the range of $1 - P_M(T')$, then $u \notin \mathcal{U}_M$, and (S4) yields

$$\|P_M u\| \leqq \alpha Q D_\zeta \, e^{M\zeta}\|u\| ,$$

hence

$$\|P_M(1-P_M(T'))\| \leqq \frac{\|T'-T\|}{\delta_1} QD_\zeta \, e^{M\zeta} .$$

Taking $\zeta < \frac{1}{2}\eta$ we may use in this formula $\eta' = \eta - 2\zeta$ instead of η . We may thus replace $\|T'-T\|$ by

$$\sup_{n\geqq M}\|T_n' - T_n\| \, e^{3n(\eta-2\zeta)} \leqq \|T'-T\| \, e^{-6M\zeta} ,$$

so that

$$\|P_M(1-P_M(T'))\| \leqq C\|T'-T\| \, e^{-5M\zeta} .$$

For sufficiently small $\|T'-T\|$ there exist thus isometries $U_M : V_M \rightarrowtail \mathfrak{h}$ with range $P_M(T')\mathfrak{h}$ such that

$$\|U_M - P_M\| \leqq C'\|T'-T\| \, e^{-5M\zeta} .$$

We write $\check{T}_n = P_n T_n P_{n-1}$, $\check{T}_n' = U_n^* T_n' U_{n-1}$ and obtain

$$\|\check{T}'_n - \check{T}_n\| \leqq C'' \|T'-T\| e^{-4M\zeta} \quad .$$

Therefore also

$$\|\check{T}'^{*-1}_n - \check{T}^{*-1}_n\| \leqq C''' \|T'-T\| e^{-3M\zeta} \quad .$$

We may thus apply Theorem 4.1. to the sequences $(\check{T}^{*-1}_n)$, $(\check{T}'^{*-1}_n)$ and obtain (4.4.b) as a consequence of (4.4.a) (with new choices of δ and B_ε) .

<u>Proof that</u> (S) <u>holds for</u> (T'_n) :

From the above we obtain also

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|\check{T}'^n \xi_0^{(1)} \wedge \ldots \wedge \check{T}'^n \xi_0^{(Q)}\|}{\|T'^n \xi_0^{(1)}\| \ldots \|T'^n \xi_0^{(Q)}\|} = 0 \quad .$$

This corresponds to (3.7) with (T_n) replaced by (T'_n) , and implies therefore that (T'_n) satisfies (S4) . In the proof of (4.4.a) we have noted that (S1) holds for (T'_n) and in the proof of (4.5) we have obtained (S.2), (S.3). Therefore (T'_n) again satisfies (S) .

<u>Estimates of</u> δ^{-1} , A , B_ε <u>for</u> $T_{(N)}$.

If we replace $(T_n)_{n>0}$ by $(T_{n+N})_{n>0}$, and $\xi_0^{(1)}, \ldots, \xi_0^{(Q)}$ by $\xi_N^{(1)}, \ldots, \xi_N^{(Q)}$ in Proposition 4.2., we may in the proof make the replacements $\varepsilon(n) \longmapsto (1+\frac{N}{n})\varepsilon(n+N)$, $C \longmapsto Ce^{N\eta}$, $D \longmapsto De^{N\eta}$. Then, according to (4.26), δ_1^{-1} is replaced by $\delta_1^{-1} e^{2N\eta}$. If we use $T^{\wedge q}$ instead of T we have $\delta_q^{-1} \longmapsto \delta_q^{-1} e^{2N\eta'}$. For E_q , see (4.6), we may take $E_q \longmapsto E_q e^{3N(\eta-\eta')}$. Therefore $\min \delta_q/E_q$,which is the choice of δ used in the proof of (4.2), (4.3) (see (4.11)) is multiplied by $e^{-N(3\eta-\eta')} \geqq e^{-3N\eta}$.

Let $\Theta > 0$ be given. In the proof of (4.4.a) we may choose B_ε , according to (4.28), (4.29), (3.2) and (S4) , so that $B_\varepsilon \longmapsto B_\varepsilon e^{N\Theta}$. Similarly for (4.4.b).

In the proof of (4.5) we may make the replacements $C' \longmapsto C'e^{N\eta}$, $D' \longmapsto D' e^{N\eta}$. Hence by (4.38), (4.39), $\delta'^{-1} \longmapsto \delta'^{-1} e^{2N\eta}$, $A \longmapsto A e^{3N\eta}$. Since we choose δ less than $\min_q \delta_q/E_q$ and δ' , we may take $\delta^{-1} \longmapsto \delta^{-1} e^{3N\eta}$. (The replacements of η by η' or $\zeta < \eta$ in the proof of (4.4.b) do not change this).

If $0 < \Theta < 3\eta$, we have

$$\sup_n \| T'_n - T_n \| e^{n\Theta} \leqq \| T' - T \|$$

We may thus use $\Theta/3$ instead of η and arrange that δ^{-1} , A , $B_\varepsilon \longmapsto \delta^{-1} e^{N\Theta}$, $A e^{N\Theta}$, $B_\varepsilon e^{N\Theta}$ when $T \longmapsto T_{(N)}$. Therefore δ^{-1} , A , B_ε may be chosen to increase with N less fast than $e^{N\Theta}$.

5. *Local stable manifolds* :

From now on $\mathfrak{h}$ will be separable. For $R > 0$, we write $B(R) = \{u \in \mathfrak{h} : \|u\| < R\}$, and let $\overline{B}(R)$ be the closure of $B(R)$. We say that a map is of class $C^{\underset{\sim}{r},\theta}$ if its derivatives up to order $\underset{\sim}{r}$ are Hölder continuous of exponent θ ; $C^{\underset{\sim}{r},\theta}$ manifolds are defined in the obvious way.

5.1. *Theorem* : Let (M,Σ,ρ) be a probability space, and $f : M \longmapsto M$ a measurable map preserving ρ. For each $x \in M$, let $F_x : (\overline{B}(1),0) \longmapsto (\mathfrak{h},0)$ be defined, and write $F_x^n = F_{f^{n-1}x} \circ \ldots \circ F_{fx} \circ F_x$. We assume that F_x is differentiable, with derivative $T(x)$ at 0, that $T(\cdot)$ is measurable, and that $\log^+ \|T(\cdot)\| \in L^1(M,\rho)$. Let an integer-valued f-invariant measurable function $Q \geqq 0$ be given. Then $\mu^{(1)} > \mu^{(2)} > \ldots$ and $r(1), \ldots, r(Q+1)$ are defined for almost all x, such that

$$\sum_{k=1}^{q} \mu^{(r(k))} = \lim_{n\to\infty} \frac{1}{n} \log \|(T_x^n)^{\wedge q}\| \tag{5.1}$$

for $q = 1, \ldots, Q+1$ *). We assume that Q and the real f-invariant measurable function $\lambda < 0$ are so chosen that

$$\mu^{(r(Q+1))} < \lambda < \mu^{(r(Q))} \tag{5.2}$$

almost everywhere. (We let $\mu^{(r(0))} = +\infty$; $\mu^{(r(Q+1))}$ may be $-\infty$).

We consider three cases.

I. $\mathfrak{h}$ is real ; F_x is $C^{\underset{\sim}{r},\theta}$; $x \longmapsto \|F_x\|_{\underset{\sim}{r},\theta}$ is measurable and

$$\int \rho(dx) \log^+ \|F_x\|_{\underset{\sim}{r},\theta} < +\infty$$

for some $\underset{\sim}{r} \geqq 1$, $\theta \in (0,1]$.

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*) See Proposition 2.1.

II. $\mathfrak{h}$ is real ; F_x is C^∞ ; $x \longmapsto \|F_x\|_{\underset{\sim}{r}}$ is measurable and

$$\int \rho(dx) \log^+ \|F_x\|_{\underset{\sim}{r}} < +\infty$$

for all $\underset{\sim}{r} \geqq 1$.

III. $\mathfrak{h}$ is complex ; F_x is holomorphic in $B(1)$; $x \longmapsto \|F_x\|_1$ is measurable and

$$\int \rho(dx) \log^+ \|F_x\|_1 < +\infty$$

Let $\Theta > 0$. Under the above conditions there is a measurable set $\Gamma \subset M$ such that $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and there are measurable functions $\beta > \alpha > 0$ on Γ with the following properties.

(a) If $x \in \Gamma$ the set

$$\mathcal{V}_x^\lambda = \{u \in \overline{B}(\alpha(x)) : \|F_x^n u\| \leqq \beta(x) e^{n\lambda(x)} \text{ for all } n \geqq 0\}$$

is a submanifold of $\overline{B}(\alpha(x))$, tangent at 0 to $V_x^{(r(Q)+1)}$. The manifold $\mathcal{V}_x^\lambda$ is respectively $C^{\underset{\sim}{r},\theta}$, C^∞ or holomorphic in the cases I., II., III.

(b) If $\lambda' : M \longmapsto \mathbb{R}$ is f-invariant measurable and satisfies

$$\mu^{(r(Q+1))} < \lambda' < \mu^{(r(Q))}$$

there exists $\gamma > 1$ measurable on Γ such that, if $u,v \in \mathcal{V}_x^\lambda$, then

$$\|F_x^n u - F_x^n v\| \leqq \gamma(x) \|u-v\| e^{n\lambda'(x)} \quad .$$

This applies in particular to $\lambda' = \lambda$. .

(c) If $x \in \Gamma$, then $\alpha(f^N x)$, $\beta(f^N x)$, $\gamma(f^N x)^{-1}$ decrease less fast than the exponential $e^{-N\Theta}$ when $N \to \infty$.

We first study the case I. with $r = 1$. By Proposition 2.1., we may take

$\Gamma \subset M$ such that $f\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and (5.1), (5.2) hold on Γ. We further assume that, if $x \in \Gamma$,

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \| F_{f^{n-1}x} \|_{1,\theta} = 0 \tag{5.3}$$

This is possible by the ergodic theorem, and implies

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \| T(f^{n-1}x) \| = 0 \quad .$$

The assumptions of Theorem 1.1. hold thus with $T_n = T(f^{n-1}x)$ and $Q = Q(x)$. If necessary we modify the definition of $\mu^{(r(Q+1))}$ to make it finite ; we replace thus the original $\mu^{(r(Q+1))}$ by a larger f-invariant function $x \longmapsto \mu^{(S+1)} < \lambda, \lambda'$. With this notation, Proposition 3.2. shows that we may assume (S) to hold for all $x \in \Gamma$. Notice that the linear space $V_x^{(r(Q)+1)}$ in part (a) of the Theorem is, in the notation of Section 3.1., $V_0 = V^{(S+1)}$.

We are now in position to apply the perturbation theorem 4.1. We choose η in this theorem satisfying $0 < 4\eta \leqq -\theta\lambda$. (We write λ, λ', ... , instead of $\lambda(x)$, $\lambda'(x)$, ...). Using (5.3) we may then write

$$\left. \begin{aligned} G = \sup_n \| F_{f^{n-1}x} \|_{1,\theta} \exp(-n\eta - \theta\lambda) < +\infty \\ \| F_{f^{n-1}x} \|_{1,\theta} \exp[n(\theta\lambda + 3\eta) - \theta\lambda] \leqq G \end{aligned} \right\} \tag{5.4}$$

We also define $\varepsilon > 0$ by

$$\varepsilon = \min(\lambda, \lambda') - \mu^{(S+1)} \tag{5.5}$$

With these choices, let $\delta, A > 0$ and $B_\varepsilon > 1$ be as in Theorem 4.1. We can make δ smaller such that

$$A\delta < 1/\sqrt{2} \tag{5.6}$$

and then define α, β, γ satisfying

$$\left.\begin{array}{ll} & \alpha = \beta/B_\varepsilon < \beta \\ 0 < \beta < 1 \quad , & G\beta^\theta < \delta \\ & \gamma = B_\varepsilon > 1 \end{array}\right\} \tag{5.7}$$

The functions $x \longmapsto \delta, A, B_\varepsilon$ may be assumed measurable as follows from their (essentially) explicit construction in the proof of Theorem 4.1. Therefore also $x \longmapsto \alpha, \beta, \gamma$ may be assumed measurable. We prove case I. $r = 1$, of Theorem 5.1. with the above choices of $\Gamma, \alpha, \beta, \gamma$.

If $0 < R \leqq 1$ we write

$$\left.\begin{array}{l} S^\nu(R) = \{u \in \overline{B}(1) : \|F_x^n u\| \leqq R\, e^{n\lambda} \text{ for } 0 \leqq n \leqq \nu\} \\ S(R) = \{u \in \overline{B}(1) : \|F_x^n u\| \leqq R\, e^{n\lambda} \text{ for all } n \geqq 0\} \end{array}\right\} \tag{5.8}$$

Taking $\kappa > 1$ such that $\kappa\beta \leqq 1$, $G\cdot(\kappa\beta)^\theta \leqq \delta$, we show now that

$$\begin{aligned} &\overline{B}(\alpha) \cap S^\nu(\beta) \cap \{u \in \mathfrak{H} : \|T^{n,\nu} F_x^\nu u\| \leqq \beta e^{n\lambda} \text{ for all } n > \nu\} \\ &\quad = \overline{B}(\alpha) \cap S^\nu(\kappa\beta) \cap (F_x^\nu)^{-1} V_\nu \end{aligned} \tag{5.9}$$

(V_ν is defined in Section 3.1.) . Let indeed $u \in S^\nu(\kappa\beta) \cap (F_x^\nu)^{-1} V_\nu$. The bounded operators

$$T'_n = \int_0^1 dt\, DF_{f^{n-1}x}(t\, F_x^{n-1} u) \qquad \text{if } n \leqq \nu$$

$$T'_n = T_n \qquad \text{if } n > \nu$$

are such that

$$T'^n u = T'_n \cdot \ldots \cdot T'_1 u = F^n_x u \qquad \text{if} \quad n \leqq \nu$$

and using (5.8), (5.4), we have

$$\|T'-T\| = \sup_n \|T'_n - T_n\| \, e^{3n\eta}$$

$$\leqq \sup_{n \leqq \nu} \|DF_{f^{n-1}x}\|_\theta \, (\kappa\beta)^\theta \exp[(n-1)\theta\lambda + 3n\eta] \leqq G(\kappa\beta)^\theta \leqq \delta$$

Therefore Theorem 4.1. applies. In particular u is in the range of $P^{(S+1)}(T')$. From (4.4a) and (5.5). We get thus

$$\|T'^n u\| \leqq B_\varepsilon \, e^{n\lambda} \|u\|$$

for all ν and all $u \in S^\nu(\kappa\beta) \cap (F^\nu_x)^{-1} V_\nu$. Since $\alpha = \beta/B_\varepsilon$, the right-hand side of (5.9) is thus contained in the left-hand side. The converse inclusion is immediate.

Let D^ν be the set defined by (5.9). Since the boundary of $S^\nu(\kappa\beta)$ is disjoint from $S^\nu(\beta)$, and hence from D^ν, we conclude from (5.9) that D^ν is open and closed in $\overline{B}(\alpha) \cap (F^\nu_x)^{-1} V_\nu$. In fact D^ν <u>is a C^1 submanifold of $\overline{B}(\alpha)$</u>. To see this it suffices to show that if $v \in D^\nu$ the range of $DF^\nu_x(v)$ together with V_ν span $\mathfrak{h}$ (transversality). Writing

$$T'_n = DF_{f^{n-1}x}(F^{n-1}_x v) \qquad \text{if} \quad n \leqq \nu$$

$$T'_n = T_n \qquad \text{if} \quad n > \nu$$

we have $\|T'-T\| < \delta$ and the transversality condition is that the range of T'^ν together with V_ν span $\mathfrak{h}$. If u is a nontrivial linear combination of $\xi^{(1)}_0, \ldots, \xi^{(Q)}_0$ if follows from Proposition 4.2. that $T'^\nu u \notin V_\nu$. This implies transversality because V_ν has codimension Q by (S2). Furthermore, <u>the tangent space to D^ν at v is the range of $P^{(S+1)}(T')$</u>, a fact which we shall use later.

Let now $u,v \in D^{\nu}$ or $u,v \in \overline{B}(\alpha) \cap S(\beta) = V_x^{\lambda}$ (in the latter case, write $\nu = \infty$) . The bounded operators

$$T'_n = \int_0^1 dt\ DF_{f^{n-1}x}(t\ F_x^{n-1}u+(1-t)\ F_x^{n-1}v) \quad \text{if} \quad n \leqq \nu$$

$$T'_n = T_n \quad \text{if} \quad n > \nu$$

are such that

$$T'^n(u-v) = T'_n \cdot \ldots \cdot T'_1(u-v) = F_x^n u - F_x^n v \quad \text{if} \quad n \leqq \nu$$

and we have

$$||T'-T|| < \delta$$

Therefore Theorem 4.1. applies, and $u-v$ is in the range of $P^{(S+1)}(T')$. From (4.4.a), (5.5) and (5.7) we get thus

$$||F_x^n u - F_x^n v|| \leqq B_\varepsilon \exp n(\mu^{(S+1)}+\varepsilon)\ ||u-v|| \leqq \gamma\ e^{n\lambda'}||u-v|| \tag{5.10}$$

which proves part (b) of the Theorem.

From (4.5) we obtain

$$\begin{aligned} ||(1-P^{(S+1)}(T))(u-v)|| &= ||(P^{(S+1)}(T')-P^{(S+1)}(T))(u-v)|| \\ & A\delta\,||u-v|| = (\sin\psi)\ ||u-v|| \end{aligned} \tag{5.11}$$

where we have written $A\delta = \sin\psi$ with $0 < \psi < \frac{\pi}{4}$ by (5.6). This implies

$$||(1-P^{(S+1)}(T))(u-v)|| \leqq (\tan\psi)\ ||P^{(S+1)}(T)(u-v)|| \tag{5.12}$$

Define $\Phi : [V^{(S+1)} \cap \overline{B}(\alpha)] \times [V^{(S+1)\perp} \cap \overline{B}(\alpha)] \longmapsto \overline{B}(\alpha)$ by

$$\Phi(u_1,u_2) = \frac{u_1}{\alpha}\sqrt{\alpha^2 - \|u_2\|^2} + u_2 \quad .$$

Let $\Phi(u_1,u_2)$, $\Phi(v_1,v_2) \in D^\nu$ or $\overline{B}(\alpha) \cap S(\beta)$. Then (5.11) yields $\|u_2\|$, $\|v_2\| \leqq A\delta\alpha$ and, by (5.12) ,

$$(\tan\psi)^{-1}\|u_2-v_2\| \leqq \left\|\frac{u_1}{\alpha}\sqrt{\alpha^2-\|u_2\|^2} - \frac{v_1}{\alpha}\sqrt{\alpha^2-\|v_2\|^2}\right\|$$

$$\leqq \left|\sqrt{\alpha^2-\|u_2\|^2} - \sqrt{\alpha^2-\|v_2\|^2}\right|\frac{\|u_1\|}{\alpha} + \frac{1}{\alpha}\sqrt{\alpha^2-\|v_2\|^2}\cdot\|u_1-v_1\|$$

$$\leqq \frac{\left|\|u_2\| - \|v_2\|\right|}{\sqrt{\alpha^2-(A\delta\alpha)^2}}\cdot A\delta\alpha + \|u_1-v_1\|$$

$$\leqq \|u_2 - v_2\| \tan\psi + \|u_1 - v_1\|$$

so that

$$\|u_2-v_2\|\,((\tan\psi)^{-1} - \tan\psi) \leqq \|u_1-v_1\| \tag{5.13}$$

where $(\tan\psi)^{-1} - \tan\psi > 0$ since $0 < \psi < \frac{\pi}{4}$. Since D^ν is a C^1 submanifold of $\overline{B}(\alpha)$, $\Phi^{-1}D^\nu$ is the graph of a C^1 function $\varphi^\nu : V^{(S+1)} \cap \overline{B}(\alpha) \longmapsto V^{(S+1)\perp} \cap \overline{B}(\alpha)$ with derivative bounded uniformly with respect to ν .

Let φ be the limit of a subsequence of (φ^ν) converging on a countable dense subset of $V^{(S+1)} \cap \overline{B}(\alpha)$. The subsequence then converges everywhere and the limit φ is Lipschitz. Since $\Phi(\text{graph } \varphi^\nu) = D^\nu \subset \overline{B}(\alpha) \cap S^\nu(\beta)$ we have $\Phi(\text{graph } \varphi) \subset \overline{B}(\alpha) \cap S(\beta)$. The converse inclusion follows from (5.13) applied to $\overline{B}(\alpha) \cap S(\beta)$. Therefore

$$\Phi\ (\text{graph } \varphi) = \overline{B}(\alpha) \cap S(\beta) = V_x^\lambda$$

and, by uniqueness of φ ,

$$\lim_{\nu\to\infty} \varphi^\nu = \varphi$$

everywhere on $V^{(S+1)} \cap \overline{B}(\alpha)$.

Let $u,v \in D^\nu$ and define bounded operators $T'_n = DF_{f^{n-1}x}(F_x^{n-1}u)$, $T''_n = DF_{f^{n-1}x}(F_x^{n-1}v)$ if $n \leqq \nu$ and

$$T'_n = T''_n = T_n \qquad \text{if } n > \nu$$

Then $\|T'-T\|$, $\|T''-T\| < \delta$ and, using (5.10),

$$\|T'_n - T''_n\| \leqq \|DF_{f^{n-1}x}\|_\theta \cdot \|F_x^{n-1}u - F_x^{n-1}v\|^\theta$$

$$\leqq \|F_{f^{n-1}x}\|_{1,\theta}\, \gamma^\theta \quad e^{(n-1)\theta\lambda} \|u-v\|^\theta$$

if $n \leqq \nu$, hence

$$\|T'-T''\| \leqq G\gamma^\theta \|u-v\|^\theta$$

Therefore (4.5) gives

$$\|P^{(S+1)}(T') - P^{(S+1)}(T'')\| \leqq AG\gamma^\theta \|u-v\|^\theta \tag{5.14}$$

where the ranges of $P^{(S+1)}(T')$, $P^{(S+1)}(T'')$ are the tangent spaces to D^ν at u and v, as remarked earlier.

When $\nu \to \infty$, let u tend to $\hat{u} \in V_x^\lambda$. Then the range of $P^{(S+1)}(T')$ tends to the range of $P^{(S+1)}(\hat{T})$ where

$$\hat{T}_n = DF_{f^{n-1}x}(F_x^{n-1}\hat{u}) \qquad \text{for all } n .$$

This is because of (4.5) and the fact that $\|T'-\hat{T}\| \to 0$ when $\nu \to \infty$ (use (5.3)).

If $\Phi(u_1,u_2) = u$, the derivative of φ^ν at u, is

$$\xi \longmapsto \frac{\sqrt{\alpha^2 - \|u_2\|^2}}{\alpha} Q\xi - \left[1 + \frac{(u_2, Qu_1)}{\alpha\sqrt{\alpha^2 - \|u_2\|^2}}\right]^{-1} \frac{(u_2, Q\xi)}{\alpha^2} Qu_1$$

where $Q = -(1+P^{(S+1)}(T) - P^{(S+1)}(T'))^{-1}(1-P^{(S+1)}(T'))$ and (5.14) shows that this derivative is Hölder continuous with respect to u_1 , uniformly with respect to ν . In particular

$$\|\varphi^{\nu}(u_1+\xi) - \varphi^{\nu}(u_1) - D\varphi^{\nu}(u_1)\xi\|$$

$$= \|\int_0^1 dt\ [D\varphi^{\nu}(u_1+t\xi) - D\varphi^{\nu}(u_1)]\xi\| \leqq \text{const.}\|\xi\|^{1+\theta}$$

If $\Delta(u_1)$ is the limit of $D^{\nu}(u_1)$ when $\nu\to\infty$,

$$\|\varphi(u_1+\xi) - \varphi(u_1) - \Delta(u_1)\xi\| \leqq \text{const.}\|\xi\|^{1+\theta}$$

so that $\Delta(u_1)$ is the derivative of φ at u_1 . This derivative is Hölder continuous of exponent θ . We have also shown that the tangent space to V_x^{λ} at $\hat{u}$ is the range of $P^{(S+1)}(\hat{T})$. In particular the tangent at 0 is the range of $P^{(S+1)}(T)$, i.e. $V_0 = V^{(S+1)}$. This concludes the proof of (a) in case I., $r = 1$.

From Theorem 4.1. and (5.6) it follows that we may assume δ^{-1} and B_{ε} at $f^N x$ to increase less fast than $e^{N\Theta}$ when $N\to\infty$. In view of (5.4), G increases less fast than $e^{N\Theta}$ provided we take $\eta<\Theta$. Therefore, by (5.7), we can take β^{-1} to increase less fast than $e^{2N\Theta/\theta}$ and α^{-1} less fast than $e^{N\Theta(2/\theta+1)}$. Changing Θ , this proves (c) in case I., $r = 1$.

We prove that V_x^{λ} is of class $C^{r,\theta}$ by induction on r for $r > 1$. Let $\check{F}_x : \overline{B}(1)\oplus\mathfrak{h}\to\mathfrak{h}\oplus\mathfrak{h}$ be the $C^{r-1,\theta}$ map defined by

$$\check{F}_x(u,v) = (F_x u, DF_x(u)v) \quad .$$

The results obtained for (F_x) can now be applied to $(\check{F}_x)$. In particular $S(\beta)$ is replaced by $\check{S}(\check{\beta})\subset\mathfrak{h}\oplus\mathfrak{h}$. Since we have shown that the tangent space to $B(\alpha)\cap S(\beta)$ at $\hat{u}$ is the range of $P^{(S+1)}(\hat{T})$ the condition

$$(u,v)\in\check{S}(\check{\beta}) \quad \text{and} \quad \|u\|^2 + \|v\|^2 \leqq \check{\alpha}^2 \tag{5.15}$$

means $u \in \overline{B}(\check{\alpha}) \cap S(\check{\beta})$, and v is tangent to $S(\check{\beta})$ at u and sufficiently small. The set defined by (5.15) is $C^{r-1,\theta}$ by induction, therefore $\overline{B}(\alpha') \cap S(\check{\beta})$ is $C^{r,\theta}$ if $\alpha' < \check{\alpha}$. We may thus choose Γ, α, β such that Theorem 5.1. holds in case I. for general r .

In the case II. we may make a choice Γ_r , α_r , β_r of Γ, α, β for every $r \geqq 1$ and some fixed $\theta \in (0,1]$ (for instance $\theta = \frac{1}{2}$) . We assume, as we may, that $\Theta < |\lambda|$. It then follows from (c) that, if $x \in \Gamma_r$, there exists $\nu \geqq 0$ such that

$$\beta_1(x)\ e^{\nu\lambda} < \alpha_r(f^{\nu} x) \qquad .$$

Thus F_x^{ν} maps $\mathcal{V}_{x,1}^{\lambda}$ (i.e. $\mathcal{V}_x^{\lambda}$ defined with α_1 and β_1) into the C^r manifold $\mathcal{V}_{f^{\nu}x,r}^{\lambda}$ (i.e. $\mathcal{V}_{f^{\nu}x}^{\lambda}$ defined with α_r and β_r). Since F_x^{ν} is C^r and satisfies a transversality condition which we shall presently discuss, $\mathcal{V}_{x,1}^{\lambda}$ is also C^r . The transversality requirement is that, if $\hat{u} \in \mathcal{V}_{x,1}^{\lambda}$ the range of $DF_x^{\nu}(\hat{u})$, together with the tangent to $\mathcal{V}_{f^{\nu}x}^{\lambda}$ at $F_x^{\nu}\hat{u}$, span $\mathfrak{h}$.

Let $\hat{T}_n = DF_{f^{n-1}}(F_x^{n-1}\hat{u})$, then the condition is that the range of $\hat{T}^{\nu}$ together with

$$\hat{V}_{\nu} = \{u \in \mathfrak{h} : \limsup_{n\to\infty} \frac{1}{n} \log ||\hat{T}^{n,\nu} u|| \leqq \mu^{(S+1)}\}$$

span $\mathfrak{h}$. This follows from (4.2) and the fact that $\tilde{V}_{\nu}$ has codimension Q .

Let now $\Gamma_{\infty} = \bigcap_{r=1}^{\infty} \Gamma_r$, then $f\Gamma_{\infty} \subset \Gamma_{\infty}$ and $\rho(\Gamma_{\infty}) = 1$. The C^{∞} version of our theorem is obtained if we take for α, β the restriction of α_1, β_1 to $\Gamma = \Gamma_{\infty}$.

In case III. our assumption implies

$$\int \rho(dx)\ \log^{+} ||F_x||_2' < +\infty$$

where $||\cdot||_2'$ is the C^2 norm on a ball of radius < 1 . In view of case I.

of the Theorem a $C^{1,1}$ manifold $\mathcal{V}_x^{\lambda}$ can be defined. It is a limit of holomorphic manifolds D^{ν} defined by (5.9), and therefore it is holomorphic.

5.2. *Corollary* : Keeping the notation of Theorem 5.1. we let (5.1) hold for $q = 1, \ldots, \max(Q,Q^*)+1$ and assume $r(Q^*+1) = r(Q^*)+1$ where $Q^* \geqq 0$ is an integer-valued f-invariant measurable function. Write $\hat{T}_n = DF_{f^{n-1}x}(F_x^{n-1}u)$ for $u \in \mathcal{V}_x^{\lambda}$, $x \in \Gamma$, and denote by $V^*(u)$ the null space of

$$\hat{\Lambda}_{Q^*} = \lim_{n\to\infty}([\hat{T}^{n*}\hat{T}^n]_{Q^*})^{1/2n}$$

If $Q^* = Q$, $V^*(u)$ is the tangent space to $\mathcal{V}_x^{\lambda}$ at u. At the expense of replacing α, β, Γ by $\alpha^*, \beta^*, \Gamma^*$ with the same general properties in the definition of $\mathcal{V}_x^{\lambda}$, we may assume that the function $V^* : \mathcal{V}_x^{\lambda} \longmapsto$ Grassmannian of $\mathfrak{h}$ is respectively $C^{r_m-1,\theta}$, C^{∞} or holomorphic on $\mathcal{V}_x^{\lambda}$ in cases I., II., III.

Notice that Q^* may be smaller or larger than Q. When $Q^* = Q$ the Corollary has been proved in the course of the demonstration of Theorem 5.1. In general we have to go back to that demonstration and adapt it. If $Q^* > Q$ we may have to modify the definition of $\mu^{(r(Q^*)+1)}$ (instead of $\mu^{(r(Q)+1)}$) to make it finite. In case I., $r_m = 1$, the proof for $Q^* = Q$ is based on (5.14) and extends immediately to general Q^*. For $r_m > 1$, let the real f-invariant measurable function λ^* satisfy

$$\mu^{(r(Q^*)+1)} < \lambda^* < \mu^{(r(Q^*))}$$

Let a $C^{r_m-1,\theta}$ map $\mathcal{F}_x : \overline{B}(1) \oplus \mathfrak{h} \longrightarrow \mathfrak{h} \oplus \mathfrak{h}$ be defined by

$$\mathcal{F}_x(u,v) = (F_x u \,,\, e^{\lambda-\lambda^*} DF_x(u)v) \quad .$$

We may apply Theorem 5.1. to $(\mathcal{F}_x)$ (we obtain $(\lambda-\lambda^*)^+ \in L^1$ by going to subsets of Γ) and the Corollary results immediately.

5.3. *Remarks* : (a) One could give a version of Theorem 5.1. for a single sequence $(F_{(n)})$ of $C^{r,\theta}$ maps $(\overline{B}(1),0) \longmapsto (\mathfrak{h},0)$ satisfying conditions corresponding to (5.1), (5.2), (5.3) and (S).

(b) Assuming further measurability properties of $x \longmapsto F_x$ would imply measurability properties of $x \longmapsto \mathcal{V}_x^\lambda$. Such properties follow from the fact that $\mathcal{V}_x^\lambda$ is the limit, as $\nu \longmapsto \infty$, of the connected component D^ν of 0 in $\overline{B}(\alpha) \cap (F_x^\nu)^{-1} V_\nu$.

(c) Property speaking $\mathcal{V}_x^\lambda$ is a local stable manifold when Q is so chosen that $\mu^{(r(Q))} \geqq 0$.

6. *Local unstable manifolds*

6.1. Theorem : Let (M,Σ,ρ) be a probability space, and $f : M \longmapsto M$ a measurable map preserving ρ. For each $x \in M$, let $\tilde{F}_x : (\overline{B}(1),0) \longmapsto (\mathfrak{h},0)$ be defined. We assume that $\tilde{F}_x$ is differentiable, with derivative $T(x)^*$ at 0, that $T(\cdot)$ is measurable, and that $\log^+\|T(\cdot)\| \in L^1(M,\rho)$. Let an integer-valued f-invariant measurable function $Q \geqq 0$ be given. Then $\mu^{(1)} > \mu^{(2)} > \ldots$ and $r(1), \ldots, r(Q+1)$ are defined for almost all x, such that

$$\sum_{k=1}^{q} \mu^{(r(k))} = \lim_{n\to\infty} \frac{1}{n} \log\|(T_x^n)^{\wedge q}\|$$

for $q = 1, \ldots, Q+1$. We assume that Q and the real f-invariant measurable function $\lambda > 0$ are so chosen that

$$\mu^{(r(Q+1))} < \lambda < \mu^{(r(Q))}$$

almost everywhere. (We let $\mu^{(r(0))} = +\infty$; $\mu^{(r(Q+1))}$ may be $-\infty$).

We consider three cases.

I. $\mathfrak{h}$ is real ; $\tilde{F}_x$ is $C^{\underset{\sim}{r},\theta}$; $x \longmapsto \|\tilde{F}_x\|_{\underset{\sim}{r},\theta}$ is measurable and

$$\int \rho(dx) \log^+ \|\tilde{F}_x\|_{\underset{\sim}{r},\theta} < +\infty$$

for some $\underset{\sim}{r} \geqq 1$, $\theta \in (0,1]$.

II. $\mathfrak{h}$ is real ; $\tilde{F}_x$ is C^∞ ; $x \longmapsto \|\tilde{F}_x\|_{\underset{\sim}{r}}$ is measurable and

$$\int \rho(dx) \log^+ \|\tilde{F}_x\|_{\underset{\sim}{r}} < +\infty$$

for all $\underset{\sim}{r} \geqq 1$.

III. $\mathfrak{h}$ is complex ; $\tilde{F}_x$ is holomorphic in $B(1)$; $x \longmapsto \|\tilde{F}_x\|_1$ is measurable and

$$\int \rho(dx)\ \log^+ \|F_x\|_1 < +\infty$$

Let $\Theta > 0$. Under the above conditions there is a measurable set $\tilde{\Gamma} \subset M$ such that $f\tilde{\Gamma} \subset \tilde{\Gamma}$, $\rho(\tilde{\Gamma}) = 1$, and there are measurable functions $\tilde{\beta} > \tilde{\alpha} > 0$ on $\tilde{\Gamma}$ with the following properties.

(a) If $x \in \tilde{\Gamma}$ the set

$$\tilde{V}_x^\lambda = \{u_0 \in \overline{B}(\tilde{\alpha}(x)) : \exists (u_n)_{n \geqq 0} \text{ with } \tilde{F}_{f^n x} u_{n+1} = u_n \text{ and } \|u_n\| \leqq \tilde{\beta}(x) e^{-n\lambda(x)}\} \quad (6.1)$$

is a submanifold of $\overline{B}(\alpha(x))$, tangent at 0 to $\tilde{V}_x^{(r(Q))}$, and (u_n) is uniquely determined by u_0 . The manifold $\tilde{V}_x^\lambda$ is respectively $C^{r,\theta}$, C^∞ or holomorphic in the cases I., II., III.

(b) If $\lambda' : M \longmapsto \mathbb{R}$ is f-invariant measurable and satisfies

$$\mu^{(r(Q+1))} < \lambda' < \mu^{(r(Q))}$$

there exists $\tilde{\gamma} > 1$ measurable on $\tilde{\Gamma}$ such that, if (u_n), (v_n) satisfy $\tilde{F}_{f^n x} u_{n+1} = u_n$, $\tilde{F}_{f^n x} v_{n+1} = v_n$; $\|u_n\|, \|v_n\| \leqq \tilde{\beta}(x)\ e^{-n\lambda(x)}$; $u_0, v_0 \in \tilde{V}_x^\lambda$, then

$$\|u_n - v_n\| \leqq \tilde{\gamma}(x)\ \|u_0 - v_0\|\ e^{-n\lambda'(x)} \quad .$$

This applies in particular to $\lambda' = \lambda$.

(c) If $x \in \tilde{\Gamma}$, then $\tilde{\alpha}(f^N x)$, $\tilde{\beta}(f^N x)$, $\tilde{\gamma}(f^N x)^{-1}$ decrease less fast than the exponential $e^{-N\Theta}$ when $N \to \infty$.

6.2. *Corollary* : Let $Q^* \geqq 0$ satisfy $r(Q^*+1) = r(Q^*)+1$. Write $\hat{T}_n^* = D\tilde{F}_{f^{n-1}x}(u_n)$ for $u_0 \in \tilde{V}_x^\lambda$, (u_n) as in (6.1), $x \in \tilde{\Gamma}$, and denote by $\tilde{V}^*(u_0)$ the (Q^*-dimensional) range of

$$\hat{\Lambda}_{Q^*} = \lim_{n\to\infty}([\hat{T}^{n*}\ \hat{T}^n]_{Q^*})^{1/2n} \quad .$$

If $Q^* = Q$, $\tilde{V}^*(u_0)$ is the tangent space to $\tilde{\mathcal{V}}_x^\lambda$ at u_0. At the expense of replacing $\tilde{\alpha}, \tilde{\beta}, \tilde{\Gamma}$ by $\tilde{\alpha}^*, \tilde{\beta}^*, \tilde{\Gamma}^*$ with the same general properties in the definition of $\tilde{\mathcal{V}}_x^\lambda$, we may assume that $u_0 \longmapsto \tilde{V}^*(u_0)$ is respectively $C^{r-1,\theta}$, C^∞, or holomorphic on $\tilde{\mathcal{V}}_x^\lambda$ in cases I., II., III.

The proofs of Theorem 6.1. and Corollary 6.2. are parallel to the proofs of 5.1. and 5.2. The idea is to replace F_x by $\tilde{F}_x^{-1}$ and $T(x)$ by $T(x)^{*-1}$ (with some caution because the inverses $\tilde{F}_x^{-1}$, $T(x)^{*-1}$ may not be well defined). The changes are easy, if Proposition 3.3. is used as multiplicative ergodic theorem, and if (4.4.b) is used instead of (4.4.a). To obtain the existence of the limit $\varphi^\nu \longmapsto \varphi$ one uses the fact that the intersections of translates of the range of $P^{(S+1)}(T)$ with

$$S^\nu = \{(u_n) : \|u_n\| \leqq \tilde{\beta}(x)\, e^{-n\lambda(x)} \quad \text{for} \quad 0 \leqq n \leqq \nu\}$$

have diameters which tend to zero when $\nu \to \infty$. [Taking u_0, v_0 in such an intersection, one constructs (T'_n) such that $(T'^\nu)^*\,(u_\nu - v_\nu) = u_0 - v_0$ and applies (4.5), (4.4.a) to get

$$\|u_0 - v_0\|\,(1 - A\,\|T - T'\|) \leqq \|P^{(S+1)}(T')(u_0 - v_0)\|$$

$$= \|P^{(S+1)}(T')(T'^\nu)^*(u_\nu - v_\nu)\| \leqq \|u_\nu - v_\nu\| \cdot \|T'^\nu P^{(S+1)}(T')\|$$

$$\leqq 2\tilde{\beta}(x)\, B_\varepsilon\, \exp(-\nu\,(\lambda(x) - \mu^{(S+1)} - \varepsilon)) \to 0$$

where we have assumed $\varepsilon < \lambda(x) - \mu^{(S+1)}$]. Further details are left to the reader.

The remarks 5.3. have obvious counterparts for unstable manifolds.

7. *Applications*

We show here how the results of sections 5 and 6 can be applied to Hilbert bundle maps and flows. This will lead to the definition of stable and unstable manifolds for differentiable maps and flows in Hilbert manifolds. We indicate the arguments in discussive manner, rather than stating lengthy theorems. The results on finite dimensional manifolds given in Ruelle and Shub [15] and those on semiflows in Hilbert space of Section 0.1. follow as special cases.

7.1. *Hilbert bundles*

Let M be a separable metrizable space and $f : M \longmapsto M$ a continuous map. Let $\pi : E \longmapsto M$ be a continuous bundle of separable Hilbert spaces over M. For simplicity we assume that the fiber has constant dimension. Finally let $T : E \longmapsto E$ be a continuous bundle map over f. We indicate how to adapt the previous "abstract" theory to the present "topological" situation and deal simultaneously with all f-invariant probability measures on M.

First, we may trivialize the bundle E by using a countable Borel partition (M_i) of M, and maps $\chi_i : \pi^{-1}M_i \longmapsto M_i \times \mathfrak{h}$. We may assume that χ_i and χ_i^{-1} have norm < 2. Let us write $\mathring{T}(x) = \chi_j\, T(x)\, \chi_i^{-1}$ if $x \in M_i$ and $fx \in M_j$, and $T_x^n = T(f^{n-1}x) \ldots T(x)$, $\mathring{T}_x^n = \mathring{T}(f^{n-1}x) \ldots \mathring{T}(x)$. Then

$$\lim_{n\to\infty} \left[\frac{1}{n} \log \|T_x^n u\| - \frac{1}{n} \log \|\mathring{T}_x^n\, \chi_i u\|\right] = 0 \quad .$$

From this it is clear that questions about T acting on E are simply related to questions about $\mathring{T}$ acting on $M \times \mathfrak{h}$.

<u>We assume that the set</u> $\Lambda = \bigcap_{n \geq 0} f^n M$ <u>is compact</u>. <u>Consider the set</u> Γ^+ <u>of those</u> $x \in M$ <u>for which there is an</u> f-<u>ergodic probability measure</u> ρ_x <u>such that</u>

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^k x) = \rho_x(\varphi) \tag{7.1}$$

<u>for all continuous functions</u> $\varphi : M \longmapsto \mathbb{R}$, <u>and</u>

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log \| (T^n_{f^k x})^{\wedge q} \| \\ = \quad &\lim_{n\to\infty} \frac{1}{n} \int \log \| (T^n_y)^{\wedge q} \| \, \rho_x \, (dy) \\ = \quad &\inf_n \frac{1}{n} \int \log \| (T^n_y)^{\wedge q} \| \, \rho_x \, (dy) \end{aligned} \tag{7.2}$$

<u>for all integers</u> $k \geqq 0$, $q > 0$. <u>Then</u> Γ^+ <u>is a Borel set,</u> $f\Gamma^+ \subset \Gamma^+$, <u>and</u> $\sigma(\Gamma^+) = 1$ <u>for every</u> f-<u>invariant Borel probability measure</u> σ <u>on</u> M .

The set of $x \in M$ for which there is an ergodic ρ_x satisfying (7.1), (7.2) is Borel, and $f\Gamma^+ \subset \Gamma^+$. Since the support of any f-invariant probability measure ρ is contained in Λ , it remains to show that $\rho(\Gamma^+ \cap \Lambda) = 1$ for every f-invariant probability measure ρ on Λ . We have thus reduced the problem to the case of a compact space. For that case $\rho(\Gamma^+) = 1$ is proved in Appendix D of [14] .

Proposition 2.1. is satisfied with Γ^+ chosen as above, and the multiplicative ergodic theorems 2.2., 2.3. also have a topological version, where Γ is Borel and $\rho(\Gamma) = 1$ for every invariant probability measure ρ .

Similarly for Proposition 3.2., provided Q is chosen Borel; the vectors $\xi_0^{(1)}, \dots, \xi_0^{(Q)}$, and κ_ε , D_ε now are Borel functions of x . Corollary 3.4. holds with Γ Borel.

In the topological version of the local stable manifold theorem 5.1. (and 5.2), λ , λ', Q (and Q^*) are assumed to be Borel, with

$$\mu^{(r(Q+1))} < (\lambda \text{ and } \lambda') < \mu^{(r(Q))}$$

on a Borel set of invariant measure 1 . Then, $\Gamma, \alpha, \beta, \gamma$ (and $\Gamma^*, \alpha^*, \beta^*$) may be taken Borel. Similarly for the local unstable manifold theorem 6.1. (and 6.2).

7.2. *Tangent bundle to a Hilbert manifold*

Let M be a C^1 Hilbert manifold, real or complex, Hausdorff and with a countable basis of open sets. (For instance M may be an open subset of a separable Hilbert space, or a finite-dimensional manifold with countable basis of open sets). Let $f : M \longmapsto M$ be a differentiable map. Then TM is a continuous bundle of Hilbert spaces, Tf a continuous bundle map over f, and the remarks of Section 7.1. apply *) . In particular one obtains a topological version of the multiplicative ergodic theorems of Sections 2 and 3 .

We shall now discuss stable and unstable manifold theorems, where the stable and unstable manifolds are subsets of M (rather than TM) . We assume that M is a $C^{r,\theta}_{\mathrm{m}}$ manifold and consider a $C^{r,\theta}_{\mathrm{m}}$ map $f : M \longmapsto M$ ("case I" of Sections 5 and 6). One can handle similarly the C^∞ case, the holomorphic case, and the real analytic case (by using local holomorphic extensions).

Let again $\Lambda = \bigcap_{n \geqq 0} f^n M$ be compact. Define a metric d on M by a Hilbert norm $\|\cdot\|_x$ on $T_x M$ depending continuously on x . We may suppose that for some continuous $\varepsilon(\cdot) > 0$, a continuous map $(x,u) \longmapsto \psi_x(u)$ of $T_\Lambda M(\varepsilon(\cdot)) = \{(x,u) : x \in M , u \in T_x M , \|u\|_x \leqq \varepsilon(x)\}$ to M is defined with the following properties.

(a) ψ_x <u>is a</u> $C^{r,\theta}_{\mathrm{m}}$ <u>diffeomorphism from the</u> $\varepsilon(x)$-<u>ball</u> $T_x M(\varepsilon(x))$ <u>to</u> $\{y \in M : d(x,y) < 2\varepsilon\}$ <u>with</u> $D\psi_x(0)$ = <u>identity</u>.

(b) $\|D\psi_x\|$ <u>and</u> $\|T\psi_x^{-1}\|$ <u>are bounded uniformly for</u> $x \in \Lambda$ (i.e. ψ_x, ψ_x^{-1}

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*) It may be desirable to replace the Hilbert topology on M by some other separable metrizable topology.

are uniformly Lipschitz).

(c) $\|\psi_{fx}^{-1} f\psi_x\|_{r,\theta}$ is bounded uniformly for $x \in \Lambda$.

We may thus write $F_x = \xi^{-1}\psi_{fx}^{-1} \ f\psi_x\xi$ where ξ denotes multiplication by a sufficiently small scalar, and apply case I of Theorem 5.1. In particular for suitable α', β', the local stable manifold

$$V_x^\lambda = \{y \in M : d(x,y) \leqq \alpha'(x) \text{ and}$$

$$d(f^n x, f^n y) \leqq \beta'(x)\, e^{n\lambda(x)} \text{ for all } n > 0\}$$

will be a $C^{r,\theta}$ submanifold of $\{y \in M : d(x,y) \leqq \alpha'(x)\}$ if $x \in \Gamma$, where $f\Gamma \subset \Gamma$, and $\rho(\Gamma) = 1$ for every invariant probability measure ρ .

A differential global stable manifold can be defined if the operators $Tf(x)$ have dense range. We shall discuss below in more details the case of global unstable manifolds.

To study local unstable manifolds, we write $\tilde{M} = \{(x_n)_{n\geqq 0} \in M^{\mathbb{N}} : fx_{n+1} = x_n$ for all $n \geqq 0\}$, and define $\tilde{f} : \tilde{M} \longrightarrow \tilde{M}$, $\tilde{\pi}: \tilde{M} \longmapsto M$ by $\tilde{f}(x_n) = (x_{n+1})$, $\tilde{\pi}(x_n) = x_0$ so that $f\circ\tilde{\pi} = \tilde{\pi}\circ\tilde{f}^{-1}$. The definition of Λ implies that, if $\tilde{x} = (x_n) \in \tilde{M}$ we have $x_n \in \Lambda$ for all $n \geqq 0$. Therefore $\tilde{M}$ is compact metrizable ; f and π are continuous.

Furthermore the map $\tilde{\pi}$ induces a bijection of $\tilde{f}$-invariant probability measures to f-invariant probability measures. We have a Hilbert bundle over $\tilde{M}$ with fiber $T_{\tilde{\pi}\tilde{x}} M$ over $\tilde{x}$, and a continuous bundle map

$$\tilde{F}_x = \xi^{-1}\, \psi_{x_0}^{-1}\, f\psi_{x_1}\, \xi : T_{\tilde{\pi}(\tilde{f}\tilde{x})} M(1) \longmapsto T_{\tilde{\pi}\tilde{x}} M$$

where (ψ_x) and ξ satisfy the same assumptions as above. We shall apply the topological version of Theorem 6.1. with $\tilde{M}$, $\tilde{f}$ replacing M, f . Notice that for almost all $\tilde{x}$ (with respect to any $\tilde{f}$-invariant probability measure ρ), the characteristic exponents and multiplicities which occur at $\tilde{x}$ for unstable manifolds are the same which occur at $\widetilde{\pi x}$ for stable manifolds (see Section 3.5). In view of Theorem 6.1., there is a Borel set $\tilde{\Gamma} \subset \tilde{M}$ such that $\tilde{f}\tilde{\Gamma} \subset \tilde{\Gamma}$, and $\rho(\tilde{\Gamma}) = 1$ for every $\tilde{f}$-invariant probability measure on $\tilde{M}$, and there are Borel functions $\tilde{\alpha}' > \tilde{\beta}' > 0$ such that the local unstable manifold

$$\begin{aligned} \tilde{V}^{\lambda}_{\tilde{x}} = \{ y_0 \in M : d(x_0, y_0) \leqq \tilde{\alpha}'(\tilde{x}) \text{ and } \exists (y_n)_{n \geqq 0} \quad \text{with} \\ f y_{n+1} = y_n \text{ and } d(x_n, y_n) \leqq \tilde{\beta}'(\tilde{x})\, e^{-n\lambda(\tilde{x})} \} \end{aligned} \tag{7.3}$$

is a $C^{r,\theta}$ submanifold of $\{y \in M : d(x,y) \leqq \tilde{\alpha}'(\tilde{x})\}$.

7.3. *Global unstable manifolds*

We retain the assumptions of Section 7.2. and further assume that $T_x f$ is injective for all $x \in \Lambda$. We have here $T^n_{\tilde{x}} f = (T_{x_n} g)^* \cdot \ldots \cdot (T x_1 f)^* = (T_{x_n} f^n)^*$ and $T^{n+k}_{\tilde{f}^{-k}\tilde{x}} = (T_{x_n} f^n)^* (T_{x_0} f^k)^*$. Let $Q \geqq 0$ be an $\tilde{f}$-invariant integer-valued Borel function on $\tilde{M}$. Given an $\tilde{f}$-ergodic measure ρ on $\tilde{M}$, we let $\tilde{\Gamma}^+_\rho$ consist of those $\tilde{x} \in \tilde{M}$ satisfying the following properties.

(a) for all continuous functions $\varphi : M \longmapsto \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(\tilde{f}^k \tilde{x}) = \rho(\varphi) \quad .$$

(b) for $q = 1, \ldots , Q(\tilde{x})$, and all integers $k \geqq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T^n_{\tilde{f}^k \tilde{x}})^{\wedge q}\| = \inf_n \frac{1}{n} \int \log \|(T^n_{\tilde{y}})^{\wedge q}\| \quad \rho(d\tilde{y})$$

(c) for all integers $k \geqq 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(T^n_{\tilde{f}^k \tilde{x}})^{\wedge (Q+1)}\| \leqq \inf_n \frac{1}{n} \int \log \|(T^n_{\tilde{y}})^{\wedge (Q+1)}\| \, \rho(d\tilde{y})$$

Notice that $\tilde{f}\, \tilde{\Gamma}^+_\rho = \tilde{\Gamma}^+_\rho$ (to check (b) use the injectivity of $T_x f$ and Theorem 1.1). We define

$$\widetilde{\Gamma}_\rho = \{\widetilde{x} \in \widetilde{\Gamma}^+_\rho : \text{(S) is satisfied}\}$$

where $T_n = (\chi_j \cdot (T_{x_n} f) \cdot \chi_i^{-1})^*$ with the notation of Section 7.1. It follows readily that $\widetilde{f}\widetilde{\Gamma}_\rho = \widetilde{\Gamma}_\rho$.

Let now $\widetilde{x} = (x_n) \in \widetilde{\Gamma}_\rho$, $\widetilde{y} = (y_n) \in \widetilde{M}$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log d(x_n,y_n) < -\lambda(\widetilde{x}) \quad .$$

Since $\widetilde{\alpha}'(\widetilde{f}^n\widetilde{x})$, $\widetilde{\beta}'(\widetilde{f}^n\widetilde{x})$ decrease less fast than $e^{-n\Theta}$ when $n\to\infty$, we have $y_n \in \widetilde{\mathcal{V}}^\lambda_{\widetilde{f}^n\widetilde{x}}$ for sufficiently large n (we have taken $\Theta < \lambda(x)$) . By application of Theorem 4.1. we get $\widetilde{f}^n\widetilde{y} \in \widetilde{\Gamma}_\rho$ and therefore $\widetilde{y} \in \widetilde{\Gamma}_\rho$.

Writing $\widetilde{\Gamma} = \bigcup_\rho \widetilde{\Gamma}_\rho$, we have $\widetilde{f}\widetilde{\Gamma} = \widetilde{\Gamma}$, and $\rho(\widetilde{\Gamma}) = 1$ for every $\widetilde{f}$-invariant probability measure on $\widetilde{M}$ (see Section 7.1). If $\widetilde{x} \in \widetilde{\Gamma}_\rho$ the "global unstable manifold" *)

$$\widetilde{\mathcal{W}}^\lambda_{\widetilde{x}} = \{y_0 \in M : \exists\widetilde{y} = (y_n) \in \widetilde{M} \text{ with}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log d(x_n,y_n) < -\lambda(\widetilde{x})\}$$

is given by

$$\widetilde{\mathcal{W}}^\lambda_{\widetilde{x}} = \bigcup_{n\geqq 0} f^n \widetilde{\mathcal{V}}^\lambda_{\widetilde{f}^n\widetilde{x}}$$

and is therefore an immersed $C^{r,\theta}$ manifold $\subset \Gamma_\rho$.

If f restricted to Λ is injective, $\widetilde{\pi}$ identifies $\widetilde{M}$ to Λ , and $\widetilde{\mathcal{W}}^\lambda_{\widetilde{x}}$ is the image of $\widetilde{\mathcal{V}}^\lambda_{\widetilde{x}}$ by an injective $C^{r,\theta}$ immersion $I_{\widetilde{x}}$ tangent to the identity at x_0 **) .

If $T_x f$ is compact when $x \in \Lambda$, one may replace (b), (c) in the definition of $\widetilde{\Gamma}_\rho$ by

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*) Properly speaking, the global unstable manifold corresponds to the case where $\mu^{(r(Q+1))} \leqq 0$. One can then also write

$$\widetilde{\mathcal{W}}^\lambda_{\widetilde{x}} = \{y_0 \in M : \exists\widetilde{y} = (y_n) \in \widetilde{M} \text{ with } \limsup_{n\to\infty} \frac{1}{n} \log d(x_n,y_n) < 0\}$$

**) This is again true in the C^∞ case (easy) and the C^ω case (by a theorem of (b') For all integers $q \geqq 0$, $k \geqq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T^n_{\tilde{f}^k\tilde{x}})^{\wedge q}\| = \inf_n \frac{1}{n} \int \log \|(T^n_{\tilde{y}})^{\wedge q}\| \ \rho(d\tilde{y})$$

One can then again show that $\tilde{x} \in \tilde{\Gamma}_\rho$ implies $\tilde{\mathcal{W}}^\lambda_{\tilde{x}} \subset \tilde{\Gamma}_\rho$ (use Corollary 6.2. with with arbitrarily large Q^*) .

7.4. *The case of non-vanishing characteristic exponents*

Let ρ be an $\tilde{f}$-invariant probability measure on $\tilde{M}$ such that the characteristic exponents $\mu^{(r)}$ are almost everywhere nonzero. We may assume that

$$\mu^{(r(Q+1))} < \lambda < 0 < \tilde{\lambda} < \mu^{(r(Q))}$$

(the argument $\tilde{x}$ or $\widetilde{\pi x}$ is omitted, $\mu^{(r(0))} = +\infty$ and $\mu^{(r(Q+1))}$ may be $-\infty$). We construct then local stable and unstable manifolds (in the strict sense) $\mathcal{V}^\lambda_x$ and $\tilde{\mathcal{V}}^{\tilde{\lambda}}_{\tilde{x}}$. From section 3.5. it follows that for ρ-almost all $\tilde{x}$, codim $\mathcal{V}^\lambda_{\widetilde{\pi x}} = \dim \tilde{\mathcal{V}}^{\tilde{\lambda}}_{\tilde{x}}$ and $\mathcal{V}^\lambda_{\widetilde{\pi x}}$, $\tilde{\mathcal{V}}^{\tilde{\lambda}}_{\tilde{x}}$ intersect transversally at $\widetilde{\pi x}$. Furthermore the "angle of intersection" of $\mathcal{V}^\lambda_{\tilde{\pi}\tilde{f}^k\tilde{x}}$ and $\tilde{\mathcal{V}}^{\tilde{\lambda}}_{\tilde{f}^k\tilde{x}}$ at $\tilde{\pi}\,\tilde{f}^k\tilde{x}$ cannot tend to zero exponentially when $k \to \pm\infty$ (see (3.16)).

If $\mu^{(1)} < 0$ almost everywhere with respect to the f-ergodic measure ρ then ρ is carried by an attracting periodic orbit. (The proof is the same as in finite dimension, see [14]).

7.5. *Flows*

The results discussed for iterates of a map f extend easily to a semi-flow $(f^t)_{t \geqq 0}$. To generalize the situation of Section 7.2., it suffices to add to (a) , (b), (c) the new condition

(d) $\|T_x f^t\|$ <u>is bounded uniformly for</u> $x \in \Lambda$, $t \in [0,1]$.

If (d) holds, the conditions of exponential decrease of $||Tf^t u||$, $d(f^t x, f^t y)$ or $d(x_t, y_t)$ hold for all real $t \geqq 0$ if and only if they hold for integer t . One is thus reduced to the study of a map. The case where f^t is defined only for $t \geqq T_0 > 0$ is dealt with similarly.

The more general situation of Section 7.1. can be treated in the same manner.

Appendix

A.1. *Theorem* (Subadditive ergodic theorem).

(M,Σ,ρ) denotes a probability space, and $f : M \longmapsto M$ a measurable map preserving ρ .

Let $(F_n)_{n>0}$ be a sequence of measurable functions $M \longmapsto \mathbb{R} \cup \{-\infty\}$ satisfying the conditions

(a) integrability : $F_1^+ \in L^1(M,\rho)$

(b) subadditivity : $F_{m+n} \leqq F_m + F_n \circ f^m$ a.e.

Then there exists a f-invariant measurable function $F : M \longmapsto \mathbb{R}$ $\{-\infty\}$ such that $F^+ \in L^1(M,\rho)$,

$$\lim_{n\to\infty} \frac{1}{n} F_n = F \qquad \text{a.e.} \tag{A.1}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int F_n(x)\, \rho(dx) = \inf \frac{1}{n} \int F_n(x)\, \rho(dx) = \int F(x)\, \rho(dx) \tag{A.2}$$

This is one version of Kingsman's theorem (see [7] Theorem 1.8). Kingman's subadditive ergodic theorem now has rather simple proofs (see for instance Derriennic [2]) .

A.2. *Corollary* : Keeping the notation and assumptions of Theorem A.1., let $G \geqq F$ be an f-invariant measurable finite-valued function. For every $\varepsilon > 0$ there is K_ε measurable finite-valued such that

$$F_{n-m} \circ f^m(x) \leqq (n-m)\, G(x) + n\varepsilon + K_\varepsilon(x)$$

for almost all x , and $m < n$.

It suffices to treat the case where $G^+ \in L^1(M,\rho)$. Define $G_n(x)$ $G_n(x) = \max \{F_n(x), n\, G(x)\}$. Theorem A.1. holds with (G_n) instead of (F_n) and G instead of F . It suffices therefore to prove the corollary with $F = G$ finite-valued.

Define $F_1'(x) = F_1(x)$ and, for $n \geqq 2$,

$$F_n'(x) = \max_{0<m<n} [F_m(x) + F_{n-m} \circ f^m(x)]$$

The sequence (F_n') again satisfies the conditions of Theorem A.1., hence

$$\lim_{n\to\infty} \frac{1}{n} F_n' = F'$$

exists almost everywhere. Since $F_n \leqq F_n'$, we have $F \leqq F'$. We shall prove that $F = F'$. To do this we may assume that f has a measurable inverse (this can be achieved by a canonical extension of the dynamical system (M,Σ,ρ,f)) . The functions $F_n \circ f^{-n}$ satisfy the conditions of Theorem A.1, with f replaced by f^{-1} , and it is clear that

$$\lim_{n\to\infty} \frac{1}{n} F_n \circ f^{-n} = F \tag{A.3}$$

(by (A.2) the left and the right hand side have the same integral on every invariant set).

Let $\alpha > 0$ and $\beta > 0$ be given. In view of (A.1), (A.3), there exist A, $\ell > 0$ and a set Δ such that $\rho(\Delta) > 1-\beta$, and $x \in \Delta$ implies

$$\left.\begin{array}{l} \frac{1}{m} F_m(x) \\[2ex] \frac{1}{m} F_m \circ f^{-m}(x) \end{array}\right\} \leqq F(x) + A \qquad \text{for all } m$$

$$\left.\begin{array}{l} \frac{1}{m} F_m(x) \\[2ex] \frac{1}{m} F_m \circ f^{-m}(x) \end{array}\right\} \leqq F(x) + \alpha \qquad \text{for } m \geqq \ell$$

Assume $n \geqq 2\ell$. We have

$$\frac{1}{n} F'_n(x) = \frac{1}{n} [F_m(x) + F_{n-m} \circ f^{-(n-m)}(f^n x)]$$

for some m depending on x , $0 < m < n$. Thus, for $x \in \Delta \cap f^{-n}\Delta$,

$$\frac{1}{n} F'_n(x) \leqq F(x) + \alpha + \frac{\ell}{n} A \qquad .$$

If $n > \frac{\ell A}{\alpha}$, this implies

$$\frac{1}{n} F'_n(x) < F(x) + 2\alpha$$

when x belongs to the set $\Delta \cap f^{-n}\Delta$, of measure $> 1-2\beta$. Therefore $F' \leqq F + 2\alpha$ on a set of measure $\geqq 1 - 2\beta$, and since α, β are arbitrary, $F' \leqq F$ almost everywhere :

$$\lim_{n\to\infty} \frac{1}{n} F'_n = F \qquad \text{a.e.}$$

Given $\varepsilon > 0$ we can thus find N measurable such that, for almost all x ,

$$\frac{1}{n} F_n(x) \geqq F(x) - \frac{\varepsilon}{2} \;, \quad \frac{1}{n} F'_n(x) \leqq F(x) + \frac{\varepsilon}{2}$$

if $n \geqq N(x)$. There is therefore K_ε measurable such that, for all n ,

$$F_n(x) \geqq n F(x) - n \frac{\varepsilon}{2} - \frac{1}{2} K_\varepsilon(x)$$

$$F'_n(x) \geqq n F(x) + n \frac{\varepsilon}{2} + \frac{1}{2} K_\varepsilon(x)$$

hence

$$F_{n-m} \circ f^m x \leqq F'_n(x) - F_m(x) \leqq (n-m) F(x) + n\varepsilon + K_\varepsilon(x)$$

for almost all x , proving the corollary.

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