ON FEIGENBAUM'S FUNCTIONAL EQUATION

$g \circ g(\lambda x) + \lambda g(x) = 0$

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Numerical studies by M. Feigenbaum have exhibited what appears to be a new codimension bifurcation for maps $f : [-1,1] \longmapsto [-1,1]$. Feigenbaum's heuristic approach (see [4],[5]) is in the process of being rigorized (see [3],[7],[1]) and extended to diffeomorphisms and flows in several dimensions (see [2],[6]) . We refer to [3] for a lucid introduction to the problem. We shall here be concerned only with Feigenbaum's first step, which was to solve the equation

$$\left.\begin{array}{l} g\circ g(\lambda x) + \lambda g(x) = 0 \\ g : [-1,1] \longmapsto [-1,1] \quad \text{even} \quad , \quad g(0) = 1 \end{array}\right\} \quad (1)$$

Feigenbaum showed numerically that there is $\lambda = 0{\cdot}39953528\ldots$ such that (1) has a solution g which behaves like $1\text{-const. } x^2$ at the origin. This has been made rigorous by O. Lanford [7] who found that (1) has an analytic solution. Lanford first guesses (numerically) a good approximation to g by a polynomial of order 40 . Then he proves by Newton's method that (1) has a solution close to the guessed approximation. This is simple and perfectly rigorous, but involves calculations beyond human ability (they are done by computer). In the present note a method for solving (1) is outlined, which does not involve superhuman calculations (although a small computer was used in fact to do them). The details are in [1] . The solution which we discuss is Feigenbaum's solution, shown in Figure 1. If numerical computations are to be trusted, Figure 2 presents another solution h behaving like $1\text{-const. } x^4$ at the origin. Figure 3 shows $x \longmapsto h(\sqrt{x})^2$ which is again a solution, but corresponding to negative λ .

We look for a solution g of (1) satisfying also

$$g \quad \text{smooth}^{*)} \quad \text{and} \quad g''(0) < 0 \tag{2}$$

Our basic idea is that the functional equation for f_2

$$f_2(x) = \varphi \circ f_2(\lambda x) \tag{3}$$

(where λ, φ are given) is relatively easy to analyze. [This equation just says that the graph of f_2 is invariant under $(x,y) \longmapsto (\lambda^{-1}x, \varphi(y))$].
We replace therefore (1), (2) by the problem

$$f_1 \circ f_2(\lambda x) + \lambda f_2(x) = 0 \tag{4}$$

$$f_1 = f_2 \tag{5}$$

$$f_2 : [-1,1] \longmapsto [-1,1] \text{ smooth, even, } f_2(0) = 1, \; f_2''(0) < 0 \tag{6}$$

The solvability of (4) with respect to f_2 (with $f_2(0) = 1$, $f_2''(0) \neq 0$) requires

$$f_1(1) + \lambda = 0 \tag{7}$$

$$\lambda f_1'(1) + 1 = 0 \tag{8}$$

Modulo (7) we may rewrite (4) as

$$f_1 \circ f_2(\lambda x) + \lambda f_2(x) = f_1(1) + \lambda \tag{4a}$$

(which is again of the form (3)) . We shall try to solve the system (4a), (5), (6) , adjust λ such that $f_1(1) + \lambda = 0$, and take $g = f_1 = f_2$.

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*) We shall later take $g(x)$ of class C^3 as a function of x^2 . There exist many C^1 solutions. In particular, the existence of a solution which behaves like $1\text{-const}|x|^{1+\varepsilon}$ is established in [3] for small ε , and suitable $\lambda(\varepsilon)$.

The condition $f_1(1) + \lambda = 0$ shows that λ is not arbitrary : our problem is a non linear eigenvalue problem. Let f_2 be a solution of (4a) for given f_1 , λ . Then $x \longmapsto f_2(kx)$ is again a solution. In view of (5), (8) we shall lift this ambiguity by choosing the solution f_2 such that $\lambda f_2'(1) + 1 = 0$.

Notice that (4a) determines $f_2(x)$ for x near 0 in terms of $f_1(y)$ for y near 1 *) . In view of these dissimilar roles of f_1 and f_2 it is convenient to introduce new variables. Let us write

$$F(x) = \lambda^{-1}[f_1(1-x) - f_1(1)]$$

$$f_2(x) = 1 - \psi(x^2)$$

Then, (4a), (5) become

$$\psi(t) = F\circ\psi(\lambda^2 t)$$

$$\left.\begin{array}{l} G(x) \equiv \lambda^{-1}[-\psi((1-x)^2) + \psi(1)] \\ F = G \end{array}\right\} \qquad (5b)$$

where it is assumed that $F(0) = 0$, $F'(0) = \lambda^{-2}$. One looks for a solution ψ of (4b) satisfying

$$2\lambda\psi'(1) = 1 \qquad (8b)$$

and imposes (5b) . If λ is such that

$$\psi(1) = 1 + \lambda \qquad (7b)$$

we have a solution of the original problem.

We may reformulate the problem as that of finding a fixed point

*) In particular one cannot hope to determine simply from (1) the coefficients of the power series expansion of g at the origin.

F of the map $\Phi_\lambda : F \longmapsto \Psi \longmapsto G$ where Ψ is defined by

$$\Psi(t) = F(\Psi(\lambda^2 t)) \;, \quad \Psi(0) = 0 \;, \quad \Psi'(0) = 1 \tag{9}$$

and

$$G(x) = \lambda^{-1} [\Psi(\alpha) - \Psi(\alpha(1-x)^2)] \tag{10}$$

where α is determined by

$$2\alpha\lambda\,\Psi'(\alpha) = 1$$

[in this notation $\psi(t) = \Psi(\alpha t)$] . Finally determine λ such that $\Psi(\alpha) = 1 + \lambda$.

From (9) and the assumed smoothness one gets formulae such as

$$\Psi'(t) = \prod_{n=1}^{\infty} (\lambda^2 F'(\Psi(\lambda^{2n} t)))$$

$$\frac{\Psi''(t)}{\Psi'(t)} = \sum_{n=1}^{\infty} \lambda^{2n} \Psi'(\lambda^{2n} t) \cdot \frac{F''(\Psi(\lambda^{2n} t))}{F'(\Psi(\lambda^{2n} t))}$$

$$(S\Psi)(t) = \sum_{n=1}^{\infty} \lambda^{4n} [\Psi'(\lambda^{2n} t)]^2 (SF)(\Psi(\lambda^{2n} t))$$

where $Sf = (f''/f')' - \frac{1}{2}(f''/f')^2$ is the Schwarzian derivative. These formulae give a good control on Ψ . Notice that these formulae require the knowledge of F only on the range of $t \longmapsto \Psi(\lambda^2 t)$, $t \in [0,\alpha]$. For the purpose of finding fixed points of Φ_λ , it will thus be possible to consider functions F on $[0,A]$ with A smaller than 1 .

The strategy will now be the following. We choose an interval

J of values of λ and for each $\lambda \in J$ define a nonempty set M_λ of functions F on some interval $[0,A]$ such that $\Phi_\lambda M_\lambda \subset M_\lambda$ and Φ_λ is a contraction on M_λ with respect to some metric d. The map Φ_λ has thus a unique fixed point F_λ in the closure of M_λ. Uniqueness implies continuity of $\lambda \longmapsto F_\lambda$ and thus of $\lambda \longmapsto \Psi(\alpha) - 1 - \lambda$. Finally one checks that $\Psi(\alpha) - 1 - \lambda$ has different signs at both ends of the interval J. Therefore there is at least one $\lambda \in J$ for which $\Psi(\alpha) = 1 + \lambda$, and this yields a solution of our original problem (1). A priori, F_λ is only in the closure of M_λ, there may thus be an annoying loss of differentiability. A little miracle occurs however which saves the situation : M_λ contains analytic functions, and Φ_λ is analyticity improving. The fixed point F_λ is thus real analytic, and the same is true of the solution g of (1).

Implementing the details of the above program is real work (see [1]), and involves in particular numerical computations. Here we give only general indication. The interval J is chosen as $[\sqrt{.152}, \sqrt{.165}]$. Then A is chosen as a function of λ (piecewise constant and $\leqq .261$). The set M_λ is convex and defined in terms of a set M'_λ such that $F \in M_\lambda \Leftrightarrow \frac{F''}{F'} \in M'_\lambda$ (notice that if $s = \frac{F''}{F'}$, then $F(x) = \int_0^x dy\, \lambda^{-2} \exp \int_0^y s(z)dz$). The convex set M'_λ consists of the C^1 functions on $[0,A]$ such that

$$\left.\begin{aligned} &\frac{1}{1-x} - \ell_1(1-x) - \ell_3(1-x)^3 \leqq -s(x) \leqq \frac{1}{1-x} - c_1(1-x) - c_3(1-x)^3 \\ &s'(x) + s(x)^2 \leqq 0 \end{aligned}\right\} \tag{11}$$

$$-s'(x) \leqq L \tag{12}$$

where ℓ_1, ℓ_3, c_1, c_3, L are given as piecewise constant functions of λ, $0 \leqq c_1 \leqq \ell_1$, $0 \leqq c_3 \leqq \ell_3$, $\ell_1 + \ell_3 < 1$. It turns out that if $F \in M_\lambda$, then G''/G' satisfies (11) on $[0,1]$ (not just $[0,A]$). In particular,

$G''/G' \leqq 0$ and $G'''/G' \leqq 0$. Since $G'(0) = \lambda^{-2}$, we have $G' \geqq 0$, $G'' \leqq 0$, $G''' \leqq 0$ on $[0,1]$. The metric d on M_λ is given by the following norm on M'_λ :

$$||s|| = \sup_{0 \leqq x \leqq A} |(1-x)^{-1} s(x)| .$$

As to the analiticity improving character of Φ_λ , one shows that if $F \in M_\lambda$ and

$$\left|\frac{1}{n!} \left(\frac{d}{dx}\right)^n F(x)\right| \leqq \lambda^{-2} B^{n-1} \qquad \text{for } x \in [0,1] \text{ , } n \geqq 1$$

with $B \geqq 1.8$, then

$$\left|\frac{1}{n!} \left(\frac{d}{dx}\right)^n F(x)\right| \leqq \lambda^{-2} \tilde{B}^{n-1} \qquad \text{for } x \in [0,1] \text{ , } n \geqq 1$$

with $\tilde{B} < B$.

Theorem : There is at least one number $\lambda \in [\sqrt{.152},\sqrt{.165}]$ for which the functional equation

$$g \circ g(\lambda x) + \lambda g(x) = 0 \quad , \quad g(0) = 1$$

has an even smooth solution on $[-1,1]$. The solution found has the following further properties

$$g''(0) < 0$$

$$g(1) + \lambda = 0 \text{ , } \lambda g'(1) + 1 = 0$$

$$g'(x) \leqq 0 \text{ , } g''(x) \leqq 0 \text{ , } g'''(x) \geqq 0 \qquad \text{on } [0,1]$$

$$\left|\frac{1}{n!} \left(\frac{d}{dx}\right)^n g(x)\right| \leqq \lambda^{-1} (1.8)^{n-1} \qquad \text{for } x \in [-1,1] \text{ , } n \geqq 1 .$$

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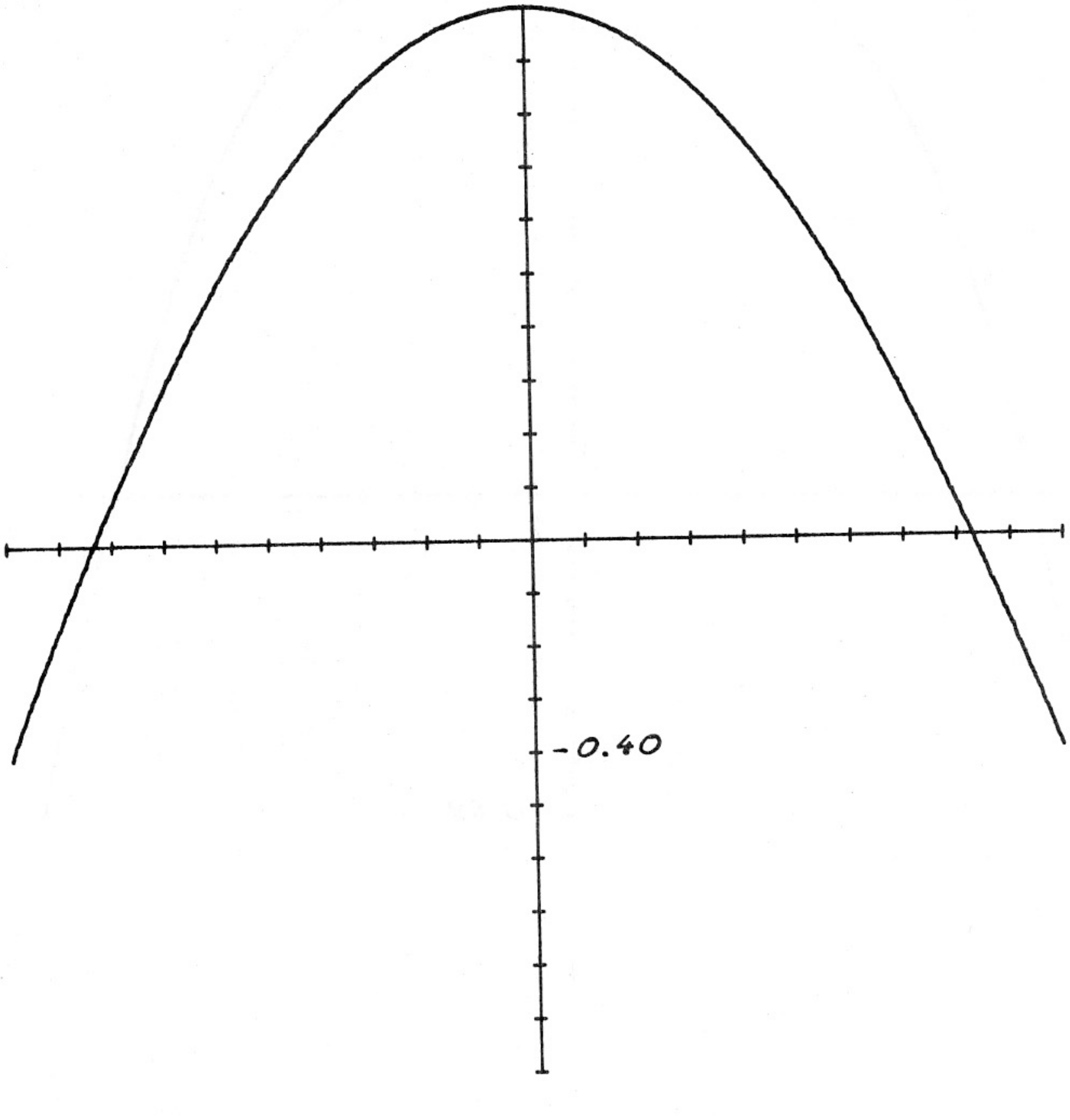

Figure 1

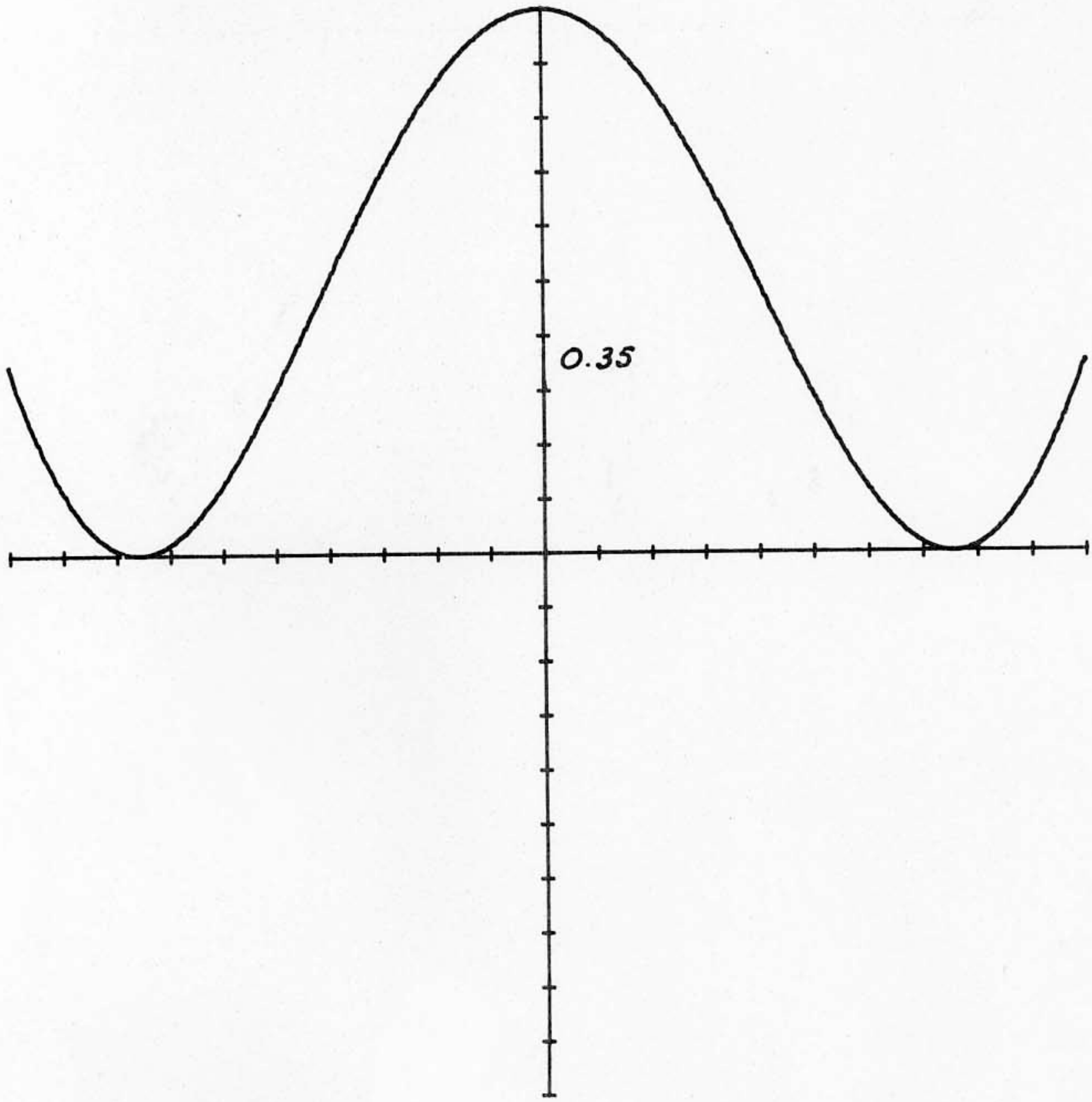

Figure 3